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An explicit formula for the intersection of two polynomials of regular languages

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Abstract. Let $\mathcal{L}$ be a set of regular languages of A^* . An $\mathcal{L}$ -polynomial is a finite union of products of the form $L_0 a_1 L_1 \cdots a_n L_n$, where each a_i is a letter of A and each L_i is a language of $\mathcal{L}$. We give an explicit formula for computing the intersection of two $\mathcal{L}$ -polynomials. Contrary to Arfi's formula (1991) for the same purpose, our formula does not use complementation and only requires union, intersection and quotients. Our result also implies that if $\mathcal{L}$ is closed under union, intersection and quotient, then its polynomial closure, its unambiguous polynomial closure and its left [right] deterministic polynomial closure are closed under the same operations.

1 Introduction

Let $\mathcal{L}$ be a set of regular languages of A^* . An $\mathcal{L}$ -polynomial is a finite union of products of the form $L_0 a_1 L_1 \cdots a_n L_n$, where each a_i is a letter of A and each L_i is a language of $\mathcal{L}$. The *polynomial closure* of $\mathcal{L}$, denoted by $\text{Pol}(\mathcal{L})$, is the set of all $\mathcal{L}$ -polynomials.

It was proved by Arfi [1] that if $\mathcal{L}$ is closed under Boolean operations and quotient, then $\text{Pol}(\mathcal{L})$ is closed under intersection. This result was obtained by giving an explicit formula for computing the intersection of two polynomials of regular languages.

It follows from the main theorem of [6] that Arfi's result can be extended to the case where $\mathcal{L}$ is only closed under union, intersection and quotient. However, this stronger statement is obtained as a consequence of a sophisticated result involving profinite equations and it is natural to ask for a more elementary proof.

The objective of this paper is to give a new explicit formula for computing the intersection of two $\mathcal{L}$ -polynomials. Contrary to the formula given in [1], our formula only requires using union, intersection and quotients of languages of $\mathcal{L}$. Our proof is mainly combinatorial, but relies heavily on the notion of syntactic ordered monoid, a notion first introduced by Schützenberger [14] (see also [10]). The main difficulty lies in finding appropriate notation to state the formula, but then its proof is merely a verification.

Our result also leads to the following result, that appears to be new: if $\mathcal{L}$ is closed under union, intersection and quotient, then its unambiguous polynomial

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closure and its left [right] deterministic polynomial closure are closed under the same operations.

Let us mention also that our algorithm can be readily extended to the setting of infinite words by using syntactic ordered ω -semigroups [8].

2 Background and notation

2.1 Syntactic order

The *syntactic congruence* of a language L of A^* is the congruence on A^* defined by $u \sim_L v$ if and only if, for every $x, y \in A^*$,

$$xuy \in L \iff xvy \in L$$

The monoid $M = A^*/\sim_L$ is the *syntactic monoid* of L and the natural morphism $\eta : A^* \rightarrow M$ is called the *syntactic morphism* of L . It is a well-known fact that a language is regular if and only if its syntactic monoid is finite.

The *syntactic preorder*¹ of a language L is the relation $\leq_L$ over A^* defined by $u \leq_L v$ if and only if, for every $x, y \in A^*$, $xuy \in L$ implies $xvy \in L$. The associated equivalence relation is the syntactic congruence $\sim_L$. Further, $\leq_L$ induces a partial order on the syntactic monoid M of L . This partial order $\leq$ is compatible with the product and can also be defined directly on M as follows: given $s, t \in M$, one has $s \leq t$ if and only if, for all $x, y \in M$, $xsy \in \eta(L)$ implies $xt y \in \eta(L)$. The ordered monoid $(M, \leq)$ is called the *syntactic ordered monoid* of L .

Let us remind an elementary but useful fact: if $v \in L$ and $\eta(u) \leq \eta(v)$, then $u \in L$. This follows immediately from the definition of the syntactic order by taking $x = y = 1$.

2.2 Quotients

Recall that if L is a language of A^* and x is a word, the *left quotient* of L by x is the language $x^{-1}L = \{z \in A^* \mid xz \in L\}$. The *right quotient* Ly^{-1} is defined in a symmetrical way. Right and left quotients commute, and thus $x^{-1}Ly^{-1}$ denotes either $x^{-1}(Ly^{-1})$ or $(x^{-1}L)y^{-1}$. For each word v , let us set

$$\begin{aligned} [L]_{\uparrow v} &= \{u \in A^* \mid \eta(v) \leq \eta(u)\} \\ [L]_{=v} &= \{u \in A^* \mid \eta(u) = \eta(v)\} \end{aligned}$$

¹ In earlier papers [6,10,13], I used the opposite preorder, but it seems preferable to go back to Schützenberger's original definition.

Proposition 2.1. *The following formulas hold:*

$$[L]_{\uparrow v} = \bigcap_{\{(x,y) \in A^* \times A^* \mid v \in x^{-1}Ly^{-1}\}} x^{-1}Ly^{-1} \quad (1)$$

$$[L]_{=v} = [L]_{\uparrow v} - \bigcup_{\eta(v) < \eta(u)} [L]_{\uparrow u} \quad (2)$$

$$[L]_{\uparrow v} = \bigcup_{\eta(v) \leq \eta(u)} [L]_{=u} \quad (3)$$

Proof. A word u belongs to the right hand side of (1) if and only if the condition $v \in x^{-1}Ly^{-1}$ implies $u \in x^{-1}Ly^{-1}$, which is equivalent to stating that $v \leq_L u$, or $\eta(v) \leq \eta(u)$, or yet $u \in [L]_{\uparrow v}$. This proves (1). Formulas (2) and (3) are obvious. $\square$

Let us make precise a few critical points. First, v always belongs to $[L]_{\uparrow v}$. This is the case even if v cannot be completed into a word of L , that is, if v does not belong to any quotient $x^{-1}Ly^{-1}$. In this case, the intersection on the right hand side of (1) is indexed by the empty set and is therefore equal to A^* .

Secondly, the intersection occurring on the right hand side of (1) and the union occurring on the right hand side of (2) are potentially infinite, but they are finite if L is a regular language, since a regular language has only finitely many quotients.

