What has Instrumental Variable method to offer for system identification?
Marion Gilson

To cite this version:
Marion Gilson. What has Instrumental Variable method to offer for system identification?. 8th IFAC International Conference on Mathematical Modelling, MATHMOD 2015, Feb 2015, Vienna, Austria. 2015, <10.1016/j.ifacol.2015.05.176>. <hal-01242758>

HAL Id: hal-01242758
https://hal.archives-ouvertes.fr/hal-01242758
Submitted on 14 Dec 2015

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.
What has Instrumental Variable method to offer for system identification?

M. Gilson\textsuperscript{1,2}

\textsuperscript{1} University of Lorraine, CRAN, UMR 7039, 2 rue Jean Lamour, F-54519 Vandoeuvre-les-Nancy, France
\textsuperscript{2} CNRS, CRAN, UMR 7039, France, marion.gilson@univ-lorraine.fr

Abstract: This paper gathers several experiences of using instrumental variable method in different contexts: closed-loop system identification, LPV model, frequency domain framework.

Keywords: instrumental variable, system identification, closed-loop, LPV model, frequency domain

1. INTRODUCTION

Mathematical models of dynamic systems are required in most area of scientific enquiry and take various forms, such as differential equations, difference equations, state-space equations and transfer functions. The most widely used approach to mathematical modeling involves the construction of mathematical equations based on physical laws that are known to govern the behaviour of the system. While the advantage of these methods relies on the deep physical insight of the resulting model, their main drawback is the complexity of the model that makes them difficult to be used in applications such as control system design, prediction or decision making.

An alternative to physically-based mathematical modeling is the so-called data-based system identification, which can be applied to any system where experimental data are available. A large scope of system identification approaches has been developed over the past decades. Amongst these, we can cite the prediction error and maximum-likelihood frameworks (see e.g. Ljung (1999); Söderström and Stoica (1989); Young (2011)); the subspace-based identification (see e.g. Van Overschee and De Moor (1996); Katayama (2005)); the frequency-domain identification (see e.g. Pintelon and Schoukens (2001); Mc Kelvey (2002), the closed-loop identification case (see e.g. Van den Hof (1998); Forssell and Ljung (1999); Ninness and Hjalmarsson (2005); Gilson and Van den Hof (2005)).

Most physical systems are continuous-time (CT) whereas, mainly due to the advent of digital computers, research on system identification has concentrated on discrete-time (DT) models from underlying CT systems input/output samples. Recently, interest in identification of CT systems from DT data has arisen (see e.g. Sinha and Rao (1991); Unbehauen and Rao (1987); Garnier and Wang (2008) and references herein) and offer a clever solution in many cases such as irregularly sampled data.

Moreover, systems encountered in practice are often nonlinear or present a time-varying nature. Unlike linearity, non-linearity is a non-property and therefore, non-linearity cannot be defined in a general way. A common framework for the identification of nonlinear models has nevertheless been presented in Sjöberg et al. (1995) and Juditsky et al. (1995). Usually, nonlinear models are classified into two classes: non-parametric models and parametric models. However, another type of models has more recently arose the attention of the system identification community and form an intermediate step between Linear Time-Invariant (LTI) systems and nonlinear/time-varying plants: the model class of Linear Parameter-Varying (LPV) systems (Bamieh and Giarré (2002); Töth (2010)).

When considering methods that can be used to identify (linear or non linear, CT or DT) models of systems operating in open- or closed-loop, instrumental variable (IV) techniques are rather attractive since they are normally simple or iterative modifications of the linear regression algorithm. For instance, when dealing with complex processes, it can be attractive to rely on methods, such as these, that do not require non-convex optimization algorithms. In addition to this computationally attractive property, IV methods also have the potential advantage that they can yield consistent and asymptotically unbiased estimates of the plant model parameters if the noise does not have rational spectral density or if the noise model is mis-specified; or even if the control system is non-linear and/or time-varying, in the closed-loop framework (Gilson and Van den Hof (2005); Gilson et al. (2011)). Even if several works arise these last ten years (e.g. Young (2011); Dankers et al. (2014); Van Herpen et al. (2014); Laurain et al. (2010); Douma (2006)), IV methods have not yet really received the attention that it deserves.

