On the Regularization of Chattering Executions in Real Time Simulation of Hybrid Systems

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Abstract: In this paper we present a new method to perform the higher order sliding modes analysis of trajectories of hybrid systems with chattering behavior. This method improves our previous work [AC15] as it modifies numerical simulation algorithms to make them compute the higher order terms of the normal unit vectors of the systems dynamics whenever the 1^{st} order sliding mode theory cannot be applied. Such modification does not affect the generality of our previous contribution in [AC15]. Our algorithm is general enough to handle both chattering on a single $\mathbb{R}^{n-1}$ switching manifold (i.e. chattering between two dynamics) as well as chattering on the intersection of finitely many intersected $\mathbb{R}^{n-1}$ switching manifolds. In this last case, we show by a special hierarchical application of convex combinations, that unique solutions can be found in general cases when the switching function takes the form of finitely many intersecting manifolds so that an efficient numerical treatment of the sliding motion constrained on the entire discontinuity region (including the switching intersection) is guaranteed. Illustrations of the techniques developed in this article are given on representative examples.

1 Introduction

Because of their heterogeneous composition, the word hybrid is attached to dynamical systems which contain state variables that are capable of evolving continuously (flowing) and/or evolving discontinuously (jumping) [CGST07]. That is, the presence of two different behaviors, continuous and discrete, is the cause of heterogeneity. Systems of this type are common in embedded computation, robotics, mechatronics, avionics, and process control [ZJLS01] [CGST07]. Hybrid systems also arise naturally in control systems where the value of a control variable may jump or whenever the laws of physics are dicontinuous. Typically, the continuous dynamics of the system in the different operation modes are described by sets of ordinary differential equations or differential-algebraic equations. The changing between different operation modes is modeled by discrete transitions resulting in switching between sets of equations describing each operation mode [LA09].
However, the interaction of continuous-time and discrete-time dynamics emerging from its components and/or their interconnection may lead to *chattering executions* [ZYM08]. Similar behavior appears in variable structure control systems and in relay control systems [JBÅ02]. Chattering executions can be defined as solutions to the system having infinitely many discrete transitions in finite time. Although physical systems do not show chattering behavior, models of real systems may be chattering due to modeling over-

abstraction. Physically, chattering behaviour occurs if equal thresholds for the transition conditions of different modes are given and the system starts to oscillate around them. On the other hand, numerical errors may lead to numerical chattering as transition conditions may be satisfied due to local errors. The numerical solution of a hybrid system exhibiting chattering behavior requires high computational costs as small step-sizes are required to restart the integration after each mode change. In the worst case, the numerical integration breaks down, as it does not proceed in time, but chatters between modes. The chattering behavior has to be treated in an appropriate way to ensure that the numerical integration terminates in a reasonable time. To deal with chattering executions of hybrid systems one needs to detect regions on the switching manifold, on which chattering can occur, and force the solution trajectory to slide on the manifold in these regions [dBBC$^+$08] [WKH14] [GST11]. An additional mode, the so-called sliding mode, can be inserted into the hybrid system to represent the dynamics during sliding, and thus, replaces the chattering. The sliding mode became the principle operation mode in so-called variable structures systems. A variable structure system consists of a set of continuous subsystems with control actions are discontinuous functions of the system state, disturbances (if they are accessible for measurement), and reference inputs. Filippovs differential inclusion method (the so-called equivalent dynamics) [Fil88] [BHJ13] is a method that was developed by Filippov to define the system dynamics on the switching surface in such a way that the state trajectory moves along the surface. In this method, regularizations of the solution trajectories on both sides in a small neighborhood around the surface are used to determine the average velocity on the surface. Another approach called equivalent control was presented by Utkin [Utk92]. For linear control, this is an identical approximation to the equivalent dynamics. However, nonlinear control may derive different behaviors since the true system behavior near the sliding surface can be attributed to hysteresis phenomena. The method of equivalent dynamics derives sliding behavior closer to the true dynamics than the method of equivalent control in these situations. In some cases, where system behavior is not well-behaved near the switching surface, i.e. the case of a saturated high gain amplifier, when system variable values tends to infinity close to the discontinuity, equivalent control may generate better approximations. The reason is that there are no higher order hysteresis effects, therefore, modeling with equivalent dynamics, which assumes hysteresis, results in the generation of deviant behaviors. However, the computation of the equivalent dynamics turns of to be difficult whenever the systems chatters between more than two dynamics or modes of operations, a scenario which may appears in control applications whenever there are multiple discontinuous control variables. Indeed, the computation of equivalent dynamics is a challenging task in special classes of hybrid systems where the data constraints in the system do not allow to determine the existence of chattering execution using the 1^{st} order theory of sliding modes.

As an extension to our previous work in [AC15], we propose in this aricle an adequate technique to detect the chattering set "on the fly " in real time simulation of hybrid systems using the higher order theory of sliding modes, and therefore circumvent it by appropriately regularization the execution of the system beyond the limit time of the infinitely fast discrete transitions. Our approach is based on mixing compile-time transformations of hybrid programs (generating what is necessary to compute the smooth equivalent dynamics), the decision at run-time of the necessary and sufficient conditions for entering and exiting

a sliding mode, and the computation, at run-time, of the smooth equivalent dynamics.
The rest of this article is organized as follows: In Section 2, we present our formalism of hyrbid systems and hybrid solution trajectory as well as the chattering execution. Then, in Section 3, we explain how we could detect and regularize the chattering execution for the most general case when the chattering set belongs to the intersection of p transversally intersected $\mathbb{R}^{n-1}$ switching manifolds in at least p dimensions for any finite (positive) integer p. Section 4 presents the higher order sliding mode analysis with application to control problems with relay feedback. Finally, the simulation results and conclusion of the study are given in Sections 5 and 6 respectively.

2 Preliminaries

In this section we provide a brief introduction to hybrid systems, their executions, and chattering behavior.