3 Infiltration product and infiltration maps

The definition below is a special case of a more general definition given in [7]. A word $c_1 \cdots c_r$ belongs to the *infiltration product* of two words $a_1 \cdots a_p$ and $v = b_1 \cdots b_q$, if there are two order preserving maps $\alpha : \{1, \dots, p\} \rightarrow \{1, \dots, r\}$ and $\beta : \{1, \dots, q\} \rightarrow \{1, \dots, r\}$ such that

- (1) for each $i \in \{1, \dots, p\}$, $a_i = c_{\alpha(i)}$,
- (2) for each $i \in \{1, \dots, q\}$, $b_i = c_{\beta(i)}$,
- (3) the union of the ranges of α and β is $\{1, \dots, r\}$.

For instance, the set $\{ab, aab, abb, aabb, abab\}$ is the infiltration product of ab and ab and the set $\{aba, bab, abab, abba, baab, baba\}$ is the infiltration product of ab and ba .

A pair of maps (α, β) satisfying Conditions (1)–(3) is called a *pair of infiltration maps*. Note that these conditions imply that $p + q \leq r$.

In the example pictured in Figure 1, one has $p = 4$, $q = 2$ and $r = 5$. The infiltration maps α and β are given by $\alpha(1) = 1$, $\alpha(2) = 2$, $\alpha(3) = 3$, $\alpha(4) = 4$ and $\beta(1) = 3$, $\beta(2) = 5$.

Fig. 1. A pair of infiltration maps.

In order to state our main theorem in a precise way, we need to handle the intervals of the form $\{\alpha(i)+1, \dots, \alpha(i+1)-1\}$, but also the two extremal intervals $\{1, \dots, \alpha(1)-1\}$ and $\{\alpha(p)+1, \dots, r\}$. As a means to get a uniform notation, it is convenient to extend α and β to mappings $\alpha : \{0, \dots, p+1\} \rightarrow \{0, \dots, r+1\}$ and $\beta : \{0, \dots, q+1\} \rightarrow \{0, \dots, r+1\}$ by setting $\alpha(0) = \beta(0) = 0$ and $\alpha(p+1) = \beta(q+1) = r+1$. The two extremal intervals are now of the standard form $\{\alpha(i)+1, \dots, \alpha(i+1)-1\}$, with $i = 0$ and $i = p$, respectively. Further, we introduce the two maps $\bar{\alpha} : \{0, \dots, r\} \rightarrow \{0, \dots, p\}$ and $\bar{\beta} : \{0, \dots, r\} \rightarrow \{0, \dots, q\}$ defined by

$$\bar{\alpha}(i) = \max\{k \mid \alpha(k) \leq i\} \quad \text{and} \quad \bar{\beta}(i) = \max\{k \mid \beta(k) \leq i\}.$$

For instance, one gets for our example:

$$\begin{array}{cccccc} \bar{\alpha}(0) = 0 & \bar{\alpha}(1) = 1 & \bar{\alpha}(2) = 2 & \bar{\alpha}(3) = 3 & \bar{\alpha}(4) = 4 & \bar{\alpha}(5) = 4 \\ \bar{\beta}(0) = 0 & \bar{\beta}(1) = 0 & \bar{\beta}(2) = 0 & \bar{\beta}(3) = 1 & \bar{\beta}(4) = 1 & \bar{\beta}(5) = 2 \end{array}$$

These two functions are conveniently represented in Figure 2

Fig. 2. Graphs of $\bar{\alpha}$ and $\bar{\beta}$: for instance, $\bar{\alpha}(3) = 3$ and $\bar{\beta}(3) = 1$.

The next lemmas summarize the connections between α and $\bar{\alpha}$. Of course, similar properties hold for β and $\bar{\beta}$.

Lemma 3.1. *The following properties hold:*

- (1) $\bar{\alpha}(\alpha(k)) = k$, for $0 \leq k \leq p$.
- (2) $\bar{\alpha}(s+1) \leq \bar{\alpha}(s) + 1$, for $0 \leq s \leq r-1$.
- (3) $k \leq \bar{\alpha}(s)$ if and only if $\alpha(k) \leq s$, for $0 \leq k \leq p$ and $0 \leq s \leq r$.
- (4) $k \geq \bar{\alpha}(s)$ if and only if $\alpha(k+1) \geq s+1$, for $0 \leq k \leq p-1$ and $0 \leq s \leq r-1$.

Proof. These properties follow immediately from the definition of $\bar{\alpha}$. $\square$

Lemma 3.2. *For $0 \leq s \leq r-1$, the conditions $\bar{\alpha}(s+1) = \bar{\alpha}(s) + 1$ and $\alpha(\bar{\alpha}(s+1)) = s+1$ are equivalent.*

Proof. Put $k = \bar{\alpha}(s)$ and suppose that $\bar{\alpha}(s+1) = k+1$. Since $k+1 \leq \bar{\alpha}(s+1)$, Lemma 3.1 (3) shows that $\alpha(k+1) \leq s+1$. Further, since $k \geq \bar{\alpha}(s)$, Lemma 3.1 (4) shows that $\alpha(k+1) \geq s+1$. Therefore $\alpha(k+1) = s+1$ and finally $\alpha(\bar{\alpha}(s+1)) = s+1$.

Conversely, suppose that $\alpha(\bar{\alpha}(s+1)) = s+1$. Putting $\bar{\alpha}(s+1) = k+1$, one gets $\alpha(k+1) = s+1$ and Lemma 3.1 (4) shows that $k \geq \bar{\alpha}(s)$. By Lemma 3.1 (2), one gets $\bar{\alpha}(s+1) \leq \bar{\alpha}(s) + 1$ and hence $k \leq \bar{\alpha}(s)$. Thus $\bar{\alpha}(s) = k$ and $\bar{\alpha}(s+1) = \bar{\alpha}(s) + 1$. $\square$

Let us denote by $P_\alpha(s)$ the property $\bar{\alpha}(s+1) = \bar{\alpha}(s) + 1$.

Lemma 3.3. *For $0 \leq s \leq r-1$, one of $P_\alpha(s)$ or $P_\beta(s)$ holds.*

Proof. Since the union of the ranges of α and β is $\{1, \dots, r\}$, there is an integer $k \geq 0$ such that either $\alpha(k+1) = s+1$ or $\beta(k+1) = s+1$. In the first case, one gets $\bar{\alpha}(s+1) = \bar{\alpha}(\alpha(k+1)) = k+1$ and Lemma 3.1 (3) shows that $\bar{\alpha}(s) \leq k$. Since $\bar{\alpha}(s+1) \leq \bar{\alpha}(s) + 1$ by Lemma 3.1 (2), one also has $k \leq \bar{\alpha}(s)$ and finally $\bar{\alpha}(s) = k$, which proves $P_\alpha(s)$. In the latter case, one gets $P_\beta(s)$ by a similar argument. $\square$

4 Main result

Let $a_1, \dots, a_p, b_1, \dots, b_q$ be letters of A and let $K_0, \dots, K_p, L_0, \dots, L_q$ be languages of A^* . Let $K = K_0 a_1 K_1 \cdots a_p K_p$ and $L = L_0 b_1 L_1 \cdots b_q L_q$.

