Application of Conditional Random Fields and Sparse Polynomial Chaos Expansions to Geotechnical Problems

Roland Schöbi, Bruno Sudret

Application of Conditional Random Fields and Sparse Polynomial Chaos Expansions to Geotechnical Problems

Roland SCHÖBI [a] , Bruno SUDRET [a]
[a] *Department of Structural Engineering, ETH Zurich, Switzerland*

Abstract. In geotechnical applications, mechanical properties of soil vary spatially within the soil mass and they are often represented by random fields. When data at certain locations of the soil mass are available, conditional random fields may be used to incorporate them. In this paper, we combine conditional random fields with sparse polynomial chaos expansions to analyze response quantities with otherwise too expensive Monte Carlo-based techniques, such as reliability and sensitivity analysis.

Keywords. conditional random fields, EOLE, meta-modelling, sparse polynomial chaos expansions, spatial variability

1. Introduction

Mechanical properties of soils are uncertain due to their natural spatial variability within the soil mass. In engineering applications, however, they are often inferred from a limited number of measurements taken in boreholes drilled through the region of interest. Between the boreholes, however, the material properties remain uncertain. A common approach to represent them in geotechnical applications is through the use of random fields based on geostatistical information. Several approaches are available in the literature to construct unconditional random fields, such as the expansion optimal linear estimation method (EOLE) (Li and Der Kiureghian, 1993). Accounting for available borehole information, however, requires the modelling of *conditional* random fields (Hoffmann and Ribak, 1991).

Typically, the analysis of geotechnical problems is carried out through the use of finite element models (FEM). In this context, analyses which require a large number of FE model evaluations, such as Monte Carlo-based reliability analysis, can become intractable. To reduce the associated costs, the expensive FE model may be replaced by a meta-model, *e.g.* Al-Bittar and Soubra (2013), Vorechovsky (2008) and Cho et al. (2013), where the framework of Sparse Polynomial Chaos Expansions (SPCE) (Blatman and Sudret, 2011) is applied.

The combination of conditional random fields and meta-modelling, however, has not been addressed in the geotechnical literature yet. In this paper we combine the idea of conditional random fields (see Section 2) and the framework of SPCE (see Section 3) into an efficient framework for analyzing response quantities in geotechnical problems. The algorithm is illustrated in Section 4 on the problem of a strip foundation located on a two-layer soil mass.

2. Spatial Variability

2.1. Random Fields

2.1.1. Definition

Consider a probability space defined by the tuple $(\Omega, \mathcal{F}, \mathbb{P})$ where Ω is the event space equipped with σ-algebra $\mathcal{F}$ and probability measure $\mathbb{P}$. In this context, a random variable X is denoted by the mapping $X: (\Omega, \mathcal{F}, \mathbb{P}) \rightarrow \mathbb{R}$. The collection of various random variables X_i is described by a random vector $\boldsymbol{X} = [X_1, \dots, X_M]$.

A random field $H(\boldsymbol{x}, \omega)$ is then defined as a curve in $\boldsymbol{x}$ in the vector space of functions with finite second moments $\mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$ where $\omega \in \Omega$ (Lin, 1967; Vanmarcke, 1983; Sudret and Der

Kiureghian, 2000). The domain $\mathcal{D}_{\boldsymbol{X}} \in \mathbb{R}^d$ describes the geometry of the system. Consequently, $\boldsymbol{x}$ denotes the geographical location, *i.e.* the coordinates of the system. Note that for a given location $\boldsymbol{x} \in \mathcal{D}_{\boldsymbol{X}}$ the random field reduces to a random variable, whereas for a given $\omega \in \Omega$ the output is a trajectory, *i.e.* a curve.

Gaussian random fields are an important family of random fields. They are completely described by their mean value $\mu(\boldsymbol{x})$, variance $\sigma^2(\boldsymbol{x})$ and autocorrelation function $\rho(\boldsymbol{x}, \boldsymbol{x}')$.

2.1.2. EOLE

A Gaussian random field $H(\boldsymbol{x}, \omega)$ is defined implicitly by the properties listed above. It is not straightforward, however, to sample it so as to obtain realizations of H.