Definition 1. (Hybrid Dynamical System) Define a hybrid system as a tuple

$$\mathcal{H} = (Q, D, E, G, R, F)$$

where

- $Q = \{1, ..., M\} \subset \mathbb{N}$ is a finite set of *discrete states*,
- $D = \{D_q\}_{q \in Q}$ is a set of *domains (or invariants)*, where D_q, a compact subset of $\mathbb{R}^n$, describes the conditions that the continuous state x has to satisfy at the discrete state $q \in Q$,
- $E \subset Q \times Q$ is a set of *discrete transitions (or edges)*, which define the connection beween states by identifying the pairs (q, q'), where for each $e = (q, q') \in E$ we denote its source $s(e) = q$ and its target $t(e) = q'$,
- $G = \{G_e\}_{e \in E}$ is a set of *guards*, where $G_e \subseteq D_{s(e)}$,
- $R = \{\phi_e\}_{e \in E}$ is a set of *reset maps*, where for each $e = (q, q') \in E$, $R_e : G_e \subseteq D_{s(e)} \to D_{t(e)}$,
- $F = \{f_q\}_{q \in Q}$ is a set of *vector fields*, where for each $q \in Q$, $f_q : D_q \to \mathbb{R}^n$ is Lipschitz on $\mathbb{R}^n$ and describes through a differential equation ODE the continuous evolution of the continuous state variables in $q \in Q$. The solution to the ODE is denoted by $x_i(t)$, where $x_i(t_0) = x_0$.

Definition 2. (Execution of a Hybrid System) An *execution* or *hybrid trajectory* of a hybrid system is a tuple

$$\chi = (\tau, \xi, \rho)$$

where

- $\tau = \{\tau_i\}_{i \in \mathbb{N}}$ such that $0 = \tau_0 \leq \tau_1 \leq \cdots \leq \tau_i \leq \cdots$ is a set of *events (or switching) times*.
- $\xi = \{\xi_i\}_{i \in \mathbb{N}}$ is a set of *initial conditions* with $\xi_i \in D_q$ for some $q \in Q$.
- $\eta = \{\eta_i\}_{i \in \mathbb{N}}$ with $\eta_i \in E$ is a *hybrid edge sequence*. An execution χ must satisfy the conditions for $i \in \mathbb{N}$

  1. $\xi_i = x_{s(\eta_i)}(\tau_i)$
  2. $\tau_{i+1} = min\{t \geq \tau_i : x_{s(\eta_i)}(t) \in G_{\eta_i}\}$
  3. $s(\eta_{i+1}) = t(\eta_i)$
  4. $\xi_{i+1} = R_{\eta_i}(x_{s(\eta_i)}(\tau_{i+1}))$

  The first and second conditions say that an event must occur at time τ_{i+1}. The thrid condition says that the discrete evolution map must evolve in a way that is consistent with the edges. The fourth condition says that the initial conditions must be in the image of the guards under the reset maps. We also require that the flow must stay in the domain D_{η_i} (i.e. $x_{s(\eta_i)}(t) \in D_{s(\eta_i)}$) for all time in $[\tau_i, \tau_{i+1}]$.

Definition 3. (Lie Derivative) Assume the flow map f_q is analytic in its second argument, the *Lie derivatives* $\mathcal{L}^k_{f_q} g_q : \mathbb{R}^n \rightarrow \mathbb{R}^n$ of a function g_q, also analytic in its second argument, along f_q, for $k > 0$, is defined by:

$$\mathcal{L}^k_{f_q} g_q(x(t)) = \left(\frac{\partial \mathcal{L}^{k-1}_{f_q} g_q(x(t))}{\partial x(t)} \right) \cdot f_q(x(t)) \tag{1}$$

with

$$\mathcal{L}^0_{f_q} g_q(x(t)) = g_q(x(t)) \tag{2}$$

Definition 4. (Pointwise relative degree)
We define the relative degree $n_q(x) : \mathbb{R}^n \rightarrow \mathbb{N}$ by:

$$n_q(x) = k \; if \bigwedge_{j<k} \mathcal{L}^j_{f_q} g_q(x) = 0 \wedge \mathcal{L}^k_{f_q} g_q(x) \neq 0 \tag{3}$$

Definition 5. (Chattering Hybrid System)
A hybrid system $\mathcal{H}$ is chattering if for some execution χ of $\mathcal{H}$ there exist finite constants τ_∞ and C such that

$$\lim_{i \to \infty} \tau_i = \sum_{i=0}^{\infty} (\tau_{i+1} - \tau_i) = \tau_\infty \tag{4}$$

$$\forall i \geq C : \;\; \tau_{i+1} - \tau_i = 0 \tag{5}$$

3 Robust Detection and Regularization of Chattering Executions

Following our contribution in [AC15], we consider a hybrid automaton $\mathcal{H}$ with a finite set of discrete states $q \in Q$ with transverse invariants where the state space is split into 2^p open convex regions (sub-domains) $C_q \in \mathbb{R}^n$, $q = 1, ...2^p$, and p switching manifold $\gamma_j(x) \in \mathbb{R}^{n-1}$, $j = 1, 2, ..., p$ by the intersection of p transversally intersected $\mathbb{R}^{n-1}$ switching manifolds Γ_j defined as the zeros of a set of scalar functions $\gamma_j(x)$ for $j = 1, 2, ..., p$,

$$\Gamma_j = \{x \in \mathbb{R}^n : \ \gamma_j(x) = 0 \ ; \quad j = 1, 2, ..., p\} \tag{6}$$

The zero crossing in opposite directions defines the switching between two adjacent flow sets. We will assume that all γ_j are assumed to be analytic in their second arguments so the normal unit vector $\perp_j$ for each one of the intersected switching manifolds Γ_j is well defined. Moreover, $\perp_j$ are linearly independent for all the $\mathbb{R}^{(n-r)}$ intersections where $r \in \{2, 3, ..., n\}$.

We use the multi-valued function $\alpha_j(x)$ such that the convex set is given for all $\Gamma_j|_{j=1,2,...,p}$ and $C_i|_{i=1,2,...,2^p}$ by:

$$\dot{x} \in \sum_{i=1}^{2^p} \left((\prod_{j=1}^{p} \frac{1 + 2\Psi_{j,i} \cdot \alpha_j(x) - \Psi_{j,i}}{2}) \cdot f_i(x) \right) \tag{7}$$

$$\alpha_j(x) = \left\{ \begin{array}{cc} 1 & for \ \ \gamma_j(x) > 0 \\ [0, 1] & for \ \ \gamma_j(x) = 0 \\ 0 & for \ \ \gamma_j(x) < 0 \end{array} \right\} \tag{8}$$

$$\sum_{i=1}^{2^p} \left(\prod_{j=1}^{p} \frac{1 + 2\Psi_{j,i} \cdot \alpha_j - \Psi_{j,i}}{2} \right) = 1 \tag{9}$$

where $\Psi_{j,i}$ gives the sign of the switching function $\gamma_j(x)$ in the domain D_i. Equations (7), (8), and (9) yield in

$$\dot{x} \in (1 - \alpha_j) \cdot \sum_{i=1}^{2^p} (R_1 \cdot f_i(x)) + \alpha_j \cdot \sum_{i=1}^{2^p} (R_2 \cdot f_i(x)) \tag{10}$$

$$R_1 = \prod_{k=1; k \neq j; \Psi_{k,i} = -1}^{p} \left(\frac{1 + 2\Psi_{k,i} \cdot \alpha_k - \Psi_{k,i}}{2} \right) \tag{11}$$

$$R_2 = \prod_{k=1; k \neq j; \Psi_{k,i} = 1}^{p} \left(\frac{1 + 2\Psi_{k,i} \cdot \alpha_k - \Psi_{k,i}}{2} \right) \tag{12}$$