A word of $K \cap L$ can be factorized as $u_0 a_1 u_1 \cdots a_p u_p$, with $u_0 \in K_0, \dots, u_p \in K_p$ and as $v_0 b_1 v_1 \cdots b_q v_q$, with $v_0 \in L_0, \dots, v_q \in L_q$. These two factorizations can be refined into a single factorization of the form $z_0 c_1 z_1 \cdots c_r z_r$, where $c_1 \cdots c_r$ belongs to the infiltration product of $a_1 \cdots a_p$ and $b_1 \cdots b_q$.

For instance, for $p = 4$ and $q = 2$, one could have $r = 5$, with the relations $c_1 = a_1, c_2 = a_2, c_3 = a_3 = b_1, c_4 = a_4$ and $c_5 = b_2$, leading to the factorization $z_0 c_1 z_1 c_2 z_2 c_3 z_3 c_4 z_4 c_5 z_5$, as pictured in Figure 3.

u_0	a_1	u_1	a_2	u_2	a_3	u_3	a_4	u_4
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z_0	c_1	z_1	c_2	z_2	c_3	z_3	c_4	z_4	c_5	z_5
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v_0	b_1	v_1	b_2	v_2
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Fig. 3. A word of $K \cap L$ and its factorizations.

The associated pair of infiltration maps (α, β) is given by

$$\begin{aligned} \alpha(1) &= 1 & \alpha(2) &= 2 & \alpha(3) &= 3 & \alpha(4) &= 4 \\ \beta(1) &= 3 & \beta(2) &= 5 \end{aligned}$$

Two series of constraints will be imposed on the words z_i :

$$\begin{aligned} z_0 \in K_0, z_1 \in K_1, z_2 \in K_2, z_3 \in K_3 \text{ and } z_4 c_5 z_5 \in K_4, \\ z_0 c_1 z_1 c_2 z_2 \in L_0, z_3 c_4 z_4 \in L_1 \text{ and } z_5 \in L_2. \end{aligned}$$

We are now ready to state our main result. Let us denote by $I(p, q)$ the set of pairs of infiltration maps (α, β) with domain $\{1, \dots, p\}$ and $\{1, \dots, q\}$, respectively. Since $r \leq p + q$, the set $I(p, q)$ is finite.

Theorem 4.1. *Let $K = K_0 a_1 K_1 \dots a_p K_p$ and $L = L_0 b_1 L_1 \dots b_q L_q$ be two products of languages. Then their intersection is given by the formulas*

$$K \cap L = \bigcup_{(\alpha, \beta) \in I(p, q)} U(\alpha, \beta) \quad (4)$$

where

$$U(\alpha, \beta) = \bigcup_{(z_0, \dots, z_r) \in C(\alpha, \beta)} U_0 c_1 U_1 \dots c_r U_r \quad (5)$$

and, for $0 \leq i \leq r$,

$$U_i = [K_{\bar{\alpha}(i)}]_{\uparrow z_i} \cap [L_{\bar{\beta}(i)}]_{\uparrow z_i} \quad (6)$$

and $C(\alpha, \beta)$ is the set of $(r+1)$ -tuples $(z_0, \dots, z_r)$ of words such that

- (C₁) for $0 \leq k \leq p$, $z_{\alpha(k)} c_{\alpha(k)+1} z_{\alpha(k)+1} \dots c_{\alpha(k+1)-1} z_{\alpha(k+1)-1} \in K_k$,
- (C₂) for $0 \leq k \leq q$, $z_{\beta(k)} c_{\beta(k)+1} z_{\beta(k)+1} \dots c_{\beta(k+1)-1} z_{\beta(k+1)-1} \in L_k$.

For instance, if (α, β) is the pair of infiltration maps of our example, one would have

$$\begin{aligned} U(\alpha, \beta) = & \bigcup_{(z_0, \dots, z_5) \in C(\alpha, \beta)} ([K_0]_{\uparrow z_0} \cap [L_0]_{\uparrow z_0}) a_1 ([K_1]_{\uparrow z_1} \cap [L_0]_{\uparrow z_1}) a_2 \\ & ([K_2]_{\uparrow z_2} \cap [L_0]_{\uparrow z_2}) b_1 ([K_3]_{\uparrow z_3} \cap [L_1]_{\uparrow z_3}) a_4 ([K_4]_{\uparrow z_4} \cap [L_1]_{\uparrow z_4}) b_2 ([K_4]_{\uparrow z_5} \cap [L_2]_{\uparrow z_5}) \end{aligned}$$

and the conditions (C₁) and (C₂) would be

$$(C_1) \ z_0 \in K_0, z_1 \in K_1, z_2 \in K_2, z_3 \in K_3, z_4 c_5 z_5 \in K_4,$$

$$(C_2) \ z_0 c_1 z_1 c_2 z_2 \in L_0, z_3 c_4 z_4 \in L_1 \text{ and } z_5 \in L_2.$$

Before proving the theorem, it is important to note that if the languages $K_0, \dots, K_p, L_0, \dots, L_q$ are regular, the union indexed by $C(\alpha, \beta)$ is actually a finite union. Indeed, Proposition 2.1 shows that, if R is a regular language, there are only finitely many languages of the form $[R]_z$.

Proof. Let U be the right hand side of (4). We first prove that $K \cap L$ is a subset of U . Let z be a word of $K \cap L$. Then z can be factorized as $u_0 a_1 u_1 \dots a_p u_p$, with $u_0 \in K_0, \dots, u_p \in K_p$ and as $v_0 b_1 v_1 \dots b_q v_q$, with $v_0 \in L_0, \dots, v_q \in L_q$. The common refinement of these two factorizations leads to a factorization of the form $z_0 c_1 z_1 \dots c_r z_r$, where each letter c_k is either equal to some a_i or to some b_j or both. This naturally defines a pair of infiltration maps $\alpha : \{1, \dots, p\} \rightarrow \{1, \dots, r\}$ and $\beta : \{1, \dots, q\} \rightarrow \{1, \dots, r\}$. Conditions (C₁) and (C₂) just say that the factorization $z_0 c_1 z_1 \dots c_r z_r$ is a refinement of the two other ones. Now, since, for $0 \leq i \leq r$, the word z_i belongs to $[K_{\bar{\alpha}(i)}]_{\uparrow z_i} \cap [L_{\bar{\beta}(i)}]_{\uparrow z_i}$, the word z belongs to U . Thus $K \cap L \subseteq U$.

