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Jorge Almeida, Antonio Cano, Ondrej Klíma, Jean-Eric Pin

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J. ALMEIDA, A. CANO, O. KLÍMA, AND J.-É. PIN

Dedicated to S.W. Margolis for his 60th birthday

Abstract. Lower subsets of an ordered semigroup form in a natural way an ordered semigroup. This lower set operator gives an analogue of the power operator already studied in semigroup theory. We present a complete description of the lower set operator applied to varieties of ordered semigroups. We also obtain large families of fixed points for this operator applied to pseudovarieties of ordered semigroups, including all examples found in the literature. This is achieved by constructing six types of inequalities that are preserved by the lower set operator. These types of inequalities are shown to be independent in a certain sense. Several applications are also presented, including the preservation of the period for a pseudovariety of ordered semigroups whose image under the lower set operator is proper.

Contents

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1 Introduction

This paper is a contribution to the following problem: *What are the fixed points of the lower set operator on pseudovarieties of ordered semigroups?* Before stating our results, a brief historical survey is relevant to understand the meaning, the origin and the scope of this problem.

1.1 Historical background

Power semigroups form an interesting research topic on its own, but over the past thirty-five years, their study has been strongly motivated by applications to formal languages. The significance of power semigroups in automata theory can be explained in many ways. For instance, it is known [14, 18] that if a language L is recognized by a semigroup S and if φ is a length-preserving homomorphism, then $\varphi(L)$ is recognized by $\mathcal{P}(S)$. Power semigroups are also used in the algebraic study of transductions [4].

In 1976, Eilenberg stated his variety theorem, and soon after that, the connection between power semigroups and languages was generalised to the variety level. For this, the power operator was extended to an operator P on pseudovarieties of semigroups and the corresponding operation on varieties of languages was identified and thoroughly studied. See [1, 2] for an overview of the known results.

Ordered semigroups came into the picture in 1995, with the extension of Eilenberg's theorem proposed by the fourth author [9]: pseudovarieties of ordered semigroups correspond to positive varieties of languages. For an ordered semigroup, the natural power structure is obtained by considering the set of its lower subsets. This structure was defined independently by

Polák [13] and Cano and Pin [8]. It gives an analogue of the power operator on pseudovarieties, the so-called *lower set (power) operator*. This operator, denoted $\mathsf{P}^{\downarrow}$ in this paper, also has a natural counterpart for positive varieties of languages [13, 8].

1.2 Main questions on power operators

There are two main questions regarding power operators on pseudovarieties: describe the power pseudovarieties and identify the fixed points.

The first problem, which subsumes the second one, is still out of reach, even for pseudovarieties of monoids, although much progress has been done [2]. For instance the action of the power operator on finite groups is well understood, is the topic of a very active research area for finite $\mathcal{J}$-trivial monoids [12] and remains an open problem for bands.

The second problem is completely solved for monoids. There is a bijective correspondence between the proper fixed points of the power operator and the commutative pseudovarieties of groups. More precisely, given a pseudovariety of groups H, the pseudovariety of all commutative monoids whose subgroups belong to H is a fixed point of P and all proper fixed points are of this form. The problem is more difficult for semigroups, but is also completely solved: the proper fixed points of the power operator are necessarily permutative pseudovarieties and a complete description of them is known. See [1, Theorems 11.5.8 and 11.6.8] and [2, Theorems 5.6 and 5.7] for more details. A consequence of these results is the existence of a unique maximal proper fixed point of the power operator, both in the monoid and in the semigroup case.

What about ordered semigroups? First of all, the operator $\mathsf{P}^{\downarrow}$ also has a unique maximal proper fixed point. This is proved for pseudovarieties of ordered monoids in [8] but the result can be easily adapted to semigroups as well. This maximal fixed point plays an important role in the theory of regular languages, in particular in the study of the shuffle product [8]. Furthermore, it is shown in [6] that the operator $\mathsf{P}^{\downarrow}$ is idempotent. It follows that the first and the second problems are equivalent for $\mathsf{P}^{\downarrow}$, since a pseudovariety of the form $\mathsf{P}^{\downarrow}\mathsf{V}$ is necessarily a fixed point of $\mathsf{P}^{\downarrow}$. Moreover, several fixed points were identified in [7].

1.3 Our main results

We now come to the results of this paper. We first consider power operators applied to varieties. By [1, Theorem 11.3.7], linear identities that are satisfied by a variety of semigroups characterize its power variety. It follows that power varieties can always be defined by linear identities. Our Theorem 5.4 extends this result to varieties of ordered semigroups by substituting semilinear inequalities for linear identities.

Our main contributions concern the fixed points of the operator $\mathsf{P}^{\downarrow}$ on pseudovarieties of ordered semigroups. We describe six basic types of such fixed points, from which many more may be constructed since the set of fixed points is closed under intersection. We actually conjecture that all fixed points of $\mathsf{P}^{\downarrow}$ can be obtained in this way, which would solve the two

problems mentioned above. In any case, our results seem to cover all known fixed points. We also show that $\mathsf{P}^{\downarrow}$ has $2^{\aleph_0}$ fixed points.

Furthermore, we establish the independence of our six types, by providing examples of fixed points of a given type that cannot be obtained as an intersection of fixed points of the five other types.

As an application of these results, we establish that the semigroups of a given profinite period of a proper fixed point of $\mathsf{P}^{\downarrow}$ also form a fixed point. We also give a refined basis of inequalities for the pseudovariety $\mathsf{V}_{3/2}$.

2 Preliminaries

2.1 Finite semigroups

The following well-known combinatorial result [1, Proposition 5.4.1] will be useful in this paper.

Lemma 2.1. *Let S be a finite semigroup with p elements and let $s_1, \ldots, s_p$ be elements of S. Then there exist indices i, j such that $1 \leqslant i < j \leqslant p$ and $s_1 \cdots s_p = s_1 \cdots s_{i-1}(s_i \cdots s_{j-1})^k s_j \cdots s_p$ for every $k \geqslant 1$.*

Let u be a word. The *content* of u is the set, denoted $c(u)$, of all letters occurring in u. For each letter a, we denote by $|u|_a$ the number of occurrences of a in u. Thus, for $A = \{a, b\}$ and $u = abaab$, one has $|u|_a = 3$ and $|u|_b = 2$. The sum

$$|u| = \sum_{a \in A} |u|_a$$

is the *length* of u.

2.2 Ordered sets

Let $(S, \leqslant)$ be a partially ordered set. A subset L of S is a *lower set* if for all $s, t \in L$, the conditions $s \in L$ and $t \leqslant s$ imply $t \in L$. Given an element s of S, the set

$$\downarrow s = \{t \in S \mid t \leqslant s\}$$

is a lower set, called the *lower set generated by* s. More generally, if X is a subset of S, *the lower set generated by* X is the set

$$\downarrow X = \bigcup_{s \in X} \downarrow s = \{t \in S \mid \text{there exists } s \in X \text{ such that } t \leqslant s\}.$$

A partial function $\varphi : S \to T$ between two partially ordered sets is said to be *(forward) order-preserving* if, for all $s_1, s_2 \in \mathrm{Dom}(\varphi)$, $s_1 \leqslant s_2$ implies $\varphi(s_1) \leqslant \varphi(s_2)$. An injective function φ is said to be *backward order-preserving* if the partial function φ^{-1} is order-preserving. A mapping that is both forward and backward order-preserving is also called an *order-embedding*.

2.3 Ordered semigroups

An *ordered semigroup* is a semigroup endowed with a stable partial order. Let S and T be two ordered semigroups. A *homomorphism of ordered semigroups* is a function $\varphi : S \to T$ between two such structures that preserves both the semigroup operation and the order, in the forward direction. By an ordered subsemigroup of an ordered semigroup S we mean a subsemigroup

T of S ordered by the induced order; equivalently, the inclusion mapping $T \hookrightarrow S$ is a semigroup homomorphism that is also an order-embedding. The product of ordered semigroups is defined as the structure of the same kind on their Cartesian product whose multiplication and order are determined component-wise.

We say that S *divides* T if S is a quotient, meaning a homomorphic image, of a subsemigroup of T.

2.4 Syntactic ordered semigroup

An important example of ordered semigroup is the *ordered syntactic semigroup* of a language. This notion was first introduced by Schützenberger in 1956 [15], but he apparently only made use of the syntactic semigroup later on. The notion was rediscovered in [9], where the opposite order was used. We are back to Schützenberger's original order in this paper.

Let L be a language of A^+. The *syntactic quasiorder* (or *preorder*) of L is the relation $\preccurlyeq_L$ defined on A^+ by $u \preccurlyeq_L v$ if and only if, for every $x, y \in A^*$,

$$xuy \in L \Rightarrow xvy \in L. \tag{1}$$

The *syntactic congruence* of L is the associated equivalence relation $\sim_L$, defined by $u \sim_L v$ if and only if $u \preccurlyeq_L v$ and $v \preccurlyeq_L u$.

The syntactic quasiorder may also be described in terms of contexts as follows. The *context* of a word $u \in A^+$ with respect to a language L is the set $C(u)$ of all pairs (x, y) of elements of A^* such that $xuy \in L$. Then, the relation $u \preccurlyeq_L v$ is equivalent to $C(u) \subseteq C(v)$.

The *syntactic semigroup* of L is the quotient semigroup $\mathrm{Synt}(L)$ of A^+ by $\sim_L$ and the natural homomorphism $\eta : A^+ \to \mathrm{Synt}(L)$ is called the *syntactic homomorphism* of L. The syntactic quasiorder $\preccurlyeq_L$ induces an order $\leqslant_L$ on $\mathrm{Synt}(L)$. The resulting ordered semigroup is called the *syntactic ordered semigroup* of L. By Kleene's theorem, a language is regular if and only if its syntactic semigroup is finite.

2.5 Profinite completions

We refer the reader to [1, 3, 20] for detailed information on profinite completions and we just recall here a few useful facts. Let d be the profinite metric on the free monoid A^*. The completion of the metric space (A^*, d) is denoted by $\widehat{A^*}$ and its elements are called *profinite words*. Moreover, the product on A^* is uniformly continuous and hence has a unique continuous extension to $\widehat{A^*}$. It follows that $\widehat{A^*}$ is a monoid, called the *free profinite monoid* on A.

Let $\mathbb{N}$ be the semiring of nonnegative integers. We denote by $\widehat{\mathbb{N}}$ the profinite completion of the semiring $\mathbb{N}$ and by $\widehat{\mathbb{N}}_+$ the set $\widehat{\mathbb{N}} \setminus \{0\}$. The elements of $\widehat{\mathbb{N}}$ are called *profinite natural numbers* and those of $\widehat{\mathbb{N}}_+$ are called *positive profinite natural numbers*.

The *content of a profinite word* can be defined by extending by continuity the content homomorphism c from the metric space (A^*, d) to the discrete semigroup $(\mathcal{P}(A), \cup)$.

In the same way, the length map and the letter frequency mappings defined in Section 2.1 are uniformly continuous functions from A^* to $\mathbb{N}$ with

unique continuous extensions from $\widehat{A^*}$ to $\widehat{\mathbb{N}}$. For each profinite word x, the map $n \to x^n$ from $\mathbb{N}$ to $\widehat{A^*}$ also has a unique continuous extension from $\widehat{\mathbb{N}}$ to $\widehat{A^*}$. It follows that if α is a profinite natural number, x^α is well-defined.

Observe that $\widehat{\mathbb{N}}_+$ has exactly one additive idempotent, namely ω, but has many multiplicative idempotents, such as 1, ω, $\omega+1$ or n^ω for every $n \in \mathbb{N}_+$. Furthermore, $\omega + 1$ is the multiplicative identity of $\widehat{\mathbb{N}} \setminus \mathbb{N}_+$, that is, for all $\alpha \in \widehat{\mathbb{N}} \setminus \mathbb{N}_+$

$$\alpha(\omega + 1) = (\omega + 1)\alpha = \alpha. \tag{2}$$

The set $\widehat{\mathbb{N}} \setminus \mathbb{N}$ constitutes an ideal of $\widehat{\mathbb{N}}$ that is actually a ring, for which ω is the additive identity. Thus, for every $\alpha \in \widehat{\mathbb{N}}$, $\omega + \alpha$ has an additive inverse in $\widehat{\mathbb{N}} \setminus \mathbb{N}$, which is denoted $\omega - \alpha$. More generally, if $\alpha \in \widehat{\mathbb{N}} \setminus \mathbb{N}$ and $\beta \in \widehat{\mathbb{N}}$, then we let $\alpha - \beta = \alpha + (\omega - \beta)$. Note that, for all $\alpha \in \widehat{\mathbb{N}} \setminus \mathbb{N}$, $\beta \in \widehat{\mathbb{N}}$, and $\gamma \in \widehat{\mathbb{N}}_+$, the equality $\gamma(\alpha - \beta) = \gamma\alpha - \gamma\beta$ holds.

3 Varieties and inequalities

In this section we briefly review the notions of varieties and pseudovarieties and their characterizations by inequalities.

3.1 Varieties and pseudovarieties

Given a class $\mathcal{C}$ of ordered semigroups, we denote by $\mathrm{H}\mathcal{C}$ the class of all homomorphic images of members of $\mathcal{C}$, by $\mathrm{S}\mathcal{C}$ the class of all ordered subsemigroups of members of $\mathcal{C}$, and by $\mathrm{P}\mathcal{C}$ (respectively $\mathrm{P}_{\mathrm{fin}}\mathcal{C}$) the class of all (respectively finite) direct products of members of $\mathcal{C}$. These operators are idempotent and (pointwise) satisfy the following relations:

$$\mathrm{SH} \subseteq \mathrm{HS}, \quad \mathrm{PH} \subseteq \mathrm{HP}, \quad \mathrm{PS} \subseteq \mathrm{SP}, \quad \mathrm{P}_{\mathrm{fin}}\mathrm{H} \subseteq \mathrm{HP}_{\mathrm{fin}}, \quad \mathrm{P}_{\mathrm{fin}}\mathrm{S} \subseteq \mathrm{SP}_{\mathrm{fin}}. \tag{3}$$

A *variety of ordered semigroups* is a nonempty class of such structures that is closed under the operators H, S, and P. A *pseudovariety of ordered semigroups* is a nonempty class of finite such structures that is closed under the operators H, S, and $\mathrm{P}_{\mathrm{fin}}$. It follows from the relations (3) that the pseudovariety $\langle\mathcal{C}\rangle$ generated by a given class $\mathcal{C}$ of finite ordered semigroups is the class $\mathrm{HSP}_{\mathrm{fin}}\mathcal{C}$.