Define a matrix F of the normal projections $f_i^{\perp_j}(x)$ for $j = 1, 2, ..., p$ and $i = 1, 2, ..., 2^p$

as

$$F = \left\{ \begin{array}{cccc} f_1^{\perp_1}(x) & f_1^{\perp_2}(x) & \cdots & f_1^{\perp_p}(x) \\ f_2^{\perp_1}(x) & f_2^{\perp_2}(x) & \cdots & f_2^{\perp_p}(x) \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ f_{2^p}^{\perp_1}(x) & f_{2^p}^{\perp_2}(x) & \cdots & f_{2^p}^{\perp_p}(x) \end{array} \right\} \tag{13}$$

where

$$f_i^{\perp_j}(x) = \mathcal{L}_{f_i}\gamma_j(x) = \left(\frac{\partial \gamma_j(x)}{\partial x}\right)^T \cdot f_i(x) \tag{14}$$

In agreement with the sign matrix Ψ, the attractive chattering on any $\mathbb{R}^{(n-r)}$ switching manifold for $r = 2, 3, ..., n$ can be easily observed by checking the signs of the matrices F and Ψ.

Lemma 1:

The sufficient condition for having an attractive chattering on any switching intersection in the system's state space requires a nodal attractivity towards the intersection itself, for all the flow maps f_i in the $\mathbb{R}^n$ regions C_i associated to this intersection. That is, the following constraint should be satisfied

$$\forall i, j: \quad sgn(f_i^{\perp_j}(x)) = -sgn(\Psi_{j,i}) \tag{15}$$

To keep the solution trajectory in a sliding motion on the intersection as long as the attractive chattering condition is satisfied we impose

$$\forall j = 1, ...p: \quad \sum_{i=1}^{2^p}(\prod_{j=1}^{p} \frac{1 + 2\Psi_{j,i} \cdot \alpha_j - \Psi_{j,i}}{2}) \cdot f_i^{\perp_j}(x) = 0 \tag{16}$$

so that

$$\alpha_j = \frac{W_1}{W_1 - W_2} \tag{17}$$

$$W_1 = \sum_{i=1}^{2^p}(\prod_{k=1;k\neq j;\Psi_{k,i}=-1}^{p} \frac{1 + 2\Psi_{k,i} \cdot \alpha_k - \Psi_{k,i}}{2}) \cdot f_i^{\perp_j} \tag{18}$$

$$W_2 = \sum_{i=1}^{2^p}(\prod_{k=1;k\neq j;\Psi_{k,i}=1}^{p} \frac{1 + 2\Psi_{k,i} \cdot \alpha_k - \Psi_{k,i}}{2}) \cdot f_i^{\perp_j} \tag{19}$$

For all $\alpha_k \in (0, 1)$, the product term in (18) (respectively (19)) takes always a value in (0,1) since it is always a product of $(1 - \alpha_k)$ (respectively α_k). It holds always that $W_1 > 0 \wedge W_2 < 0$ as long as an attractive chattering takes place at $x \in \left(\bigcup_{j=1}^{p} \Gamma_j\right) \cap C$, where C is the entire flow set in the system phase space (i.e. $C = \bigcup_{q\in Q}\{D_q\}$). This gives us a hypercube convex hull of sign coordiantes (±1, ±1, $\cdots$, ±1) with an edge of length 2 and $\mathbb{R}^n$ volume 2^p. Therefore, a solution to the fixed point non-linear problem

(16) exists. However, the uniqueness of the solution is no guaranteed. To deal with non-uniqueness on the intersection−on which the attractive chattering occurs−we propose to give an equivalent to the product term in (16) so that the sliding parameters are given in term of a rational function of coefficients κ_i:

$$\prod_{j=1}^{p} \frac{1 + 2\Psi_{j,i} \cdot \alpha_j - \Psi_{j,i}}{2} = \frac{\kappa_i}{\sum_{k=1}^{2^p} \kappa_k} \tag{20}$$

where

$$\kappa_i = \frac{\left(\prod_{l=1;l\neq i}^{2^p} (\Omega_l)\right)^{\frac{1}{2^p-1}}}{\left(\prod_{l=1;l\neq i}^{2^p} (\Omega_l)\right)^{\frac{1}{2^p-1}} - (\Omega_i)} \tag{21}$$

$$\Omega_i = [(b_i)_1 \ (b_i)_2 \ \cdots \ (b_i)_p] \cdot \begin{bmatrix} \mathcal{L}_{f_i}\gamma_1(x) \\ \mathcal{L}_{f_i}\gamma_2(x) \\ \vdots \\ \mathcal{L}_{f_i}\gamma_p(x) \end{bmatrix} \tag{22}$$

$$\Omega_l = [(b_l)_1 \ (b_l)_2 \ \cdots \ (b_l)_p] \cdot \begin{bmatrix} \mathcal{L}_{f_l}\gamma_1(x) \\ \mathcal{L}_{f_l}\gamma_2(x) \\ \vdots \\ \mathcal{L}_{f_l}\gamma_p(x) \end{bmatrix} \tag{23}$$