This paper is dedicated to the use of IV methods in several cases of system identification. After an introduction of the IV principles in Section 2, the focus is made on closed-loop system in Section 3, on LPV models with an application on rainfall-flow modeling in Section 4 and on frequency domain framework in Section 5.

2. INSTRUMENTAL VARIABLE METHOD

System identification is based on three main ingredients: data (experiment design), model set selection, identification criterion, which are used to estimate a model of a given system. In this paper, we will mainly focus on the identification criterion named Instrumental Variable (IV).

IV is a criterion aiming at minimizing the prediction error. Consider a stable, linear, Single Input Single Output (SISO) data-generating system assumed to be described as

\[ y(t_k) = G_0(q)u(t_k) + H_0(q)e(t_k) \quad (1) \]

The plant is denoted by \( G_0(q) = B_0(q^{-1})/A_0(q^{-1}) \) with the numerator and denominator degree equals to \( n_0 \). \( q^{-1} \) is the
delay operator with \( g^{-1}x(t_k) = x(t_{k-1}) \), \( u \) describes the plant input signal, \( y \) the plant output signal. A colored disturbance \( \zeta_0(t_k) = H_0(q)e_0(t_k) \) is assumed to affect the system, where \( e_0 \) is a white noise, with zero mean and variance \( \sigma_e^2 \).

The following general model structure and parameter plant model are chosen to model the system
\[
\mathcal{M} : \quad y(t_k) = G(q, \theta)u(t_k) + H(q, \theta)e(t_k) \tag{2}
\]
\[
\mathcal{G} : \quad G(q, \theta) = \frac{B(q^{-1})}{A(q^{-1})} \tag{3}
\]
In the prediction error method (PEM), the parameters are computed by minimizing the criterion function (see Ljung (1999))
\[
V(q, \theta) = \frac{1}{N} \sum_{k=1}^{N} |e(t_k, \theta)|^2 \tag{4}
\]
where \( e(t_k, \theta) = y(t_k) - \hat{y}(t_k, \theta) \) is the prediction error. Therefore, the parameters are given as
\[
\hat{\theta} = \arg \min \frac{1}{N} \sum_{k=1}^{N} (y(t_k) - \varphi^T(t_k)\theta)^2 \tag{5}
\]
where \( \varphi(t_k) \) denotes the regressor.

The optimal IV algorithm providing the minimum value of the covariance matrix is known to be obtained for (see Söderström and Stoica (1983); Young (2011, 2014))
\[
\hat{\varphi}(t_k) = \frac{1}{A_0(q^{-1})}H_0(q) \tag{6}
\]
\[
L^{opt}(q) = \frac{1}{A_0(q^{-1})}H_0(q) \tag{7}
\]
\[
\hat{\zeta}_\theta = \frac{1}{N} \sum_{t=1}^{N} \zeta_f(t_k)\varphi_f(t_k) \tag{8}
\]
\[
\hat{\varphi}(t_k) = \varphi(t_k) \tag{9}
\]
\[
L^{opt}(q) = \frac{1}{A_0(q^{-1})}H_0(q) \tag{10}
\]
\[
\hat{\phi}(t_k) = \frac{1}{A_0(q^{-1})}H_0(q) \tag{11}
\]
\[
\hat{\zeta}_\theta = \frac{1}{N} \sum_{t=1}^{N} \zeta_f(t_k)\varphi_f(t_k) \tag{12}
\]
\[
\hat{\varphi}(t_k) = \varphi(t_k) \tag{13}
\]
\[
L^{opt}(q) = \frac{1}{A_0(q^{-1})}H_0(q) \tag{14}
\]
\[
\hat{\phi}(t_k) = \frac{1}{A_0(q^{-1})}H_0(q) \tag{15}
\]

The notation \( \mathbb{E}[\cdot] \) is adopted from the prediction error framework of Ljung (1999).
variables are dealt with in Gilson et al. (2011). It has been shown that the optimal solution is given by (11), however, since not only the output but also the input are corrupted by the noise, the computation of the noise-free regressor (9) has to take into account the noise-free part of \( y \) and \( u \).