It is convenient to use different fonts to distinguish between varieties and pseudovarieties. Our convention in this paper is to use $\mathcal{V}$ for varieties and V for pseudovarieties.

3.2 Birkhoff's and Reiterman's theorems

Birkhoff's theorem states that a class of semigroups is a variety if and only if it can be defined by a set of identities. Formally, an *identity* is an equality between two words of the free semigroup on a countable alphabet A. A semigroup S *satisfies the identity* $u = v$ if and only if $\varphi(u) = \varphi(v)$ for every homomorphism $\varphi : A^+ \to S$.

Similarly, Reiterman's theorem states that a class of finite semigroups is a pseudovariety if and only if it can be defined by a set of *pseudoidentities*, which are formal equalities between profinite words.

It was proved by Bloom [5] that a precise analogue of Birkhoff's theorem holds for varieties of ordered semigroups, if one considers identities of the

form $u \leqslant v$. Similarly, a precise analogue of Reiterman's theorem holds for pseudovarieties of ordered semigroups [10].

Given a set Σ of inequalities between words, we denote by $[\Sigma]$ the variety defined by the set of inequalities Σ. Similarly, if Σ is a set of inequalities between profinite words, we denote by $[\![\Sigma]\!]$ the pseudovariety defined by this set of inequalities.

3.3 Ordered versus unordered semigroups

Every semigroup may be considered as an ordered semigroup for the equality ordering.

Let U be a pseudovariety of semigroups and consider the pseudovariety V generated by all semigroups from U, ordered by equality. Note that, forgetting the order, all semigroups from V belong to U. In fact, V consists precisely of all semigroups from U, endowed with arbitrary stable partial orders. There is hence no essential harm in using the same notation for U and V, a convention that has become standard in the literature.

The preceding convention is coherent with interpreting a definition of a pseudovariety by pseudoidentities. Indeed, for ordered semigroups, the pseudoidentity $u = v$ may be regarded as an abbreviation of the pair of inequalities $u \leqslant v$ and $v \leqslant u$. Thus for a set Σ of pseudoidentities, one may interpret $[\![\Sigma]\!]$ in two ways, namely as a pseudovariety U of semigroups and as a pseudovariety V of ordered semigroups, with the above convention. It is immediate to observe that V is precisely the pseudovariety associated to U in the preceding paragraph.

3.4 Inequalities on a one-letter alphabet

We now establish a few elementary results on inequalities of profinite words on a one-letter alphabet. We first show that the inequalities of the form $x \leqslant x^\alpha$ are not independent.

Lemma 3.1. *Let α and β be two positive profinite natural numbers such that $\alpha\alpha = \alpha$, $\alpha\beta = \alpha$ and $\beta + \omega = \beta$. Then the pseudovariety $[\![x \leqslant x^\alpha]\!]$ is contained in $[\![x^\alpha = x^\beta]\!]$ and therefore also in $[\![x \leqslant x^\beta]\!]$.*

Proof. We exclude the trivial case $\alpha = \beta = 1$. Then $\alpha \in \widehat{\mathbb{N}} \setminus \mathbb{N}$, from which the equality $\alpha(\alpha - \beta) = \alpha - \alpha = \omega$ follows.

Substituting in $x \leqslant x^\alpha$ the variable x respectively by x^β and $x^{\alpha-\beta}$, we obtain the inequalities $x^\beta \leqslant x^{\alpha\beta} = x^\alpha$ and $x^{\alpha-\beta} \leqslant x^{\alpha(\alpha-\beta)} = x^\omega$. Multiplying both sides of the latter inequality by x^β yields $x^\alpha \leqslant x^\beta$. Hence $[\![x \leqslant x^\alpha]\!] \subseteq [\![x^\alpha = x^\beta]\!]$ and the other inclusion follows trivially. $\square$

Example 3.1. Let $\alpha = 2^\omega$ and $\beta = \omega + 1$. Then α and β satisfy the conditions $\alpha\alpha = \alpha$, $\alpha\beta = \alpha$ and $\beta + \omega = \beta$ and thus by Lemma 3.1, $[\![x \leqslant x^{2^\omega}]\!]$ is contained in $[\![x^{2^\omega} = x^{\omega+1}]\!]$ and in $[\![x \leqslant x^{\omega+1}]\!]$.

Our second result deals with inequalities of the forms $x^\omega \leqslant x^{\omega+\alpha}$ and $x^\omega \leqslant x^{\omega-\alpha}$.

Lemma 3.2. *Let α be a profinite natural number. Then the inequalities $x^\omega \leqslant x^{\omega+\alpha}$ and $x^\omega \leqslant x^{\omega-\alpha}$, as well as their duals, are all equivalent. In particular, the equalities $[\![x^\omega \leqslant x^{\omega+\alpha}]\!] = [\![x^\omega = x^{\omega+\alpha}]\!] = [\![x^{\omega+\alpha} \leqslant x^\omega]\!]$ hold.*

Proof. Each inequality $x^\omega \leqslant x^{\omega+\alpha}$ and $x^\omega \leqslant x^{\omega-\alpha}$ is obtained from the other by substituting x by $x^{\omega-1}$. The same holds for their duals. On the other hand multiplying both sides of the inequality $x^\omega \leqslant x^{\omega+\alpha}$ by $x^{\omega-\alpha}$ leads to the inequality $x^{\omega-\alpha} \leqslant x^\omega$ and hence to the equivalent inequality $x^{\omega+\alpha} \leqslant x^\omega$. □

4 Power operators

4.1 Lower set semigroups

Let $(S, \leqslant)$ be an ordered semigroup and let $\mathcal{P}^{\downarrow}(S)$ be the set of all nonempty lower sets of S.[1] Let us define the *product of two lower sets* X and Y as the lower set

$$XY = \{z \in S \mid \text{there exist } x \in X \text{ and } y \in Y \text{ such that } z \leqslant xy\}.$$

This operation makes $\mathcal{P}^{\downarrow}(S)$ a semigroup. Furthermore, set inclusion is compatible with this product and thus $(\mathcal{P}^{\downarrow}(S), \subseteq)$ is an ordered semigroup, called the *lower set semigroup* of S.

Note that, for an ordered semigroup $(S, \leqslant)$, the mapping $\alpha : S \to \mathcal{P}^{\downarrow}(S)$ given by the rule $\alpha(s) = {\downarrow}s$ is both a semigroup embedding and an order-embedding.

Given a mapping $\varphi : S \to T$ between two ordered semigroups, let us define a mapping $\varphi^{\downarrow} : \mathcal{P}^{\downarrow}(S) \to \mathcal{P}^{\downarrow}(T)$ by setting $\varphi^{\downarrow}(X) = {\downarrow}\varphi(X)$.

Proposition 4.1. *Let $\varphi : S \to T$ be a mapping between two ordered semigroups.*

(1) *The mapping $\varphi^{\downarrow}$ is order-preserving.*
(2) *If φ is order-preserving and surjective then $\varphi^{\downarrow}$ is surjective.*
(3) *If φ is backward order-preserving, then so is $\varphi^{\downarrow}$.*
(4) *If φ is an order embedding, then so is $\varphi^{\downarrow}$.*
(5) *If φ is an order-preserving homomorphism, then so is $\varphi^{\downarrow}$.*

Proof. Throughout the proof, X and Y denote arbitrary elements of $\mathcal{P}^{\downarrow}(S)$.

(1) If $X \subseteq Y$, then $\varphi(X) \subseteq \varphi(Y)$ and so

$$\varphi^{\downarrow}(X) = {\downarrow}\varphi(X) \subseteq {\downarrow}\varphi(Y) = \varphi^{\downarrow}(Y).$$

Thus $\varphi^{\downarrow}$ is order-preserving.

(2) Suppose that φ is order-preserving and surjective and let $Z \in \mathcal{P}^{\downarrow}(T)$. Then $\varphi^{-1}(Z)$ belongs to $\mathcal{P}^{\downarrow}(S)$ and the sequence of equalities

$$Z = \varphi(\varphi^{-1}(Z)) = {\downarrow}\varphi(\varphi^{-1}(Z)) = \varphi^{\downarrow}(\varphi^{-1}(Z))$$

show that $\varphi^{\downarrow}$ is also surjective.

(3) Suppose that φ is backward order-preserving. If $\varphi^{\downarrow}(X) \subseteq \varphi^{\downarrow}(Y)$ and $x \in X$, then $\varphi(x)$ belongs to ${\downarrow}\varphi(X)$ and therefore also to ${\downarrow}\varphi(Y)$. Hence there exists $y \in Y$ such that $\varphi(x) \leqslant \varphi(y)$ and consequently $x \leqslant y$. Since Y is a lower set, it follows that $x \in Y$, which shows that $X \subseteq Y$, and so $\varphi^{\downarrow}$ is also backward order-preserving.

Statement (4) is immediate in view of (1) and (3).

[1] This is the construction introduced in [13] and the dual of that considered in [8].

(5) For the purpose of this proof let us write UV for the product in $\mathcal{P}(S)$ and $U \circ V$ for the product in $\mathcal{P}^{\downarrow}(S)$. Thus,

$$UV = \{uv \mid u \in U \text{ and } v \in V\}$$
$$U \circ V = \{s \in S \mid \text{there exist } u \in U \text{ and } v \in V \text{ such that } s \leqslant uv\}.$$

Note that, for all $U, V \in \mathcal{P}(S)$, we have $\downarrow(UV) = \downarrow U \circ \downarrow V$.

If $\varphi : S \to T$ is a homomorphism, then $\varphi(XY) = \varphi(X)\varphi(Y)$. Furthermore, if φ is order-preserving, then $\downarrow\varphi(X) = \downarrow\varphi(\downarrow X)$.

Using the above remarks and the definition of $\circ$, we deduce that, if φ is an order-preserving homomorphism, then the following equalities hold:

$$\downarrow\varphi(X \circ Y) = \downarrow\varphi(X) \circ \downarrow\varphi(Y).$$

Consequently, $\varphi^{\downarrow}$ is an order-preserving homomorphism. □

Remark If φ is a non-order-preserving homomorphism, $\downarrow\varphi$ need not be a homomorphism. For instance, let S and T be the three-element monogenic aperiodic semigroup $\{a, a^2, 0\}$, with S ordered by $0 < a^2$ and T by equality, and let φ be the identity mapping. Then, for $X = Y = \{a\}$, the equality $\downarrow\varphi(X \circ Y) = \downarrow\varphi(X) \circ \downarrow\varphi(Y)$ fails.

Corollary 4.2. *The following properties hold:*

(1) *If S is a subsemigroup of T, then $\mathcal{P}^{\downarrow}(S)$ is a subsemigroup of $\mathcal{P}^{\downarrow}(T)$.*
(2) *If S is a quotient of T, then $\mathcal{P}^{\downarrow}(S)$ is a quotient of $\mathcal{P}^{\downarrow}(T)$.*
(3) *If S divides T, then $\mathcal{P}^{\downarrow}(S)$ divides $\mathcal{P}^{\downarrow}(T)$.*
(4) *If $(S_i)_{i\in I}$ is a family of semigroups, then the product $\prod_{i\in I} \mathcal{P}^{\downarrow}(S_i)$ is a subsemigroup of $\mathcal{P}^{\downarrow}(\prod_{i\in I} S_i)$.*

Proof. Properties (1), (2) and (3) are consequences of Proposition 4.1. For (4), it suffices to observe that the natural mapping from $\prod_{i\in I} \mathcal{P}^{\downarrow}(S_i)$ into $\mathcal{P}^{\downarrow}(\prod_{i\in I} S_i)$ is an order and semigroup embedding. □

4.2 Power operators applied to varieties and pseudovarieties

Given a variety $\mathcal{V}$ of ordered semigroups, we denote by $\mathcal{P}^{\downarrow}\mathcal{V}$ the variety of ordered semigroups generated by the class of all ordered semigroups $\mathcal{P}^{\downarrow}(S)$ such that $S \in \mathcal{V}$. For a pseudovariety V of ordered semigroups, the pseudovariety $\mathsf{P}^{\downarrow}\mathsf{V}$ is defined similarly.

Proposition 4.3. *If $\mathcal{C}$ is a class of finite ordered semigroups, then*

$$\mathsf{P}^{\downarrow}\langle\mathcal{C}\rangle = \mathrm{HS}\{\mathcal{P}^{\downarrow}(S_1 \times \cdots \times S_n) \mid S_1, \ldots, S_n \in \mathcal{C}\}. \tag{4}$$

Furthermore, if $\mathcal{C}$ is closed under finite products, then

$$\mathsf{P}^{\downarrow}\langle\mathcal{C}\rangle = \mathrm{HS}\{\mathcal{P}^{\downarrow}(S) \mid S \in \mathcal{C}\}.$$

Proof. Let V denote the right hand side of (4). The inclusion $\mathsf{V} \subseteq \mathsf{P}^{\downarrow}\langle\mathcal{C}\rangle$ follows immediately from the definition of pseudovariety.

For the opposite inclusion, first recall that

$$\mathsf{P}^{\downarrow}\langle\mathcal{C}\rangle = \mathrm{HSP}_{\mathrm{fin}}\{\mathcal{P}^{\downarrow}(S) \mid S \in \langle\mathcal{C}\rangle\}.$$

If $S \in \langle \mathcal{C} \rangle$, then S divides an ordered semigroup $T \in \mathrm{P}_{\mathrm{fin}}\mathcal{C}$ and by Corollary 4.2 (3), $\mathcal{P}^{\downarrow}(S)$ divides $\mathcal{P}^{\downarrow}(T)$. It follows that

$$\{\mathcal{P}^{\downarrow}(S) \mid S \in \langle \mathcal{C} \rangle\} \subseteq \mathrm{HS}\{\mathcal{P}^{\downarrow}(T) \mid T \in \mathrm{P}_{\mathrm{fin}}\mathcal{C}\}$$

and by (3) we get

$$\mathsf{P}^{\downarrow}\langle \mathcal{C} \rangle \subseteq \mathrm{HSP}_{\mathrm{fin}}\{\mathcal{P}^{\downarrow}(T) \mid T \in \mathrm{P}_{\mathrm{fin}}\mathcal{C}\}.$$

Now we have by Corollary 4.2 (4)

$$\begin{aligned}\mathrm{P}_{\mathrm{fin}}\{\mathcal{P}^{\downarrow}(T) \mid T \in \mathrm{P}_{\mathrm{fin}}\mathcal{C}\} &\subseteq \{\mathcal{P}^{\downarrow}(T_1 \times \cdots \times T_m) \mid T_1, \ldots, T_m \in \mathrm{P}_{\mathrm{fin}}\mathcal{C}\} \\ &= \{\mathcal{P}^{\downarrow}(S_1 \times \cdots \times S_n) \mid S_1, \ldots, S_n \in \mathcal{C}\}\end{aligned}$$

which gives the inclusion $\mathsf{P}^{\downarrow}\langle \mathcal{C} \rangle \subseteq \mathsf{V}$.