The vectors b_n for $n = i, l$ are given as sign permutations of coordinates $[\pm 1, \pm 1, \cdots, \pm 1]^T$ under the constraint:

$$sgn(b_n)_j = \left\{ \begin{array}{ll} sgn(\mathcal{L}_{f_n}\gamma_j(x)) & for \ \ n = 1 \\ -sgn(\mathcal{L}_{f_n}\gamma_j(x)) & for \ \ n = 2, 3, ..., 2^p \end{array} \right\} \tag{24}$$

where $j = 1, 2, ..., p$ and p is the number of the intersected $\mathbb{R}^{(n-1)}$ switching manifolds. This gives always $\Omega_1 > 0$ and $\Omega_n < 0$ for all $n \in \{2, 3, .., 2^p\}$ which is exactly what we want. Another advantage of using the signs constraint in (24) is that $\kappa_j = 0$ for all $j \neq i$ when $\kappa_i = 1$ for a given index $i \in \{1, 2, ..., 2^p\}$, this allows us to detect when a switching regime of different dimension has been reached by the solution trajectory, and then, to select the appropriate vector fields on this regime. Moreover, the parameter κ_i takes always a value $0 \leq \kappa_i \leq 1$ for $i = 1, 2, ..., 2^p$, yields in $\sum_{i=1}^{2^p} \left(\frac{\kappa_i}{\sum_{k=1}^{2^p} \kappa_k} \right) = 1$, which is consistent with the approach of Filippov differential inclusion.
However, special structures in control problems may lead to very complicated situations. In particular, in linear control problems with relay feedback, the existence of 1^{st} order sliding modes can simply be determined from studying the normal projections $f_1^{\perp}(x(t)) = CAx + CB$ and $f_2^{\perp}(x(t)) = CAx - CB$ close to hyper switching manifold Γ. We see that depending on the value CB we can decide whether we should expect to have 1^{st} order sliding modes or not. Roughly speaking, if the data in the system are given such that $CB = 0$, that is, the sliding region Γ_s vanishes so that $f_1^{\perp}(x(t)) = f_2^{\perp}(x(t)) = 0$ then it necessary to deal with higher order conditions for both f_1 and f_2. We provide in the following the necessary and sufficient condition for the existence of multiple fast switches in linear control problems with relay feedback using higher order sliding modes analysis.

4 Sliding with Higher Order Conditions: Application to Control Problems

Relay feedback exists in a lot of control application such that automatic tuning of PID controllers, modeling of quantization errors in digital control, and the analysis of sigma-delta converters, friction models. By simply replacing the controller by a relay, measuring the amplitude and frequency of the possible oscillation to derive the controller parameters, a robust control design method is obtained. The relay feedback system consists of a dynamical system and a sign function connected in feedback. The sign function leads to a discontinuous differential equation. The problem given as a linear system with relay feedback is

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= C^T x(t) \\ u(t) &= -sgn(y(t)) = \left\{ \begin{array}{rl} 1 & for\ \ y(t) < 0 \\ -1 & for\ \ y(t) > 0 \end{array} \right\} \end{aligned} \tag{25}$$

The sign function is discontinuous at $y(t) = 0$, therefore, we have a hybrid system of two discrete states q_1 and q_2 where the phase space of the system is split by a single hyper switching surface $\Gamma = \{x \in \mathbb{R}^n :\ \gamma(x(t)) = 0\}$ into two domain: $D_1 = \{x \in \mathbb{R}^n :\ \gamma(x(t)) \leq 0\}$ and $D_2 = \{x \in \mathbb{R}^n :\ \gamma(x(t)) \geq 0\}$ so that opposed zero crossing of the switching function γ defines the switching from q_1 to q_2 and vice-versa. In our notation, we have $\gamma = C^T x(t)$ and $\Gamma = \Gamma_1 \cap \Gamma_2 = \{x \in \mathbb{R}^n :\ C^T x(t) = 0\}$. It is assumed that $\frac{\partial \gamma(x(t))}{\partial x(t)} = C^T \neq 0$ for all $x \in \Gamma$.

The system dynamics, including the sliding dynamics $f_s = \frac{1-\delta(x)}{2} \cdot f_1(x) + \frac{1+\delta(x)}{2} \cdot f_2(x)$ with $\delta(x) \in [-1, 1]$ on the sliding surface $\Gamma_s \subset \Gamma$, are given by

$$\dot{x} = \left\{ \begin{array}{lll} f_1(x(t)) = Ax(t) + \ \ B & for\ \ x(t) \in D_1 \\ f_s(x(t)) = Ax(t) - \delta B & for\ \ x(t) \in \Gamma_s \\ f_2(x(t)) = Ax(t) - \ \ B & for\ \ x(t) \in D_2 \end{array} \right\} \tag{26}$$

Suppose that $x_m = x(t_m) \in \Gamma$, so the 1^{st} order Lie derivatives $f_1^{\perp}(x_m)$ and $f_2^{\perp}(x_m)$ are given by

$$f_1^{\perp}(x_m) = \frac{\partial \gamma(x_m)}{\partial x_m} \cdot f_1(x_m) = C^T A x_m + C^T B \tag{27}$$

$$f_2^{\perp}(x_m) = \frac{\partial \gamma(x_m)}{\partial x_m} \cdot f_2(x_m) = C^T A x_m - C^T B \tag{28}$$

We have to consider the following two cases:

Case I. $C^T B \neq 0$: For which an attractive sliding motion takes place

1. at all $x_m \in \Gamma$ (i.e. $C^T x_m = 0$) if and only if $f_1^{\perp}(x_m) > 0 \wedge f_2^{\perp}(x_m) < 0$ satisfied by the constraint $|C^T A x_m| < C^T B$. Let's denote ζ_1 and ζ_2 to the two boundaries of the sliding surface Γ_s. These two boundaries are defined explicitely then by

$$\zeta_1 = \{x_m \in \Gamma : C^T A x_m = -C^T B\}; \quad \zeta_2 = \{x_m \in \Gamma : C^T A x_m = \ \ C^T B\}$$

2. at ζ_1 if and only if $\exists k > 1 \in \mathbb{N}$ such that $\left(\bigwedge_{i=0}^{1} f_1^{\perp^{(i)}}(x(t_m)) = 0 \wedge f_1^{\perp^{(k)}}(x(t_m)) > 0 \right)$ satisfied by the constraint

   $$(C^T x_m = 0) \wedge (C^T A x_m = -C^T B) \wedge (\exists k > 1 \in \mathbb{N} : C^T A^k x_m > -C^T A^{k-1} B)$$

   Note that, since $C^T A x_m = -C^T B$ then $C^T A x_m < C^T B$ (i.e. $f_2^{\perp}(x_m) < 0$), and therefore, the sufficient condition for attractive chattering is already violated.

