Construction of Gevrey functions with compact support using the Bray-Mandelbrojt iterative process and applications to the moment method in control theory

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Abstract

In this paper, we construct some interesting Gevrey functions of order α for every $\alpha > 1$ with compact support by a clever use of the Bray-Mandelbrojt iterative process. We then apply these results to the moment method, which will enable us to derive some upper bounds for the cost of fast boundary controls for a class of linear equations of parabolic or dispersive type that partially improve the existing results proved in [P. Lissy, On the Cost of Fast Controls for Some Families of Dispersive or Parabolic Equations in One Space Dimension SIAM J. Control Optim., 52(4), 2651-2676]. However this construction fails to improve the results of [G. Tenenbaum and M. Tucsnak, New blow-up rates of fast controls for the Schrödinger and heat equations, Journal of Differential Equations, 243 (2007), 70-100] in the precise case of the usual heat and Schrödinger equation.

1 Introduction

1.1 Presentation

The main motivation of this paper is to continue the study of [11] concerning the estimation for the cost of fast "boundary" controls for a class of linear equations of parabolic or dispersive type. The precise scope of the paper will be made more precise later.

Let us introduce some usual notations. Let H be some Hilbert space and U be another Hilbert space. Let $A : \mathcal{D}(A) \to H$ be a self-adjoint operator with compact resolvent. The eigenvalues of A (which are here supposed to be different from 0 without loss of generality) are called $(\lambda_k)_{k\geqslant 1}$ and supposed to be with multiplicity 1 in all what follows. To each eigenvalue we associate a normalized eigenvector corresponding to the eigenvalue λ_k, which is called e_k. We assume that $-A$ generates on H a strongly continuous semigroup $S : t \mapsto S(t) = e^{-tA}$. The Hilbert space $D(A^*)'(= D(A)')$ will be equipped with the norm

$$||x||^2_{D(A)'} = \sum \frac{< x, e_k >^2_H}{\lambda_k^2}.$$

We call $B \in \mathcal{L}_c(U, \mathcal{D}(A)')$ an admissible control operator for this semigroup, i.e. such that for every time $T > 0$, there exists some constant $C(T) > 0$ such that for every $z \in \mathcal{D}(A)$, one has

$$\int_0^T ||B^* S(t)^* z||^2_U \leqslant C(T)||z||^2_H.$$

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We consider the following class of controlled semigroups:

$$y_t + Ay = Bu \text{ (parabolic case)} \tag{1}$$

and

$$y_t + iAy = Bu \text{ (dispersive case).} \tag{2}$$

In the case of Equation (1), A will moreover be supposed to be a positive operator to ensure the existence of solutions. Then, it is well-known (see for example [2, Chapter 2, Section 2.3], the operators $-A$ or $-iA$ generates a strongly continuous semigroup under the hypothesis given before thanks to the Lummer-Phillips or Stone theorems) that if $u \in L^2((0,T),U)$, System (1) or (2) with initial condition $y^0 \in H$ has a unique solution satisfying $y \in \mathcal{C}^0([0,T],H)$.

From now on and until the end of the paper, we will assume that B is a *scalar* control, that is to say of the form $Bu = bu$, where $b \in \mathcal{D}(A)'$ and $u \in L^2((0,T),\mathbb{K})$ where here $U = \mathbb{K}$ with $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. From now on, we will call

$$b_k = < b, e_k >_{(D(A)',D(A))},$$

where $<,>_{(D(A)',D(A))}$ is here the duality product between $\mathcal{D}(A)'$ and $\mathcal{D}(A)$ with pivot space H. It is well-known (see [6]) that if $||(b_k)_{k\in\mathbb{N}^*}||_\infty < +\infty$ and if $(\lambda_k)_{k\geqslant 1}$ is *regular* in the sense that

$$\inf_{m\neq n} |\lambda_m - \lambda_n| > 0,$$

then B is an admissible control operator.

Let us now introduce the notion of cost of the control. Assume that that system (1) or (2) is null controllable at some time $T_0 > 0$ (i.e. for every $y^0 \in H$, there exists some control $u \in L^2((0,T_0),U)$ such that $y(T_0,\cdot) \equiv 0$). One can then verify easily that there exists a unique control $u_{opt} \in L^2((0,T_0),U)$ with minimal $L^2((0,T_0),U)$-norm. Moreover, the map $y^0 \mapsto u_{opt}$ is linear continuous (see for example [2, Chapter 2, Section 2.3]). The norm of this operator will be from now on called the *cost of the control at time* T_0 and denoted C_{T_0} in this section. Thanks to the definition of C_{T_0}, this constant is also exactly the smallest constant $C > 0$ such that for every $y^0 \in H$, there exists some control u driving y^0 to 0 at time T_0 with

$$||u||_{L^2((0,T_0),U)} \leqslant C||y^0||_H.$$

Here we attend to give some precise upper bounds on C_T when $T \to 0$ for some families of operators A for which equations (1) and (2) are null controllable in arbitrary small time. Some applications to fractional heat and Schrödinger equations will be provided later.

Let us explain now precisely the scope of the paper:

1. We give a new family of possible multipliers for the moment method that depend on the asymptotic growth of $(\lambda_k)_{k\geqslant 1}$ and seem well-adapted (this is the main result of the paper). The author believes that it would be quite difficult to improve the construction given here, notably because the construction approximates quite well some optimal problem (see Section 3.2), but there is still some small possibility of doing better (see Remark 4) that the author did not manage to exploit suitably.

2. We manage to extend the results of [11] to the wider possible range of exponents for the asymptotic polynomial behavior of the eigenvalues $(\lambda_k)_{k\in\mathbb{N}^*}$. In section 2.3, we will explain into more details why the multiplier used in [11] was not really adapted to the case of eigenvalues λ_k behaving roughly like k^α with $\alpha \neq 2$ and how we managed to improve it.

3. As we will see later (see notably Figures 1 and 3), we will *not* improve the upper bounds in the case of the classical heat equation and Schrödinger equation given in [22]. However, we will also see that there are a wide range of families of eigenvalues for which our estimates are better than in [11], and notably when $\alpha \to \infty$ we dramatically improve the results of [11] and we are quite close to the lower bounds of [12].

2 State of the art

As far as the author know, the study of the behavior of fast control for partial differential equations started with the paper [21], where an upper bound for the cost of fast boundary control for the one-dimensional heat equation was given. A lower bound were then given in [4], proving that the cost of fast controls had to be necessarily roughly (up to fractional terms in T) of the form $\exp(K/T)$ for some $K > 0$ as $T \to 0$. The same result for the boundary control of one-dimensional Schrödinger equations was proved later in [15]. The next natural question is then to try to estimate precisely K and notably its dependence with respect to the geometry of the problem (i.e. to the behavior on the eigenvalues λ_k). In the case of the heat equation, if we call L the length of the interval on which we control our one-dimensional equation, we know that

$$L^2/2 \leqslant K \leqslant 3L^2/4.$$

The best upper bound was obtained in [22] whereas the lower bound was obtained in [12]. For the Schrödinger equation, we have

$$L^2/4 \leqslant K \leqslant 3L^2/2.$$

The upper bound was obtained in [22] and the lower bound in [15].

Let us now explain with more details what was precisely done [11]. In this article, the author proved precise upper bounds concerning the cost of the control for some large classes of linear parabolic or dispersive equations when the time T goes to 0, where the underlying "elliptic" operator was chosen to be self-adjoint or skew-adjoint with eigenvalues roughly as Rk^α or $\pm Rk^\alpha$ (only for (2)) for some $R > 0$ and $\alpha \geqslant 2$ when $k \to +\infty$. The cost of the control is proved to be bounded from above by $\exp\left(K/(RT)^{1/(\alpha-1)}\right)$ where K is some explicit constant depending on α. This does not cover all possible cases, because it is well-known that equations like (1) and (2) are controllable in arbitrary small time if and only if $\alpha > 1$.