The second part of the proposition follows immediately. □

4.3 Semigroups versus monoids

Unlike most of the literature on the ordered version of pseudovarieties, this paper deals not with monoids but with semigroups. The purpose of this section is to clarify the relationships between the two theories.

First of all, the definitions and arguments of Section 2 concerning ordered semigroups, their pseudovarieties, and the operator $\mathsf{P}^{\downarrow}$ can be adapted by simply adding the additional requirement that semigroups have an identity element and that it is preserved by homomorphisms. In general, we use the same notation for the operator $\mathsf{P}^{\downarrow}$ on pseudovarieties of both semigroups and monoids.

For a pseudovariety V of ordered semigroups, we let V_{OM} denote the class of all ordered monoids that, as ordered semigroups, are members of V. Note that V_{OM} is a pseudovariety of ordered monoids. For a pseudovariety V of ordered monoids, we denote by V_{OS} the pseudovariety of ordered semigroups that is generated by V. In order to avoid confusion, we will distinguish here the semigroup and monoid versions of the operators H, S and $\mathsf{P}^{\downarrow}$ by denoting them, respectively, by H_S, S_S, $\mathsf{P}^{\downarrow}_S$ and H_M, S_M, $\mathsf{P}^{\downarrow}_M$.

Proposition 4.4. *For each pseudovariety* V *of ordered monoids, the relation* $(\mathsf{V}_{OS})_{OM} = \mathsf{V}$ *holds.*

Proof. The result follows from the fact that every ordered monoid in $\mathrm{H}_S S_S \mathsf{V}$ is also in V. To establish this fact, consider an ordered subsemigroup S of a monoid T from V and let f be an onto order-preserving homomorphism from S onto a monoid M. Let e be an idempotent of S such that $f(e) = 1$. Since eSe is a subsemigroup of S and $f(eSe) = M$, we may assume that $S = eSe$.

If e is the identity element of T, then we are done. Otherwise, consider the ordered submonoid $R = eSe \cup \{1\}$ of T and extend f to R by mapping 1 to 1. We need to verify that any comparability relations between elements of S and 1 are preserved. For example, if $s \in S$ is such that $s \leqslant 1$, then $s = ese \leqslant e$ and so $f(s) \leqslant f(e) = f(1)$. A relation of the form $1 \leqslant s$ is handled similarly. The above now shows that M is the image of R under an order-preserving homomorphism, whence $M \in \mathsf{V}$. □

The following proposition is the ordered analogue of [1, Exercise 11.10.1].

Proposition 4.5. *Let* V *be a pseudovariety of ordered monoids. Then the following equalities hold:*

(1) $(\mathsf{P}_M^{\downarrow}\mathsf{V})_{OS} = \mathsf{P}_S^{\downarrow}(\mathsf{V}_{OS})$;

(2) $\mathsf{P}_M^{\downarrow}\mathsf{V} = \big(\mathsf{P}_S^{\downarrow}(\mathsf{V}_{OS})\big)_{OM}$.

Proof. (1) The inclusion

$$(\mathsf{P}_M^{\downarrow}\mathsf{V})_{OS} \subseteq \mathsf{P}_S^{\downarrow}(\mathsf{V}_{OS})$$

follows from the trivial relation

$$\{\mathcal{P}^{\downarrow}(M) \mid M \in \mathsf{V}\} \subseteq \{\mathcal{P}^{\downarrow}(S) \mid S \in \mathsf{V}_{OS}\}.$$

The opposite inclusion follows from Proposition 4.3, since we have:

$$\begin{aligned}\mathsf{P}_S^{\downarrow}(\mathsf{V}_{OS}) = \mathsf{P}_S^{\downarrow}\langle\mathsf{V}\rangle &= \mathrm{H}_S\mathrm{S}_S\{\mathcal{P}^{\downarrow}(M) \mid M \in \mathsf{V}\}\\ &\subseteq \mathrm{H}_S\mathrm{S}_S\mathsf{P}_M^{\downarrow}\mathsf{V} = (\mathsf{P}_M^{\downarrow}\mathsf{V})_{OS}.\end{aligned}$$

(2) It follows from Proposition 4.4 that $\mathsf{P}_M^{\downarrow}\mathsf{V} = \big((\mathsf{P}_M^{\downarrow}\mathsf{V})_{OS}\big)_{OM}$. Thus by (1) we get $\mathsf{P}_M^{\downarrow}\mathsf{V} = \big((\mathsf{P}_M^{\downarrow}\mathsf{V})_{OS}\big)_{OM} = \big(\mathsf{P}_S^{\downarrow}(\mathsf{V}_{OS})\big)_{OM}$. □

An important consequence of Proposition 4.5 is that the fixed points of $\mathsf{P}_M^{\downarrow}$ and $\mathsf{P}_S^{\downarrow}$ are closely related. Recall that a pseudovariety V of ordered semigroups is said to be *monoidal* if it is generated by its own ordered monoids, or equivalently if $\mathsf{V} = (\mathsf{V}_{OM})_{OS}$.

Proposition 4.6. *The fixed points of* $\mathsf{P}_M^{\downarrow}$ *and* $\mathsf{P}_S^{\downarrow}$ *are related as follows:*

(1) *If* V *is a fixed point of* $\mathsf{P}_M^{\downarrow}$*, then* V_{OS} *is a fixed point of* $\mathsf{P}_S^{\downarrow}$.

(2) *If* V *is a fixed point of* $\mathsf{P}_S^{\downarrow}$*, then* V_{OM} *is a fixed point of* $\mathsf{P}_M^{\downarrow}$.

(3) *Every fixed point of* $\mathsf{P}_M^{\downarrow}$ *is of the form* V_{OM}*, where* V *is monoidal and a fixed point of* $\mathsf{P}_S^{\downarrow}$.

Proof. (1) is a trivial consequence of Proposition 4.5 (1).

(2) Let $M \in \mathsf{V}_{OM}$. Then $M \in \mathsf{V}$ and since V is a fixed point of $\mathsf{P}_S^{\downarrow}$, we also have $\mathcal{P}^{\downarrow}(M) \in \mathsf{V}$. Finally, $\mathcal{P}^{\downarrow}(M) \in \mathsf{V}_{OM}$ since $\mathcal{P}^{\downarrow}(M)$ is a monoid.

(3) If V be a fixed point of $\mathsf{P}_M^{\downarrow}$, then V_{OS} is a fixed point of $\mathsf{P}_S^{\downarrow}$ by (1) and $\mathsf{V} = (\mathsf{V}_{OS})_{OM}$ by Proposition 4.4. It follows that $\mathsf{V}_{OS} = ((\mathsf{V}_{OS})_{OM})_{OS}$ and thus V_{OS} is monoidal. □

Proposition 4.6 shows that the study of fixed points of $\mathsf{P}_S^{\downarrow}$ subsumes that of $\mathsf{P}_M^{\downarrow}$. Moreover, mutatis mutandis, similar results hold for varieties of ordered semigroups versus varieties of ordered monoids.

4.4 Bases of inequalities

Another aspect that may lead to confusion when dealing with pseudovarieties of semigroups or monoids involves their definition through inequalities. When Σ is a set of inequalities, it is ambiguous whether $[\![\Sigma]\!]$ represents a pseudovariety of semigroups or monoids. This ambiguity should be resolved from the context in which such an expression appears. In most cases, it is harmless. Indeed, note that a basis of inequalities for a pseudovariety V of ordered semigroups is also a basis of inequalities for V_{OM}. We proceed to

show that a basis of inequalities for a pseudovariety V of ordered monoids can be automatically transformed into a basis of inequalities for V_{OS}.

Let Σ be a set of monoid inequalities. We define a set Σ' of semigroup inequalities as follows:

(i) let Γ consist of all monoid inequalities that can be obtained from those from Σ by substituting 1 for some of the variables;
(ii) let Δ consist of all monoid inequalities of the form $u \leqslant w$ such that $u \leqslant 1$ and $1 \leqslant v$ belong to $\Sigma \cup \Gamma$, $u \neq 1$ and $v \neq 1$, and w is obtained from v by mapping each variable in $c(v) \cap c(u)$ to a new variable and fixing all other variables;
(iii) the set Σ' is obtained by replacing in $\Sigma \cup \Gamma \cup \Delta$ all inequalities of the form $u \leqslant 1$ by the inequalities $ux \leqslant x$ and $xu \leqslant x$, and all inequalities of the form $1 \leqslant u$ by the inequalities $x \leqslant ux$ and $x \leqslant xu$, where, in both cases, x is a variable such that $x \notin c(u)$.

Proposition 4.7. *If Σ is a set of monoid inequalities, then $[\![\Sigma]\!]_{OS} = [\![\Sigma']\!]$.*

Proof. Let S be a finite ordered semigroup. We should show that the condition that S divides a finite ordered monoid that satisfies Σ is necessary and sufficient for S to satisfy Σ'. That the condition is sufficient follows from the fact that, if a finite ordered monoid satisfies Σ then it also satisfies Σ'. For necessity, suppose that S satisfies Σ'. If S is a monoid, then S satisfies Σ, and we are done, since S divides itself. Otherwise, consider the monoid S^1 that is obtained from S by adding an identity element, denoted 1. We claim that one may suitably extend the partial order of S to S^1 so that S^1 satisfies Σ.

For each inequality of the form $u \leqslant 1$ from $\Sigma \cup \Gamma$, set $s \leqslant 1$ whenever $s \in S$ is such that there is some $s' \in S$ that can be obtained by evaluating u in S and that satisfies $s \leqslant s'$. Dually, for each inequality of the form $1 \leqslant u$ from $\Sigma \cup \Gamma$, set $1 \leqslant s$ whenever $s \in S$ is such that there is some $s' \in S$ that can be obtained by evaluating u in S and that satisfies $s' \leqslant s$. This extends the partial order of S to S^1 in such a way that S^1 clearly satisfies Σ, provided we show that S^1 is an ordered monoid.

Indeed, if $s < 1$ and $t \in S$, then there is in $\Sigma \cup \Gamma$ an inequality of the form $u \leqslant 1$ such that u may be evaluated to s' in S with $s \leqslant s'$. Hence the inequality $ux \leqslant x$ belongs to Σ' and so S satisfies it. By extending the evaluation of the variables that maps u to s' by evaluating x to t, we deduce that $st \leqslant s't \leqslant t$. All remaining stability cases can be verified similarly.

We next check that the extended order is transitive. For cases of the form $s \leqslant t \leqslant 1$, this is achieved automatically by the choice of when the relation $s \leqslant 1$ holds. The cases of the form $1 \leqslant s \leqslant t$ are similar. For the remaining cases, it suffices to observe that, if $s \leqslant 1 \leqslant t$ in S^1, then $s \leqslant t$ in S. Indeed, there exist inequalities $u \leqslant 1$ and $1 \leqslant v$ in $\Sigma \cup \Gamma$ and evaluations of u and v respectively to some $s', t' \in S$ such that $s \leqslant s'$ and $t' \leqslant t$. By the definition of the set Δ, it contains an inequality of the form $u \leqslant w$, where $c(u) \cap c(w) = \emptyset$ and w is obtained from v by renaming the variables. Hence, we may evaluate, simultaneously, u and w to respectively s' and t'. Since S satisfies the inequality $u \leqslant w$, it follows that $s \leqslant s' \leqslant t' \leqslant t$ in S.

Finally, we show that the extended order is anti-symmetric. The only cases to be considered are those of the forms $s \leqslant 1 \leqslant s$, with $s \in S$. The

argument in the preceding paragraph shows that there is an inequality $u \leqslant w$ in Δ such that u and w may be simultaneously evaluated to elements s' and s'', respectively, such that $s \leqslant s' \leqslant s'' \leqslant s$. Hence, $s = s' = s''$. Moreover, by the definition of Δ, there is a profinite word v that is obtained from w by renaming the variables such that the inequalities $u \leqslant 1$ and $1 \leqslant v$ are both in $\Sigma \cup \Gamma$. Hence, the inequalities $ux \leqslant x$ and $x \leqslant vx$ are both in Σ', which implies that $st \leqslant t \leqslant st$ for every $t \in S$, since $S \models \Sigma'$, so that s is a left identity in S. Similarly, s is a right identity in S and, therefore, S has an identity element, in contradiction with our assumption that S is not a monoid. Hence the inequalities $s \leqslant 1$ and $1 \leqslant s$ cannot simultaneously hold with $s \in S$. □

Hence, for a set Σ of monoid inequalities, we have the following equalities of pseudovarieties of ordered monoids:

$$[\![\Sigma']\!] = [\![\Sigma']\!]_{OM} = \big([\![\Sigma]\!]_{OS}\big)_{OM} = [\![\Sigma]\!].$$

In this chain of equalities, the first $[\![\Sigma']\!]$ denotes the pseudovariety of ordered monoids defined by Σ' and the second is its semigroup counterpart, while the two $[\![\Sigma]\!]$ stand for the pseudovariety of ordered monoids defined by Σ.

5 Inequalities defining $\mathcal{P}^{\downarrow}\mathcal{V}$

Let $\mathcal{V}$ be a variety of ordered semigroups. The main result of this section, Theorem 5.4, provides a set of inequalities defining $\mathcal{P}^{\downarrow}\mathcal{V}$ in terms of inequalities satisfied by $\mathcal{V}$. This result and its proof are analogous to [1, Theorem 11.3.7] and rely on two propositions of independent interest. The first one, Proposition 5.1, gives a sufficient condition for an inequality satisfied by $\mathcal{V}$ to be also satisfied by $\mathcal{P}^{\downarrow}\mathcal{V}$. The second one, Proposition 5.3, shows that each inequality satisfied by $\mathcal{P}^{\downarrow}\mathcal{V}$ is a consequence of some inequality satisfied by $\mathcal{V}$.

A word u is *linear* if every letter occurs at most once in the word u. For instance *acedb* is a linear word, but *abcad* is not. An inequality $u \leqslant v$ is *semilinear* if the word u is linear.