As a result, the closed-loop optimal IV estimator is dependent on unknown system properties, i.e. the plant as well as the noise dynamics. Whereas dependency of plant dynamics could be taken care of by an iterative procedure where the instrument and prefilter are constructed on the basis of a previous plant model estimate \( \hat{\theta}^{i-1} \), knowledge of the noise dynamics is generally missing in an extended IV estimator as it is not particularly estimated.

Therefore the next step to an optimal IV method is to extend the estimator (8) with a procedure to estimate an appropriate noise model, to be used as a basis for constructing the optimal prefilter \( L^{opt}(q) \) given in (10).

Then several solutions have been proposed in a unified way, depending on the structure of the process and noise models (see Gilson et al. (2011)): ARX, ARARX, OE, BJ models are analyzed to develop the refined IV method dedicated to the closed-loop framework.

**Simulation examples**

The simulation model is based on the relations (12), where

\[
G_0(q) = \frac{0.0997q^{-1} - 0.0902q^{-2}}{1 - 1.8858q^{-1} + 0.9048q^{-2}}, \quad n = 2
\]
\[
C_r(q) = \frac{10.75 - 9.25q^{-1}}{1 - q^{-1}}.
\]

The excitation signal \( r(t) \) is a pseudo random binary signal of maximal length, with the number of stages for the shift register set to 9 and the clock period set to 8; \( e_0(t) \) is a white noise uncorrelated with \( r(t) \). The case of colored noise acting on the loop is illustrated on this example, with

\[
H_0(q) = \frac{1 + 0.5q^{-1}}{1 - 0.85q^{-1}}.
\]

The following algorithms are used to estimate this model Gilson et al. (2011):

- **clivr** method (with \( G_0 \in G, S \notin M \)), the 1st closed-loop IV method, where \( \zeta(t_k) = \) delayed version of the reference signal \( r(t_k) \), method initially developed by Söderström et al. (1987).
- **clivrARX** method (with \( G_0 \in G, S \notin M \)), optimal IV, using ARX structure, controller known or not.
- **clivrBJ** method (with \( S \in M \)), optimal IV, using BJ structure, controller known or not.
- **pem** algorithm (with \( S \in M \)), applied to the closed-loop data \( (u(t_k) \, y(t_k)) \) known to be theoretically efficient in the \( S \in M \) case (see Forssell and Ljung (1999)).

The plant parameters are estimated on the basis of closed-loop data of length \( N = 4088 \). A Monte-Carlo simulation of 100 runs is performed for a signal to noise ratio SNR = 25dB (with new noise at each run).

It can be seen from figure 2, that even with a wrong assumption on the noise, the two IV methods clivr and clivrARX provide unbiased results with a lower variance thanks to the use of clivrARX. In the case where \( S \in M \), optimal estimates (no bias, minimum variance) are provided by the clivrBJ method whereas some initialization problems may occur with pem.

![Fig. 2. True (yellow) and estimated (red) Bode diagrams (gain and phase (degree)) of the plant model \( G(q, \theta) \) over the 100 MCS, colored noise](image)

**4. LPV MODEL IDENTIFICATION, APPLICATION TO RAINFALL/FLOW MODELING**

When dealing with NL model, the first question to address is which kind of non linearity is to be used. In this paper, a focus is made on the use of NL model for environmental data: rainfall and flow in a rural catchment. The most common modeling for rainfall/flow relationship is the parametric block representation named Hammerstein structure (see Figure 3).

In this case, the system behavior can be described in the following way : the catchment reacts as a sponge which retains the water until it reaches its full retaining capacity and the runoff starts following a linear tank model. Consequently in the Hammerstein case, the static non-linearity represents the loss of rain which does not reach the outlet of the catchment.

Nonetheless, the sponge effect can be translated into a dynamic change more than in a static non-linearity. Consequently, aside from the block models, another type of models drew the attention of the system identification community lately: the model class of Linear Parameter-Varying (LPV) systems (see Figure 4). In LPV systems the signal relations are considered to be linear just as in the LTI case, but the parameters are assumed to be function of a measurable time-varying signal, the so-called scheduling variable.