3. at ζ_2 if and only if $\exists k > 1 \in \mathbb{N}$ such that $\left(\bigwedge_{i=0}^{1} f_2^{\perp^{(i)}}(x(t_m)) = 0 \wedge f_2^{\perp^{(k)}}(x(t_m)) < 0 \right)$ satisfied by the constraint

   $$(C^T x_m = 0) \wedge (C^T A x_m = C^T B) \wedge (\exists k > 1 \in \mathbb{N} : C^T A^k x_m < C^T A^{k-1} B)$$

   Similarly, since $C^T A x_m = C^T B$ then $C^T A x_m > -C^T B$ (i.e. $f_1^{\perp}(x_m) > 0$), and therefore, the sufficient condition for attractive chattering is already violated.

Case II. $C^T B = 0$: This case is considerably more complicated to decide whether we should expect to leave Γ or to slide on it by the standard 1^{st} order theory of sliding concept since we have from (27) and (28): $f_1^{\perp}(x_m) = f_2^{\perp}(x_m) = C^T A x_m$. One way to treat this special case is to consider the higher order conditions (i.e. the higher order norms of the projections of both f_1 and f_2 normal onto Γ). It is clear to realize that it is not allowed to have neither transversality nor attractive sliding motion on Γ unless $f_1^{\perp}(x_m) = f_2^{\perp}(x_m) = 0$ satisfied by the constraint $C^T A x_m = 0$. This constraint on the system dynamics represents the necessary (but not sufficient) condition for the existence of 2^{nd} order transversality/sliding on Γ.
Recalling (1), (2), and (25) with $x_m = x(t_m) \in \Gamma$, the 2^{nd} order normal projections $f_1^{\perp^{(2)}}(x_m)$ and $f_2^{\perp^{(2)}}(x_m)$ are given by

$$f_1^{\perp^{(2)}}(x_m) = \frac{\partial f_1^{\perp}(x_m)}{\partial x_m} \cdot f_1(x_m) = C^T A^2 x_m + C^T A B \tag{29}$$

$$f_2^{\perp^{(2)}}(x_m) = \frac{\partial f_2^{\perp}(x_m)}{\partial x_m} \cdot f_2(x_m) = C^T A^2 x_m - C^T A B \tag{30}$$

Similarly, we should consider the following two cases:

Case I. $C^T A B \neq 0$: For which a 2^{nd} order attractive sliding motion takes place

1. at all $x_m \in \Gamma$ (i.e. $C^T x_m = 0$) if and only if $f_1^{\perp^{(2)}}(x(t_m)) > 0 \wedge f_2^{\perp^{(2)}}(x(t_m)) < 0$ satisfied by the constraint $|C^T A^2 x_m| < C^T A B$. In this case, the two boundaries are defined explicitely then by

   $$\zeta_1 = \{x_m \in \Gamma : C^T A x_m = 0 \ \wedge \ C^T A^2 x_m = -C^T A B\}$$
   $$\zeta_2 = \{x_m \in \Gamma : C^T A x_m = 0 \ \wedge \ C^T A^2 x_m = \ \ C^T A B\}$$

2. at ζ_1 if and only if $\exists k > 2 \in \mathbb{N}$ such that $\left(\bigwedge_{i=0}^{2} f_1^{\perp^{(i)}}(x(t_m)) = 0 \wedge f_1^{\perp^{(k)}}(x(t_m)) > 0\right)$ satisfied by the two constraints

$$(C^T x_m = 0) \ \wedge \ (C^T A x_m = 0) \ \wedge \ (C^T A^2 x_m = -C^T AB)$$

$$\exists k > 2 \in \mathbb{N} : C^T A^k x_m > -C^T A^{k-1} B$$

3. at ζ_2 if and only if $\exists k > 2 \in \mathbb{N}$ such that $\left(\bigwedge_{i=0}^{2} f_2^{\perp^{(i)}}(x(t_m)) = 0 \wedge f_2^{\perp^{(k)}}(x(t_m)) < 0\right)$ satisfied by the two constraints

$$(C^T x_m = 0) \ \wedge \ (C^T A x_m = 0) \ \wedge \ (C^T A^2 x_m = C^T AB)$$

$$\exists k > 2 \in \mathbb{N} : C^T A^k x_m < C^T A^{k-1} B$$

Case II. $C^T B = 0$: for which we have from (29) and (30): $f_1^{\perp^{(2)}}(x_m) = f_2^{\perp^{(2)}}(x_m) = C^T A^2 x_m$, the case in which we should consider the 3^{rd} conditions to determine whether we should stay on the sliding surface Γ_s or leave it.
Lemma: In the k^{th} order sliding modes analysis of control problems with relay feedback with $k > 0 \in \mathbb{N}$, the multi-valued sliding parameter $\delta(x)$ is given by

$$\delta(x) = \frac{C \cdot A^k \cdot x}{C \cdot A^{k-1} \cdot B} \tag{31}$$

We summarize in Table 1 the general case of higher order conditions analysis for the existence of an attractive sliding motion on $\Gamma_s \subset \Gamma$. The constraints on the data as well as on the dynamics are reported under the heading "Data"and "Dynamics", respectively. In Tables 2 and 3 we summarize the higher order conditions for the staying conditions at the tangential exit points ζ_1 and ζ_2 respectively.

Table 1 - Higher-order conditions for the existence of sliding on $\Gamma_s \subset \Gamma$

Order	Data	Dynamics	$\delta(x)$	Sliding on $\Gamma_s \subset \Gamma$
1	$CB \neq 0$	$Cx(t) = 0$	$CAx(t)/CB$	$\lvert\delta(x)\rvert < 1$
2	$CB = 0 \wedge$ $CAB \neq 0$	$Cx(t) = 0 \wedge$ $CAx(t) = 0$	$CA^2x(t)/CAB$	$\lvert\delta(x)\rvert < 1$
3	$CB = 0 \wedge$ $CAB = 0 \wedge$ $CA^2B \neq 0$	$Cx(t) = 0 \wedge$ $CAx(t) = 0 \wedge$ $CA^2x(t) = 0$	$CA^3x(t)/CA^2B$	$\lvert\delta(x)\rvert < 1$
...	...	...	...	...
k	$\bigwedge_{i=0}^{k-2} CA^iB = 0$ $\wedge\, CA^{k-1}B \neq 0$	$\bigwedge_{i=0}^{k-1} CA^ix = 0$	$CA^kx(t)/CA^{k-1}B$	$\lvert\delta(x)\rvert < 1$