Proposition 5.1. *Every semilinear inequality satisfied by $\mathcal{V}$ is also satisfied by $\mathcal{P}^{\downarrow}\mathcal{V}$.*

Proof. Suppose that $\mathcal{V}$ satisfies the semilinear inequality

$$x_1 \cdots x_n \leqslant x_{\tau(1)} \cdots x_{\tau(m)} \tag{5}$$

where τ is a map from $\{1, \ldots, m\}$ into $\{1, \ldots, r\}$, with $r \geqslant n$. Let $S \in \mathcal{V}$ and let $A_1, \ldots, A_r$ be arbitrary nonempty lower subsets of S. We claim that $A_1 \cdots A_n \subseteq A_{\tau(1)} \cdots A_{\tau(m)}$. Indeed, let $s \in A_1 \cdots A_n$. Then by definition, $s \leqslant a_1 \cdots a_n$ for some $a_1 \in A_1, \ldots, a_n \in A_n$. Since S satisfies (5), we may extend the choice of the $a_i \in A_i$, with $i = 1, \ldots, n$ to $i = 1, \ldots, r$ such that $a_1 \cdots a_n \leqslant a_{\tau(1)} \cdots a_{\tau(m)}$ and thus $s \in A_{\tau(1)} \cdots A_{\tau(m)}$, which proves the claim. It follows that $\mathcal{P}^{\downarrow}\mathcal{V}$ satisfies (5). □

Corollary 5.2. *The equality $\mathcal{P}^{\downarrow}[u \leqslant v] = [u \leqslant v]$ holds for each semilinear inequality $u \leqslant v$.*

Proof. The inclusion $\mathcal{P}^{\downarrow}[u \leqslant v] \subseteq [u \leqslant v]$ follows from Proposition 5.1 and the opposite inclusion is trivial. $\square$

An inequality $u \leqslant v$ *infers* another inequality $u' \leqslant v'$ if there is a semigroup homomorphism $\varphi : c(uv)^+ \to c(u'v')^+$ such that $\varphi(u) = u'$ and $\varphi(v) = v'$. If φ can be chosen to be length preserving, then $u \leqslant v$ is said to *elementarily infer* $u' \leqslant v'$.

For instance, the inequality $x_1x_2x_3x_4x_5x_6 \leqslant x_3y_1x_2x_6x_5x_2x_1y_2$ elementarily infers the inequality $x_1x_2x_1x_2x_1x_3 \leqslant x_1y_1x_2x_3x_1x_2x_1y_2$. To see this, it suffices to take the homomorphism φ defined by $\varphi(x_1) = x_1$, $\varphi(x_2) = x_2$, $\varphi(x_3) = x_1$, $\varphi(x_4) = x_2$, $\varphi(x_5) = x_1$, $\varphi(x_6) = x_3$, $\varphi(y_1) = y_1$ and $\varphi(y_2) = y_2$.

Proposition 5.3. *Every inequality satisfied by $\mathcal{P}^{\downarrow}\mathcal{V}$ is elementarily inferred by a semilinear inequality satisfied by $\mathcal{V}$.*

Proof. Suppose that $\mathcal{P}^{\downarrow}\mathcal{V}$ satisfies the inequality $u \leqslant v$ with

$$u = x_{\sigma(1)} \cdots x_{\sigma(p)} \text{ and } v = x_{\tau(1)} \cdots x_{\tau(q)} \tag{6}$$

where σ and τ are partial transformations of the set $\{1, \ldots, n\}$. Let $X = \{x_1, \ldots, x_n\}$. We equip the free semigroup X^+ with the stable quasiorder $\preccurlyeq_{\mathcal{V}}$ defined by $s \preccurlyeq_{\mathcal{V}} t$ if $\mathcal{V}$ satisfies the inequality $s \leqslant t$. Let $\sim_{\mathcal{V}}$ denote the associated equivalence relation, which is therefore a congruence on X^+. In the quotient $X^+/{\sim_{\mathcal{V}}}$, the quasiorder becomes a stable partial order, denoted by $\leqslant_{\mathcal{V}}$. The resulting ordered semigroup $(X^+/{\sim_{\mathcal{V}}}, \leqslant_{\mathcal{V}})$ belongs to $\mathcal{V}$. We denote the $\sim_{\mathcal{V}}$-class of a word w by $\overline{w}$.

For each $i \in \{1, \ldots, p\}$, we define the subset A_i of $X^+/{\sim_{\mathcal{V}}}$ by setting

$$A_i = \{\bar{x}_j \mid \sigma(j) = \sigma(i)\}$$

and consider the homomorphism $\varphi : X^+ \to \mathcal{P}^{\downarrow}(X^+/{\sim_{\mathcal{V}}})$ defined by

$$\varphi(x_k) = \begin{cases} \downarrow A_i & \text{if } k = \sigma(i) \text{ for some } i \\ \downarrow \bar{x}_k & \text{otherwise.} \end{cases}$$

We have $\bar{x}_1 \cdots \bar{x}_p \in \downarrow A_1 \cdots \downarrow A_p = \varphi(u)$. Since $\mathcal{P}^{\downarrow}(X^+/{\sim_{\mathcal{V}}}) \in \mathcal{P}^{\downarrow}\mathcal{V}$ and $\mathcal{P}^{\downarrow}\mathcal{V}$ satisfies $u \leqslant v$ we have $\bar{x}_1 \cdots \bar{x}_p \in \varphi(v)$. From the definition of multiplication of lower subsets, the set $\varphi(v)$ is the lower set generated by the product $\varphi(x_{\tau(1)}) \cdots \varphi(x_{\tau(q)})$. It follows that $\varphi(v)$ is the lower set generated by a product $B_1 \cdots B_q$ where

$$B_j = \begin{cases} A_i & \text{if } \tau(j) = \sigma(i) \text{ for some } i \\ \bar{x}_{\tau(j)} & \text{otherwise.} \end{cases}$$

Thus, every element of $\varphi(v)$, in particular $\bar{x}_1 \cdots \bar{x}_p$, is $\leqslant_{\mathcal{V}}$-dominated by some element of the form $\bar{x}_{\rho(1)} \cdots \bar{x}_{\rho(q)}$ where each $\bar{x}_{\rho(j)} \in B_j$, which translates to the conditions

$$\begin{cases} \bar{x}_{\rho(j)} \in A_i & \text{whenever } \tau(j) = \sigma(i) \text{ for some } i \\ \rho(j) = \tau(j) & \text{otherwise.} \end{cases} \tag{7}$$

The conclusion is therefore that

$$\bar{x}_1 \cdots \bar{x}_p \leqslant_{\mathcal{V}} \bar{x}_{\rho(1)} \cdots \bar{x}_{\rho(q)} \tag{8}$$

for some function $\rho : \{1, \ldots, q\} \to \{1, \ldots, n\}$ satisfying

$$\tau(j) = \sigma(i) \implies \sigma(\rho(j)) = \tau(j). \tag{9}$$

By the definition of the partial order $\leqslant_{\mathcal{V}}$, (8) is equivalent to the relation

$$x_1 \cdots x_p \preccurlyeq_{\mathcal{V}} x_{\rho(1)} \cdots x_{\rho(q)} \tag{10}$$

which shows that $\mathcal{V}$ satisfies the semilinear inequality

$$x_1 \cdots x_p \leqslant x_{\rho(1)} \cdots x_{\rho(q)}.$$

Let $z = x_{\rho(1)} \cdots x_{\rho(q)}$ and let ψ be the homomorphism that maps each variable x_i to $x_{\sigma(i)}$ for i in $\{1, \ldots, p\}$ and fixes every other variable in X. Then $\psi(x_1 \cdots x_p) = u$ and by (7) and (9), we also have $\psi(z) = v$.

By (10), the variety $\mathcal{V}$ satisfies the inequality $x_1 \cdots x_p \leqslant z$ and by Proposition 5.1, so does $\mathcal{P}^{\downarrow}\mathcal{V}$. Since the inequality $x_1 \cdots x_p \leqslant z$ elementarily infers the original inequality $u \leqslant v$, the proof is complete. □

Putting Proposition 5.1 and 5.3 together, we get the following result.

Theorem 5.4. *The variety $\mathcal{P}^{\downarrow}\mathcal{V}$ is defined by the semilinear inequalities satisfied by $\mathcal{V}$.*

6 Fixpoints of $\mathsf{P}^{\downarrow}$

Several fixed points of $\mathsf{P}^{\downarrow}$ have already been discovered. For instance, it was proved in [7] that the following pseudovarieties of monoids are fixed points of $\mathsf{P}^{\downarrow}$: $[\![xy = yx, 1 \leqslant x]\!]$, $[\![x^\omega y = yx^\omega, x \leqslant x^2]\!]$, $[\![x^\omega y = yx^\omega, 1 \leqslant x]\!]$, $[\![x^\omega y^\omega = y^\omega x^\omega, 1 \leqslant x]\!]$.

In this section, we present several infinite families of fixed points of the operator $\mathsf{P}^{\downarrow}$. We start with a general observation concerning fixed points which we will use without further reference from hereon.

Recall that a partially ordered set $(I, \leqslant)$ is *directed* if for any $i, j \in I$, there exists $k \in I$ such that $i \leqslant k$ and $j \leqslant k$. We also say that a family of pseudovarieties $(\mathsf{V}_i)_{i \in I}$ is *directed* if $(I, \leqslant)$ is a directed partially ordered set such that $i \leqslant j$ implies $\mathsf{V}_i \subseteq \mathsf{V}_j$.

Proposition 6.1. *Every intersection and every directed union of fixed points of $\mathsf{P}^{\downarrow}$ is also a fixed point for $\mathsf{P}^{\downarrow}$.*

Proof. Let $(\mathsf{V}_i)_{i \in I}$ be a family of fixed points for $\mathsf{P}^{\downarrow}$ and let $\mathsf{V} = \bigcap_{i \in I} \mathsf{V}_i$. Since the operator $\mathsf{P}^{\downarrow}$ preserves inclusion and it is expansive, in the sense that $\mathsf{U} \subseteq \mathsf{P}^{\downarrow}\mathsf{U}$ for every pseudovariety U, we deduce that

$$\mathsf{P}^{\downarrow}\mathsf{V} \subseteq \bigcap_{i \in I} \mathsf{P}^{\downarrow}\mathsf{V}_i = \bigcap_{i \in I} \mathsf{V}_i = \mathsf{V} \subseteq \mathsf{P}^{\downarrow}\mathsf{V},$$

which shows that $\mathsf{P}^{\downarrow}\mathsf{V} = \mathsf{V}$.

Let $(\mathsf{V}_i)_{i \in I}$ be a directed family of fixed points of $\mathsf{P}^{\downarrow}$. Let $\mathsf{V} = \bigcup_{i \in I} \mathsf{V}_i$. It is easy to deduce that V is a pseudovariety. Moreover, Proposition 4.3 shows that

$$\mathsf{P}^{\downarrow}\mathsf{V} = \mathrm{HS}\{\mathcal{P}^{\downarrow}(S) \mid S \in \mathsf{V}\}.$$

As $(\mathsf{V}_i)_{i \in I}$ is a directed family, there exists $i \in I$ such that $S \in \mathsf{V}_i$. Since V_i is a fixed point of $\mathsf{P}^{\downarrow}$, the ordered semigroup $\mathcal{P}^{\downarrow}(S)$ also belongs to V_i and hence to V. It follows that V is a fixed point of $\mathsf{P}^{\downarrow}$. □

6.1 Semilinear inequalities

By extension, we say that an inequality $u \leqslant v$ of profinite words is *semilinear* if u is a finite linear word. The following is an application of Corollary 5.2.

Theorem 6.2. *Let $u \leqslant v$ be a semilinear inequality of profinite words. Then the pseudovariety $[\![u \leqslant v]\!]$ is a fixed point for $\mathsf{P}^{\downarrow}$.*

Proof. If v is a finite word, then $\mathcal{P}^{\downarrow}[u \leqslant v] = [u \leqslant v]$ by Corollary 5.2. Furthermore, since $\mathsf{P}^{\downarrow}[\![u \leqslant v]\!] \subseteq \mathcal{P}^{\downarrow}[u \leqslant v]$, we get $\mathsf{P}^{\downarrow}[\![u \leqslant v]\!] \subseteq [u \leqslant v]$ and thus $\mathsf{P}^{\downarrow}[\![u \leqslant v]\!] \subseteq [\![u \leqslant v]\!]$, which proves the theorem in this case.

Suppose now that v is a profinite word and let S be a semigroup of the pseudovariety $[\![u \leqslant v]\!]$. Let w be a finite word such that $\mathcal{P}^{\downarrow}(S)$ satisfies the pseudoidentity $v = w$ (whence so does S). Then S satisfies the inequality $u \leqslant w$ and since $\mathsf{P}^{\downarrow}[\![u \leqslant w]\!] = [\![u \leqslant w]\!]$ by the preceding paragraph, $\mathcal{P}^{\downarrow}(S)$ satisfies $u \leqslant w$ and hence also $u \leqslant v$. It follows that $[\![u \leqslant v]\!]$ is a fixed point for $\mathsf{P}^{\downarrow}$. □

Corollary 6.3. *For each profinite word v, the pseudovariety $[\![x \leqslant v]\!]$ is a fixed point for $\mathsf{P}^{\downarrow}$.*

Given a profinite word u, an r-tuple of distinct variables $\boldsymbol{y} = (y_1, \ldots, y_r)$, and an r-tuple $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_r)$ of positive profinite natural numbers, we denote by $(u)_{\boldsymbol{y}}^{\boldsymbol{\alpha}}$ the profinite word that is obtained from u by replacing each variable y_i by $y_i^{\alpha_i}$. Note that, if each α_i is a multiplicative idempotent, then the equality $\big((u)_{\boldsymbol{y}}^{\boldsymbol{\alpha}}\big)_{\boldsymbol{y}}^{\boldsymbol{\alpha}} = (u)_{\boldsymbol{y}}^{\boldsymbol{\alpha}}$ holds in $\widehat{X^+}$ since it holds when u is a variable. For an inequality $u \leqslant v$, we may also write $(u \leqslant v)_{\boldsymbol{y}}^{\boldsymbol{\alpha}}$ instead of $(u)_{\boldsymbol{y}}^{\boldsymbol{\alpha}} \leqslant (v)_{\boldsymbol{y}}^{\boldsymbol{\alpha}}$. For a set Σ of inequalities, we denote by $(\Sigma)_{\boldsymbol{y}}^{\boldsymbol{\alpha}}$ the set of all inequalities of the form $(u \leqslant v)_{\boldsymbol{y}}^{\boldsymbol{\alpha}}$ with $u \leqslant v$ in Σ.