Explicit lower bounds have later been derived in [12] for all controllable in small time fractional heat or Schrödinger equations. Let us mention that in the case of the Schrödinger equation (which corresponds to $\lambda_k = k^2$ and equation (2)), the lower bound proved in [12] is exactly the same as the one given by Miller in [15], whereas the lower bound for the heat equation controlled on the boundary (which corresponds to the case $\lambda_k = k^2$ and equation (1)) is twice the one obtained by Miller in [16] and was conjectured to be to exact behavior for the cost of the control until now.

To finish, let us mention that the study of the behavior of fast controls may also be applied in some cases to study the uniform controllability of convection-diffusion equations in the vanishing viscosity limit as explained in [9] and [10], however the results given in the present article does not enable us to deduce directly new results for this problem.

2.1 Main results and comments

In this section, we are going to give the main results of this paper and some additional comments. Let us first set some notations. In all what follows, $f \lesssim g$ (with f and g some complex valued functions depending on some variable x in some set $\mathcal{S}$) means that there exists some constant $C > 0$ such that for all $x \in \mathcal{S}$, one has $|f(x)| \leqslant C|g(x)|$, (such a C is called an implicit constant in the inequality $f \lesssim g$), and $f \simeq g$ means that we have both $f \lesssim g$ and $g \lesssim f$. Sometimes, when it is needed, we might detail the dependence of the implicit constant with respect to some parameters.

Theorem 2.1 *Assume that $(\lambda_n)_{n\geqslant 1}$ is a regular increasing sequence of positive numbers verifying moreover that there exist some $\alpha > 1$ and some $R > 0$ such that*

$$\lambda_n = Rn^\alpha + \underset{n\to\infty}{O}(n^{\alpha-1}) \tag{3}$$

holds (as $n \to \infty$), and assume that $b_n \simeq 1$ (in the sense that the sequence $(|b_n|)_{n\in\mathbb{N}}$ is bounded from below and above by positive constants). Then system (2) is null controllable and the cost of the control C_T verifies for T small enough:

1. *If $\alpha \geqslant 2$, then*

$$C_T \lesssim e^{\frac{K}{(RT)^{1/(\alpha-1)}}}, \text{ for every } K > C_S(\alpha) := \frac{2^{\frac{1}{\alpha-1}}\pi^{\frac{\alpha}{\alpha-1}}}{(\alpha-1)^{\frac{1}{\alpha-1}}\sin(\frac{\pi}{\alpha})^{\frac{\alpha}{\alpha-1}}}.$$

2. *If $\alpha \in (1,2)$, then*

$$C_T \lesssim e^{\frac{K}{(RT)^{1/(\alpha-1)}}}, \text{ for every } K > C_S(\alpha) := \frac{4^{\frac{1}{\alpha-1}}\pi^{\frac{\alpha}{\alpha-1}}}{(\alpha-1)^{\frac{1}{\alpha-1}}\sin(\frac{\pi}{\alpha})^{\frac{\alpha}{\alpha-1}}}.$$

(the implicit constant in the previous inequalities might depend on α or R but not on T)

We also have a theorem in the dispersive case when the eigenvalues are not supposed anymore to be positive.

Theorem 2.2 *Assume that the sequence of increasing eigenvalues $(\lambda_n)_{n\in\mathbb{Z}^*}$ of A is a regular sequence of non-zero numbers verifying moreover that there exist some $\alpha > 1$ and some constant $R > 0$ such that*

$$\begin{cases} \lambda_n = Rn^\alpha + \underset{n\to\infty}{O}(n^{\alpha-1}), & n > 0, \\ \lambda_{-n} = -Rn^\alpha + \underset{n\to\infty}{O}(n^{\alpha-1}), & n < 0, \\ \operatorname{sgn}(\lambda_n) = \operatorname{sgn}(n), \end{cases} \tag{4}$$

and assume that $b_n \simeq 1$. Then system (2) is null controllable and the cost of the control C_T verifies for T small enough:

1. *If $\alpha \geqslant 2$, then*

$$C_T \lesssim e^{\frac{K}{(RT)^{1/(\alpha-1)}}}, \text{ for every } K > C_{S+-}(\alpha) := \frac{2^{\frac{\alpha+1}{\alpha-1}}\pi^{\frac{\alpha}{\alpha-1}}}{(\alpha-1)^{\frac{1}{\alpha-1}}\sin(\frac{\pi}{\alpha})^{\frac{\alpha}{\alpha-1}}}.$$

2. *If $\alpha \in (1,2)$, then*

$$C_T \lesssim e^{\frac{K}{(RT)^{1/(\alpha-1)}}}, \text{ for every } K > C_{S+-}(\alpha) := \frac{2^{\frac{1}{\alpha-1}}(2^{\frac{1}{\alpha}}+1)^{\frac{\alpha}{\alpha-1}}\pi^{\frac{\alpha}{\alpha-1}}}{(\alpha-1)^{\frac{1}{\alpha-1}}\sin(\frac{\pi}{\alpha})^{\frac{\alpha}{\alpha-1}}}.$$

(the implicit constant in the previous inequalities might depend on α or R but not on T)

Concerning the parabolic case, we obtain the following result:

Theorem 2.3 *Assume that $(\lambda_n)_{n\geqslant 1}$ is a regular increasing sequence of positive numbers verifying moreover that there exists some $\alpha > 1$ and some constant $R > 0$ such that (3) holds. Assume that $b_n \simeq 1$. Then system (1) is null controllable. Moreover, the control can be chosen in the space $C^0([0,T],U)$ and the cost of the control C_T (in norm $L^\infty(0,T)$, so this is also true in $L^2(0,T)$) verifies for T small enough*

$$C_T \lesssim e^{\frac{K}{(RT)^{1/(\alpha-1)}}}, \text{ for every } K > \frac{\pi^{\frac{\alpha}{\alpha-1}}}{2(\alpha-1)^{\frac{1}{\alpha-1}}\sin(\frac{\pi}{2\alpha})^{\frac{\alpha}{\alpha-1}}}.$$

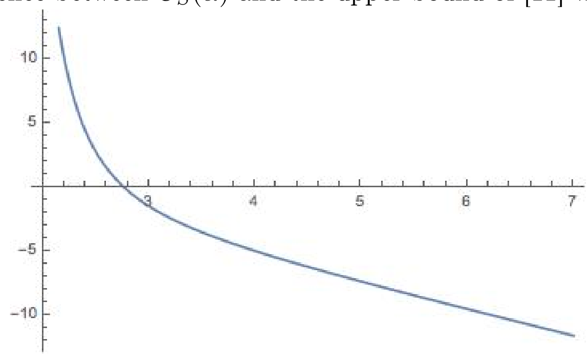
(the implicit constant in the previous inequalities might depend on α or R but not on T)

Remark 1 *1. Concerning the asymptotic behavior of $C_S(\alpha)$, we obtain by easy Taylor expansions that*

$$C_S(\alpha) \simeq \exp(1)\left(\frac{4}{(\alpha-1)^{\alpha+1}}\right)^{\frac{1}{\alpha-1}} \quad \text{as } \alpha \to 1^+,$$
$$C_S(\alpha) \simeq \alpha \text{ as } \alpha \to +\infty.$$

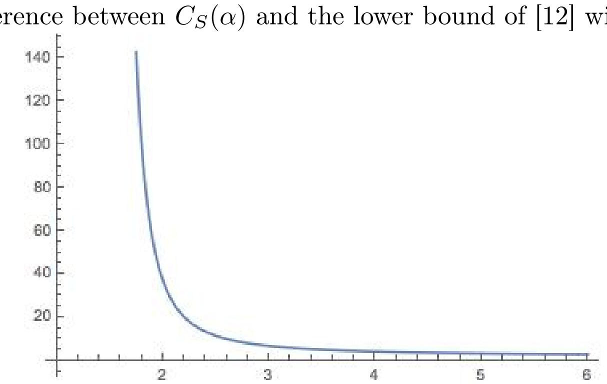
2. We observe that for $\alpha = 2$, the cost of the control is bounded by $e^{\frac{K\pi^2}{RT}}$ for every $K > C_S(2) = 2\pi^2$, which is worse than [22] which stated that we have the same result for every $K > 3\pi^2/2$. However, we see on figure 1 (which compares the upper bound $C_S(\alpha)$ and the one found in [11] in the case $\alpha \geqslant 2$) that our new bound becomes better for $\alpha \geqslant 2.76$ and is linearly better as $\alpha \to \infty$.