Table 2 - Higher-order conditions for the existence of sliding on $\zeta_1 \in \Gamma$

Order	Data	Dynamics	$\delta(x)$	Sliding on $\zeta_1 \in \Gamma$
1	$CB \neq 0$	$Cx(t) = 0$	$CAx(t)/CB$	$\delta(x) = -1 \wedge \exists k > 1 : CA^k x(t) > -CA^{k-1}B$
2	$CB = 0 \wedge$ $CAB \neq 0$	$Cx(t) = 0 \wedge$ $CAx(t) = 0$	$CA^2x(t)/CAB$	$\delta(x) = -1 \wedge \exists k > 1 : CA^k x(t) > -CA^{k-1}B$
3	$CB = 0 \wedge$ $CAB = 0 \wedge$ $CA^2B \neq 0$	$Cx(t) = 0 \wedge$ $CAx(t) = 0 \wedge$ $CA^2x(t) = 0$	$CA^3x(t)/CA^2B$	$\delta(x) = -1 \wedge \exists k > 1 : CA^k x(t) > -CA^{k-1}B$
...	...	...	...	...
k	$\bigwedge_{i=0}^{k-2} CA^iB = 0$ $\wedge CA^{k-1}B \neq 0$	$\bigwedge_{i=0}^{k-1} CA^i x = 0$	$CA^kx(t)/CA^{k-1}B$	$\delta(x) = -1 \wedge \exists k > 1 : CA^k x(t) > -CA^{k-1}B$

Table 3 - Higher-order conditions for the existence of sliding on $\zeta_2 \in \Gamma$

Order	Data	Dynamics	$\delta(x)$	Sliding on $\zeta_2 \in \Gamma$
1	$CB \neq 0$	$Cx(t) = 0$	$CAx(t)/CB$	$\delta(x) = 1 \wedge \exists k > 1 : CA^k x(t) < CA^{k-1}B$
2	$CB = 0 \wedge$ $CAB \neq 0$	$Cx(t) = 0 \wedge$ $CAx(t) = 0$	$CA^2x(t)/CAB$	$\delta(x) = 1 \wedge \exists k > 1 : CA^k x(t) < CA^{k-1}B$
3	$CB = 0 \wedge$ $CAB = 0 \wedge$ $CA^2B \neq 0$	$Cx(t) = 0 \wedge$ $CAx(t) = 0 \wedge$ $CA^2x(t) = 0$	$CA^3x(t)/CA^2B$	$\delta(x) = 1 \wedge \exists k > 1 : CA^k x(t) < CA^{k-1}B$
...	...	...	...	...
k	$\bigwedge_{i=0}^{k-2} CA^iB = 0$ $\wedge CA^{k-1}B \neq 0$	$\bigwedge_{i=0}^{k-1} CA^i x = 0$	$CA^kx(t)/CA^{k-1}B$	$\delta(x) = 1 \wedge \exists k > 1 : CA^k x(t) < CA^{k-1}B$

5 Simulation Results

In this section we carry out a series of simulation tests to illustrate the performance as well as the efficiency of the approaches developed and presented in this paper.

Example 1. Consider the following system with relay feedback

$$\dot{x} = Ax + Bu; \qquad y = Cx; \qquad u = -sgn(y) \tag{32}$$

$$A = \begin{bmatrix} -3 & 1 & 0 \\ -3 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}; \qquad B = \begin{bmatrix} 1 \\ -2\beta \\ \beta^2 \end{bmatrix}; \tag{33}$$

$$C = [1 \;\; 0 \;\; 0]; \;\; x = (x_1, ..., x_n)^T \in \mathbb{R}^n \tag{34}$$

Depending on the value of CB, a classification of the directions of the trajectories divide the switch plane into two or three regions. In this example we have $CB > 0$, therefore, there exist 1^{st} order sliding modes. Figure 1 shows a $2D$ plot of the simulation of this system with $\beta = 0.5$ and $x_0 = [0.5\; 3\; 0.1]^T$.

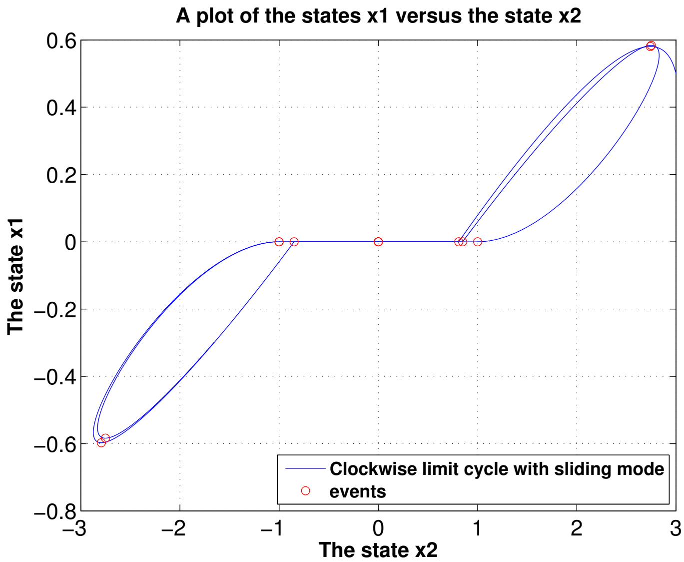


Figure 1: A clockwise limit cycle of the system with $\beta = 0.5$, and $x_0 = [0.5\; 3\; 0.1]^T$.

As it is demonstrated in Figure 1, the exit from the sliding surface is tangential at the exit points ζ_1 and ζ_2. For a simulation time of 20 seconds: (i) six relay switches have been recorded with three sliding segments, (ii) 4 detections of tangential crossing outside the hyper switching plane Γ have been recorded. Such detection of the tangential crossings gives precise information whether the gradient of the continuous time behavior of the systems trajectory is directed or not towards the switching plane. A $3D$ plot of the simulation of this system with the same initial conditions is illustrated in Figure 2.