Discretization procedures approximate the random field $H(\boldsymbol{x}, \omega)$ with some function $\widehat{H}(\boldsymbol{x}, \boldsymbol{\zeta})$, where $\boldsymbol{\zeta} = \{\zeta_i, i = 1, \dots, M\}$ is a finite set of random variables describing the randomness of the field:

$$H(\boldsymbol{x}, \omega) \approx \widehat{H}(\boldsymbol{x}, \boldsymbol{\zeta}). \tag{1}$$

The explicit function $\widehat{H}$ allows one to sample the random field H. An overview of several available discretization algorithms is given in Sudret and Der Kiureghian (2000). Among them, the *expansion optimal linear estimation method* (EOLE) is presented here briefly.

EOLE is based on the Kriging method which is popular in geostatistics. Considering a set of nodal points $\widehat{\boldsymbol{X}} = \{\widehat{\boldsymbol{x}}_1, \dots, \widehat{\boldsymbol{x}}_p\}$ in the domain $\mathcal{D}_X$, the optimal linear estimation of the random field at a point $\boldsymbol{x}$ is given by:

$$\widehat{H}(\boldsymbol{x}, \omega) = \boldsymbol{\mu}(\boldsymbol{x}) + \boldsymbol{\Sigma}_{H(x)\hat{Y}}\, \boldsymbol{\Sigma}_{\hat{Y}\hat{Y}}^{-1}\left(\widehat{\boldsymbol{Y}} - \boldsymbol{\mu}_{\hat{Y}}\right), \tag{2}$$

where $\widehat{\boldsymbol{Y}} = \{\hat{y}_1, \dots, \hat{y}_p\}$ is the set of correlated Gaussian variables associated to points $\widehat{\boldsymbol{X}}$, $\boldsymbol{\mu}_{\hat{Y}}$ and $\boldsymbol{\Sigma}_{\hat{Y}\hat{Y}}$ its mean value and covariance matrix ($\Sigma_{\hat{Y}\hat{Y}}^{k,l} = \sigma(\widehat{\boldsymbol{x}}_k)\, \sigma(\widehat{\boldsymbol{x}}_l)\, \rho(\widehat{\boldsymbol{x}}_k, \widehat{\boldsymbol{x}}_l)$), and $\boldsymbol{\Sigma}_{H(x)\hat{Y}}$ is a vector of components $\sigma(\boldsymbol{x})\, \sigma(\widehat{\boldsymbol{x}}_k)\, \rho(\boldsymbol{x}, \widehat{\boldsymbol{x}}_k)$, $k = 1, \dots, p$.

By introducing the eigenvalue decomposition of the covariance matrix:

$$\boldsymbol{\Sigma}_{\hat{Y}\hat{Y}} \boldsymbol{\phi}_i = \lambda_i \boldsymbol{\phi}_i, \tag{3}$$

one gets the EOLE approximation:

$$\widehat{H}(\boldsymbol{x}, \boldsymbol{\zeta}(\omega)) = \boldsymbol{\mu}(\boldsymbol{x}) + \sum_{i=1}^{M} \frac{\zeta_i(\omega)}{\sqrt{\lambda_i}} \boldsymbol{\phi}_i^{\mathrm{T}} \boldsymbol{\Sigma}_{H(x)\hat{Y}}, \tag{4}$$

Note than the p eigenvalues in Eq. (3) have been listed in descending order, and that only $M \leq p$ terms are retained in practice.

It can be shown that the approximation of a random field using EOLE leads to an underestimation of the variance of the random field (Sudret and Der Kiureghian, 2000) due to the finite number M of eigenvalues considered. Hence, M should be chosen large enough in order to ensure a good approximation of H (*e.g.* $\mathrm{Var}(\widehat{H}) \geq 0.95\, \mathrm{Var}(H)$).