We now prove the opposite inclusion. Let $r \leq p + q$ be an integer, let $\alpha : \{1, \dots, p\} \rightarrow \{1, \dots, r\}$ and $\beta : \{1, \dots, q\} \rightarrow \{1, \dots, r\}$ be two infiltration maps and let $(z_0, \dots, z_r) \in C(\alpha, \beta)$ and $c_1, \dots, c_r$ satisfying (C₁) and (C₂). It suffices to prove that $U_0 c_1 U_1 \dots c_r U_r$ is a subset of $K \cap L$. We need a stronger version of (C₁) and (C₂).

Lemma 4.2. *The following relations hold:*

$$(C_3) \text{ for } 0 \leq k \leq p, U_{\alpha(k)} c_{\alpha(k)+1} U_{\alpha(k)+1} \dots c_{\alpha(k+1)-1} U_{\alpha(k+1)-1} \subseteq K_k,$$

$$(C_4) \text{ for } 0 \leq k \leq q, U_{\beta(k)} c_{\beta(k)+1} U_{\beta(k)+1} \dots c_{\beta(k+1)-1} U_{\beta(k+1)-1} \subseteq L_k.$$

Coming back once again to our main example, these conditions would be

$$(C_3) \ U_0 \subseteq K_0, U_1 \subseteq K_1, U_2 \subseteq K_2, U_3 \subseteq K_3, U_4 c_4 U_5 \subseteq K_4,$$

$$(C_4) \ U_0 c_1 U_2 c_2 U_2 \subseteq L_0, U_3 c_4 U_4 \subseteq L_1, U_5 \subseteq L_5.$$

Proof. Let η_k be the syntactic morphism of K_k . To simplify notation, let us set $i = \alpha(k) + 1$ and $j = \alpha(k + 1) - 1$. Since $\alpha(k) = i - 1 < i < \dots < j < \alpha(k + 1)$, one gets $\bar{\alpha}(i - 1) = \bar{\alpha}(i) = \dots = \bar{\alpha}(j) = k$. Let $u_{i-1} \in U_{i-1}, u_i \in U_i, \dots, u_j \in U_j$. Then $u_{i-1} \in [U_k]_{\uparrow z_{i-1}}, u_i \in [U_k]_{\uparrow z_i}, \dots, u_j \in [U_k]_{\uparrow z_j}$ and by definition, $\eta_k(z_{i-1}) \leq \eta_k(u_{i-1}), \eta_k(z_i) \leq \eta_k(u_i), \dots, \eta_k(z_j) \leq \eta_k(u_j)$. Therefore we get

$$\begin{aligned} \eta_k(z_{i-1} c_i z_i \dots c_j z_j) &= \eta_k(z_{i-1}) \eta_k(c_i) \eta_k(z_i) \dots \eta_k(c_j) \eta_k(z_j) \\ &\leq \eta_k(u_{i-1}) \eta_k(c_i) \eta_k(u_i) \dots \eta_k(c_j) \eta_k(u_j) = \eta_k(u_{i-1} c_i u_i \dots c_j u_j) \end{aligned}$$

Now, since $z_{i-1} c_i z_i \dots c_j z_j \in K_k$ by (C₁), we also get $u_{i-1} c_i u_i \dots c_j u_j \in K_k$, which proves (C₃). The proof of (C₄) is similar. $\square$

Now, since $\bar{\alpha}$ and $\bar{\beta}$ are surjective, Lemma 4.2 shows that $U_0 c_1 U_1 \dots c_r U_r$ is a subset of $K \cap L$, which concludes the proof of the theorem. $\square$

Example 4.3. Let $K = b^*aA^*ba^*$ and $L = a^*bA^*ab^*$. The algorithm described in Theorem 4.1 gives for $K \cap L$ the expression $aa^*bA^*ba^*a \cup bb^*aA^*ba^*a \cup aa^*bA^*ab^*b \cup bb^*aA^*ab^*b \cup aa^*ba^*a \cup bb^*ab^*b$.

Corollary 4.4. *Let $\mathcal{L}$ be a lattice of regular languages closed under quotient. Then its polynomial closure is also a lattice closed under quotient.*

5 Some variants of the product

We consider in this section two variants of the product introduced by Schützenberger in [15]: unambiguous and deterministic products. These products were also studied in [2,3,4,5,9,11,12,13].

5.1 Unambiguous product

The marked product $L = L_0a_1L_1 \cdots a_nL_n$ of n nonempty languages $L_0, L_1, \dots, L_n$ of A^* is *unambiguous* if every word u of L admits a unique factorization of the form $u_0a_1u_1 \cdots a_nu_n$ with $u_0 \in L_0, u_1 \in L_1, \dots, u_n \in L_n$. We require the languages L_i to be nonempty to make sure that subfactorizations remain unambiguous:

Proposition 5.1. *Let $L_0a_1L_1 \cdots a_nL_n$ be an unambiguous product and let $i_1, \dots, i_k$ be a sequence of integers satisfying $0 < i_1 < \dots < i_k < n$. Finally, let $R_0 = L_0a_1L_1 \cdots a_{i_1-1}L_{i_1-1}$, $R_1 = L_{i_1}a_{i_1+1}L_{i_1+1} \cdots a_{i_2-1}L_{i_2-1}$, $\dots$, $R_k = L_{i_k}a_{i_k+1}L_{i_k+1} \cdots a_nL_n$. Then the product $R_0a_{i_1}R_1 \cdots a_{i_k}R_k$ is unambiguous.*

Proof. Trivial.

The *unambiguous polynomial closure* of a class of languages $\mathcal{L}$ of A^* is the set of languages that are finite unions of unambiguous products of the form $L_0a_1L_1 \cdots a_nL_n$, where the a_i 's are letters and the L_i 's are elements of $\mathcal{L}$. The term *closure* actually requires a short justification.