If α is a positive profinite natural number and V is a set of variables, we denote by $(u)_V^{\alpha}$ the profinite word that is obtained from u by replacing each variable z in V by z^{α}. We adopt a similar notation for inequalities and sets of inequalities. We further drop the exponent α in $(u)_V^{\alpha}$ if $\alpha = \omega$.

Lemma 6.4. *If $u \leqslant v$ is a semilinear inequality, $\boldsymbol{y} = (y_1, \ldots, y_r)$ is an r-tuple of distinct variables and $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_r)$ is an r-tuple of multiplicative idempotents of $\widehat{\mathbb{N}}_+$, then*

$$[\![(u \leqslant v)_{\boldsymbol{y}}^{\boldsymbol{\alpha}},\ x \leqslant x^{\alpha_i}\ (1 \leqslant i \leqslant r)]\!] = [\![u \leqslant (v)_{\boldsymbol{y}}^{\boldsymbol{\alpha}},\ x \leqslant x^{\alpha_i}\ (1 \leqslant i \leqslant r)]\!].$$

Proof. From $x \leqslant x^{\alpha_i}$ $(1 \leqslant i \leqslant r)$, we deduce that $u \leqslant (u)_{\boldsymbol{y}}^{\boldsymbol{\alpha}}$, which yields the inclusion from left to right. For the opposite inclusion, it suffices to apply the substitution $(\_)_{\boldsymbol{y}}^{\boldsymbol{\alpha}}$ to the inequality $u \leqslant (v)_{\boldsymbol{y}}^{\boldsymbol{\alpha}}$ to obtain $(u)_{\boldsymbol{y}}^{\boldsymbol{\alpha}} \leqslant \big((v)_{\boldsymbol{y}}^{\boldsymbol{\alpha}}\big)_{\boldsymbol{y}}^{\boldsymbol{\alpha}} = (v)_{\boldsymbol{y}}^{\boldsymbol{\alpha}}$. □

Combining Theorem 6.2 and Lemma 6.4, we obtain the following result.

Corollary 6.5. *Let Σ be a set of semilinear inequalities, $\boldsymbol{y} = (y_1, \ldots, y_r)$ an r-tuple of distinct variables and $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_r)$ an r-tuple of multiplicative idempotents of $\widehat{\mathbb{N}}_+$. Then the pseudovariety $[\![(\Sigma)_{\boldsymbol{y}}^{\boldsymbol{\alpha}}]\!] \cap [\![x \leqslant x^{\alpha_i} \mid 1 \leqslant i \leqslant r]\!]$ is a fixed point of the operator $\mathsf{P}^{\downarrow}$.* □

Example 6.1.

(1) Applying Corollary 6.5 with $\alpha = (\omega, \omega)$, we conclude that the pseudovariety $[\![x^\omega y^\omega = y^\omega x^\omega,\ x \leqslant x^\omega]\!]$ is a fixed point of $\mathsf{P}^{\downarrow}$.

(2) In the same way, taking $\alpha = (\omega + 1, 1)$ shows that the pseudovariety $[\![x^{\omega+1} y = y x^{\omega+1},\ x \leqslant x^{\omega+1}]\!]$ is a fixed point of $\mathsf{P}^{\downarrow}$.

(3) Let us give a final example to illustrate how our previous results may be combined. Taking $\alpha = (\omega + 1, 1, 2^\omega)$, we get by Equation (2), Corollary 6.5 and Lemma 3.1 that the pseudovariety $\mathsf{V} = [\![x^{\omega+1} y z^{2^\omega} = z^{2^\omega} y x^{\omega+1},\ x \leqslant x^{2^\omega}]\!]$ is a fixed point of $\mathsf{P}^{\downarrow}$. Alternatively, one could take $\alpha = (\omega + 1, 1, \omega + 1)$ to conclude that $[\![x^{\omega+1} y z^{\omega+1} = z^{\omega+1} y x^{\omega+1},\ x \leqslant x^{\omega+1}]\!]$ is a fixed point of $\mathsf{P}^{\downarrow}$. Furthermore, Corollary 6.3 shows that $[\![x \leqslant x^{2^\omega}]\!]$ is also a fixed point of $\mathsf{P}^{\downarrow}$. The intersection of these two fixed points, which is equal to V according to Example 3.1, is also a fixed point.

Proposition 6.6. *Let $u \leqslant v$ be a semilinear inequality and let V be a set of variables. Then the following properties hold:*

(1) *if the first letter of u does not belong to V, then $\mathsf{P}^{\downarrow}[\![(u \leqslant v)^\omega_V, xy \leqslant x]\!]$ satisfies the inequality $(u \leqslant v)^\omega_V$;*

(2) *if the last letter of u does not belong to V, then $\mathsf{P}^{\downarrow}[\![(u \leqslant v)^\omega_V, yx \leqslant x]\!]$ satisfies the inequality $(u \leqslant v)^\omega_V$;*

(3) *the pseudovariety $\mathsf{P}^{\downarrow}[\![(u \leqslant v)^\omega_V, x^\omega y \leqslant x^\omega, y x^\omega \leqslant x^\omega]\!]$ satisfies the inequality $(u \leqslant v)^\omega_V$.*

Proof. In each case, we need to show that, given an ordered semigroup S satisfying $(u \leqslant v)^\omega_V$ and some additional inequality or inequalities, $\mathcal{P}^{\downarrow}(S)$ also satisfies $(u \leqslant v)^\omega_V$. For this purpose, we consider an arbitrary continuous homomorphism $\varphi : \widehat{X^+} \to \mathcal{P}^{\downarrow}(S)$, where X is such that $u, v \in \widehat{X^+}$, and we show that $\varphi\big((u)^\omega_V\big) \subseteq \varphi\big((v)^\omega_V\big)$. Let $u = x_1 \cdots x_n$ and for each $i \in \{1, \ldots, n\}$, let s_i be an element of $\varphi\big((x_i)^\omega_V\big)$.

If $x_i \in V$, then $\varphi\big((x_i)^\omega_V\big) = \varphi(x_i^\omega)$ is an idempotent. In particular, for such an index i, a relation of the form $s_i \leqslant s_{i,1} \cdots s_{i,m}$ holds for some $s_{i,j} \in \varphi(x_i^\omega)$, where $m = |S|$. By Lemma 2.1, it follows that there are indices $k_i < \ell_i$ such that

$$s_i \leqslant s_{i,1} \cdots s_{i,k_i-1} (s_{i,k_i} \cdots s_{i,\ell_i-1})^\omega s_{i,\ell_i} \cdots s_{i,m}.$$

We let $s'_i = (s_{i,k_i} \cdots s_{i,\ell_i-1})^\omega$ in such a case and we put $s'_i = s_i$ in case $x_i \notin V$. Note that s'_i also belongs to $\varphi\big((x_i)^\omega_V\big)$ for $i = 1, \ldots, n$ but with the additional property that s'_i is idempotent whenever $x_i \in V$.

Let $\psi : \widehat{X^+} \to S$ be a continuous homomorphism such that $\psi(x_i) = s'_i$ for $i = 1, \ldots, n$ and $\psi(x) \in \varphi\big((x)^\omega_V\big)$ for every other $x \in X$. For $x \in V$, as $(x)^\omega_V$ is idempotent and φ is a homomorphism, $\varphi\big((x)^\omega_V\big)$ is a subsemigroup of the finite semigroup S and, therefore, it contains some idempotent. Hence, we may assume that $\psi(x)$ is idempotent for every $x \in V$. Since, in particular, $\psi(x_i)$ is idempotent for every $x_i \in V$ $(i = 1, \ldots, n)$, in fact the relations $\psi\big((x)^\omega_V\big) = \psi(x) \in \varphi\big((x)^\omega_V\big)$ hold for every $x \in X$. Since the mappings $\psi\big((\_)^\omega_V\big)$ and $\varphi\big((\_)^\omega_V\big)$ are continuous homomorphisms, it follows in particular

that $\psi\big((v)_V^\omega\big) \in \varphi\big((v)_V^\omega\big)$. Since S satisfies $(u \leqslant v)_V^\omega$, we obtain

$$s_1' \cdots s_n' = \psi(u) = \psi\big((u)_V^\omega\big) \leqslant \psi\big((v)_V^\omega\big) \in \varphi\big((v)_V^\omega\big),$$

which shows that $s_1' \cdots s_n'$ belongs to the lower set $\varphi\big((v)_V^\omega\big)$.

To conclude the proof, it remains to observe that in each case, the additional inequalities yield the relation $s_1 \cdots s_n \leqslant s_1' \cdots s_n'$ and, therefore, we also have $s_1 \cdots s_n \in \varphi\big((v)_V^\omega\big)$. □

More generally, we have the following result.

Corollary 6.7. *Let Σ be a set of semilinear inequalities and let V be a set of variables. Then each of the following pseudovarieties of ordered semigroups is a fixed point for $\mathsf{P}^{\downarrow}$:*

(1) $[\![(\Sigma)_V]\!] \cap [\![xy \leqslant x]\!]$, *provided that for each inequality $u \leqslant v$ from Σ, the first letter of u does not belong to V;*
(2) $[\![(\Sigma)_V]\!] \cap [\![yx \leqslant x]\!]$, *provided that for each inequality $u \leqslant v$ from Σ, the last letter of u does not belong to V;*
(3) $[\![(\Sigma)_V]\!] \cap [\![x^\omega y \leqslant x^\omega,\ yx^\omega \leqslant x^\omega]\!]$.

Proof. The only case that does not follow trivially from Theorem 6.2 and Proposition 6.6 is that of the pseudovariety $[\![x^\omega y \leqslant x^\omega,\ yx^\omega \leqslant x^\omega]\!]$. To treat this case, we may take in Proposition 6.6 (3) the inequality $u \leqslant v$ to be $xy \leqslant x$, respectively $yx \leqslant x$, and $V = \{x\}$, to deduce that the image of the pseudovariety under $\mathsf{P}^{\downarrow}$ satisfies $x^\omega y \leqslant x^\omega$, respectively $yx^\omega \leqslant x^\omega$. □

Note that the pseudovarieties to which Corollary 6.7 applies are contained in $[\![x^{\omega+1} \leqslant x^\omega]\!]$, which is equal to $[\![x^{\omega+1} = x^\omega]\!]$ by Lemma 3.2. We proceed with some observations concerning the pseudovariety $[\![x^\omega y \leqslant x^\omega,\ yx^\omega \leqslant x^\omega]\!]$.

The following result relates some of the inequalities in Corollary 6.7 with some important pseudovarieties of semigroups. Let R [J] be the pseudovariety of all finite $\mathcal{R}$-trivial [$\mathcal{J}$-trivial] semigroups. It is well known that $\mathsf{R} = [\![(xy)^\omega x = (xy)^\omega]\!]$.

Proposition 6.8.

(1) *The pseudovariety $[\![xy \leqslant x]\!]$ is contained in R.*
(2) *The pseudovariety $[\![x^\omega y \leqslant x^\omega]\!]$ is contained in R.*
(3) *The pseudovariety $[\![x^\omega y \leqslant x^\omega,\ yx^\omega \leqslant x^\omega]\!]$ is contained in J.*
(4) *The pseudovariety of semigroups generated by $[\![xy \leqslant x,\ yx \leqslant x]\!]$ is J.*

Proof. (1) This follows from (2) since $[\![xy \leqslant x]\!] \subseteq [\![x^\omega y \leqslant x^\omega]\!]$.

(2) In every member $(S, \leqslant)$ of $[\![x^\omega y \leqslant x^\omega]\!] \subseteq [\![x^{\omega+1} = x^\omega]\!]$, we have

$$(xy)^\omega x \leqslant (xy)^\omega \leqslant (xy)^{\omega+1} = x(yx)^\omega y \leqslant x(yx)^\omega = (xy)^\omega x,$$

by Lemma 3.2. Hence, S belongs to R.

(3) This follows from (2) together with its left-right dual.

(4) An ordered monoid belongs to $[\![xy \leqslant x,\ yx \leqslant x]\!]$ if and only if it satisfies the (semilinear) monoid inequality $x \leqslant 1$. By Simon's syntactic characterization of piecewise testable languages [17], every finite $\mathcal{J}$-trivial monoid is a homomorphic image of one that admits an order satisfying the inequality $x \leqslant 1$.[2] The result now follows by observing that adjoining an

[2]A direct algebraic proof of this result has been given by Straubing and Thérien [19].

identity element to a $\mathcal{J}$-trivial semigroup, one obtains a $\mathcal{J}$-trivial monoid. $\square$

6.2 Local inequalities

In this subsection, we consider a special type of inequalities of the form $u^\omega \leqslant u^\omega v u^\omega$, comparing an idempotent with certain elements in the local submonoid it determines.

We say that an n-tuple $\boldsymbol{e} = (e_1, \ldots, e_n)$ of elements of $\widehat{\mathbb{N}} \setminus \mathbb{N}_+$ is an $(\omega-1)$-*partition of length* n if $1+\sum_{i=1}^n e_i = \omega$. Given such an n-tuple and the linear word $u = x_1 \cdots x_n$, we consider the profinite words

$$u_{\boldsymbol{e},i} = (x_i x_{i+1} \cdots x_n x_1 \cdots x_{i-1})^{e_i} x_i.$$

Note that

$$u_{\boldsymbol{e},1} u_{\boldsymbol{e},2} \cdots u_{\boldsymbol{e},n} = u^\omega.$$

For example, for $u = xy$ and $\boldsymbol{e} = (\omega-1, \omega)$, we get $u_{\boldsymbol{e},1} = (xy)^{\omega-1}x$ and $u_{\boldsymbol{e},2} = (yx)^\omega y = y(xy)^\omega$. If we take instead $e = (0, \omega-1)$, then we get $u_{\boldsymbol{e},1} = x$ and $u_{\boldsymbol{e},2} = y(xy)^{\omega-1}$.

Let $X_m = \{x_1, ..., x_m\}$. Given a profinite word $\rho \in \widehat{X_m^+}$ and profinite words $w_1, \ldots, w_n$ with $n \leqslant m$, let us denote by $\rho(w_1, \ldots, w_n)$ the result of substituting in ρ the first n variables respectively by $w_1, \ldots, w_n$. We do not require that ρ actually depends on each of the first n variables.

Theorem 6.9. *Let u be a linear word of length n, $\boldsymbol{e}$ an $(\omega-1)$-partition of length n, and let ρ be an element of $\widehat{X_m^+}$ with $m \geqslant n$. Then the pseudovariety defined by the inequality*

$$u^\omega \leqslant u^\omega \rho(u_{\boldsymbol{e},1}, \ldots, u_{\boldsymbol{e},n}) u^\omega \tag{11}$$

is a fixed point of the operator $\mathsf{P}^\downarrow$.