2.2. Conditional Random Fields

2.2.1. Observations in Random Fields

Conditional random fields are random fields conditioned on observations. In this paper we define the observations as a set of geographical locations $\boldsymbol{\mathcal{X}} = \{\boldsymbol{x}^{(i)}, i = 1, \dots, L\}$ at which the random field values $\boldsymbol{\mathcal{Y}} = \{\mathcal{y}_i, i = 1, \dots, L\}$ are known. The conditional random field reads then:

$$H_c(\boldsymbol{x}, \omega) = H\left(\boldsymbol{x}, \omega \middle| H\left(\boldsymbol{x}^{(1)}\right) = \mathcal{y}_1, \dots, H\left(\boldsymbol{x}^{(L)}\right) = \mathcal{y}_L\right). \tag{5}$$

2.2.2. Discretization of a Conditional Random Field

The conditional random field is discretized by a finite set of nodal points. We define the random vector $\boldsymbol{Z} \in \mathbb{R}^{L+K}$ composed of the response of the observations $\boldsymbol{\mathcal{Y}}$ and the set of yet unknown responses $\widehat{\boldsymbol{Y}} = \{\hat{y}_j = H_c(\widehat{\boldsymbol{x}}_j), j = 1, \dots, K\}$:

$$\boldsymbol{Z} = \begin{bmatrix} \widehat{\boldsymbol{Y}} \\ \boldsymbol{\mathcal{Y}} \end{bmatrix}, \tag{6}$$

where $\widehat{\boldsymbol{Y}} \in \mathbb{R}^K$ and $\boldsymbol{\mathcal{Y}} \in \mathbb{R}^L$. Then $\boldsymbol{Z}$ can be represented as a Gaussian vector:

$$\boldsymbol{Z} \sim \mathcal{N}(\boldsymbol{\mu}_Z, \boldsymbol{\Sigma}_{ZZ}), \quad (7)$$

where:

$$\boldsymbol{\mu}_Z = \begin{bmatrix} \boldsymbol{\mu}_{\hat{Y}} \\ \boldsymbol{\mu}_{\mathcal{Y}} \end{bmatrix}, \quad \boldsymbol{\Sigma}_{ZZ} = \begin{bmatrix} \boldsymbol{\Sigma}_{\hat{Y}\hat{Y}} & \boldsymbol{\Sigma}_{\hat{Y}\mathcal{Y}} \\ \boldsymbol{\Sigma}_{\hat{Y}\mathcal{Y}}^{\mathrm{T}} & \boldsymbol{\Sigma}_{\mathcal{Y}\mathcal{Y}} \end{bmatrix}, \quad (8)$$

where $\boldsymbol{\Sigma}_{\hat{Y}\mathcal{Y}}$ is the covariance matrix between the set of prediction points $\widehat{\boldsymbol{X}}$ and the observation points $\boldsymbol{\mathcal{X}}$. Hence, it can be shown that $\widehat{\boldsymbol{Y}}$ conditioned on the observations $\boldsymbol{\mathcal{Y}}$ can be computed by:

$$\widehat{Y}|\boldsymbol{\mathcal{Y}} \sim \mathcal{N}(\boldsymbol{\mu}_{\hat{Y}|\mathcal{Y}}, \boldsymbol{\Sigma}_{\hat{Y}|\mathcal{Y}}), \quad (9)$$

where:

$$\boldsymbol{\mu}_{\hat{Y}|\mathcal{Y}} = \boldsymbol{\mu}_{\hat{Y}} + \boldsymbol{\Sigma}_{\hat{Y}\mathcal{Y}} \boldsymbol{\Sigma}_{\mathcal{Y}\mathcal{Y}}^{-1} (\boldsymbol{\mathcal{Y}} - \boldsymbol{\mu}_{\mathcal{Y}}), \quad (10)$$

$$\boldsymbol{\Sigma}_{\hat{Y}|\mathcal{Y}} = \boldsymbol{\Sigma}_{\hat{Y}\hat{Y}} - \boldsymbol{\Sigma}_{\hat{Y}\mathcal{Y}} \boldsymbol{\Sigma}_{\mathcal{Y}\mathcal{Y}}^{-1} \boldsymbol{\Sigma}_{\hat{Y}\mathcal{Y}}^{\mathrm{T}}. \quad (11)$$

Eq. (11) describes the covariance matrix of $\widehat{\boldsymbol{Y}}$ conditional on the observations $\boldsymbol{\mathcal{Y}}$, which can be used for sampling $\widehat{\boldsymbol{Y}}$.