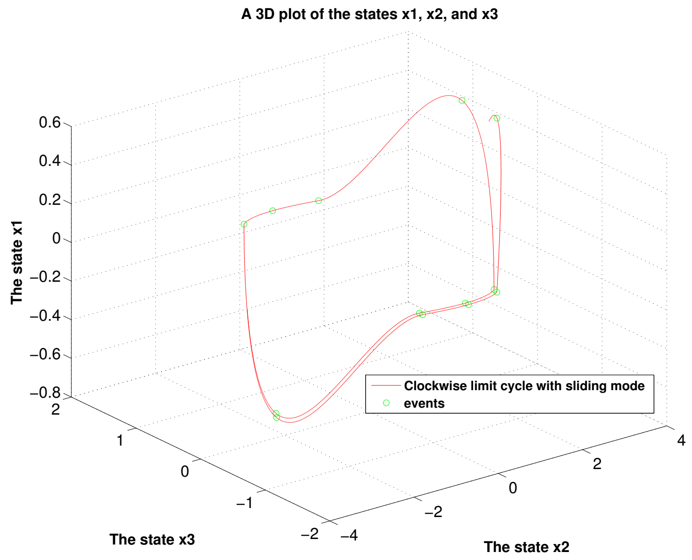


Figure 2: A $3D$ orbit of x_1 versus x_2 and x_3 with $\beta = 0.5$, and $x_0 = [0.5\ 3\ 0.1]^T$.

Example 2. Consider the following system with relay feedback

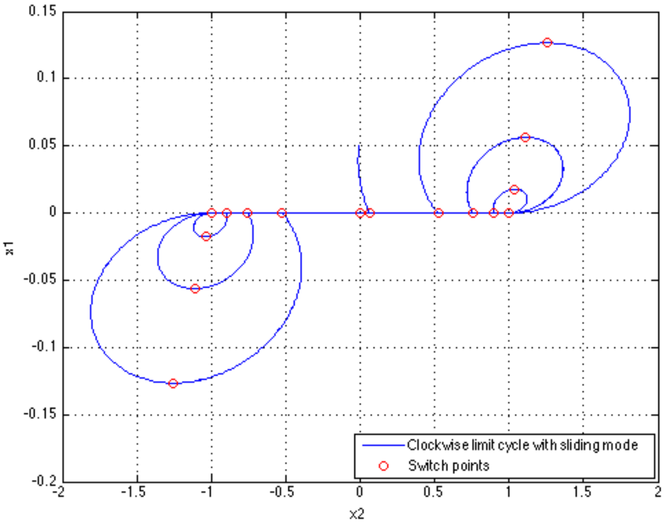
$$\dot{x} = Ax + Bu; \qquad y = Cx; \qquad u = -sgn(y) \tag{35}$$

where

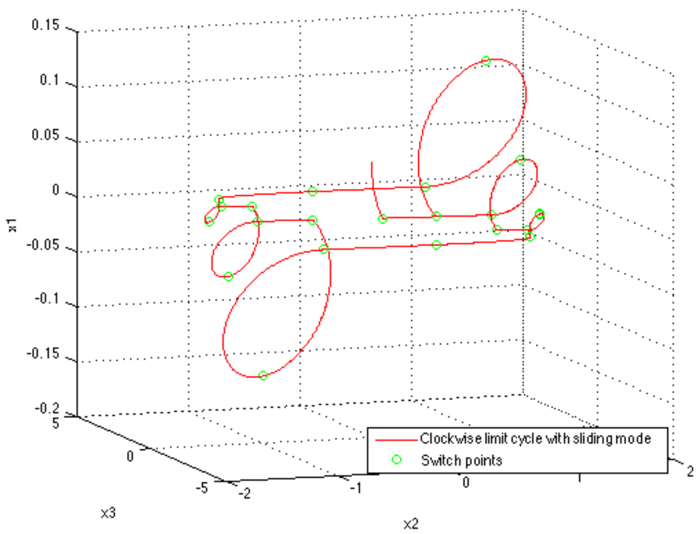
$$A = \begin{bmatrix} -(2ab+1) & 0 & 1 \\ -(2ab+b^2) & 0 & 1 \\ -b^2 & 0 & 0 \end{bmatrix}; \quad B = \begin{bmatrix} d \\ -2s \\ 1 \end{bmatrix}; \tag{36}$$

$$C = [1\ \ 0\ \ 0]; \ \ x = (x_1, ..., x_n)^T \in \mathbb{R}^n \tag{37}$$

With this data set of B, a first-order attractive sliding is expected on the switch plane $Cx = 0$ as long as d is positive. Figure 3 shows the clockwise trajectories of the system with the parameters $a = 0.05$, $b = 10$, $d = 1$, $s = -2$ and initial conditions $x_0 = [0.05\ -0.01\ 0.1]^T$. In a simulation time of 20 seconds, 68 events were detected including 20 tangential crossings outside the switching plane.



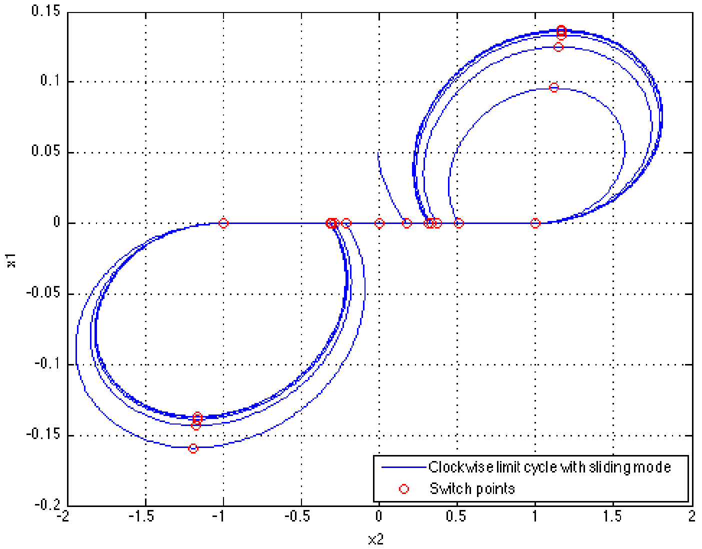
(a)