The inequalities of the form (11) will be called *local inequalities*.

Proof. Let $(S, \leqslant)$ be a finite ordered semigroup satisfying the inequality (11). We need to show that so does $\mathcal{P}^\downarrow(S)$. Let $\varphi : \widehat{X_m^+} \to \mathcal{P}^\downarrow(S)$ be a continuous homomorphism, let $T = \varphi(u^\omega)$, and let t be an element of T. We claim that t also belongs to $\varphi\big(u^\omega \rho(u_{\boldsymbol{e},1}, \ldots, u_{\boldsymbol{e},n}) u^\omega\big)$. Since T is idempotent in $\mathcal{P}^\downarrow(S)$, the element t is less than or equal to a product of $|S|$ elements of T. By Lemma 2.1, taking again into account that T is idempotent, it follows that there exist $t_1, f, t_2 \in T$ such that $t \leqslant t_1 f t_2$ and $f^2 = f$.

By hypothesis, u is of the form $u = x_1 \cdots x_n$, where the x_i are distinct variables. For each $i \in \{1, \ldots, n\}$, let p_i be a positive integer such that $\mathcal{P}^\downarrow(S)$, whence also S, satisfies the pseudoidentity $x^{e_i} = x^{p_i}$ and let $p = 1 + \sum_{i=1}^n p_i$. Note that, since $\boldsymbol{e}$ is an $(\omega-1)$-partition, $\mathcal{P}^\downarrow(S)$ satisfies the pseudoidentity $x^\omega = x^p$.

Since $T = \varphi(u^\omega) = \varphi(u^p)$, there exist in S elements a_j $(j = 1, \ldots, pn)$ such that

$$f \leqslant a_1 \cdots a_{pn} \tag{12}$$

$$a_{\ell n+i} \in \varphi(x_i) \quad (i = 1, \ldots, n;\ \ell = 0, \ldots, p-1). \tag{13}$$

For each $i \in \{1, \dots, n+1\}$, let $k_i = \sum_{h<i}(p_h n+1)$ so that, in particular, the relations $k_1 = 0$, $k_{i+1} \equiv i \pmod{n}$, for $i = 1, \dots, n$, and $k_{n+1} = pn$ hold. Consider any continuous homomorphism $\psi : \widehat{X_m^+} \to S$ such that

$$\psi(x_i) = a_{k_i+1} a_{k_i+2} \cdots a_{k_{i+1}} \quad (i = 1, \dots, n), \tag{14}$$

$$\psi(x) \in \varphi(x) \quad \text{for every variable } x \notin \{x_1, \dots, x_n\}. \tag{15}$$

From (14) we deduce that

$$\begin{aligned}\psi(u_{\boldsymbol{e},i}) &= \psi\big((x_i x_{i+1} \cdots x_n x_1 \cdots x_{i-1})^{e_i} x_i\big) = \psi\big((x_i x_{i+1} \cdots x_n x_1 \cdots x_{i-1})^{p_i} x_i\big) \\ &= (a_{k_i+1} \cdots a_{pn} a_1 \cdots a_{k_i})^{p_i} a_{k_i+1} \cdots a_{k_{i+1}}.\end{aligned}$$

Using (13), one can easily check that

$$\begin{aligned} a_{k_i+1} \cdots a_{pn} a_1 \cdots a_{k_i} &\in \varphi\big((x_i x_{i+1} \cdots x_n x_1 \cdots x_{i-1})^p\big) \\ a_{k_i+1} \cdots a_{k_{i+1}} &\in \varphi\big((x_i x_{i+1} \cdots x_n x_1 \cdots x_{i-1})^{p_i} x_i\big), \end{aligned}$$

which implies that

$$\psi(u_{\boldsymbol{e},i}) \in \varphi\big((x_i x_{i+1} \cdots x_n x_1 \cdots x_{i-1})^{p_i(p+1)} x_i\big). \tag{16}$$

On the other hand, (2) shows that $e_i = e_i(\omega + 1)$ for $i = 1, \dots, n$. Hence $\mathcal{P}^{\downarrow}(S)$ satisfies each of the pseudoidentities $x^{p_i(p+1)} = x^{e_i}$. Taking into account the definition of $u_{\boldsymbol{e},i}$ and (16), we conclude that

$$\psi(u_{\boldsymbol{e},i}) \in \varphi(u_{\boldsymbol{e},i}). \tag{17}$$

Combining (12) and (14), we obtain $f \leqslant a_1 \cdots a_{pn} = \psi(x_1 \cdots x_n) = \psi(u)$. Since f is idempotent, it follows that $f \leqslant \psi(u^\omega)$. On the other hand, since S satisfies the inequality (11), we deduce that

$$t \leqslant t_1 f t_2 \leqslant t_1 \psi(u^\omega) t_2 \leqslant t_1 \psi(u^\omega) \psi\big(\rho(u_{\boldsymbol{e},1}, \dots, u_{\boldsymbol{e},n})\big) \psi(u^\omega) t_2.$$

Finally, since t_1, t_2, and $\psi(u^\omega)$ belong to the idempotent $T = \varphi(u^\omega)$, to conclude that indeed $t \in \varphi\big(u^\omega \rho(u_{\boldsymbol{e},1}, \dots, u_{\boldsymbol{e},n}) u^\omega\big)$ it only remains to observe that

$$\psi\big(\rho(u_{\boldsymbol{e},1}, \dots, u_{\boldsymbol{e},n})\big) \in \varphi\big(\rho(u_{\boldsymbol{e},1}, \dots, u_{\boldsymbol{e},n})\big),$$

which follows from (15) and (17). □

Corollary 6.10. *Each of the following inequalities defines a fixed point of the operator* $\mathsf{P}^{\downarrow}$ *on pseudovarieties of ordered semigroups:*

(1) $x^\omega \leqslant x^\omega y x^\omega$;
(2) $(xy)^\omega \leqslant (xy)^\omega x^\alpha (xy)^\omega$ *(for each positive profinite natural number* α*);*
(3) $(xy)^\omega \leqslant (xy)^\omega z (xy)^\omega$;
(4) $(xy)^\omega \leqslant (xy)^\omega (yx)^\omega (xy)^\omega$;
(5) $(xy)^\omega \leqslant \big((xy)^\omega (yx)^\omega (xy)^\omega\big)^\omega$.

Proof. It suffices to observe that all inequalities in the statement of the corollary are of the form (11). Indeed, to obtain the first inequality, let $u = x$ be a variable, $\boldsymbol{e} = (\omega - 1)$, and $\rho(x, y) = y$. For the other inequalities, we take $u = xy$. For the second one, we let $\boldsymbol{e} = (0, \omega - 1)$ and $\rho(x, y) = x^\alpha$. Choosing $\boldsymbol{e} = (\omega - 1, \omega)$, and putting respectively $\rho(x, y, z) = z$, $\rho(x, y) = yx$, and $\rho(x, y) = (xy \cdot yx \cdot xy)^\omega$, we obtain the last three inequalities. □

6.3 Bordered idempotent inequalities

Given a set V of variables, let us consider for each variable $x \in V$ three new variables associated with x, denoted x_1, x_2, and x_3. Let X be a set containing the variables x, x_1, x_2, x_3 for all $x \in V$. Let σ_V be the continuous endomorphism of $\widehat{X^+}$ defined by

$$\sigma_V(x) = \begin{cases} x_1 x_2^\omega x_3 & \text{if } x \in V \\ x & \text{otherwise} \end{cases}$$

By definition, a *bordered idempotent inequality* is an inequality of the form $\sigma_V(u) \leqslant \sigma_V(v)$, where $u \leqslant v$ is a semilinear inequality and u contains some variable from V.

Theorem 6.11. *Every pseudovariety defined by a bordered idempotent inequality is a fixed point of the operator* $\mathsf{P}^{\downarrow}$.

Proof. Let $u \leqslant v$ be a semilinear inequality over the set of variables X and let V be a subset of X as above. We must show that, if a finite ordered semigroup S satisfies the inequality $\sigma_V(u) \leqslant \sigma_V(v)$ then so does $\mathcal{P}^{\downarrow}(S)$.

Consider a continuous homomorphism $\varphi : \widehat{X^+} \to \mathcal{P}^{\downarrow}(S)$ and let s be an arbitrary element of $\varphi(\sigma_V(u))$. We claim that s also belongs to $\varphi(\sigma_V(v))$, which will establish the theorem.

By assumption, u is a linear word, say $u = x_1 \cdots x_n$, where the x_i are distinct variables from X. Since φ is a homomorphism and by definition of the product in $\mathcal{P}^{\downarrow}(S)$, there are elements $s_1, \ldots, s_n$ in S such that $s \leqslant s_1 \ldots s_n$ and

$$s_i \in \begin{cases} \varphi(x_{i,1} x_{i,2}^\omega x_{i,3}) & \text{if } x_i \in V \\ \varphi(x_i) & \text{otherwise,} \end{cases}$$

where, for notational convenience, we write $x_{i,j}$ instead of $(x_i)_j$. In case $x_i \in V$, since $\varphi(x_{i,2}^\omega)$ is idempotent, by Lemma 2.1 we have a further inequality $s_i \leqslant a_{i,1} b_i e_i c_i a_{i,3}$, where $a_{i,j} \in \varphi(x_{i,j})$ $(j = 1, 3)$, $b_i, c_i \in \varphi(x_{i,2}^\omega)$, and $e_i \in \varphi(x_{i,2}^\omega)$ is idempotent. We choose any continuous homomorphism $\psi : \widehat{X^+} \to S$ that satisfies the following conditions:

(i) if $x_i \notin V$, then $\psi(x_i) = s_i$;
(ii) if $x_i \in V$, then $\psi(x_{i,1}) = a_{i,1} b_i$, $\psi(x_{i,2}) = e_i$, and $\psi(x_{i,3}) = c_i a_{i,3}$;
(iii) if $z \in X$ is not one of the x_i or $x_{i,j}$, then $\psi(z) \in \varphi(z)$.

For such a choice of ψ, we have $s \leqslant s_1 \cdots s_n \leqslant \psi(\sigma_V(u))$ and $\psi(\sigma_V(v)) \in \varphi(\sigma_V(v))$. On the other hand, since S satisfies the inequality $\sigma_V(u) \leqslant \sigma_V(v)$, we obtain $\psi(\sigma_V(u)) \leqslant \psi(\sigma_V(v))$. Hence $s \in \varphi(\sigma_V(v))$ since $\varphi(\sigma_V(v))$ is a lower set and $s \leqslant \psi(\sigma_V(v)) \in \varphi(\sigma_V(v))$. □

For example, the pseudovariety $[\![x^\omega y \leqslant x^\omega yz]\!]$ is a fixed point of the operator $\mathsf{P}^{\downarrow}$. Indeed, the inequality that defines it is equivalent to the inequality $x_1 x_2^\omega x_3 y \leqslant x_1 x_2^\omega x_3 yz$, which is $\sigma_{\{x\}}(xy) \leqslant \sigma_{\{x\}}(xyz)$.

The following proposition provides a necessary condition for a pseudovariety to satisfy some nontrivial bordered idempotent inequality, which will be useful in Section 8.

Proposition 6.12. *If a pseudovariety satisfies some nontrivial bordered idempotent inequality then it satisfies at least one of the following inequalities, where e, f, g stand for ω-powers of distinct variables:*

$$efege \leqslant egefe, \tag{18}$$

$$efe \leqslant (efe)^{\alpha} \quad \textit{for some positive profinite natural number } \alpha \neq 1, \tag{19}$$

$$efe \leqslant e. \tag{20}$$

Proof. Let $u \leqslant v$ be a semilinear inequality over a set of variables X and let V be a subset of X such that the corresponding bordered idempotent inequality $\sigma_V(u) \leqslant \sigma_V(v)$ is nontrivial. We distinguish several cases. In each case, we exhibit a substitution of the variables in $\sigma_V(u) \leqslant \sigma_V(v)$ transforming this inequality into another that implies one of the form (18), (19), or (20).

If v is a permutation of the word u, then there are at least two variables $x, y \in X$ that appear in u in the order x, y and in v in the reverse order. We substitute efe for x, in case $x \notin V$, f for x_2 otherwise; ege for y, in case $y \notin V$, g for y_2 otherwise; and e for every other variable. This substitution yields (18) as the resulting inequality.

Suppose next that at least one of the variables x that occurs in u occurs more than once in v, in the sense that v admits a factorization of the form $v = v_1xv_2xv_3$. Then, substituting efe for x, in case $x \notin V$, or f for x_2 otherwise, and e for every other variable, we obtain an inequality of the form (19).

If v involves some variable x that does not occur in u, then we may substitute efe for x, in case $x \notin V$, or f for x_2 otherwise, and e for every other variable. The resulting inequality has the form $e \leqslant (efe)^{\beta}$ for some positive profinite natural number β. Multiplying both sides by efe, we obtain $efe \leqslant (efe)^{\beta+1}$, which is of the form (19).

Finally, suppose that there is some variable x that occurs in u but not in v. Then we may substitute efe for x, in case $x \notin V$, or f for x_2 otherwise, and e for every other variable, to obtain the inequality (20). □

7 Applications

This section is dedicated to applications of the results from Section 6.

7.1 Number of fixed points of $\mathsf{P}^{\downarrow}$

We have shown in the previous section that the operator $\mathsf{P}^{\downarrow}$ has many fixed points. In this subsection, we determine their precise number.

Theorem 7.1. *The operator $\mathsf{P}^{\downarrow}$ has $2^{\aleph_0}$ fixed points.*

Proof. By Corollary 6.3, each of the pseudovarieties $\mathsf{V}^{\alpha} = [\![x \leqslant x^{1+\alpha}]\!]$, where α is a positive profinite natural number, is a fixed point of the operator $\mathsf{P}^{\downarrow}$. But they are not all distinct. For instance, one can easily check that $\mathsf{V}^{\omega-1} = \mathsf{V}^{\omega+1}$. However, it suffices to exhibit uncountably many values of α whose corresponding V^{α} are all distinct.

Let G denote the pseudovariety of all finite groups. It is well known that the only compatible partial order in a finite group is equality. Thus finite

groups with equality constitute a pseudovariety of ordered semigroups whose intersection with V^α is defined by the pseudoidentity[3] $x^\alpha = 1$.