2.2.3. Sampling of a Conditional Random Field

Hoffmann and Ribak (1991) and Hoffmann (2009) proposed a two-step algorithm to sample from the conditional multivariate Gaussian in Eq. (9):

1. Generate a realization of $\boldsymbol{Z}$ from the unconditional random field $H(\boldsymbol{x}, \omega)$ with EOLE ignoring the observations and denote the realization $\boldsymbol{Z}^0 = [\widehat{\boldsymbol{Y}}^0, \boldsymbol{\mathcal{Y}}^0]^{\mathrm{T}}$.
2. Compute the realization of the conditional random field using Eq. (10), which reads in the specific case $\widehat{\boldsymbol{Y}} = \widehat{\boldsymbol{Y}}^0 + \boldsymbol{\Sigma}_{\hat{Y}\mathcal{Y}} \boldsymbol{\Sigma}_{\mathcal{Y}\mathcal{Y}}^{-1} (\boldsymbol{\mathcal{Y}} - \boldsymbol{\mathcal{Y}}^0)$, Note that this is a deterministic transformation of the realization of the unconditional random field obtained in step 1.

This algorithm offers a convenient way of generating realizations of conditional random fields as a function of M random variables defined in EOLE and a set of observations $\{\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{Y}}\}$.

3. Meta-modelling

The realizations of the conditional random field are then typically plugged into a finite element model (FEM) in order to analyze a quantity of interest (*e.g.* the settlement of a foundation). Analyzing the influence of the conditional random field on the quantity of interest is often unfeasible due to the high computational costs of FEM. An alternative is then to approximate the behavior of the FEM by a meta-model.

3.1. Computational Model

Generally speaking, in engineering applications a computational model $\mathfrak{M}$, such as a FEM, is a mapping of the M-dimensional input vector $\boldsymbol{u}$ to the output scalar v, *i.e.* $\mathfrak{M}: \boldsymbol{u} \in \mathcal{D}_{\boldsymbol{U}} \in \mathbb{R}^M \to v \in \mathbb{R}$. Suppose the uncertainties in the inputs are represented by a random vector $\boldsymbol{U}$ with joint cumulative density distribution (CDF) $F_{\boldsymbol{U}}$. The components of $\boldsymbol{U} = [U_1, \dots, U_M]^{\mathrm{T}}$ are assumed independent for the sake of simplicity, hence the joint probability distribution f (PDF) of $\boldsymbol{U}$ can be written as the product of its marginals U_i, *i.e.* $f_{\boldsymbol{U}}(\boldsymbol{u}) = \prod_{i=1}^{M} f_{U_i}(u_i)$. Then the model response is a random variable $V = \mathfrak{M}(\boldsymbol{U})$.

3.2. Polynomial Chaos Expansions

A common non-intrusive meta-modelling method is Polynomial Chaos Expansions (PCE) which approximates the computational model $\mathfrak{M}$ with a sum of polynomials orthogonal with respect to the distributions of the input variables (Ghanem and Spanos, 2003; Sudret, 2015):

$$V \approx \mathfrak{M}^{(PCE)}(\boldsymbol{U}) = \sum_{\alpha \in \mathcal{A}} a_\alpha \psi_\alpha(\boldsymbol{U}), \quad (12)$$

where $a_\alpha \in \mathbb{R}$ are the polynomial coefficients corresponding to indices α in the truncated set $\mathcal{A} \subset \mathbb{N}^M$ and $\psi_\alpha(\boldsymbol{U})$ are multivariate orthonormal polynomials.

3.3. Sparse PCE

One strategy to compute efficiently the coefficients a_α in Eq. (12) is least-square minimization, as introduced by Berveiller et al. (2006). Consider a set of N samples of the input vector $\mathcal{U} = \{\boldsymbol{u}^{(i)}, i = 1, \dots, N\}$ and the corresponding responses of the exact computational model $\mathcal{V} = \{\mathcal{v}^{(i)} = \mathfrak{M}(\boldsymbol{u}^{(i)}), i = 1, \dots, N\}$. The set of coefficients $a_{\boldsymbol{\alpha}}$ can be computed through the solution of the least-squares problem:

$$\hat{\boldsymbol{a}} = \underset{\boldsymbol{\alpha}\in\mathbb{R}^{|\mathcal{A}|}}{\operatorname{argmin}} \frac{1}{N} \sum_{i=1}^{N} \left(\mathcal{v}^{(i)} - \sum_{\boldsymbol{\alpha}\in\mathcal{A}} a_{\boldsymbol{\alpha}} \psi_{\boldsymbol{\alpha}}(\boldsymbol{u}^{(i)}) \right)^2. \tag{13}$$

The efficiency of meta-modelling algorithms depends greatly on the choice of the set of polynomials $\mathcal{A}$ in Eq. (12) and (13). For this reason, algorithms have been developed to select out of a candidate set of polynomials the ones that are most influential to the system response. Following Efron et al. (2004), Blatman and Sudret (2011) introduced the least angle regression (LAR) algorithm for this purpose. LAR determines the sparse set of polynomials that best describes the behaviour of the exact computational model $\mathfrak{M}$ based on the experimental design $\mathcal{U}$, hence the name *sparse PCE*.