The main advantage of the LPV model is to represent a trade-off between LTI models and NL models. As a result, they present a wide range of behavior representation capability to be used in many practical applications. A LPV model is given as

\[
\begin{align*}
\mathcal{M} = \{ & A(p_k, q^{-1})\chi(t_k) = B(p_k, q^{-1})u(t_k) \\
y(t_k) = \chi(t_k) + H(q)e(t_k) & \}
\end{align*}
\tag{16}
\]

![Fig. 3. Hammerstein model](image)
where $\chi$ is the noise-free output, $u$ is the input, $e$ is the additive noise with bounded spectral density, $y$ is the noisy output of the system. $p$ are the so-called scheduling variables. For LTI systems, the coefficients of $A$ and $B$ are constant in time while they are time-varying, depending on $p$, for LPV models. $A(p_k, q^{-1})$ and $B(p_k, q^{-1})$ are polynomials in $q^{-1}$ of degree $n_a$ and $n_b$, respectively:

$$A(p_k, q^{-1}) = 1 + \sum_{i=1}^{n_a} a_i(p_k) q^{-i}$$
$$B(p_k, q^{-1}) = \sum_{j=0}^{n_b} b_j(p_k) q^{-j}$$

with

$$a_i(p_k) = a_{i,0} + \sum_{n,a} a_{i,n,a} b_0 \cdots b_{n}$$
$$b_j(p_k) = b_{j,0} + \sum_{l=1}^{n,b} b_{j,l} p_k$$

In this parametrization, $\{f_i\}_{i=1}^{n_a}$ and $\{g_j\}_{j=1}^{n_b}$ are functions of $p$, with static dependence, allowing the identifiability of the model (pairwise orthogonal functions for example). It can be noticed that the knowledge of $\{a_{i,j}\}_{i=1,j=1}^{n_a,n_b}$ and $\{b_{0,l}\}_{l=0,l=1}^{n_b}$ ensures the knowledge of the full model. Therefore, these model parameters are stacked column-wise in the parameter vector $\theta$ (with $n_\theta = n_a(n_a+1) + (n_b+1)(n_b+1)$),

$$\theta = [a_1 \cdots a_{n_a} b_0 \cdots b_{n_b}]^T \in \mathbb{R}^{n_\theta},$$

where $a_i = [a_{i,0} \cdots a_{i,n_a}] \in \mathbb{R}^{n_a+1}$ and

$$b_j = [b_{j,0} b_{j,1} \cdots b_{j,n_b}] \in \mathbb{R}^{n_b+1}.$$
5. FREQUENCY DOMAIN IDENTIFICATION

We consider here a linear time invariant, input output system given by a rational transfer function
\[ S : \quad G_0(s) = \frac{B_0(s)}{A_0(s)} \]  
(26)
where \( s \) denotes the Laplace transform variable. The frequency domain identification problem is to determine an estimate of this system from a measured frequency response
\[ G(j\omega_k) = G_0(j\omega_k) + V(j\omega_k), \quad k = 1, \ldots, N \]
which is available in a particular frequency grid denoted \( \omega_k \), \( k = 1, \ldots, N \) and where \( G(j\omega) \) denotes the measurement of the system frequency function \( G_0(j\omega) \), corrupted with the measurement noise \( V(j\omega) \). We will assume in this paper that we have access to the system measurements in the frequency domain.

The system is modeled by its transfer function \( G(s) \) and parametrized by a parameter vector \( \theta \), as follows
\[ G : \quad G(s, \theta) = \frac{B(s, \theta)}{A(s, \theta)} \]  
(27)
where
\[ B(s, \theta) = b_m s^m + b_{m-1} s^{m-1} + \ldots + b_1 s + b_0 \]  
(28)
\[ A(s, \theta) = s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0 \]  
(29)
with \( \theta = [a_0 \ldots a_{n-1} b_0 \ldots b_m] \in \mathbb{R}^{n+m+1} \).  
(30)
If the plant \( G_0 \) is included in the chosen model set \( G \) (i.e. \( G_0 \in G \)), the output \( G(j\omega) \) can be written as
\[ G(j\omega_k) = \Phi^*(j\omega_k) \theta_0 + A_0(j\omega_k)V(j\omega_k) \]  
(31)
with the regressor
\[ \Phi^*(j\omega_k) = [-G(j\omega_k) \ldots - (j\omega_k)^{n-1}G(j\omega_k) 1 \ldots (j\omega_k)^m] \]