Figure 1: Difference between $C_S(\alpha)$ and the upper bound of [11] with respect to α.



3. Another interesting comparison can be done between the upper bound $C_S(\alpha)$ and the lower bound given in [12]. This is what is done in figure 2. We can see that our upper bound becomes close to the lower bound given in [12] when $\alpha \to \infty$, more precisely the difference between the two quantities converges to $1 + \ln(2)$. However, the upper bound is very far from the lower bound for $\alpha \to 1^+$.

Figure 2: Difference between $C_S(\alpha)$ and the lower bound of [12] with respect to α.



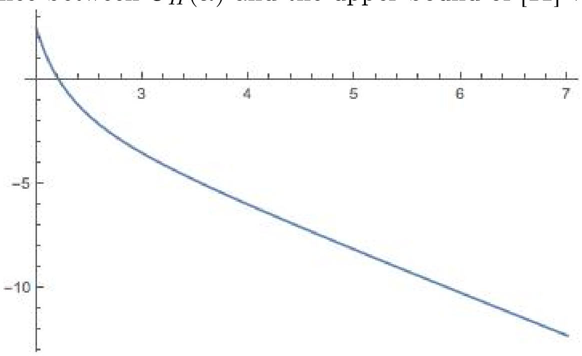
Remark 2 *1. Concerning the asymptotic behavior of $C_H(\alpha)$, we obtain by easy Taylor expan-*

sions that

$$C_H(\alpha) \simeq \frac{\pi^{\frac{\alpha}{\alpha-1}}}{2(\alpha-1)^{\frac{1}{\alpha-1}}} \text{ as } \alpha \to 1^+,$$
$$C_H(\alpha) \simeq \alpha \text{ as } \alpha \to +\infty.$$

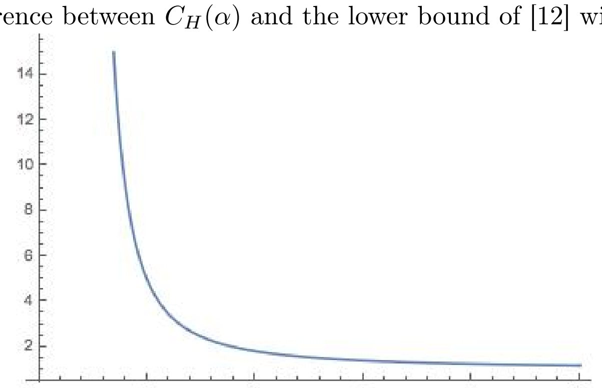
2. *We observe that for $\alpha = 2$, the cost of the control is bounded by $e^{\frac{K\pi^2}{RT}}$ for every $K > C_H(2) = \pi^2$, which is worse than [22] which stated that we have the same result for every $K > 3\pi^2/4$. However, we see on figure 3 (which compares the upper bound $C_H(\alpha)$ and the one found in [11] in the case $\alpha \geqslant 2$) that our new bound becomes better for $\alpha \geqslant 2.221$ and is linearly better as $\alpha \to \infty$.*

Figure 3: Difference between $C_H(\alpha)$ and the upper bound of [11] with respect to α.



3. *To finish, let us compare this upper bound and the lower bound given in [12]. This is what is done in figure 4. We can see that our upper bound becomes close to the lower bound given in [12] when $\alpha \to \infty$, more precisely the difference between the two quantities tends to 1. However, the upper bound is very far from the lower bound for $\alpha \to 1^+$.*

Figure 4: Difference between $C_H(\alpha)$ and the lower bound of [12] with respect to α.



2.2 Application to fractional heat or Schrödinger equations with "boundary control"

Let us give the two main applications of this result, which have already been mentioned widely. Let us consider the 1-D Laplace operator Δ with domain $D(\Delta) := H^1_0(0,L)$ and state space $H := H^{-1}(0,L)$. It is well-known that $-\Delta : D(\Delta) \to H^{-1}(0,L)$ is a positive definite operator with compact resolvent, the k-th eigenvalue is

$$\lambda_k = \frac{k^2\pi^2}{L^2},$$

with associated eigenvector

$$e_k(x) := \sin\left(\frac{k\pi x}{L}\right) / || \sin\left(\frac{k\pi x}{L}\right) ||_{H^{-1}(0,L)}.$$

Thanks to the continuous functional calculus for positive self-adjoint operators, one can define any positive power of $-\Delta$. Let us define our "boundary" control (for more explanations, see notably [11, Sections 3.3 & 3.4] $b \in (\mathcal{D}(-(-\Delta)^{\alpha/2}))'$ as follows:

$$b := (\partial_x \delta_0) \circ \Delta^{-1}.$$

We set $\mathbb{K} := \mathbb{R}$ (for (5)) or $\mathbb{C}$ (for (6)). Let us consider here some $\gamma > 1$ and two different equations, one particular case of (1), which are the fractional heat equations

$$\begin{cases} y_t = -(-\Delta)^{\gamma/2} y + bu & \text{in } (0,T) \times (0,L), \\ y(0,\cdot) = y^0 & \text{in } (0,L), \end{cases} \tag{5}$$

and one particular case of (2), which are the fractional Schrödinger equations

$$\begin{cases} y_t = -i(-\Delta)^{\gamma/2} y + bu & \text{in } (0,T) \times (0,L), \\ y(0,\cdot) = y^0 & \text{in } (0,L). \end{cases} \tag{6}$$

Equation (5) is often used to model anomaly fast or slow diffusion (see for example [14]), whereas (6) was introduced to study the energy spectrum of a 1-D fractional oscillator or for some fractional Bohr atoms (see for example [7]).

As explained in details in [11], the above equations exactly fit our abstract setting and we directly deduce from our main Theorems the following results:

Theorem 2.4 *System* (6) *is null controllable and the cost of the control C_T verifies for T small enough:*

1. *If $\gamma \geqslant 2$, then*

$$C_T \lesssim e^{\frac{KL^{2\gamma/(2\gamma-1)}}{T^{1/(2\gamma-1)}}}, \text{ for every } K > \frac{2^{\frac{1}{2\gamma-1}}}{(2\gamma-1)^{\frac{1}{2\gamma-1}} \sin(\frac{\pi}{2\gamma})^{\frac{2\gamma}{2\gamma-1}}}.$$

2. *If $\gamma \in (1,2)$, then*

$$C_T \lesssim e^{\frac{KL^{2\gamma/(2\gamma-1)}}{T^{1/(2\gamma-1)}}}, \text{ for every } K > \frac{4^{\frac{1}{2\gamma-1}}}{(2\gamma-1)^{\frac{1}{2\gamma-1}} \sin(\frac{\pi}{2\gamma})^{\frac{2\gamma}{2\gamma-1}}}.$$

System (5) *is null controllable. Moreover, the control can be chosen in the space $C^0([0,T],U)$ and the cost of the control C_T verifies for T small enough*

$$C_T \lesssim e^{\frac{KL^{2\gamma/(2\gamma-1)}}{T^{1/(2\gamma-1)}}}, \text{ for every } K > \frac{1}{2(2\gamma-1)^{\frac{1}{2\gamma-1}} \sin(\frac{\pi}{4\gamma})^{\frac{2\gamma}{2\gamma-1}}}.$$

(the implicit constant in the previous inequalities might depend on α but not on T).

2.3 Gevrey functions, the moment method and the Bray-Mandelbrojt construction

In this "heuristic" Section, our goal is to give some informal explanations on the strong link between Gevrey functions and the moment method of Fattorini and Russell [3] that will be used here to prove our theorems. Let us consider here the parabolic case (1), the dispersive case being quite similar. Let us decompose our initial condition on our Hilbert basis of eigenfunctions:

$$y^0(x) = \sum_{k=1}^{\infty} a_k e_k(x),$$

with $(a_k)_{k\in\mathbb{N}^*} \in l^2(\mathbb{N}^*)$. Then, using the notations of the previous section, it is well-known that we have for all $k \in \mathbb{N}^*$ and $t \in (0,T]$ that

$$\langle y(t), e_k\rangle_H = a_k e^{-\lambda_k t} + b_k \int_0^t e^{-\lambda_k (t-s)} u(s) ds.$$

Imposing that $y(T,\cdot) \neq 0$ is then equivalent to imposing that for every $k \in \mathbb{N}^*$, one has

$$a_k e^{-\lambda_k T} + b_k \int_0^T e^{-\lambda_k (T-s)} u(s) dt = 0,$$

i.e.

$$\int_0^T e^{-\lambda_k (T-s)} u(s) dt = \frac{-a_k e^{-\lambda_k T}}{b_k},$$

the right-hand side being in $l^2(\mathbb{N}^*)$ as soon as $b_k \simeq 1$.