3.4. PCE for Conditional Random Fields

Consider a finite element model whose input depends on the random input vector $\boldsymbol{U}$ and the realization of a random field $H(\boldsymbol{x}, \omega)$. As seen in Section 2.2, the conditional random field can be discretized by a finite set of M random variables, collected in the vector $\boldsymbol{\zeta}$. A new input vector can then be built as a combination of $\boldsymbol{\zeta}$ and the other stochastic input variables $\boldsymbol{U}$. The computational model reads then:

$$V \approx \mathfrak{M}^{+}(\boldsymbol{U}, \boldsymbol{\zeta}), \tag{14}$$

which can be approximated by a sparse PCE meta-model of dimension $M + |\boldsymbol{U}|$. The experimental design includes realizations of the stochastic variables $\boldsymbol{U}$ as well as realizations of the variables $\boldsymbol{\zeta}$ used to describe the random field.

Note that the computational model $\mathfrak{M}^{+}$ is composed of the steps of (i) generating the conditional random field realization based on EOLE, $\hat{\boldsymbol{\zeta}}$ and the observations $\{\mathcal{X}, \mathcal{Y}\}$, (ii) calculating the realization of the conditional random field and the stochastic variables $\boldsymbol{u}$ in the finite element model and (iii) computing the response value v of the finite element model.

4. Foundation Settlement on Soil Layers with Uncertain Thickness

4.1. Problem Statement

Consider the two-dimensional strip foundation sketched in Figure 1. The soil mass is composed of two layers separated by an irregular horizontal interface I. It is assumed that the soil mass is weightless and lays on a rigid bedrock at a depth of 5 m below the soil surface.

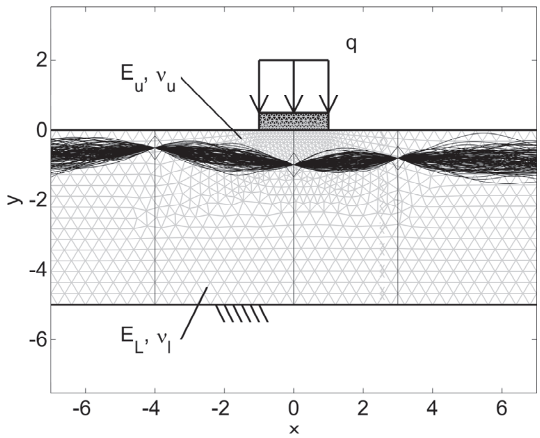

Figure 1. Foundation settlement – geometry and notations.

The interface between the two soil layers is modelled by a one-dimensional Gaussian random field I with mean value $\mu_I = -1$ m measured from the soil surface, $\sigma_I = 0.3$ m and an exponential autocorrelation function with correlation length $l_{I,x} = 5$ m. Note that there is a probability of $\Phi\left(-\frac{1}{0.3}\right) = 4.3 \cdot 10^{-4}$ that the interface is "above" the soil surface. In this case, the lower layer would reach the soil surface at that location. Further, at three locations on the horizontal axis indicated in Figure 1, the interface depth I is measured to be: $I(x = -4\text{ m}) = -0.5$ m, $I(0\text{ m}) = 1$ m, $I(3\text{ m}) = -0.8$ m. Several realizations of the conditional random field are drawn in Figure 1 to showcase its variability. The measurements clearly reduce the spatial variability of the interface in the area close to the observations.

The mechanical properties of the soil mass are modelled by a linear elastic behaviour. In particular, Young's modulus E and Poisson's ratio ν describe the linear elastic constitutive model.

On top of the soil mass is located a 2 m wide and 0.5 m thick foundation ($E_f = 25$ GPa, $\nu_f = 0.4$) which is subjected to a vertical pressure q.