5.1 IV method in the frequency domain
The determination of the parameter vector \( \theta \) on the basis of the measured frequency domain data may be achieved by minimising
\[ J(\theta) = \frac{1}{N} \sum_{k=1}^{N} |\varepsilon(j\omega_k, \theta)|^2, \]  
(32)
with \( \varepsilon(j\omega_k, \theta) = G(j\omega_k) - G(j\omega_k, \theta) \)

The Least-Square (LS) solution to this problem is known to give unbiased results since \( \Phi(j\omega_k) \) and the noise are correlated, whereas the IV method
\[ \hat{\theta}_{IV} = \arg\min_{\theta} \sum_{k=1}^{N} \zeta(j\omega_k) \left[ \frac{1}{A_0(j\omega_k)} G(j\omega_k) \right]^2 \]  
(33)
provides unbiased result whatever the noise is, under the following two conditions

- \( \frac{1}{N} \sum_{k=1}^{N} \zeta(j\omega_k)^2 \Phi^*(j\omega_k) \) is full column rank
- \( \frac{1}{N} \sum_{k=1}^{N} \zeta(j\omega_k)W(j\omega) = 0 \),

and with
\[ \zeta(j\omega_k) = [-G_0(j\omega_k) - (j\omega_k)^{n-1}G_0(j\omega_k) 1 \ldots (j\omega_k)^m], \]
\( k = 1, \ldots, N \), therefore \( \zeta(j\omega_k) \) corresponds to noise-free part of the regressor (see Gilson et al. (2013)).

As previously, the IV solution is dependent on unknown system property that has to be taken care of by an iterative procedure where the instruments and prefilter are constructed from previous model estimate. A refined IV dedicated to the frequency domain has been developed in that sense (Gilson et al. (2013)). Even though the IV algorithm offers a nice solution to remove the bias of the least squares method in case of noisy measurements, it still relies on the use of the normal matrix. A poor conditioning of this matrix results in poor or erroneous estimates of the system parameters. This problem especially occurs when identifying systems with a large dynamic range, as it is often the case in the frequency domain.

A solution has then been proposed to improve the IV method in this case. It relies on the technique developed in Welsh and Goodwin (2003), based on a particular set of basis functions, which is aimed specifically at improving the numerical properties of the normal matrix in a rational function estimation over a large dynamic range. A key point in this approach is that the method restricts the dynamic range over which each coefficient is estimated by the use of frequency localizing basis functions (FLBF) which span a desired frequency region:
\[ F_k(s) = \prod_{l=1}^{k} \frac{s^l - p_k}{s + p_l}, \quad k = 1, \ldots, n \]
These functions allow the normal matrix to take on a near block diagonal form and hence improve its conditioning. Moreover, the filters used in these functions are bandpass and hence easy to implement. In Gilson et al. (2013), a method introducing basis functions into an IV procedure has been proposed to consistently handle the frequency domain identification case.

5.2 Data-based resonant beam modeling
This IV method associated with the frequency localizing basis functions has been used to estimate the real frequency response data collected from a resonant beam. The experimental data spans approximately 2 decades and is shown in Figure 8 by the blue dots. Two methods have been used to estimate this process:

- \( \text{flbf-ls} \): least-square estimation technique associated with the FLBF (see Welsh and Goodwin (2003))
6. CONCLUSION

In conclusion, this paper has gathered several experiences of using instrumental variable technique in different contexts. It has been illustrated that IV method has a lot to offer for system identification: computationally attractive property, yields consistent estimates even in the case of mis-specified noise model, may potentially be used in many different practical situations: linear, non linear model, open-loop, closed-loop...

7. ACKNOWLEDGMENT

The results presented in this paper gathered several really interesting and fruitful working days, coffee break discussions, running sessions with my friend Hugues Garnier. It also collects joint work developed with my friends and colleagues Paul Van den Hof, Peter Young, James Welsh (especially during several visiting periods in their labs) and Vincent Laurain (my former PhD student).

REFERENCES