Hence, if we assume that we are able to exhibit a *bi-orthogonal* family to $\{t \mapsto e^{-\lambda_n (T-t)}\}$ in $L^2(0,T)$, i.e. a family of functions $\{\psi_m\}_{m\in\mathbb{N}^*}$ such that for every $(k,l) \in (\mathbb{N}^*)^2$ one has

$$\langle e^{\lambda_k t}, \psi_l\rangle_{L^2(0,T)} = \delta_{kl}, \tag{7}$$

then one can use as a control function

$$u(t) := -\sum_{k\in\mathbb{N}^*} \frac{a_k e^{-\lambda_k T}}{b_k} \psi_k(t). \tag{8}$$

A usual way (but not the only one, see notably [1]) to construct the family$\{\psi_k\}$ is to use the Paley-Wiener Theorem (see for example [20, Theorem 19.3, Page 370]), which says that ψ_k can be constructed (after some adequate translation) as the inverse Fourier Transform of a $L^2(\mathbb{R})$-function J_k of exponential type $T/2$ verifying moreover $J_k(i\lambda_l) = \delta_{kl}$. As we will see during the proof, finding such a function J_k can be decomposed into two main steps:

- First exhibit some family of Weierstrass product Ψ_k involving the eigenvalues and verifying $\Psi_k(i\lambda_l) = \delta_{kl}$ (see notably Lemma 3.1).
- Then multiply Ψ_k with some adequate function M_k (called *multiplier*) so that the product $J_k := \Psi_k M_k$ will then be of exponential type $T/2$ and belongs to $L^2(\mathbb{R})$. Moreover M_k has to verify the normalization condition

$$M_k(i\lambda_k) = 1. \tag{9}$$

The crucial point is that the behavior for $x \to \infty$ of the Weierstrass product is very bad and can be proved to be exactly of the form $\exp(K|x|^{1/\alpha})$ for some constant $K > 0$ if we assume that

$\lambda_k \simeq k^\alpha$ for some $\alpha > 1$. Hence, we observe that if we want to obtain a function J_k which is in $L^2(\mathbb{R})$, it is *necessary* that the multiplier M_k behaves at least as $\exp(-K|x|^{1/\alpha})$ at infinity. This exactly means that M_k has to be the Fourier transform of some function G_k which has to be *Gevrey of order* α, i.e. verifying

$$||G_k^{(j)}||_\infty \leqslant C_{Gev}(\alpha, T, R) R^j j!^\alpha, \tag{10}$$

for some constant $R > 0 > 0$ (see for example [19, Section 1.6, Page 30]). Moreover, as it will be observed during the proof of our theorems, an important point is that the constant R *determines in an optimal way* the exact behavior of the exponential decreasing of M_k (i.e. the constant K). Hence, imposing here that M_k compensates quite "exactly" the growth of Ψ_n means that this constant ν in (10) is totally imposed by the data of our problem. To finish, since we want M_k to be of exponential type $T/2$, applying one more time the Paley-Wiener Theorem, it is equivalent to saying that G_k has to be *of compact support* $[-T/2, T/2]$. Using some estimations made more precise later, the function ψ_k will also be bounded by $C_{Gev}(\alpha, T, R)$ and then using (8) we deduce that $C_H(\alpha, T)$ is also bounded by $C_{Gev}(\alpha, T, R)$. Hence, if we want to obtain a precise bound on $C_H(\alpha, T)$, it is necessary to make all our possible to obtain some constant $C_{Gev}(\alpha, T, R)$ which is *as small as possible.* Let us also mention that the normalization condition (9) can be easily replaced by the condition $\int |G_k| = 1$ (and some suitable inequality on $G_k(i\lambda_k)$ that we do not detail here for the sake of clarity, see notably (60)).

Taking into account all these remarks, the rules of the game can be gathered as the following.

Optimizing the cost of the control is closely related to the following problem: given some $T > 0$, construct a Gevrey function G of order α, with support equal to $[-T/2, T/2]$, verifying $\int |G| = 1$, with imposed coefficient R appearing in the growth of the derivative and minimizing the quantity $C_{Gev}(\alpha, T, R)$ (which has of course to explode as $T \to 0$) that is appearing in (10).

Let us now explain a possible way to construct Gevrey function (or more generally C^∞ functions) with compact support. This construction was first described by Szolem Mandelbrojt in [13, Section 13], but in this book it is mentioned that the construction comes from previous unpublished works of Hubert Bray, whence the name "Bray-Mandelbrojt construction" adopted here. The idea is to use repeated mean-values of functions, i.e. to make an infinite convolution product of rectangle functions on $[-a_k, a_k]$, $(a_k)_{k\geqslant 0}$ being supposed to be a convergent series of positive numbers (whose sum will exactly be the support of the final function). This construction can also be found in [5, Pages 19-20]. Let us mention that since the Fourier transform of a rectangle function is exactly some cardinal sine function, and that the Fourier transform changes convolution into products, this construction is totally equivalent to the ones we can find notably in [16, Proof of Lemma 4.4], which is also used in [15] and that goes back to the work of Ingham in [8] (we refer to [18] for extra explanations on the usefulness of this construction).

To finish, let us mention that in [22], the authors used to following Gevrey function to construct the multiplier:

$$G(x) := \exp(-\nu/(1-x^2)),$$

where $\nu > 0$ is some large parameter to be chosen. It can be proved that G is a Gevrey function of order exactly 2. Notably, this function is not Gevrey of order α for $\alpha \in (1, 2)$ (but it is a Gevrey function of order α for every $\alpha > 2$), and this explains why in [11] (where the same multiplier was used) we were not able to treat the case $\alpha \in (1, 2)$ and why our estimations were quite bad for large α.

3 Proofs of Theorems 2.1, 2.2 and 2.3

3.1 Estimates on the Weierstrass product involving the eigenvalues

As usual when the moment method is concerned, we first study the asymptotic behavior for large $z \in \mathbb{C}$ of the Weierstrass product constructed from the family of eigenvalues $(\lambda_n)_{n\in\mathbb{N}}$.

Lemma 3.1 *Let $(\lambda_n)_{n\geqslant 1}$ be a regular increasing sequence of positive numbers verifying moreover that there exists some $\alpha > 1$ and some constant $R > 0$ such that (3) holds.*

Let Φ_n be defined as follows:

$$\Phi_n(z) := \prod_{k\neq n}\left(1 - \frac{z}{\lambda_k - \lambda_n}\right). \tag{11}$$

Then,

1. *If $\alpha \in (1,2)$, then for every $z \in \mathbb{C}$,*

$$\Phi_n(z) \lesssim e^{\frac{2^{1/\alpha}\pi}{R^{1/(\alpha-1)}\sin(\pi/\alpha)}|z|^{\frac{1}{\alpha}}} P(\lambda_n, |z|), \tag{12}$$

where P is a polynomial.

2. *If $\alpha \geqslant 2$, then for every $z \in \mathbb{C}$,*

$$\Phi_n(z) \lesssim e^{\frac{\pi}{R^{1/(\alpha-1)}\sin(\pi/\alpha)}|z|^{\frac{1}{\alpha}}} \tilde{P}(\lambda_n, |z|), \tag{13}$$

where $\tilde{P}$ is a polynomial.

3. *If $x \in \mathbb{R}$,*

$$\Phi_n(-ix - \lambda_n) \lesssim e^{\frac{\pi}{2R^{1/(\alpha-1)}\sin(\pi/2\alpha)}|x|^{\frac{1}{\alpha}}} \overline{P}(\lambda_n, |x|), \tag{14}$$

where $\overline{P}$ is a polynomial.