Proposition 5.2. *Any unambiguous product of unambiguous products is unambiguous.*

Proof. Let

$$\begin{aligned} L_0 &= L_{0,0}a_{1,0}L_{1,0} \cdots a_{k_0,0}L_{k_0,0} \\ L_1 &= L_{0,1}a_{1,1}L_{1,1} \cdots a_{k_1,1}L_{k_1,1} \\ &\vdots \\ L_n &= L_{0,n}a_{1,n}L_{1,n} \cdots a_{k_n,n}L_{k_n,n} \end{aligned} \tag{7}$$

be unambiguous products and let $L = L_0b_1L_1 \cdots b_nL_n$ be an unambiguous product. We claim that the product

$$L_{0,0}a_{1,0}L_{1,0} \cdots a_{k_0,0}L_{k_0,0}b_1L_{0,1}a_{1,1}L_{1,1} \cdots b_nL_{0,n}a_{1,n}L_{1,n} \cdots a_{k_n,n}L_{k_n,n}$$

is unambiguous. Let u be a word of L with two factorizations

$$x_{0,0}a_{1,0}x_{1,0} \cdots a_{k_0,0}x_{k_0,0}b_1x_{0,1}a_{1,1}x_{1,1} \cdots b_nx_{0,n}a_{1,n}x_{1,n} \cdots a_{k_n,n}x_{k_n,n}$$

and

$$y_{0,0}a_{1,0}y_{1,0} \cdots a_{k_0,0}y_{k_0,0}b_1y_{0,1}a_{1,1}x_{1,1} \cdots b_ny_{0,n}a_{1,n}y_{1,n} \cdots a_{k_n,n}y_{k_n,n}$$

with $x_{0,0}, y_{0,0} \in L_{0,0}, \dots, x_{k_n,n}, y_{k_n,n} \in L_{k_n,n}$. Setting

$$\begin{aligned} x_0 &= x_{0,0}a_{1,0}x_{1,0} \cdots a_{k_0,0}x_{k_0,0} & y_0 &= y_{0,0}a_{1,0}y_{1,0} \cdots a_{k_0,0}y_{k_0,0} \\ x_1 &= x_{0,1}a_{1,1}x_{1,1} \cdots a_{k_1,1}x_{k_1,1} & y_1 &= y_{0,1}a_{1,1}y_{1,1} \cdots a_{k_1,1}y_{k_1,1} \\ &\vdots & &\vdots \\ x_n &= x_{0,n}a_{1,n}x_{1,n} \cdots a_{k_n,n}x_{k_n,n} & y_n &= y_{0,n}a_{1,n}y_{1,n} \cdots a_{k_n,n}y_{k_n,n} \end{aligned} \tag{8}$$

we get two factorizations of u : $x_0b_1x_1 \cdots b_nx_n$ and $y_0b_1y_1 \cdots b_ny_n$. Since the product $L_0b_1L_1 \cdots a_nL_n$ is unambiguous, we have $x_0 = y_0, \dots, x_n = y_n$. Each of these words has now two factorizations given by (8) and since the products of (7) are unambiguous, these factorizations are equal. This proves the claim and the proposition. $\square$

We now consider the intersection of two unambiguous products.

Theorem 5.3. *If the products $K = K_0a_1K_1 \cdots a_pK_p$ and $L = L_0b_1L_1 \cdots b_qL_q$ are unambiguous, the products occurring in Formula (4) are all unambiguous.*

Proof. Let (α, β) be a pair of infiltration maps, and let $U_i = [K_{\bar{\alpha}(i)}]_{\uparrow z_i} \cap [L_{\bar{\beta}(i)}]_{\uparrow z_i}$, for $0 \leq i \leq r$. We claim that the product $U = U_0c_1U_1 \cdots c_rU_r$ is unambiguous. Let

$$u = u_0c_1u_1 \cdots c_ru_r = u'_0c_1u'_1 \cdots c_ru'_r \tag{9}$$

be two factorizations of a word u of U such that, for $0 \leq i \leq r$, $u_i, u'_i \in U_i$. We prove by induction on s that $u_s = u'_s$.

Case $s = 0$. By the properties of α and β , we may assume without loss of generality that $\alpha(1) = 1$, which implies that $c_1 = a_1$. It follows from (C₃) that $U_0 \subseteq K_0$. Now the product $K_0a_1(K_1a_2K_2 \cdots a_pK_p)$ is unambiguous by Proposition 5.1, and by (C₃), $U_1c_2U_2 \cdots c_ru_r$ is contained in $K_1a_1K_2 \cdots a_pK_p$. Therefore, u admits the two factorizations $u_0a_1(u_1c_2u_2 \cdots c_ru_r)$ and $u'_0a_1(u'_1c_2u'_2 \cdots c_ru'_r)$ in this product. Thus $u_0 = u'_0$.

Induction step. Let $s > 0$ and suppose by induction that $u_i = u'_i$ for $0 \leq i \leq s-1$. If $s = r$, then necessarily $u_s = u'_s$. If $s < r$, we may assume without loss of generality that s is in the range of α . Thus $\alpha(k) = s$ for some k and $c_s = a_k$. We now consider two cases separately.

If $\alpha(k+1) = s+1$ (and $c_{s+1} = a_{k+1}$), it follows from (C₃) that u has two factorizations

$$\begin{aligned} (u_0c_1u_1 \cdots c_{s-1}u_{s-1})a_ku_s a_{k+1}(u_{s+1}c_{s+1}u_{s+2} \cdots c_ru_r) \text{ and} \\ (u_0c_1u_1 \cdots c_{s-1}u_{s-1})a_ku'_s a_{k+1}(u'_{s+1}c_{s+1}u'_{s+2} \cdots c_ru'_r) \end{aligned}$$

over the product $(K_0 a_1 K_1 \cdots a_{s-1} K_{s-1}) a_k K_s a_{k+1} (K_{s+1} a_{k+2} K_{s+2} \cdots a_p K_p)$. Since this product is unambiguous by Proposition 5.1, we get $u_s = u'_s$.