The probabilistic model of all stochastic variables is summarized in Table 1. Each variable is defined by its distribution function, mean value and coefficient of variation (CoV).

Table 1. Foundation settlement – input distributions (subscript *u*, *l* stand for upper and lower layer of the soil model; *I* summarizes the random field)

Variable	Distribution	Mean	CoV
E_u	Lognormal	5 MPa	10 %
E_l	Lognormal	20 MPa	10 %
ν_u, ν_l	Deterministic	0.3	-
q	Gumbel	100 kPa	20 %
I	Gaussian	-1 m	30 %

4.2. Analysis

The quantity of interest is the settlement v of the midpoint of the foundation.

The random field is discretized with $M_{RF} = 30$ random variables leading to a total dimensionality of the computational model of $M_{tot} = 30 + 3 = 33$. The experimental design consists of $N = 300$ Latin-hypercube samples. The FE model has been developed using the COMSOL Multiphysics software with a total width of the soil model of 40 m and the mesh (partially) displayed in Figure 1 (17'042 degree-of-freedom). The sparse PCE model is calibrated using the Matlab-based toolbox UQLab (Marelli and Sudret, 2014).

For estimating the accuracy of the meta-model, a validation set of $n = 1000$ Monte Carlo samples is generated. The relative mean square error between the prediction and the exact values on the validation set is $err_{gen} = 4.4\%$. This indicates an accurate approximation of the foundation settlement v obtained by $N = 300$ evaluations of the exact computational model $\mathfrak{M}$.

Due to the inexpensive-to-evaluate formulation of the sparse PCE meta-model, the PDF $f_V(v)$ of the foundation settlement v can be estimated. The solid line in Figure 2 displays the estimate of $f_V(v)$ obtained from the kernel smoothing of a large Monte Carlo ($n_{MC} = 10^6$) sample of the input vector.

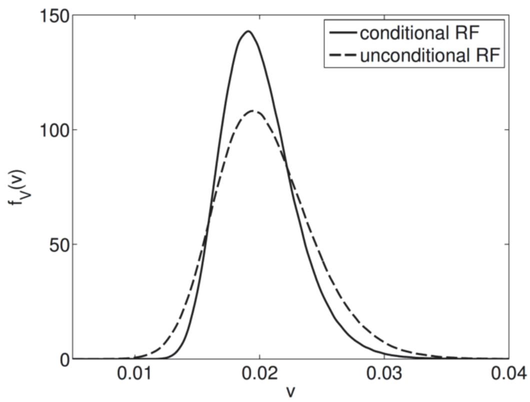

Figure 2. Foundation settlement – PDF of u.

The dashed line in Figure 2 represents the PDF of the settlement in case an unconditional random field (with the same parameters) is used *i.e.* ignoring the borehole data. The difference between the two distributions shows the influence of the borehole data, namely a significant reduction of the uncertainty in the settlement estimation. The conditional random field PDF is more peaked and narrower. The mean value and the standard deviation of the foundation settlement are $\{\mu_{v,c} = 20.0mm,$ $\sigma_{v,c} = 3.0mm\}$ and $\{\mu_v = 20.5mm,$ $\sigma_v = 3.9mm\}$ for the conditional and unconditional random field, respectively. These points indicate that the observations reduce indeed the uncertainty in the settlement prediction.

In order to analyze the influence of the random field on the foundation settlement, Sobol' indices are computed from the coefficients of the PCE meta-model (Sudret, 2008). The resulting total Sobol' indices, denoted by S_i^{tot} , are illustrated in Figure 3 for the cases of conditional (black bars) and unconditional (white bars) random field. Note that "I" stands for the total Sobol' index involving all 30 random variables describing the random field of the interface.

In the case of an unconditional random field, the influence of the random field onto the variance of the settlement is the largest amongst the input variables, whereas the influence is the smallest for the conditional random field. This indicates that the few observations (three points) significantly reduce the uncertainty regarding the location of the interface. The remaining input variables have the same relative importance to each other in terms of the Sobol' indices for both cases.