(In the previous inequalities, the implicit constant may depend on α and R but not on z, x or n)

Remark 3 *One can see numerically by taking the particular case $\lambda_k = k^\alpha$ that inequalities (13) and (14) are optimal. However, as explained in [11, Remark 3] we are unable to extend inequality (13) to the case $\alpha \in (1,2)$, where is seems numerically not to be true anymore. However, it is likely that estimate (12) is far from being optimal (because notably of the gap when $\alpha \to 2^-$ with what we have in (13) for $\alpha = 2$).*

Proof of Lemma 3.1. Without loss of generality, we can assume that $R = 1$ (one can go back to the general case by an easy scaling argument).

Let us prove inequality (12) (inequality (13) was already proved in [11]). We go back to the computations done in [11, Proof of Lemma 2.1]. One notably has

$$|\Phi_n(z)| \lesssim (1 + |z|/\gamma)^{2C} e^{\int_0^{|z|}\int_\gamma^\infty \frac{(\lambda_n+s)^{\frac{1}{\alpha}} - ((\lambda_n-s)^+)^{\frac{1}{\alpha}}}{(t+s)^2} ds dt}. \tag{15}$$

We also have

$$\begin{aligned}\int_0^{|z|}\int_\gamma^\infty \frac{(\lambda_n+s)^{\frac{1}{\alpha}} - ((\lambda_n-s)^+)^{\frac{1}{\alpha}}}{(t+s)^2} ds dt &= |z|\int_\gamma^\infty \frac{(\lambda_n+s)^{\frac{1}{\alpha}} - ((\lambda_n-s)^+)^{\frac{1}{\alpha}}}{s(s+|z|)} ds \\ &\leqslant \frac{|z|}{\lambda_n^{1-\frac{1}{\alpha}}}\left(U\left(\frac{|z|}{\lambda_n}\right) + V\left(\frac{|z|}{\lambda_n}\right)\right),\end{aligned} \tag{16}$$

where

$$U(x) := \int_0^1 \frac{(1+v)^{\frac{1}{\alpha}} - (1-v)^{\frac{1}{\alpha}}}{v(v+x)} dv$$

and

$$V(x) := \int_1^\infty \frac{(v+1)^{\frac{1}{\alpha}}}{v(v+x)} dv.$$

Using the change of variables $t = x/v$, we easily obtain

$$x^{1-1/\alpha} V(x) = \int_0^x \frac{(1/t + 1/x)^{\frac{1}{\alpha}}}{1+t} dt \tag{17}$$

and

$$x^{1-1/\alpha} U(x) = \int_x^\infty \frac{(1/t + 1/x)^{\frac{1}{\alpha}} - (1/x - 1/t)^{\frac{1}{\alpha}}}{1+t} dt. \tag{18}$$

Using expression (17), one has

$$x^{1-1/\alpha} V(x) \leqslant 2^{\frac{1}{\alpha}} \int_0^x \frac{1}{t^{\frac{1}{\alpha}}(1+t)} dt.$$

Using expression (18), we also have

$$x^{1-1/\alpha} U(x) \leqslant 2^{\frac{1}{\alpha}} \int_x^\infty \frac{1}{t^{\frac{1}{\alpha}}(1+t)} dt.$$

Hence we deduce (12) taking into account that (see for example [11, Lemma 2.2])

$$\int_0^\infty \frac{dt}{t^{\frac{1}{\alpha}}(1+t)} = \sin(\frac{\pi}{\alpha}). \tag{19}$$

Concerning (14), one verifies that the proof provided in [11, Page 2661] for $\alpha \geqslant 2$ is also valid for $\alpha \in (1,2)$ because we can use (19), replacing α by 2α (> 1). ■

In the dispersive case, we also need some estimate in the case where the eigenvalues are not supposed to be positive anymore.

Lemma 3.2 *Let $(\lambda_n)_{n\geqslant 1}$ be a regular increasing sequence of positive numbers verifying moreover that there exists some $\alpha > 1$ and some constant $R > 0$ such that (4) holds.*

Let Φ_n be defined as follows:

$$\Phi_n(z) := \prod_{k\neq n} (1 - \frac{z}{\lambda_k - \lambda_n}). \tag{20}$$

Then,

1. *If $\alpha \in (1,2)$, then for every $z \in \mathbb{C}$,*

$$\Phi_n(-z - \lambda_n) \lesssim e^{\frac{(2^{1/\alpha}+1)\pi}{R^{1/(\alpha-1)} \sin(\pi/\alpha)} |z|^{\frac{1}{\alpha}}} P(\lambda_n, |z|), \tag{21}$$

where P is a polynomial.

2. If $\alpha \geqslant 2$, then for every $z \in \mathbb{C}$,

$$\Phi_n(-z-\lambda_n) \lesssim e^{\frac{2\pi}{R^{1/(\alpha-1)}\sin(\pi/\alpha)}|z|^{\frac{1}{\alpha}}}\,\tilde{P}(\lambda_n,|z|), \tag{22}$$

(In the previous inequalities, the implicit constant may depend on α and R but not on z, x or n)

This Proposition was already proved in [11] for the case $\alpha \geqslant 2$. For $\alpha \in (1,2)$, the proof is exactly the same as the corresponding case in Lemma 3.1 that we combine with the computations made in the proof of [11, Lemma 2.4] (see Pages $2666-2668$ in this reference) and will be omitted.

3.2 Construction of adequate multipliers

As explained before, we now have to construct an adequate multiplier, which has to be the Fourier transform of a Gevrey function with compact support. Let us emphasize that the main contribution of this paper is the construction given in (43).

To begin, let us give some useful estimates concerning the Bray-Mendelbrojt construction of functions with compact support (see [13, Section 13]).

Proposition 3.1 *Let $(a_k)_{k\geqslant 0}$ be a decreasing sequence of positive numbers such that*

$$a := \sum_k a_k < \infty.$$

Then, there exists a nonnegative function u with compact support included in $[-a,a]$ verifying

$$\int_{-a}^{a} u = 1, \tag{23}$$

$$u(0) \leqslant \frac{1}{2a_0}, \tag{24}$$

$$u \text{ is even}, \tag{25}$$

$$u \text{ is increasing on } [-a,0], \tag{26}$$

such that for every $j \in \mathbb{N}$, one has

$$||u^{(j)}||_\infty \leqslant \frac{1}{\prod_{i=0}^{j} a_i}, \tag{27}$$

Proof of Proposition 3.1 We follow step by step the proof given in [5, Pages 19-20]. For every $b>0$ we call

$$H_b := \frac{\mathbb{1}_{[-b,b]}}{2b}. \tag{28}$$

Let us remark that

$$\int_{\mathbb{R}} H_b = \int_{-b}^{b} H_b = 1 \tag{29}$$

and

$$H_b \text{ is even.} \tag{30}$$

We then consider

$$u_n := H_{a_0} * H_{a_1} * \cdots * H_{a_n},$$

where $*$ represents the convolution product. u_n is of class C^{n-1} and

$$Supp(u_n)[-\sum_1^n a_i, \sum_1^n a_i](\subset [-a,a]). \tag{31}$$

Moreover, one easily verifies that, for $j \leqslant n-1$ we have

$$u_n^{(j)}(x) = \left(\prod_0^{j-1} \frac{1}{2a_i}(\tau_{-a_i} - \tau_{a_i}) \right) H_{a_j} * \cdots * H_{a_n}(x), \tag{32}$$

where

$$\tau_b : u \mapsto (x \mapsto u(x-b)).$$

Using (29), the fact that for u and v some regular enough functions one has $\int u * v = \int u \int v$, and $|u * v| \leqslant ||u||_\infty ||v||_1$, we deduce, taking into account (32), that

$$||u_n^{(j)}||_\infty \leqslant \frac{1}{\prod_{i=0}^{j} a_i}.$$

Let us also remark that

$$u_n \text{ is even} \tag{33}$$

and verifies

$$\int_{-a}^{a} u_n(x)dx = \int H_{a_1} \cdots \int H_{a_n} = 1, \tag{34}$$

moreover

$$\begin{aligned} u_n(0) &= \int_{-a}^{a} H_{a_0}(s) H_{a_1} * \cdots * H_{a_n}(-s)ds = \frac{1}{2a_0} \int_{-a_0}^{a_0} H_{a_1} * \cdots * H_{a_n}(-s)ds \\ &\leqslant \frac{1}{2a_0} \int_{-a}^{a} H_{a_1} \cdots \int_{-a}^{a} H_{a_n} = \frac{1}{2a_0}. \end{aligned} \tag{35}$$