If $\alpha(k+1) \neq s+1$, then $s+1 = \beta(t+1)$ for some t and $c_{s+1} = b_{t+1}$. Setting $i = \beta(t)$, we get $c_i = b_t$ and it follows from (C₄) that u has two factorizations

$$(u_0 c_1 u_1 \cdots c_{i-1} u_{i-1}) b_t (u_i c_{i+1} u_{i+1} \cdots c_s u_s) b_{t+1} (u_{s+1} c_{s+2} u_{s+2} \cdots c_r u_r) \text{ and} \\ (u_0 c_1 u_1 \cdots c_{i-1} u_{i-1}) b_t (u'_i c_{i+1} u'_{i+1} \cdots c_s u'_s) b_{t+1} (u'_{s+1} c_{s+2} u'_{s+2} \cdots c_r u'_r)$$

over the product $(L_0 b_1 L_1 \cdots b_{t-1} L_{t-1}) b_t L_t b_{t+1} (L_{t+1} b_{t+2} L_{t+2} \cdots b_p L_p)$. This product is unambiguous by Proposition 5.1, and thus

$$u_i c_{i+1} u_{i+1} \cdots c_s u_s = u'_i c_{i+1} u'_{i+1} \cdots c_s u'_s$$

Now the induction hypothesis gives $u_i = u'_i, \dots, u_{s-1} = u'_{s-1}$ and one finally gets $u_s = u'_s$. $\square$

We state separately another interesting property.

Theorem 5.4. *Let $K = K_0 a_1 K_1 \cdots a_p K_p$ and $L = L_0 b_1 L_1 \cdots b_q L_q$ be two unambiguous products and let (α, β) and (α', β') be two pairs of infiltration maps of $I(p, q)$. If the sets $U(\alpha, \beta)$ and $U(\alpha', \beta')$ meet, then $\alpha = \alpha'$ and $\beta = \beta'$.*

Proof. Suppose that a word u belongs to $U(\alpha, \beta)$ and to $U(\alpha', \beta')$. Then u has two decompositions of the form

$$u = u_0 c_1 u_1 \cdots c_r u_r = u'_0 c'_1 u'_1 \cdots c'_{r'} u'_{r'}$$

Condition (C₁) [(C₂)] and the unambiguity of the product $K_0 a_1 K_1 \cdots a_p K_p$ [$L_0 b_1 L_1 \cdots b_q L_q$] show that, for $0 \leq i \leq p$ and for $0 \leq j \leq q$,

$$u_{\alpha(i)} c_{\alpha(i)+1} u_{\alpha(i)+1} \cdots c_{\alpha(i+1)-1} u_{\alpha(i+1)-1} = \\ u'_{\alpha'(i)} c'_{\alpha'(i)+1} u'_{\alpha'(i)+1} \cdots c'_{\alpha'(i+1)-1} u'_{\alpha'(i+1)-1} \in K_i \quad (10)$$

$$u_{\beta(j)} c_{\beta(j)+1} u_{\beta(j)+1} \cdots c_{\beta(j+1)-1} u_{\beta(j+1)-1} = \\ u'_{\beta'(j)} c'_{\beta'(j)+1} u'_{\beta'(j)+1} \cdots c'_{\beta'(j+1)-1} u'_{\beta'(j+1)-1} \in L_j \quad (11)$$

We prove by induction on s that, for $1 \leq s \leq \min(r, r')$, the following properties hold:

$$E_1(s) : u_{s-1} = u'_{s-1} \text{ and } c_s = c'_s,$$

$$E_2(s) : \bar{\alpha}(s) = \bar{\alpha}'(s) \text{ and } \bar{\beta}(s) = \bar{\beta}'(s),$$

$$E_3(s) : \text{for } i \leq \bar{\alpha}(s), \alpha(i) = \alpha'(i) \text{ and for } j \leq \bar{\beta}(s), \beta(j) = \beta'(j).$$

Case $s = 1$. We know that either $\alpha(1) = 1$ or $\beta(1) = 1$ and that either $\alpha'(1) = 1$ or $\beta'(1) = 1$. Suppose that $\alpha(1) = 1$. We claim that $\alpha'(1) = 1$. Otherwise, one has $\beta'(1) = 1$. Now, Formula (10) applied to $i = 0$ gives

$$u_0 = u'_0 c'_1 u'_1 \cdots c'_{\alpha'(1)-1} u'_{\alpha'(1)-1}$$

and Formula (11) applied to $j = 0$ gives

$$u_0 c_1 u_1 \cdots c_{\beta(1)-1} u_{\beta(1)-1} = u'_0.$$

Therefore $u_0 = u'_0$ and $\alpha'(1) = 1$, which proves the claim. It follows also that $a_1 = c_{\alpha(1)} = c_{\alpha'(1)}$ and thus $c_1 = c'_1$. We also have in this case $\bar{\alpha}(1) = \bar{\alpha}'(1) = 1$. A similar argument shows that if $\alpha'(1) = 1$, then $\alpha(1) = 1$. Therefore, the conditions $\alpha(1) = 1$ and $\alpha'(1) = 1$ are equivalent and it follows that $\bar{\alpha}(1) = \bar{\alpha}'(1)$. A dual argument would prove that the conditions $\beta(1) = 1$ and $\beta'(1) = 1$ are equivalent and that $\bar{\beta}(1) = \bar{\beta}'(1)$.

Induction step. Let s be such that $1 \leq s+1 \leq \min(r, r')$ and suppose by induction that the properties $E_1(i)$, $E_2(i)$, $E_3(i)$ hold for $1 \leq i \leq s$.

Lemma 5.5. *Suppose that $P_\alpha(s)$ holds and let $k = \bar{\alpha}(s)$. Then*

$$s \leq \alpha'(k+1) - 1 \quad (12)$$

and

$$u_s = u'_s c'_{s+1} u'_{s+1} \cdots c_{\alpha'(k+1)-1} u'_{\alpha'(k+1)-1} \quad (13)$$

Proof. Applying (10) with $i = k$, we get

$$u_{\alpha(k)} c_{\alpha(k)+1} u_{\alpha(k)+1} \cdots c_s u_s = u'_{\alpha'(k)} c'_{\alpha'(k)+1} u'_{\alpha'(k)+1} \cdots c_{\alpha'(k+1)-1} u'_{\alpha'(k+1)-1} \quad (14)$$

Since $\bar{\alpha}(s) = \bar{\alpha}'(s)$ by $E_2(s)$, one has $\bar{\alpha}'(s) = k$ and $\alpha'(k+1) \geq s+1$ by Lemma 3.1, which gives (12). Further, since $k = \bar{\alpha}(s)$, it follows from $E_3(s)$ that $\alpha(k) = \alpha'(k)$. Now, for $i \leq s$, $E_1(i)$ implies that $u_{i-1} = u'_{i-1}$ and $c_i = c'_i$. It follows that the word $u_{\alpha(k)} c_{\alpha(k)+1} u_{\alpha(k)+1} \cdots c_s$ is a prefix of both sides of (14). Therefore, this prefix can be deleted from both sides of (14), which gives (13). $\square$

We now establish $E_1(s+1)$.

