Why Intuitionistic Relevant Logic Cannot Be a Core Logic
Joseph Vidal-Rosset

To cite this version:
Joseph Vidal-Rosset. Why Intuitionistic Relevant Logic Cannot Be a Core Logic. 2014. hal-01240886

HAL Id: hal-01240886
https://hal.archives-ouvertes.fr/hal-01240886
Submitted on 9 Dec 2015

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers. L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
Why Intuitionistic Relevant Logic Cannot Be a Core Logic

Joseph Vidal-Rosset

Abstract
At the end of the eighties Tennant invented a logical system that he called “intuitionistic relevant logic” (for short IR). Now he calls this same system “Core logic”. In section 1, by reference to the rules of natural deduction for IR, I explain why IR is a relevant logic in a subtle way. Sections 2, 3 and 4 give three reasons to assert that IR cannot be a Core logic.

1 IR in a nutshell

1.1 Motivation of IR. Intuitionistic Relevant logic (for short IR) is a logical system invented by Tennant at the end of the eighties in order to prove all theorems of Heyting logic without accepting the intuitionistic absurdity rule $\perp E$, i.e. $\perp_E$.

Tennant claims that, in his logical system, neither the “First Lewis Paradox” i.e. the sequent provable in intuitionistic logic by application of the intuitionistic absurdity rule $\perp E$, i.e. Ex Contradictione Quodlibet $\neg A, A \vdash B$ (ECQ) nor the negate-conclusion version of the First Lewis Paradox, i.e. the following sequent provable in minimal logic:

$\neg A, A \vdash_m \neg B$ (ECQ$\neg$)

are provable in IR. The subtle goal of IR can be understood thanks to Tennant’s slogan [8, p. 711]:

If one needs a slogan to help with orientation, ours is a method of relevantizing “at the level of the turnstile.” As a result, there will be some tweaking of the rules for the logical operators (in the natural deduction or sequent setting); but these tweakings are not as to change their established meanings.

2010 Mathematics Subject Classification: Primary 03B20, 03B47
Keywords: Intuitionistic logic, Relevant logic, Minimal logic
1.2 Tweaking the rules with IR. Before seeing how IR tweaks the rules of intuitionistic logic, note the unchanged rules of minimal logic belonging also to IR:

\[
\begin{align*}
\neg A & \quad \perp \quad 1.E \\
A & \quad B \quad \land I \\
A \lor B & \quad \lor I \\
A \lor B & \quad \lor I
\end{align*}
\]

Now let see the “tweaking” that Tennant \[8, 7, 5\] has made. In the group of introduction rules there is an additional rule \( \rightarrow I_{ir} \). (Labels \( \rightarrow I_{ir} \) and \( \lor E_{ir} \) are mines.) The other changes consist in notification of obligatory or permissible discharge:

\[
\begin{align*}
A & \quad \square (i) \\
\vdots & \quad \vdots \\
\neg A & \quad \neg I_{(i)} \\
A \rightarrow B & \quad \rightarrow I_{(i)} \\
A & \quad B \rightarrow A \rightarrow B \rightarrow I_{(i)}
\end{align*}
\]

The symbol \( \square \) means that the discharge of \( A \) is obligatory. If you can deduce a contradiction from an open assumption \( A \), then either you can conclude \( \neg A \) by application of rule \( \neg I \), or you can introduce a conditional with \( A \) as antecedent and any well formed formula as consequent, like \( B \), but in both cases you have to discharge \( A \). In the other case, when you can deduce a formula \( B \) from \( A \), if \( B \) is not a contradiction, the discharge of \( A \) is \( \Diamond \) i.e. just permissible. I must specify now a crucial meta-rule of IR:

**Definition 1.1 (Proof in IR)** A deduction \( \mathcal{D} \) is a proof in IR if and only if only IR-rules of inference are used in \( \mathcal{D} \) and if and only if \( \mathcal{D} \) is in normal form. According to Prawitz \[3, p. 34\], a proof is in normal form “if no formula occurrence is both the consequence of an application of an I-rule and major premise of an application of an E-rule”. A proof is said to be “in normal form” if it is “redex free” i.e. if no introduction rule for a connective is followed immediately by an elimination rule of the same connective. If a deduction is not in normal form, it is not a proof in IR.