Let us now prove that u_n is increasing on $[-a,0]$ for every $n \in \mathbb{N}$. Let us prove this by induction. it is true for $n=0$ and $n=1$. Let us assume that u_n is increasing on $[-a,0]$. Then we compute the derivative of u_{n+1}, we obtain

$$u'_{n+1}(x) = \frac{1}{2a_{n+1}}(\tau_{-a_{n+1}} - \tau_{a_{n+1}})(H_{a_0} * \cdots * H_{a_n}(x)) = \frac{1}{2a_{n+1}}(u_n(x+a_{n+1}) - u_n(x-a_{n+1})). \tag{36}$$

Let $x \in [-a,0]$. Then, we distinguish two cases:

1. If $x \leqslant -a_{n+1}$, then $x - a_{n+1} \leqslant x + a_{n+1} \leqslant 0$, hence, by using the fact that u_n is increasing on $[-a,0]$, it is clear that
$$u_n(x - a_{n+1}) \leqslant u_n(x + a_{n+1}),$$
and then by (36), $u'_{n+1}(x) \geqslant 0$.

2. If $x \geqslant -a_{n+1}$, then we have $-a_{n+1} - x \geqslant 0$. But we also have $a_{n+1} - x \geqslant -x - a_{n+1} \geqslant 0$. Using the fact that u_n is even, we know by induction that u_n is decreasing on $[0,a]$, hence we deduce that
$$u_n(x + a_{n+1}) = u_n(-x - a_{n+1}) \geqslant u_n(a_{n+1} - x) = u_n(x - a_{n+1}),$$
and then by (36), $u'_{n+1}(x) \geqslant 0$.

We conclude as in [5] by letting $n \to \infty$, u being the limit (that exists) of the u_n.

■

According to Section 2.3 and to inequality (27) (that has to be verified for every n), we see that the problem of minimizing what we called $C_{Gev}(\alpha, T, R)$ in the case of the Bray-Mandelbrojt construction can be reformulated as maximizing the following quantity with respect to the sequence $(a_n)_{n\in\mathbb{N}}$:

$$\max_{j\in\mathbb{N}} \prod_{i=0}^{j} \left(\frac{(i+1)^\alpha a_i}{\nu^{\alpha-1}} \right),$$

for some large parameter $\nu > 0$ to be chosen later. It is clear that this problem is equivalent to maximizing

$$\prod_{i=0}^{\infty} \left(\frac{(i+1)^\alpha a_i}{\nu^{\alpha-1}} \right).$$

Taking into account all these considerations, it is quite natural to investigate the following optimization Problem.

Problem (1) Let $\nu > 0$ and $a > 0$. Find the sequence $(a_k)_{k\in\mathbb{N}}$ that maximizes the quantity

$$\sum_{k=0}^{\infty} \ln \left(\frac{(k+1)^\alpha a_k}{\nu^{\alpha-1}} \right)$$

with the constraints:

1. $a_k > 0$ for every $k \in \mathbb{N}$.
2. $(a_k)_{k\in\mathbb{N}}$ is non-increasing.
3. $$\sum_{k=0}^{\infty} a_k = a. \tag{37}$$

Unfortunately, we have the following proposition:

Proposition 3.2 *There are no solutions to Problem (1).*

Proof of Proposition 3.2. Let

$$f((a_k)) := \sum_{k=0}^{\infty} \ln \left(\frac{(k+1)^\alpha a_k}{v^{\alpha-1}} \right). \tag{38}$$

One has

$$\frac{\partial f}{\partial a_i}((a_k)) = \frac{1}{a_k}. \tag{39}$$

Let

$$g((a_k)) := \sum_{k=0}^{\infty} a_k. \tag{40}$$

One has

$$\frac{\partial g}{\partial a_i}((a_k)) = 1. \tag{41}$$

We want to maximize f under the constraint $g((a_k))$ for every $j \in \mathbb{N}$ (the constraint $a_k > 0$ can be forgotten here). We do not need here to use that we want $(a_k)_{k\in\mathbb{N}}$ to be decreasing.

Let $\lambda \in \mathbb{R}$ and $(\mu_j)_{j\in\mathbb{N}}$, we consider the Lagrangian

$$\mathcal{L}((a_k)) := f((a_k)) - \lambda g((a_k)). \tag{42}$$

We remark that for every $j \in \mathbb{N}$, we have $\frac{\partial \mathcal{L}}{\partial a_j} = 0$ if and only if $a_j = \lambda$ according to (39) and (41), hence the sequence (a_j) has to be constant. If we take into account the fact that we want the constraint $g((a_k)) = a$ to be verified, we see that Problem (1) cannot have any solution. ■

We observe thanks to the previous study that if we want to "mimic" in some sense an optimal sequence, a good idea would be to consider some sequence $(a_k)_{k\in\mathbb{N}}$ which is constant at least for small k and then decreases in some suitable way such that it becomes summable. Hence, taking into account this remark, we set $\nu > 0$ some parameter (that is destined to compensate the bad growth of the Weierstrass product (11), see (59), and to be very large so we will always assume $\nu \geqslant C > 0$), and we consider the following sequence $(a_k)_{k\in\mathbb{N}}$ defined by

$$\begin{cases} a_k & = \frac{1}{\nu} \quad \text{if } k \leqslant \lfloor \nu \rfloor - 1, \\ a_k & = \frac{\nu^{\alpha-1}}{(1+k)^{\alpha}} \quad \text{if } k \geqslant \lfloor \nu \rfloor , \end{cases} \tag{43}$$

The main advantage of this construction is that for $k \geqslant \lfloor \nu \rfloor$, we have

$$\frac{(k+1)^{\alpha} a_k}{\upsilon^{\alpha-1}} = 1,$$

which will simplify a lot the forthcoming computations.

Remark 4 *The second part of the construction of* (43) *may seem to be chosen quite arbitrarily, however the author did not find a more suitable construction. It might be possible that another appropriate decreasing that would enable to improve the bounds given in this article.*

We consider the corresponding function σ_ν constructed as in the proof of Lemma 3.1 from the sequence $(a_k)_{k\in\mathbb{N}}$. We call a_ν its support, that is to say

$$a_\nu := 1 + \sum_{k=\lfloor \nu \rfloor}^{\infty} \frac{\nu^{\alpha-1}}{(1+k)^{\alpha}}. \tag{44}$$

It may seem quite strange that we do not impose the support to be $[-T/2, T/2]$ here, however for the sake of clarity we prefer to adjust the support thanks to the parameter β to be introduced later (see (58)).