Lemma 5.6. *One has $u_s = u'_s$ and $c_{s+1} = c'_{s+1}$. Further, $P_\alpha(s)$ and $P_{\alpha'}(s)$ are equivalent and $P_\beta(s)$ and $P_{\beta'}(s)$ are equivalent.*

Proof. Let us prove that u'_s is a prefix of u_s . By Lemma 3.3, either $P_\alpha(s)$ or $P_\beta(s)$ holds. Suppose that $P_\alpha(s)$ holds. Then by Lemma 5.5, u'_s is a prefix of u_s . If $P_\beta(s)$ holds, we arrive to the same conclusion by using (11) in place of (10) in the proof of Lemma 5.5.

Now, a symmetrical argument using the pair $(\bar{\alpha}', \bar{\beta}')$ would show that u_s is a prefix of u'_s . Therefore, $u_s = u'_s$. Coming back to (13), we obtain $\alpha'(k+1) = s+1$ and since by $E_2(s)$, $k = \bar{\alpha}(s) = \bar{\alpha}'(s)$, one gets $\alpha'(\bar{\alpha}'(s) + 1) = s+1$, which, by Lemma 3.2, is equivalent to $P_{\alpha'}(s)$. Thus $P_\alpha(s)$ implies $P_{\alpha'}(s)$ and a dual argument would prove the opposite implication.

We also have $c_{s+1} = c_{\alpha(k+1)} = a_{k+1} = c'_{\alpha'(k+1)} = c'_{s+1}$ and thus $c_{s+1} = c'_{s+1}$. Finally, a similar argument works for β . $\square$

We now come to the proof of $E_2(s+1)$ and $E_3(s+1)$. Since $P_\alpha(s)$ and $P_{\alpha'}(s)$ are equivalent, the next two lemma cover all cases.

Lemma 5.7. *If neither $P_\alpha(s)$ nor $P_{\alpha'}(s)$ hold, then $\bar{\alpha}(s+1) = \bar{\alpha}'(s+1)$ and for $i \leq \bar{\alpha}(s+1)$, $\alpha(i) = \alpha'(i)$. Similarly, if neither $P_\beta(s)$ nor $P_{\beta'}(s)$ hold, then $\bar{\beta}(s+1) = \bar{\beta}'(s+1)$ and for $i \leq \bar{\beta}(s+1)$, $\beta(i) = \beta'(i)$.*

Proof. We just prove the “ α part” of the lemma. If neither $P_\alpha(s)$ nor $P_{\alpha'}(s)$ hold, then $\bar{\alpha}(s+1) = \bar{\alpha}(s)$ and $\bar{\alpha}'(s+1) = \bar{\alpha}'(s)$. Since $\bar{\alpha}(s) = \bar{\alpha}'(s)$ by $E_2(s)$, one gets $\bar{\alpha}(s+1) = \bar{\alpha}'(s+1)$. The second property is an immediate consequence of $E_3(s)$. $\square$

Lemma 5.8. *If both $P_\alpha(s)$ and $P_{\alpha'}(s)$ hold, then $\bar{\alpha}(s+1) = \bar{\alpha}'(s+1)$ and for $i \leq \bar{\alpha}(s+1)$, $\alpha(i) = \alpha'(i)$. Similarly, if both $P_\beta(s)$ and $P_{\beta'}(s)$ hold, then $\bar{\beta}(s+1) = \bar{\beta}'(s+1)$ and for $i \leq \bar{\beta}(s+1)$, $\beta(i) = \beta'(i)$.*

Proof. Again, we just prove the “ α part” of the lemma. If both $P_\alpha(s)$ and $P_{\alpha'}(s)$ hold, then $\bar{\alpha}(s+1) = \bar{\alpha}(s) + 1$ and $\bar{\alpha}'(s+1) = \bar{\alpha}'(s) + 1$. Since $\bar{\alpha}(s) = \bar{\alpha}'(s)$ by $E_2(s)$, one gets $\bar{\alpha}(s+1) = \bar{\alpha}'(s+1)$. Property $E_3(s)$ shows that for $i \leq \bar{\alpha}(s)$, $\alpha(i) = \alpha'(i)$. Since $\bar{\alpha}(s+1) = \bar{\alpha}(s) + 1$, it just remains to prove that

$$\alpha(\bar{\alpha}(s+1)) = \alpha'(\bar{\alpha}'(s+1)) \quad (15)$$

But Lemma 3.2 shows that $\alpha(\bar{\alpha}(s+1)) = s+1$ and $\alpha'(\bar{\alpha}'(s+1)) = s+1$, which proves (15) since $\bar{\alpha}(s+1) = \bar{\alpha}'(s+1)$. $\square$

This concludes the induction step and the proof of Theorem 5.4. $\square$

Corollary 5.9. *Let $\mathcal{L}$ be a lattice of regular languages closed under quotient. Then its unambiguous polynomial closure is also a lattice closed under quotient.*

If $\mathcal{L}$ is a Boolean algebra, then one can be more precise.

Corollary 5.10. *Let $\mathcal{L}$ be a Boolean algebra of regular languages closed under quotient. Then its unambiguous polynomial closure is also a Boolean algebra closed under quotient.*

Let us conclude with an example which shows that, under the assumptions of Theorem 5.4, the sets $U(\alpha, \beta)$ cannot be further decomposed as a disjoint union of unambiguous products.

Let $K = K_0 a K_1$ and $L = L_0 a L_1$ with $K_0 = L_1 = 1 + b + c + c^2$ and $L_0 = K_1 = a + ab + ba + ac + ca + ac^2 + bab + cac + cac^2$. Then

$$\begin{aligned} K \cap L = & aa + aab + aba + aac + aca + aac^2 + abab + acac + acac^2 + \\ & baa + baab + baba + baac + baac^2 + babab + caa + \\ & caab + caac + caca + caac^2 + cacac + cacac^2 \end{aligned}$$

One can write for instance $K \cap L$ as $(1 + b + c)aa(1 + b + c + c^2) + (1 + b)a(1 + b)a(1 + b) + (1 + c)a(1 + c)a(1 + c + c^2)$ but the three components of this language

are not disjoint, since they all contain aa . Note that the words $acab$, $abac$, $baca$ and $caba$ are not in $K \cap L$.