Let $\boldsymbol{p} = (p_n)_n$ be a strictly increasing sequence of prime numbers and let $\alpha_{\boldsymbol{p}}$ be an accumulation point of the sequence $(p_1 p_2 \cdots p_n)_n$ in $\widehat{\mathbb{N}}_+$. Then, for a prime number q, the cyclic group of order q satisfies the pseudoidentity $x^{\alpha_{\boldsymbol{p}}} = 1$ if and only if $q = p_n$ for some n. Hence, distinct sequences $\boldsymbol{p}$ produce different pseudovarieties $[\![x^{\alpha_{\boldsymbol{p}}} = 1]\!]$ and so, by the preceding paragraph, also distinct pseudovarieties $\mathsf{V}^{\alpha_{\boldsymbol{p}}}$. To conclude the proof, it remains to observe that there are $2^{\aleph_0}$ such sequences of prime numbers. □

7.2 Preservation of period

Let V be a pseudovariety of semigroups. Let $\mathrm{Per}(\mathsf{V})$ be the set of profinite natural numbers α such that V satisfies the identity $x^\omega = x^{\omega+\alpha}$.

Proposition 7.2. *The set* $\mathrm{Per}(\mathsf{V})$ *is a closed ideal of the semiring* $\widehat{\mathbb{N}}$*. Furthermore, there exists a profinite natural number that generates* $\mathrm{Per}(\mathsf{V})$ *as a closed ideal.*

Proof. It is clear that $\mathrm{Per}(\mathsf{V})$ is a closed ideal of $\widehat{\mathbb{N}}$. Furthermore $\mathrm{Per}(\mathsf{V})$ coincides with $\mathrm{Per}(\mathsf{V} \cap \mathsf{Ab})$, where Ab is the pseudovariety of Abelian groups. By [1, Corollary 3.7.8], there exists a profinite natural number α such that

$$\mathsf{V} \cap \mathsf{Ab} = [\![x^\alpha = 1, xy = yx]\!].$$

It follows that a cyclic group C_n belongs to V if and only if n divides α.

By considering the natural homomorphisms from $\widehat{\mathbb{N}}$ to finite cyclic groups, it is easy to show that every β in $\widehat{\mathbb{N}}$ is the limit of a sequence of natural numbers dividing it. Thus β divides γ if and only if every finite divisor of β divides γ. Therefore $\mathrm{Per}(\mathsf{V})$ consists precisely of all multiples of α if and only if $\mathsf{V} \cap \mathsf{Ab}$ is defined by $x^\alpha = 1$, $xy = yx$. □

A profinite natural number generating $\mathrm{Per}(\mathsf{V})$ as a closed ideal is called a *period* of V.

Let W be the pseudovariety of ordered monoids defined in [8]. In view of the discussion in Section 4.3, we also denote by W the pseudovariety of ordered semigroups that it generates. Here are the key characteristic properties of this pseudovariety, which combine the results of [8], dealing with pseudovarieties of ordered monoids, with the observations of Section 4.3:

(W1) W is the largest proper fixed point of the operator $\mathsf{P}^{\downarrow}$ on pseudovarieties of ordered semigroups;

(W2) W is the largest pseudovariety of ordered semigroups that does not contain the ordered syntactic monoid of the language $(ab)^*$ over the alphabet $\{a, b\}$;

(W3) given any element ρ in the minimum ideal of the free profinite semigroup $\widehat{X_2^+}$, the pseudovariety W is defined by the inequality

$$(xy)^\omega \leqslant \Big((xy)^\omega \rho\big((xy)^{\omega-1}x, y(xy)^\omega\big)(xy)^\omega\Big)^\omega. \tag{21}$$

[3]Here we use the convention for semigroups detailed in Section 4.4: $u = 1$ is an abbreviation for the pseudoidentities $ux = xu = x$, where x is a variable not occurring in u.

Note that, if we choose for $\rho(x_1, x_2)$ an idempotent that starts with x_1 and ends with x_2, then the right hand side of (21) becomes $\rho\big((xy)^{\omega-1}x, y(xy)^\omega\big)$. For shortness, we will call *special* such an idempotent ρ.

Proposition 7.3. *The following equality holds for every special idempotent e:*

$$\mathsf{W} = [\![(xy)^\omega \leqslant e\big((xy)^{\omega-1}x, y(xy)^\omega\big)]\!].$$

Note that it follows from both (W3) and Proposition 7.3 that W is a fixed point of the operator $\mathsf{P}^{\downarrow}$ (a property that is included in (W1)) since both inequalities defining W are of the form (11) of Theorem 6.9.

Theorem 7.4. *Given a special idempotent e, let $f = e((xy)^{\omega-1}x, y(xy)^\omega)$. Then for every positive profinite natural number α, the following equality holds:*

$$\mathsf{W} \cap [\![x^{\omega+\alpha} = x^\omega]\!] = [\![(xy)^\omega \leqslant \big(fy(xy)^\omega f\big)^\alpha]\!]. \tag{22}$$

In particular, the pseudovariety $\mathsf{W} \cap [\![x^{\omega+\alpha} = x^\omega]\!]$ is a fixed point of $\mathsf{P}^{\downarrow}$.

Proof. Let V denote the pseudovariety on the right hand side of the equality (22).

($\supseteq$) By Theorem 6.9, V is a fixed point of $\mathsf{P}^{\downarrow}$. Hence, by property (W1), V is contained in W. On the other hand, substituting y by x in the inequality

$$(xy)^\omega \leqslant \big(fy(xy)^\omega f\big)^\alpha \tag{23}$$

yields the inequality $x^\omega \leqslant x^{\omega+\alpha}$, which is therefore valid in V. The inclusion ($\supseteq$) now follows from Lemma 3.2.

($\subseteq$) Suppose that S is an ordered semigroup from W that satisfies the pseudoidentity $x^{\omega+\alpha} = x^\omega$ and let $\varphi : \widehat{X_2^+} \to S$ be a continuous homomorphism. Then $\varphi(f)$ lies in the minimum ideal of the subsemigroup generated by $\varphi\big((xy)^{\omega-1}x\big)$ and $\varphi\big(y(xy)^\omega\big)$ and therefore $\varphi\big(fy(xy)^\omega f\big)$ is an element of the maximal subgroup with idempotent $\varphi(f)$. Since S satisfies the pseudoidentity $x^{\omega+\alpha} = x^\omega$, we must have $\varphi\big(\big(fy(xy)^\omega f\big)^\alpha\big) = \varphi(f)$. Hence the inequality (23) holds in S by Proposition 7.3. □

The main result of this subsection is an easy consequence of Theorem 7.4 and (W1).

Corollary 7.5. *Let α be a positive profinite natural number. If V is a proper pseudovariety of ordered semigroups that is a fixed point of $\mathsf{P}^{\downarrow}$, then so is $\mathsf{V} \cap [\![x^{\omega+\alpha} = x^\omega]\!]$.* □

In particular, for each positive profinite natural number α, the pseudovariety

$$\mathsf{Com}_\alpha = [\![xy = yx,\ x^{\omega+\alpha} = x^\omega]\!]$$

is a fixed point of P. This provides an alternative example of a continuum of fixed points for $\mathsf{P}^{\downarrow}$.

7.3 The pseudovariety $\mathsf{V}_{3/2}$

This subsection concerns another application of Theorem 6.9. The pseudovariety $\mathsf{V}_{3/2}$ is one of the lowest levels of a hierarchy of pseudovarieties of ordered monoids introduced in [11] as a refinement of the Straubing-Thérien

hierarchy. It is defined by the inequalities of the following form [11, Theorem 8.7]:

$$u^\omega \leqslant u^\omega v u^\omega \text{ with } c(v) \subseteq c(u). \tag{24}$$

As in the preceding subsection, we also denote by $\mathsf{V}_{3/2}$ the pseudovariety of ordered semigroups generated by the pseudovariety of ordered monoids $\mathsf{V}_{3/2}$, which is defined by the same basis. We next show that one can refine this basis of inequalities for $\mathsf{V}_{3/2}$.

Proposition 7.6. *The pseudovariety $\mathsf{V}_{3/2}$ is defined by the set of all inequalities of the form* (24) *where u and v are linear words such that the last letter of u is not in $c(v)$.*

Proof. Let S be a finite ordered semigroup and let u and v be two profinite words. Then there are finite words u' and v' such that the pseudoidentities $u = u'$ and $v = v'$ hold in S and $c(u) = c(u')$ and $c(v) = c(v')$. This observation shows that the pseudovariety $\mathsf{V}_{3/2}$ can be defined by the inequalities of the form (24), where u and v are finite words such that $c(v) \subseteq c(u)$.

Let now u and v be two finite words such that $c(v) \subseteq c(u)$. Let $p = |u|$ and let $q = \max_{x \in c(v)} |v|_x$ be the maximum number of occurrences of any given letter in v. Let $\tilde{u} = x_1 \cdots x_n$ be a linear word of length $n = pq + 1$ and let $\tilde{v}$ be the linear word obtained from v by replacing the k-th occurrence of each letter x by the letter $x_{p(k-1)+i}$, where i is the first position of x in u. Then $c(\tilde{v}) \subseteq c(\tilde{u})$ but x_n does not occur in $\tilde{v}$. For example, if $u = xyx^2z^2$ and $v = zxyxz$, then we have $p = 6$, $q = 2$, $\tilde{u} = x_1 \cdots x_{13}$ and $\tilde{v} = x_5x_1x_2x_7x_{11}$.

Now, the homomorphism that maps the variable $x_{p(k-1)+i}$ to the i-th variable in u for $1 \leqslant k \leqslant q$ and $1 \leqslant i \leqslant p$ and the variable x_n to u sends $\tilde{u}$ to u^{q+1} and $\tilde{v}$ to v. Since $u^\omega = (u^{q+1})^\omega$, it follows that the inequality $u^\omega \leqslant u^\omega v u^\omega$ is a consequence of the inequality $\tilde{u}^\omega \leqslant \tilde{u}^\omega \tilde{v} \tilde{u}^\omega$. □

Corollary 7.7. *The pseudovariety $\mathsf{V}_{3/2}$ is a fixed point of $\mathsf{P}^\downarrow$.*

Proof. Let $u = x_1 \cdots x_n$ and v be linear words such that x_n is not in $c(v)$. Consider the $(\omega - 1)$-partition

$$\boldsymbol{e} = (\underbrace{0, \ldots, 0}_{n-1 \text{ times}}, \omega - 1).$$

Then, with the notation of Section 6.2, one has $u_{\boldsymbol{e},i} = x_i$ for $1 \leqslant i \leqslant n-1$ and $u_{\boldsymbol{e},n} = (x_n x_1 \cdots x_{n-1})^{\omega-1} x_n$. Then since $v(u_{\boldsymbol{e},1}, \ldots, u_{\boldsymbol{e},n}) = v$, the inequality $u^\omega \leqslant u^\omega v(u_{\boldsymbol{e},1}, \ldots, u_{\boldsymbol{e},n}) u^\omega$ can be simply written as $u^\omega \leqslant u^\omega v u^\omega$. Therefore, it suffices to apply Theorem 6.9 to show that $\mathsf{V}_{3/2}$ is a fixed point of $\mathsf{P}^\downarrow$. □

An alternative proof of Corollary 7.7 is obtained by applying the language-theoretic characterizations of the operator $\mathsf{P}^\downarrow$ [13, 8, 6]. Indeed, it suffices to show that the corresponding variety of languages is closed under literal homomorphisms, that is under homomorphisms that map letters to letters. This is clear since a language over a finite alphabet A has its syntactic monoid in $\mathsf{V}_{3/2}$ if and only if it is a union of languages of the form $A_0^* a_1 A_1^* \cdots a_n A_n^*$, where each $A_i \subseteq A$ and each $a_i \in A$.

8 Independence

In Section 6, we found several types of inequalities defining fixed points for the operator $\mathsf{P}^{\downarrow}$. The main result of this section, Theorem 8.3, summarizes these results and establishes their independence, using also some of the applications in Section 7.

We need two elementary results. Let U_1 be the two-element semilattice $\{0,1\}$, ordered by equality. The same semilattice, ordered by $0 < 1$, is denoted by U_1^+. For the reverse order, we denote it U_1^-. We denote by $B(m,n)$ the $m \times n$ rectangular band, ordered by equality. The proof of the following lemma is immediate.

Lemma 8.1. *The following properties hold for a given inequality $u \leqslant v$ of profinite words:*

(1) *it holds in U_1^+ if and only if $c(v) \subseteq c(u)$;*
(2) *it holds in U_1^- if and only if $c(u) \subseteq c(v)$;*
(3) *it holds in U_1 if and only if $c(u) = c(v)$;*
(4) *it holds in $B(2,1)$ if and only if u and v have the same first letter;*
(5) *it holds in $B(1,2)$ if and only if u and v have the same last letter.*

It follows in particular from Lemma 8.1 that the pseudovariety

$$\mathsf{SI}^+ = [\![xy = yx, x^2 = x, xy \leqslant y]\!],$$

is generated by U_1^+.

Denote by N the pseudovariety consisting of all finite nilpotent semigroups.

Lemma 8.2. *The pseudovariety N satisfies no nontrivial semilinear inequalities.*

Proof. Let $u \leqslant v$ be a semilinear inequality that holds in N and let $k = |u| + 1$ and A be a finite set such that $u, v \in \widehat{A^+}$. Let S be the Rees quotient $A^+/(A^kA^*)$, ordered by equality, and let $\pi : \widehat{A^+} \to S$ be the natural homomorphism. Then $\pi(u) = \pi(v)$, but $\pi(u) \neq 0$. Now the inequality $\pi(v) \neq 0$ implies that v is also a finite word of length at most k and the equality $\pi(u) = \pi(v)$ shows that $u = v$. □

We can now state our main result.

Theorem 8.3. *The following types of inequalities define fixed points for the operator $\mathsf{P}^{\downarrow}$:*

(1) *semilinear inequalities;*
(2) *in the presence of the inequality $xy \leqslant x$, inequalities of the form $(u \leqslant v)_V$, where u is a linear word whose first letter does not belong to V;*
(3) *in the presence of the inequality $xy \leqslant y$, inequalities of the form $(u \leqslant v)_V$, where u is a linear word whose last letter does not belong to V;*
(4) *in the presence of the inequalities $x^\omega y \leqslant x^\omega$ and $xy^\omega \leqslant y^\omega$, inequalities of the form $(u \leqslant v)_V$, where u is a linear word;*
(5) *local inequalities;*
(6) *bordered idempotent inequalities.*

Moreover, these types of inequalities are independent in the sense that, for any of them, there is a set of inequalities of that type defining a pseudovariety that cannot be defined by a combination of inequalities of the other types.