Taking into account the definition of a_ν, we may observe that a_ν does not depend too much on a_ν in the sense that

$$a_\nu \sim \frac{\alpha}{\alpha - 1} \text{ as } \nu \to \infty. \tag{45}$$

Let us remark that

$$\int_{a_\nu}^{a_\nu} \sigma_\nu = 1, \tag{46}$$

$$\sigma_\nu(0) \leqslant \frac{\nu}{2}, \tag{47}$$

$$\sigma_\nu \text{ is even}, \tag{48}$$

$$\sigma_\nu \text{ is increasing on } [-a_\upsilon, 0]. \tag{49}$$

Then, using (27), we obtain

$$||u^{(j)}||_\infty \leqslant \frac{1}{\prod_{i=0}^{j} a_i}. \tag{50}$$

We can then deduce the following crucial estimate :

Lemma 3.3 *One has for every $j \in \mathbb{N}$*

$$||u^{(j)}||_\infty \lesssim \frac{e^{\alpha\nu}(j!)^{\alpha}}{(\nu^{\alpha-1})^{j}}. \tag{51}$$

Proof of Lemma 3.3. Using the expression of a_k given in (43) and (50), we deduce that for $j \geqslant \lfloor \nu \rfloor$ we have

$$
\begin{aligned}
||u^{(j)}||_\infty &\leqslant \frac{1}{\prod_{i=0}^{j} a_i} && (52)\\
&\lesssim \prod_{i=0}^{j} \frac{\nu^{\alpha-1}/(i+1)^\alpha}{(i+1)^\alpha a_i/\nu^{\alpha-1}} && (53)\\
&\lesssim \frac{(j!)^\alpha}{(\nu^{\alpha-1})^j} \prod_{i=0}^{j} \frac{\nu^{\alpha-1}}{(i+1)^\alpha a_i} && (54)\\
&\lesssim \frac{(j!)^\alpha}{(v^{\alpha-1})^j} \prod_{i=0}^{\lfloor \nu \rfloor -1} \frac{\nu^\alpha}{(i+1)^\alpha}. && (55)
\end{aligned}
$$

Now, using an usual lower bound on the factorial, we obtain

$$
\prod_{i=0}^{\lfloor \nu \rfloor -1} \frac{\nu^\alpha}{(i+1)^\alpha} = \frac{\nu^{\nu\alpha}}{((\lfloor \nu \rfloor)!)^\alpha} \lesssim \frac{\nu^{\nu\alpha}}{\nu^{\nu\alpha} e^{-\alpha\nu}},
$$

from which we deduce (51).

The same estimation is also clearly true for $j \leqslant \lfloor \nu \rfloor$ since in this case we have

$$
||u^{(j)}||_\infty \leqslant \nu^j.
$$

■

Let

$$
H_\beta(z) := \int_{-a_\nu}^{a_\nu} \sigma_\nu(t) e^{-i\beta t z} dt, \tag{56}
$$

which will be our multiplier (up to some homothety). We have the following properties:

Lemma 3.4 *For every $\delta > 0$ small enough, one has*

$$
\begin{aligned}
H_\beta(0) &= 1, && (57)\\
|H_\beta(z)| &\leqslant e^{a_\nu \beta |Im(z)|}, && (58)\\
|H_\beta(x)| &\lesssim e^{\alpha\nu} e^{-\alpha((\beta\nu^{\alpha-1})|x|)^{1/\alpha} + \delta/(2\sin(\pi/(\alpha)))|x|^{1/\alpha}},\ x \in \mathbb{R}. && (59)\\
|H_\beta(ix)| &\gtrsim \frac{1}{\nu} e^{\frac{\beta x}{4\nu}},\ x > 0. && (60)
\end{aligned}
$$

Proof of Lemma 3.4. Equality (57) comes from (46). Inequality (58) is an immediate consequence of the definition of H_β given in (56). Let us now prove (59). Since all derivatives of σ_ν vanish at $t = -a_\nu$ and $t = a_\nu$, we have

$$
|H_\beta(x)| \lesssim \frac{||\sigma_\nu^{(j)}||_\infty}{(\beta x)^j}, \tag{61}
$$

for every $x > 0$ and $j \in \mathbb{N}$. Combining (61) and (51), we deduce that

$$
|H_\beta(x)| \lesssim \frac{e^{\alpha\nu}(j!)^\alpha}{(\beta\nu^{\alpha-1}x)^j}.\ j \in \mathbb{N}. \tag{62}
$$

We set

$$j := \lfloor (\beta \nu^{\alpha-1} x)^{1/\alpha} \rfloor. \tag{63}$$

Then we have

$$\beta \nu^{\alpha-1} x \geqslant (aj)^{\gamma}. \tag{64}$$

Using (64) and (62) we obtain

$$|H_\beta(x)| \lesssim e^{\alpha\nu} \frac{(j!)^\alpha}{j^{\alpha j}}. \tag{65}$$

Combining (65), (63), and inequality

$$(j!)^\alpha \lesssim j^{\alpha/2} j^{\alpha j} e^{-\alpha j},$$

we deduce

$$|H_\beta(x)| \lesssim e^{\alpha\nu} e^{-\alpha j} j^{\alpha/2} \lesssim e^{\alpha\nu} e^{-\alpha(\beta\nu^{\alpha-1}x)^{1/\alpha} + \delta/(2\sin(\pi/(\alpha)))x^{1/\alpha}},$$

which proves the desired estimate.

Let us know prove (60). We have, since $\sigma_\nu \geqslant 0$ and using (49) together with (48),

$$|H_\beta(ix)| = \int_{-a_\nu}^{a_\nu} \sigma_\nu(t) e^{\beta x t} dt \geqslant \int_{1/(4\nu)}^{1/(2\nu)} \sigma_\nu(t) e^{\beta x t} dt \geqslant \frac{1}{4\nu} \sigma_\nu(\frac{1}{2\nu}) e^{\frac{x}{4\nu}}. \tag{66}$$

Using (46), (47), (48) and (49), we have

$$1 = 2\int_{1/(2\nu)}^{a_\nu} \sigma_\nu + 2\int_0^{1/(2\nu)} \sigma_\nu \leqslant 2(a_v - 1/(2\nu))\sigma_\nu(\frac{1}{2\nu}) + \frac{2\sigma_\nu(0)}{2\nu} \leqslant 2(a_v - 1/(2\nu))\sigma_\nu(\frac{1}{2\nu}) + \frac{1}{2}.$$

We deduce that for ν large enough (which is equivalent to T small enough, see (69))

$$\sigma_\nu(\frac{1}{2\nu}) \gtrsim \frac{1}{a_v - 1/(2\nu)} \gtrsim 1,$$

which gives the desired result thanks to (66).

■

Proof of Theorem 2.1. The proof follows the one of [22, Theorem 3.1 and 3.4]. We still assume without loss of generality that $R = 1$. Let us first consider the dispersive case (Equation (2)) and the case $\alpha \geqslant 2$. Let $\delta > 0$ a small enough parameter. We call

$$g_n(z) := \Phi_n(-z - \lambda_n) H_\beta(z + \lambda_n), \tag{67}$$

so that one has $g_n(-\lambda_k) = \delta_{kn}$ by (57) and (11). We want to apply at the end the Paley-Wiener Theorem (see estimate (58)) in an optimal way, so we want $a_\nu \beta$ to be close to $T/2$. From now on we will always consider ν large enough such that (see (45))

$$|a_\nu - \frac{\alpha}{\alpha - 1}| \leqslant \delta/4.$$

Assume that $a_\nu \beta < T/2$ and close to $T/2$, for example

$$\beta = \frac{T(\alpha-1)(1-\delta/2)}{2\alpha}. \tag{68}$$

Now we choose ν such that

$$\alpha^\alpha(\beta\nu^{\alpha-1}) > \frac{\pi}{\sin(\frac{\pi}{\alpha})}$$

and close to $\pi/\sin(\pi/\alpha)$ (see estimates (13) and (59)), for example

$$\nu := \frac{(\pi+\delta)^{\alpha/(\alpha-1)}}{(\alpha\sin(\pi/\alpha))^{\alpha/(\alpha-1)}\beta^{1/(\alpha-1)}}. \tag{69}$$

Then, using (67), (13), (68), (69) and (59), we obtain

$$|g_n(x)| \lesssim e^{\alpha\nu+\pi/\sin(\pi/\alpha)|x+\lambda_n|^{\frac{1}{\alpha}}-(\pi+\delta/2)/\sin(\pi/\alpha)|x+\lambda_n|^{\frac{1}{\alpha}}}\tilde{P}(|x+\lambda_n|).$$

It is clear that

$$e^{\pi/\sin(\pi/\alpha)|x+\lambda_n|^{\frac{1}{\alpha}}-(\pi+\delta/2)/\sin(\pi/\alpha)|x+\lambda_n|^{\frac{1}{\alpha}}}\tilde{P}(|x+\lambda_n|) \lesssim \frac{1}{1+(x+\lambda_n)^2},$$

hence we deduce using (69) and (68) that

$$\begin{aligned}|g_n(x)| &\lesssim \frac{e^{\alpha\nu}}{1+(x+\lambda_n)^2} \\ &\lesssim \frac{e^{\frac{\alpha(2\alpha)^{1/(\alpha-1)}(\pi+\delta)^{\alpha/(\alpha-1)}}{(\alpha\sin(\pi/\alpha))^{\alpha/(\alpha-1)}(T(\alpha-1)(1-\delta/2))^{1/(\alpha-1)}}}}{1+(x+\lambda_n)^2} \\ &\lesssim \frac{e^{\frac{2^{1/(\alpha-1)}(\pi+\delta)^{\alpha/(\alpha-1)}}{(\sin(\pi/\alpha))^{\alpha/(\alpha-1)}(T(\alpha-1)(1-\delta/2))^{1/(\alpha-1)}}}}{1+(x+\lambda_n)^2}.\end{aligned}$$