1.2.1 Why (ECQ) is claimed to be unprovable in IR
At this stage, we do not need to know more about the rules of IR to understand why Tennant claims that (ECQ \( \neg \)) and (ECQ) are not among the deducibilities of IR, while for example the formula expressing the *ex falso quodlibet* in the axiomatic of Heyting logic i.e.

\[
\vdash \neg A \rightarrow (A \rightarrow B)
\]

is nevertheless also theorem of IR:

Proof

\[
\begin{align*}
\neg A & \quad 1 \\
A & \quad 2 \\
A \rightarrow B & \quad \rightarrow I_{(2)} \\
\neg A \rightarrow (A \rightarrow B) & \quad \rightarrow I_{(1)}
\end{align*}
\]

Note that this proof of (1) does not prove (ECQ) and that it is impossible to use a conditional elimination rule in order to deduce (ECQ): such a proof would be an abnormal proof i.e. a non-proof in IR (see definition 1.1).
1.2.2 Why (ECQ\neg) is claimed to be unprovable in IR
To prevent a proof (ECQ\neg) in IR, vacuous discharge (or Weakening) is banned from IR when the assumptions base is reducible to contradiction. Indeed, suppose that there is no restriction on the use of vacuous discharge, as in minimal logic and in intuitionistic logic, then the following deduction schema
\[
\begin{array}{c}
\vdash \\
B \rightarrow \bot \rightarrow I
\end{array}
\]
is a sufficient proof of (ECQ\neg) because by the intuitionistic definition of negation the formula B \rightarrow \bot is \neg B. But one must add that if vacuous discharge is replaced by the following rule of Weakening,\(^1\) defined by David et al. [1, p. 38]:
\[
\begin{array}{c}
\vdash \\
A \\
\hline
B \quad (i) \\
\hline
A \quad wk
\end{array}
\]
then this deduction schema shows that (ECQ\neg) is always provable:
\[
\begin{array}{c}
\vdash \\
\hline
\neg A \\
\hline
A \quad 1 \\
\hline
B \quad wk \\
\hline
\neg B \quad 1, 1
\end{array}
\]
To avoid the provability in IR, one must therefore ban this instance of Weakening\(^2\):
\[
\begin{array}{c}
\vdash \\
\hline
\bot \quad (i) \\
\hline
B \quad wk
\end{array}
\]

1.2.3 Tweaking of elimination rules and acceptance of Disjunctive Syllogism
For computational reasons explained in Autologic [5], Tennant replaces the usual rules \land E and rule \rightarrow E by these “parallelized” rules:
\[
\begin{array}{c}
\square(i)
\hline
A \land B
\hline
\square (i)
\hline
B
\end{array}
\]
\[
\begin{array}{c}
\vdash \\
A \rightarrow B \\
\hline
A \quad C
\hline
C \land E (i)
\hline
\vdash \\
\hline
A \quad C
\hline
\rightarrow E (i)
\hline
C
\end{array}
\]
Two \square: at least one of A, B must have been used

Remark 1.2
The usual (i.e. non-parallelized) rules \land E and \rightarrow E of natural deduction are trivially derivable in IR, by replacing C by either A or B in the former and by B in the latter.

Last, the following liberalized rule \lor E\(_\lor\)
\[
\begin{array}{c}
\vdash \\
\hline
A \\
\hline
\square (i)
\hline
B \quad \square (i)
\end{array}
\]
\[
\begin{array}{c}
\vdash \\
\hline
\bot / C \\
\hline
\bot / C \quad \lor E\(_\lor\) (i)
\end{array}
\]
has been invented by Tennant in order to allow in IR the proof of Disjunctive Syllogism i.e.
\[
(A \lor B), \neg A \vdash B \quad (\text{DS})
\]
without making use of the intuitionistic rule \bot E.