Proof. It remains to establish the independence statement. This is proved by exhibiting, for each of our types of inequalities, a suitable pseudovariety that shows the independence of that type from the others.

Consider the pseudovariety $\mathsf{U}_1 = [\![x \leqslant xy]\!]$, which is defined by an inequality of type (1).

Lemma 8.4. *The pseudovariety* U_1 *is not defined by any combination of inequalities of types* (2)–(6).

Proof. By Lemma 8.1, since U_1^- and $B(2,1)$ belong to U_1 but fail, respectively, the inequalities $xy \leqslant x$, and $xy \leqslant y$ and $xy^\omega \leqslant y^\omega$, we may not use inequalities of types (2), (3) and (4). Clearly $x \leqslant xy$ entails the inequality $x^\omega \leqslant x^\omega y x^\omega$, and therefore every local inequality, of which the latter is the strongest one. Consider the ordered syntactic semigroup S of the language ab over the alphabet $\{a, b\}$, which has four elements, namely the syntactic classes of a, b, ab, ba, the latter being a zero; the zero is a minimum and there are no other nontrivial comparability relations. Then S satisfies $x^\omega \leqslant x^\omega y x^\omega$ as well as every inequality of type (6), since, under any evaluation of the variables, both sides gives zero. Yet, S fails the inequality $x \leqslant xy$. Hence U_1 cannot be defined by any combination of inequalities of types (5) and (6). □

Consider next the pseudovariety $\mathsf{U}_2 = [\![xy \leqslant x, xy^\omega z t^\omega \leqslant x t^\omega z y^\omega]\!]$, which is defined by an inequality of type (2).

Lemma 8.5. *Let* A_n *be the alphabet consisting of the letters* $a_1, \ldots, a_n, b$ *and let* S_n *be the ordered syntactic semigroup of the language*

$$L_n = a_1 b^* \{1, a_2\} b^* \cdots \{1, a_n\} b^*$$

over the alphabet A_n*. Let* $u \leqslant v$ *be a semilinear inequality such that* $|u| \leqslant n$ *and let* V *be a (possibly empty) set of variables. Then* S_n *belongs to* U_2 *and, if it satisfies* $\sigma_V(u) \leqslant \sigma_V(v)$ *then* $[\![xy \leqslant x]\!]$ *satisfies* $u \leqslant v$*.*

Proof. For a word $w \in A_n^+$, denote by $\overline{w}$ its syntactic class. We first observe that S_n has a zero, which is the syntactic class of all words with empty context. This zero is also the minimum of S_n. Furthermore, $\bar{b}$ is the only nonzero idempotent of S_n and S_n satisfies the identity $x^3 = x^2$. Hence, S_n satisfies the inequality $xy^\omega z t^\omega \leqslant x t^\omega z y^\omega$ because, if both idempotents y^ω and t^ω evaluate to a nonzero idempotent, they evaluate to the same idempotent. Since in a word from L_n, every letter but the first can be removed without leaving the language, the inequality $xy \leqslant x$ also holds in S_n.

Suppose next that S_n satisfies the inequality $\sigma_V(u) \leqslant \sigma_V(v)$. Let $u = x_1 \cdots x_r$, where the x_i are distinct variables and $r \leqslant n$. Let X be the set of variables that appear in $\sigma_V(u) \leqslant \sigma_V(v)$ and consider the continuous homomorphism $\varphi : \widehat{X^+} \to S_n$ that sends each $x_i \notin V$ to $\bar{a}_i$, each $x_{i,1}$ to $\bar{a}_i$, every other letter in $\sigma_V(u)$ to $\bar{b}$, and the remaining letters, if there are any, to zero. Since $a_i b^k$ and a_i have the same context, the equality $\varphi(\sigma_V(u)) = \bar{a}_1 \cdots \bar{a}_r$ holds and $\varphi(\sigma_V(u)) \neq 0$ since the context of the word

$a_1 \cdots a_r$ includes the pair $(1, a_{r+1} \cdots a_n)$. Since S_n satisfies the inequality $\sigma_V(u) \leqslant \sigma_V(v)$, it follows also that $\varphi(\sigma_V(v)) \neq 0$. Hence no x_i can occur in v more than once, they must appear in v in the same order as in u, and v must start with x_1. This implies that the inequality $u \leqslant v$ is a consequence of $xy \leqslant x$. □

Lemma 8.6. *The pseudovariety* U_2 *is not defined by any combination of inequalities of types* (1) *and* (3)–(6).

Proof. By Lemma 8.5, every inequality of the form $\sigma_V(u) \leqslant \sigma_V(v)$ that is valid in U_2 is also valid in $[\![xy \leqslant x]\!]$. Thus, there is no sense in using inequalities of types (1) and (6) other than $xy \leqslant x$ itself. By Lemma 8.1, since $B(2,1)$ belongs to U_2 and fails the inequalities $xy \leqslant y$ and $xy^\omega \leqslant y^\omega$, we may not use inequalities of types (3) and (4). Also by Lemma 8.1, since U_1^+ belongs to U_2, for every local inequality $u \leqslant v$ valid in U_2, we must have $c(u) = c(v)$. By Proposition 7.6, such inequalities define $\mathsf{V}_{3/2}$. On the other hand, by Proposition 6.8(1), the pseudovariety $[\![xy \leqslant x]\!]$ is contained in R, whence also in $\mathsf{V}_{3/2}$. Thus, the only way the lemma can fail is if $\mathsf{U}_2 = [\![xy \leqslant x]\!]$. We leave it to the reader to check that the syntactic ordered semigroup of the language $ab^*\{1,c\}d^*$ satisfies the inequality $xy \leqslant x$ but fails $xy^\omega zt^\omega \leqslant xt^\omega zy^\omega$. □

Consider next the pseudovariety $\mathsf{U}_4 = [\![x^\omega y \leqslant x^\omega,\ yx^\omega \leqslant x^\omega]\!]$, which is defined by inequalities of type (4) (see Corollary 6.7).

Lemma 8.7. *The pseudovariety* U_4 *is not defined by any combination of inequalities of types* (1)–(3), (5), *and* (6).

Proof. Since $\mathsf{N} \subseteq \mathsf{U}_4$, it follows from Lemma 8.2 that U_4 satisfies no nontrivial semilinear inequalities, and so we may not use inequalities of types (1)–(3). By Lemma 8.1, U_1^+ belongs to U_4 and so every local inequality $u \leqslant v$ that is valid in U_4 satisfies $c(u) = c(v)$. Since, by the same lemma, U_1, which is not in U_4, satisfies all inequalities of the form $u \leqslant v$ with $c(u) = c(v)$, if the lemma fails then there is at least one bordered idempotent inequality $\sigma_V(u) \leqslant \sigma_V(v)$ that holds in U_4 and such that $c(v)$ is a strict subset of $c(u)$. We are going to show that this is impossible. We may as well assume that $V = c(u)$.

Let $u = x_1 \cdots x_n$. Let $A = \{a_i, b, c_i \mid i = 1, \ldots, n\}$ and consider the homomorphism $\varphi : V^+ \to P(A^+)$ defined by

$$\varphi(x_i) = a_i b^+ c_i \cup a_i b^+ \cup b^+ c_i \cup b^+.$$

Consider the language $L = \varphi(u)$ and let S be its ordered syntactic semigroup over the alphabet A. We show that $S \in \mathsf{U}_4$ and that S fails the inequality $\sigma_V(u) \leqslant \sigma_V(v)$.

For a word $w \in A^+$, we denote by $\overline{w}$ its syntactic class. Note that S has a zero, which is also the minimum element of S: it is the syntactic class of all words with empty context. In particular,

every word w in which some letter different from b occurs at least twice, or some c_i occurs before some a_j with $j < i$, or has a factor of the form $c_i a_j$ with $j \neq i+1$, satisfies $\overline{w} = 0$. (25)

This leaves only one possible nonzero idempotent in S, namely $e = (\bar{b})^\omega$, which is indeed nonzero since $(b^n, 1)$ is a context of b^k for every $k \geqslant 1$. Thus, to verify that $S \in \mathsf{U}_4$, one only needs to consider substitutions in which the idempotents are mapped to e. By symmetry, we may as well deal only with the inequality $yx^\omega \leqslant x^\omega$. It suffices to show that $C(db^m) \subseteq C(b^m)$ for $d \in A$ and m sufficiently large. The case $d = b$ is clear since the language L is star-free. Since the other cases are similar, we consider only the case where $d = a_i$ for some i. Suppose that $(p, q) \in C(a_i b)$. Then $p \in \varphi(x_1 \cdots x_{i-1})$ and $bq \in (b^+ c_i \cup b^+)\varphi(x_{i+1} \cdots x_n) \subseteq \varphi(x_i \cdots x_n)$, whence $(p, q) \in C(b)$.

It remains to show that S fails the inequality $\sigma_V(u) \leqslant \sigma_V(v)$. Consider the alphabet $Z = \{x_{i,j} \mid i = 1, \ldots, n;\ j = 1, 2, 3\}$ and the continuous homomorphism $\psi : \widehat{Z^+} \to S$ defined by $\psi(x_{i,1}) = \bar{a}_i$, $\psi(x_{i,2}) = \bar{b}$ and $\psi(x_{i,3}) = \bar{c}_i$. Then we have $\psi(\sigma_V(u)) = \bar{a}_1 e \bar{c}_1 \cdots \bar{a}_n e \bar{c}_n > 0$ because $(1, 1) \in C(a_1 b^m c_1 \cdots a_n b^m c_n)$ for all $m \geqslant 1$. If some variable x_i appears more than once in v, then $\psi(\sigma_V(v)) = 0$ by (25), and the inequality $\sigma_V(u) \leqslant \sigma_V(v)$ fails in S. Similarly, also using (25), we may assume that the variables appear in v in the same order as in u, without gaps. The remaining case is when v is a proper factor of u. Consider the representatives w_1 and w_2, respectively of the syntactic classes $\psi(\sigma_V(u))$ and $\psi(\sigma_V(v))$, that are given by choosing as representatives for the factors $\bar{a}_i$, $\bar{c}_i$, and e respectively the words a_i, c_i, and b^m for a large enough exponent m. Then the pair $(1, 1)$ belongs to $C(w_1)$ but not to $C(w_2)$, thereby showing that S fails the inequality $\sigma_V(u) \leqslant \sigma_V(v)$. □

Recall that, by Proposition 7.3, the pseudovariety W is defined by an inequality of type (5).

Lemma 8.8. *The pseudovariety* W *is not defined by any combination of inequalities of types* (1)–(4) *and* (6).

Proof. Since N is contained in W, Lemma 8.2 implies that W satisfies no nontrivial semilinear inequalities. On the other hand, W is not contained in the pseudovariety $[\![x^\omega y \leqslant x^\omega, xy^\omega \leqslant y^\omega]\!]$ since the latter is contained in J by Proposition 6.8(3) while $\mathsf{DS} \subseteq \mathsf{W}$ by [8, Corollary 3.9] and $\mathsf{DS} \nsubseteq \mathsf{J}$. It remains to show that W is not defined by inequalities of type (6). Since DA is contained in W, again by [8, Corollary 3.9], it suffices to exhibit, for each inequality from Proposition 6.12, a DA-recognizable language L whose ordered syntactic semigroup $\mathrm{Synt}(L)$ fails the inequality.

For the inequality (18), we consider the language $L = a^+ b^+ a^+ c^+ a^+$ over the alphabet $\{a, b, c\}$. Since L is clearly an unambiguous product of languages of the form B^* $(B \subseteq A)$ and $\{x\}$ $(x \in A)$, it is DA-recognizable by a well-known theorem of Schützenberger [16]. Denote by $\overline{w}$ the syntactic class of each word w. For each letter x, the element $\bar{x}$ is an idempotent. The word *acaba* has empty context and, therefore, its syntactic class is zero, which is the minimum in the syntactic order. Now, we may evaluate the variables in (18) so that e maps to $\bar{a}$, f to $\bar{b}$ and g to $\bar{c}$. So, if (18) were valid in $\mathrm{Synt}(L)$ then we would have $\bar{a}\bar{b}\bar{a}\bar{c}\bar{a} = 0$, which is not the case since the word *abaca* has nonempty context.

The inequalities (19) and (20) are handled similarly by considering the language $a^+b^+a^+$ over the alphabet $\{a, b\}$. Indeed the pair $(1, 1)$ belongs to $C(aba)$ but neither to $C((aba)^n)$, for $n > 1$, nor to $C(a)$. □

Finally, consider the pseudovariety $\mathsf{U}_6 = [\![x^\omega y \leqslant x^\omega yz]\!]$, which is, as observed in Subsection 6.3, defined by an inequality of type (6).

Lemma 8.9. *The pseudovariety* U_6 *is not defined by any combination of inequalities of types* (1)–(5).

Proof. Since $\mathsf{N} \subseteq \mathsf{U}_6$, Lemma 8.2 shows that U_6 satisfies no nontrivial semilinear inequalities. By Lemma 8.1, U_1^- belongs to U_6 but fails the inequality $x^\omega y \leqslant x^\omega$. Hence U_6 satisfies no inequality of type (4).

Substituting in $x^\omega y \leqslant x^\omega yz$ the variable y by x^ω and z by yx^ω, we deduce that U_6 satisfies the inequality $x^\omega \leqslant x^\omega yx^\omega$ and, therefore, every local inequality. On the other hand, $B(1, 2)$ ordered by equality satisfies the inequality $x^\omega \leqslant x^\omega yx^\omega$ but fails $x^\omega y \leqslant x^\omega yz$. Hence U_6 cannot be defined by a combination of inequalities of types (1)–(5). □

In view of Lemmas 8.4, 8.6 and its dual, 8.7, 8.8, and 8.9, and the remarks at the beginning of the proof, the proof of Theorem 8.3 is now complete. □

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CMUP, Dep. Matemática, Faculdade de Ciências, Universidade do Porto, Rua do Campo Alegre 687, 4169-007 Porto, Portugal
E-mail address: `jalmeida@fc.up.pt`

Departamento de Sistemas Informáticos y Computación, Universidad Politécnica de Valencia, Camino de Vera s/n, P.O. Box: 22012, E-46020 - Valencia.
E-mail address: `acano@dsic.upv.es`

Dept. of Mathematics and Statistics, Masaryk University, Kotlářská 2, 611 37 Brno, Czech Republic
E-mail address: `klima@math.muni.cz`

LIAFA, Université Paris-Diderot and CNRS, Case 7014, 75205 Paris Cedex 13, France
E-mail address: `Jean-Eric.Pin@liafa.univ-paris-diderot.fr`