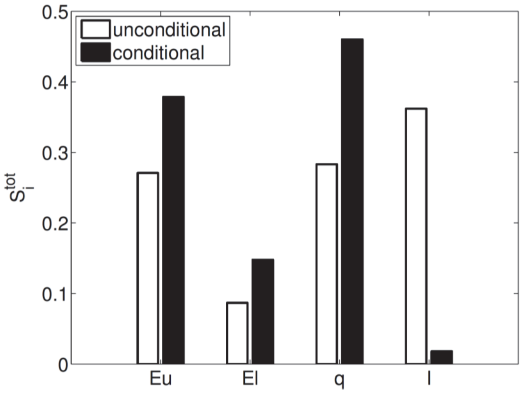

Figure 3. Foundation settlement – total Sobol' indices of the PCE meta-models

5. Conclusions

Geotechnical problems are often solved with expensive-to-evaluate computer models, *e.g.* finite element models (FEM). When accounting for the variability in the input variables, as well as for the spatial variability of the model parameters, the analysis of the model response can become computationally intractable.

In this paper we combine the idea of surrogating the computer model with sparse polynomial chaos expansions (PCE) and the framework of conditional random fields. EOLE is used to approximate the conditional random field and parametrize it with a finite number of variables used as inputs for the PCE.

The combined approach is capable of accurately predicting the probability distribution of a foundation settlement with only 300 runs of the FE model. In addition, global sensitivity analysis can be carried out as post-processing of the PCE. It quantitatively demonstrates how even a small number of observations can substantially reduce the uncertainty related to the spatial variability of the model parameters.

References

Al-Bittar, T., Soubra, A.-H. (2013). Bearing capacity of strip footings on spatially random soils using sparse polynomial chaos expansion. *Int. J. Num. Anal. Meth. Geomech*. **37**, 2039–2060.

Berveiller, M., Sudret, B., Lemaire, M. (2006). Stochastic finite elements: a non intrusive approach by regression. *Eur. J. Comput. Mech.* **15**(1-3), 81–92.

Blatman, G., Sudret, B. (2011). Adaptive sparse polynomial chaos expansion based on Least Angle Regression. *J. Comput. Phys* **230**, 2345–2367.

Cho, H., Venturi, D., Karniadakis, G. (2013). Karhunen Loève expansion for multi-correlated stochastic processes. *Probabilistic Eng. Mech*. **34**, 157–167.

Efron, B., Hastie, T., Johnstone, I., Tibshirani, R. (2004). Least angle regression. *Annals of Statistics* **32**, 407–499.

Ghanem, R., Spanos, P. (2003). *Stochastic Finite Elements: A Spectral Approach (2nd ed.)*. Dover Publications, Mineola.

Hoffmann, Y. (2009). Gaussian fields and constrained simulation of the large-scale structure. In *Data Anal. Cosmol. Lect. Notes Phys*., pp. 565–583. Berlin: Springer Verlag.

Hoffmann, Y., Ribak, E. (1991). Constrained realizations of Gaussian fields - a simple algorithm. *Astrophys. J*. **380**, L5–L8.

Li, C.C., Der Kiureghian, A. (1993). Optimal discretization of random fields. *J. Eng. Mech*. **119**(6), 1136–1154.

Lin, Y.-K. (1967). *Probabilistic theory of structural dynamics*. McGraw-Hill.

Marelli, S., Sudret, B. (2014). UQLab: a framework for uncertainty quantification in MATLAB. *Proc. 2nd Int. Conf. on Vulnerability, Risk Analysis and Management (ICVRAM2014)*, Liverpool, United Kingdom.

Sudret, B. (2015). Polynomials chaos expansions and stochastic finite element methods. In K.-K. Phoon, Ching, J. (Eds.), *Risk Reliab. Geotech. Eng*., Chapter 5. Taylor and Francis.

Sudret, B. (2008). Global sensitivity analysis using polynomial chaos expansions. *Reliab. Eng. Sys. Safety*. **93**, 964-979.

Sudret, B., Der Kiureghian, A. (2000). *Stochastic finite elements and reliability: a state-of-the-art report*. Technical Report UCB/SEMM-2000/08, University of California, Berkeley. 173 pages.

Vanmarcke, E. (1983). *Random fields: analysis and synthesis*. Cambridge, Massachusetts: MIT Press.

Vorechovsky, M. (2008). Simulation of simply cross correlated random fields by series expansion methods. *Struct. Safety* **30**(4), 337–363.