Let us fix some

$$K > \frac{2^{1/(\alpha-1)}\pi^{\alpha/(\alpha-1)}}{(\alpha-1)^{1/(\alpha-1)}\sin(\pi/\alpha)^{\alpha/(\alpha-1)}T^{1/(\alpha-1)}}.$$

Considering δ as close as 0 as needed, we deduce that

$$|g_n(x)| \lesssim \frac{e^{\frac{K}{T^{1/(\alpha-1)}}}}{1+(x+\lambda_n)^2} \tag{70}$$

This notably proves that $g_n \in L^2(\mathbb{R})$. Moreover, using (13), (67), (68) and (58), we obtain

$$|g_n(z)| \lesssim e^{T|z|/2}.$$

Hence, using the Paley-Wiener Theorem, g_n is the Fourier transform of a function $f_n \in L^2(\mathbb{R})$ with compact support $[-T/2, T/2]$. Moreover, by construction $\{f_n\}$ is biorthogonal to the family $\{e^{i\lambda_n t}\}$. Then, one can create the control thanks to the family $\{f_n\}$. Let us consider $y^0 = \sum a_k e_k$ the initial condition, we call

$$u(t) := -\sum_{k\in\mathbb{N}}(a_k/b_k)e^{-iT\lambda_k/2}f_k(t-T/2). \tag{71}$$

This expression is meaningful since $b_k \simeq 1$, moreover the corresponding solution y of (2) verifies $y(T,\cdot) \equiv 0$. We obtain by proceeding as in [22, Page 87-88] that

$$\begin{aligned}||u(t)||_{L^2(0,T)} &\lesssim e^{\frac{K}{T^{1/(\alpha-1)}}}\sum|a_k|^2 \\ &\lesssim e^{\frac{K}{T^{1/(\alpha-1)}}}||y^0||_H.\end{aligned}$$

The case $\alpha \in (1,2)$ follows by doing exactly the same proof, the only difference being that we use (12) instead of (13).

■

Proof of Theorem 2.2. The proof is exactly the same as the one of Theorem 2.1, using (21) instead of (12) for $\alpha \in (1, 2)$ and (22) instead of (13) for $\alpha \geqslant 2$, and will hence be omitted.

■

Proof of Theorem 2.3. We now consider the parabolic case (Equation (1)). Let $\delta > 0$ a small enough parameter. We call

$$h_n(z) := \Phi_n(-iz - \lambda_n) \frac{H_\beta(z)}{H_\beta(i\lambda_n)}. \tag{72}$$

One has, using (11), that $h_n(i\lambda_k) = \delta_{kn}$. We want to apply at the end the Paley-Wiener Theorem (see estimate (58)) in an optimal way, so we want $a_\nu \beta$ to be close to $T/2$. Let us consider ν large enough such that (see (45))

$$|a_\nu - \frac{\alpha}{\alpha - 1}| \leqslant \delta/4.$$

Assume that $a_\nu \beta < T/2$ and close to $T/2$, for example

$$\beta = \frac{T(\alpha - 1)(1 - \delta/2)}{2\alpha}. \tag{73}$$

Now we choose ν such that

$$\alpha^\alpha (\beta \nu^{\alpha - 1}) > \pi/(2 \sin(\pi/2\alpha))$$

and close to $\pi/(2\sin(\pi/(2\alpha)))$ (see estimates (13) and (59)), for example

$$\nu := \frac{(\pi + \delta)^{\alpha/(\alpha-1)}}{(2\alpha \sin(\pi/2\alpha))^{\alpha/(\alpha-1)} \beta^{1/(\alpha-1)}}. \tag{74}$$

Moreover, thanks to (72), (14), (59), (60), (74) and (73), one has

$$\begin{aligned}
|h_n(x)| &\lesssim \nu e^{\alpha\nu + \pi/(2\sin(\pi/2\alpha))|x|^{\frac{1}{\alpha}} - ((\pi+\delta/2)/(2\sin(\pi/2\alpha)))|x|^{\frac{1}{\alpha}} - \frac{\beta\lambda_n}{4\nu}} \overline{P}(|x|, |\lambda_n|) \\
&\lesssim \nu e^{\alpha\nu - \delta/(2\sin(\pi/2\alpha))|x|^{\frac{1}{\alpha}} - \frac{\beta\lambda_n}{4\nu}} \overline{P}(|x|, \lambda_n|) \\
&\lesssim \nu \frac{e^{\alpha \frac{(\pi+\delta)^{\alpha/(\alpha-1)}}{(2\alpha\sin(\pi/2\alpha))^{\alpha/(\alpha-1)}\beta^{1/(\alpha-1)}}}}{(1 + (x + \lambda_n)^2)} \\
&\lesssim \nu \frac{e^{\alpha \frac{(2\alpha)^{1/(\alpha-1)}(\pi+\delta)^{\alpha/(\alpha-1)}}{(2\alpha\sin(\pi/2\alpha))^{\alpha/(\alpha-1)}(T(\alpha-1)(1-\delta/2))^{1/(\alpha-1)}}}}{(1 + (x + \lambda_n)^2)} \\
&\lesssim \nu \frac{e^{\frac{2^{1/(\alpha-1)}(\pi+\delta)^{\alpha/(\alpha-1)}}{(2\sin(\pi/2\alpha))^{\alpha/(\alpha-1)}(T(\alpha-1)(1-\delta/2))^{1/(\alpha-1)}}}}{(1 + (x + \lambda_n)^2)}.
\end{aligned}$$

Let us fix some

$$K > \frac{\pi^{\alpha/(\alpha-1)}}{2(\sin(\pi/2\alpha))^{\alpha/(\alpha-1)}(\alpha - 1)^{1/(\alpha-1)}}.$$

Considering δ as close as 0 as needed, we deduce that

$$|h_n(x)| \lesssim \frac{e^{\frac{K}{T^{1/(\alpha-1)}}}}{(1 + (x + \lambda_n)^2)}, \tag{75}$$

This notably implies that $h_n(x) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and

$$||h_n||_{L^1(\mathbb{R})} \lesssim e^{\frac{K}{T^{1/(\alpha-1)}}}. \tag{76}$$

Moreover, using (13), (72), (58) and (73)

$$|h_n(z)| \lesssim e^{T|z|/2},$$

so using the Paley-Wiener Theorem, h_n is the Fourier transform of a function $w_n \in L^2(\mathbb{R})$ with compact support $[-T/2, T/2]$. Moreover, by construction $\{w_n\}$ is biorthogonal to the family $\{e^{\lambda_n t}\}$. Then, one can create the control thanks to the family $\{h_n\}$. Let us consider $y^0 = \sum a_k e_k$ the initial condition, we call

$$u(t) := -\sum (a_k/b_k) e^{-T\lambda_k/2} w_k(t - T/2), \tag{77}$$

This expression is meaningful since $b_k \simeq 1$, moreover the corresponding solution y of (1) verifies $y(T, \cdot) \equiv 0$. One easily verifies that $u \in C^0([0,T], \mathbb{R})$. Using (77), $|b_k| \simeq 1$ and inequality (76), we obtain

$$||u(t)||_{L^\infty(0,T)} \lesssim e^{\frac{K}{T^{1/(\alpha-1)}}} \sum |a_k| e^{-T\lambda_k/2}.$$

Using the Cauchy-Schwarz inequality, one deduces

$$||u(t)||_{L^\infty(0,T)} \lesssim e^{\frac{K}{T^{1/(\alpha-1)}}} ||y^0||_H.$$

■

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