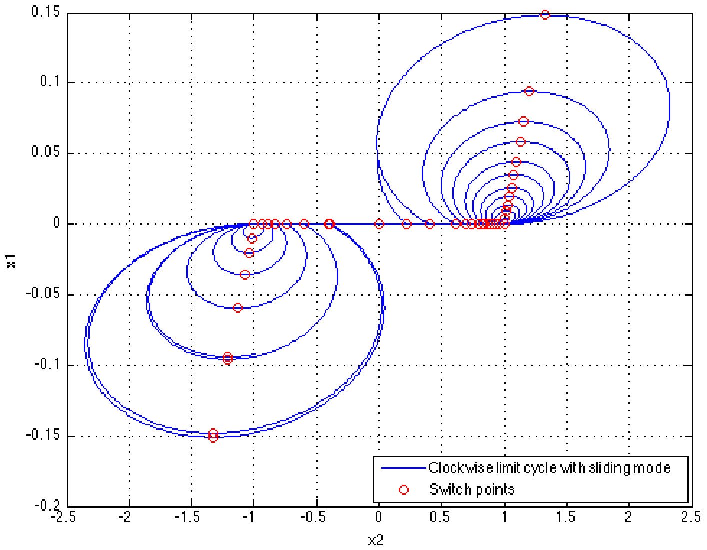
(b)

Figure 3: Simulation of the system (35) for $a = 0.05$, $b = 10$, $d = 1$, $s = -2$, and $x_0 = [0.05\ -0.01\ 0.1]^T$: (a). The evolution of x_2 versus x_1. (b). A $3D$ view of the Periodic orbit of x_1 versus x_2 and x_3.

Two other dynamical scenarios were simulated for this system. It has been observed that increasing b and decreasing s reduces the number of the sliding segments (Figure 4 (a)). Large enough b and small enough s will result in a transversality switching (Figure 4 (b)).



(a)



(b)

Figure 4: Clockwise trajectories of the system (35): (a). The evolution of x_2 versus x_1 for $a = 0.01$, $b = 10$, $d = 1$, $s = -3$, and $x_0 = [0.05\ -0,01\ 0.1]^T$. (b). The evolution of x_2 versus x_1 for $a = 0.01$, $b = 10$, $d = 1$, $s = -3$, and $x_0 = [0.05\ -0,01\ 0.1]^T$.

Example 3. Consider again the system in Example 2. Setting $d = 0$ yields in $CB = 0$, and therefore, applying the higher order analysis implies that multiple fast switches exist only if $CAB > 0$. In Figure 5, with the set of parameters $a = 0.03$, $b = 5$, $d = 0$, $s = -10$, we have $CAB = -2s = 20 > 0$. Note that, the sliding motion takes place on the order switching plane $\Gamma^{(2)}$ which is given by

$$\Gamma^{(2)} = \{x \in \mathbb{R}^n : \ \gamma^{(2)}(x) = -1.3x_1 + x_2 = 0\} \tag{38}$$

On the plane x_1, a transversality switching is observed. The trajectory converges to a limit cycle. It was recorded that the bigger is the parameters a and b the faster is the convergence to the limit cycle (see Figure 6).

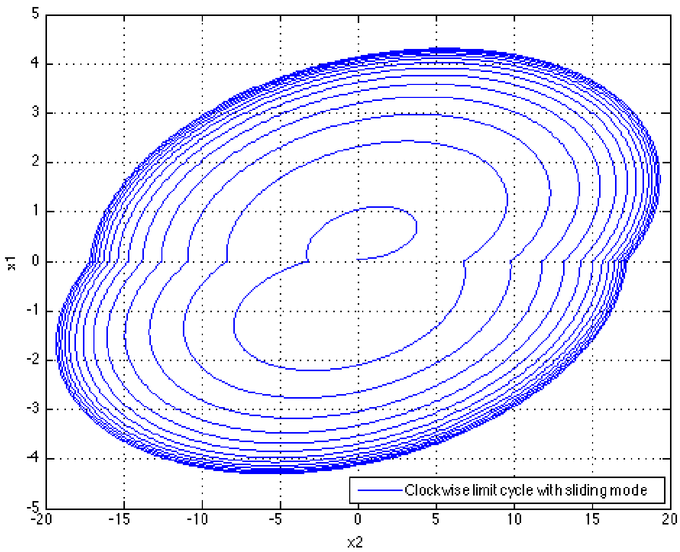


Figure 5: Clockwise trajectories of the system (35) with a 2^{nd} order sliding mode simulation for $a = 0.03$, $b = 5$, $d = 0$, $s = -10$, and $x_0 = [0.05 \ -0, 01 \ 0.1]^T$: The evolution of x_2 versus x_1.

6 Conclusions

In this paper we presented a new computational framewrork for the purpose of a robust detection of the chattering behavior "on the fly" in real-time simulation of hybrid systems as well as the treatment of chattering behavior during the numerical simulation using the higher order sliding mode simulation. The main objective of the proposed regularization technique is to switch between the transversality modes and the sliding modes simulation automatically as well as integrating each particular state appropriately and localize the structural changes in the system in an accurate way. The method presented in this paper

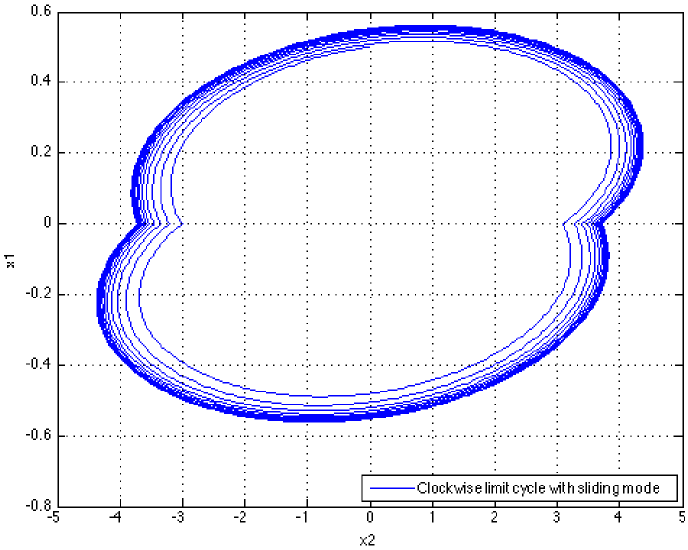


Figure 6: Clockwise trajectories of the system (35) with a 2^{nd} order sliding mode simulation for $a = 0.07$, $b = 8$, $d = 0$, $s = -10$, and $x_0 = [0.05\ \ -0, 01\ 0.1]^T$: The evolution of x_2 versus x_1.

makes use of the higher order sliding modes analysis in the appliactions when the 1^{st} order analysis cannot be applied. Finally, the simulation results - reported here on a set of representative examples - showed that our approach is efficient and precise enough to provide a chattering bath avoidance, to perform a special numerical treatment of the constrained motion along the discontinuity surface, as well as its robustness in achieving an accurate detection and localization of all the switch points.

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