Tennant [8, p. 714, in [4]] explains this rule as follows:
By stating graphically \[ \lor E_{ir} \] as we have, we are providing for the possibility that one of the case assumptions might lead to absurdity (\( \bot \)). We are then permitted to bring down as the main conclusion whatever is concluded from the other case assumption. [...] Liberalizing proof by cases in this way is entirely natural, given how we reason informally. Suppose one is told that \( A \lor B \) hold, along with certain other assumptions \( X \), and one is required to prove that \( C \) follows from the combined assumption \( X, A \lor B \). If one assumes \( A \) and discovers that it is inconsistent with \( X \), one simply stop one’s investigation of that case, and turns to the case \( B \). If \( C \) follows in the latter case, one concludes \( C \) as required. One does not go back to the conclusion of absurdity in the first case, and artificially dress it up with an application of the absurdity rule so as to make it also “yield” the conclusion \( C \).

Our paper focuses only on natural deduction for IR and does not deal with the corresponding sequent calculus for IR. But, to conclude this section, I stress on the treatment of the Deduction Theorem and its converse. About this basic theorem, Tennant [6, p. 344] says:

The orthodox Deduction Theorem for logical calculi states that:

\[ X \vdash B \text{ only if } X \{ A \} \vdash A \rightarrow B; \text{ and } X \vdash A \rightarrow B \text{ only if } X, A \vdash B \]

In IR, the first implication holds. But the converse fails, for in IR we have \( \neg A \vdash A \rightarrow B \) but not: \( \neg A, A \vdash B \). The lesson here is that we have to be careful to distinguish between asserting a conditional and making an inference. We have determined a conditional that is exactly what is needed in order to have the best possible system of relevant inference. The price of ‘unpacking’ a conditional is eternal vigilance with regard to the joint consistency of the new set of assumptions in play, and with regard to the possible logical truth of the new conclusion.

2 IR changes the meaning of disjunction elimination

Statement 2.1 It is provable that rule \( \lor E_{ir} \) changes the established meaning of \( \lor \) when \( \lor \) is on the left of the turnstile.

Proof

\[ A \lor B \vdash C \] (2)

means in minimal logic that \( C \) is a syntactical consequence of both \( A \) and \( B \), i.e. (2) is provable if and only if it is provable that \( A \vdash_m C \) and if it is provable that \( B \vdash_m C \), whatever \( A, B, C \) are. In IR the meaning of (2) changes if either \( A \) or \( B \) is equivalent to \( \bot \), because via rule \( \lor E_{ir} \)

\[ \bot \lor B \vdash B \] (3)

but only because of the axiom \( B \vdash B \) and in spite of the claim that \( \bot \not\vdash B \) in IR. Therefore IR changes the meaning of \( \lor \) when \( \lor \) is on the left of the turnstile. \( \Box \)

3 The law of substitution of logical equivalents fails in IR

Theorem 3.1 The law of substitution of logical equivalents fails in IR.

Proof

The logical equivalence in IR between \((\bot \lor B)\) and \( B \) is provable:

\[
\begin{align*}
B & \quad \vdash 1 \\
\frac{B \lor \bot}{B \lor \bot \rightarrow I, 1} & \quad \lor I \\
\frac{B \rightarrow (B \lor \bot)}{B \lor (B \lor \bot) \rightarrow I, 1} & \quad \rightarrow I, 1 \\
\frac{B}{(B \lor \bot) \leftrightarrow B} & \quad \leftrightarrow I \\
\frac{B \lor \bot}{B} & \quad \lor E_{ir}, 2, 3
\end{align*}
\]
Nevertheless, \((\bot \lor B)\) and \(B\) are not always substitutable salva veritate in \(\text{IR}\). Indeed

\[
\bot \vdash \bot \lor B \tag{4}
\]

is provable in \(\text{IR}\) (via rule \(\lor I\)), but if one replaces \(\bot \lor B\) by \(B\) one gets

\[
\bot \vdash B \tag{5}
\]

in contradiction with the claim of “relevance at the level of the turnstile” in \(\text{IR}\), i.e. \(\bot \not\vdash B\). Therefore the law of substitution of logical equivalents fails in \(\text{IR}\) because it is not true in \(\text{IR}\) that, in any context \(\Gamma\)

\[
\Gamma \vdash \bot \lor B \tag{6}
\]

if and only if

\[
\Gamma \vdash B \tag{7}
\]

\(\Box\)

4 A heavy rule of absurdity in \(\text{IR}\)

\(\text{IR}\) does not differ from minimal logic only by rejecting the Weakening rule. Contrary to minimal logic and in agreement with intuitionistic logic, \(\bot\) is in \(\text{IR}\) an inference marker occurring with contradiction and being equivalent to any contradiction; but this difference with minimal logic entails some problematic theorems, such as the following.