The syntactic ordered monoid of K_0 and L_1 has 4 elements $\{1, a, b, c\}$ and is presented by the relations $a = ba = b^2 = bc = ca = cb = 0$ and $c^2 = b$. Its syntactic order is defined by $a < b < c < 1$.

The syntactic ordered monoid of L_0 and K_1 has 13 elements:

$$\{1, a, b, c, a^2, ab, ac, ba, ca, c^2, ac^2, bab, cac\}$$

and is defined by the relations $cac^2 = bab$ and

$$b^2 = bc = cb = a^2 = aba = aca = bac = cab = c^2a = c^3 = 0.$$

The syntactic order is:

There is only one pair of infiltration maps (α, β) of $I(1, 1)$ that defines a nonempty set $U(\alpha, \beta)$. This pair is defined as follows: $\alpha(1) = 1$ and $\beta(1) = 2$. The triples (z_0, z_1, z_2) of $C(\alpha, \beta)$ are exactly the triples of words such that $z_0az_1az_2 \in K \cap L$. In particular, $z_0 \in \{1, b, c\}$, $z_1 \in \{1, b, c\}$ and $z_2 \in \{1, b, c, c^2\}$. Now, one has

$$\begin{array}{llll} [K_0]_{\uparrow 1} = 1 & [K_0]_{\uparrow b} = 1 + b + c + c^2 & [K_0]_{\uparrow c} = 1 + c & \\ [K_1]_{\uparrow 1} = 1 & [K_1]_{\uparrow b} = 1 + b & [K_1]_{\uparrow c} = 1 + c & [K_1]_{\uparrow c^2} = 1 + c + c^2 \\ [L_0]_{\uparrow 1} = 1 & [L_0]_{\uparrow b} = 1 + b & [L_0]_{\uparrow c} = 1 + c & \\ [L_1]_{\uparrow 1} = 1 & [L_1]_{\uparrow b} = 1 + b + c + c^2 & [L_1]_{\uparrow c} = 1 + c & [L_1]_{\uparrow c^2} = 1 + b + c + c^2 \end{array}$$

which gives the following possibilities for the triples (U_0, U_1, U_2) , for the following triples $z = (z_0, z_1, z_2)$:

$z = (1, 1, 1)$	$U_0 = 1$	$U_1 = 1$	$U_2 = 1$
$z = (b, b, b)$	$U_0 = 1 + b$	$U_1 = 1 + b$	$U_2 = 1 + b$
$z = (c, c, c)$	$U_0 = 1 + c$	$U_1 = 1 + c$	$U_2 = 1 + c$
$z = (b, c, c^2)$	$U_0 = 1 + b$	$U_1 = 1 + c$	$U_2 = 1 + c + c^2$
$z = (c, c, c^2)$	$U_0 = 1 + c$	$U_1 = 1 + c$	$U_2 = 1 + c + c^2$

5.2 Deterministic product

The marked product $L = L_0 a_1 L_1 \cdots a_n L_n$ of n nonempty languages $L_0, L_1, \dots, L_n$ of A^* is *left deterministic* [*right deterministic*] if, for $1 \leq i \leq n$, the set $L_0 a_1 L_1 \cdots L_{i-1} a_i [a_i L_i \cdots a_n L_n]$ is a prefix [suffix] code. This means that every word of L has a unique prefix [suffix] in $L_0 a_1 L_1 \cdots L_{i-1} a_i [a_i L_i \cdots a_n L_n]$. It is observed in [3, p. 495] that the marked product $L_0 a_1 L_1 \cdots a_n L_n$ is deterministic if and only if, for $1 \leq i \leq n$, the language $L_{i-1} a_i$ is a prefix code. Since the product of two prefix codes is a prefix code, we get the following proposition.

Proposition 5.11. *Any left [right] deterministic product of left [right] deterministic products is left [right] deterministic.*

Proof. This follows immediately from the fact that the product of two prefix codes is a prefix code. $\square$

Factorizing a deterministic product also gives a deterministic product. More precisely, one has the following result.

Proposition 5.12. *Let $L_0 a_1 L_1 \cdots a_n L_n$ be a left [right] deterministic product and let $i_1, \dots, i_k$ be a sequence of integers satisfying $0 < i_1 < \dots < i_k < n$. Finally, let $R_0 = L_0 a_1 L_1 \cdots a_{i_1-1} L_{i_1-1}$, $\dots$, $R_k = L_{i_k} a_{i_k+1} L_{i_k+1} \cdots L_{n-1} a_n L_n$. Then the product $R_0 a_{i_1} R_1 \cdots a_{i_k} R_k$ is left [right] deterministic.*

Proof. Trivial. $\square$

The *left [right] deterministic polynomial closure* of a class of languages $\mathcal{L}$ of A^* is the set of languages that are finite unions of left [right] deterministic products of the form $L_0 a_1 L_1 \cdots a_n L_n$, where the a_i 's are letters and the L_i 's are elements of $\mathcal{L}$.

We can now state the counterpart of Theorem 5.3 for deterministic products.

Theorem 5.13. *If the products $K = K_0 a_1 K_1 \cdots a_p K_p$ and $L = L_0 b_1 L_1 \cdots b_q L_q$ are deterministic, the products occurring in Formula (4) are all deterministic.*

Proof. Let $i \in \{0, \dots, r\}$. By construction, there exists $k \geq 0$ such that $i + 1 = \alpha(k + 1)$ or $i + 1 = \beta(k + 1)$. By Lemma 4.2, there exists $j \leq i$ such that either $U_j c_{j+1} U_{j+1} \cdots U_i \subseteq K_k$ and $c_{\alpha(k+1)} = a_{k+1}$ or $U_j c_{j+1} U_{j+1} \cdots U_i \subseteq L_k$ and

$c_{\alpha(k+1)} = b_{k+1}$. Suppose we are in the first case and that $U_i c_{i+1}$ is not a prefix code. Then $U_j c_{j+1} U_{j+1} \cdots U_i c_{i+1}$ is not a prefix code and thus $K_k a_{k+1}$ is not a prefix code. This yields a contradiction since the product $K_0 a_1 K_1 \cdots a_p K_p$ is deterministic. $\square$

Corollary 5.14. *Let $\mathcal{L}$ be a lattice of regular languages closed under quotient. Then its deterministic polynomial closure is also closed under quotient.*

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