**Theorem 4.1** Unlike minimal logic, any contradiction is logically equivalent to any other one in \(\text{IR}\), i.e.

\[
\vdash (\neg A \land A) \leftrightarrow (\neg B \land B) \tag{8}
\]

Proof

\[
\begin{array}{c}
\begin{array}{c}
\neg A \land A \land \neg A \\
\land E\end{array} \frac{1}{-A} \\
\neg A \\
A \land E \\
\bot \rightarrow I_{0} \\
(\neg A \land A) \rightarrow (\neg B \land B) \\
\rightarrow E_{2} \\
(\neg A \land A) \leftrightarrow (\neg B \land B)
\end{array}
\end{array}
\]

\(\Box\)

Theorem 4.1 leads to a way of avoiding in \(\text{IR}\) the relevance at the level of the turnstile:

**Theorem 4.2** In \(\text{IR}\), \(B\) is deducible “at the level of the turnstile” from the assumption of the conjunction of contradiction \(\neg A \land A\) and theorem \((\neg A \land A) \leftrightarrow (\neg B \land B)\), i.e.

\[
\neg A \land A \land ((\neg A \land A) \leftrightarrow (\neg B \land B)) \vdash B \tag{9}
\]

is provable in \(\text{IR}\) via a proof in perfect normal form, using only elimination rules.

Proof

\[
\begin{array}{c}
\begin{array}{c}
\neg A \land A \land ((\neg A \land A) \leftrightarrow (\neg B \land B)) \\
\land E
\end{array} \\
\neg A \land A \land ((\neg A \land A) \rightarrow (\neg B \land B)) \\
\land E \\
(\neg A \land A) \rightarrow (\neg B \land B) \\
\land E
\end{array}
\]

\[
\begin{array}{c}
\neg A \land A \land ((\neg B \land B) \rightarrow (\neg A \land A)) \\
\land E
\end{array}
\]

\[
\begin{array}{c}
\neg A \land A \land ((\neg A \land A) \leftrightarrow (\neg B \land B)) \\
\land E
\end{array}
\]

\[
\begin{array}{c}
\neg A \\
\rightarrow E
\end{array}
\]

\[
\begin{array}{c}
\neg B \land B \\
\land E \rightarrow E
\end{array}
\]

\[
\begin{array}{c}
\neg A \land A \\
\rightarrow E
\end{array}
\]

\[
\begin{array}{c}
B
\land E
\end{array}
\]

\(\Box\)
Remark 4.3 This proof is a proof in IR: it is in normal form, and usual rules $\land E$ and $\rightarrow E$ are rules of IR because they are derivable in IR (remark 1.2).

Remark 4.4 Sequent (9) is semantically reducible to
\[ \bot \land (\bot \leftrightarrow \bot) \vdash B \] (10)
and therefore to
\[ \bot \land \top \vdash B \] (11)
and last to
\[ \bot \vdash B \] (12)

5 Conclusion

The arguments of section 2, 3 and 4 provide three sufficient reasons to deny that IR is a Core logic.

1. We expect that a Core logic respects the univocal meaning of logical rules. It is not the case with IR: rule $\lor E$ has not the same meaning when it is applied to a sequent like
\[ A \lor \neg \neg \neg \neg A \vdash \neg \neg A \] (13)
and when it is applied to
\[ (\neg A \land A) \lor B \vdash B \] (14)
Sequent (13) is valid in IR because two sequents are provable, i.e. $A \vdash \neg \neg A$ and $\neg \neg \neg \neg A \vdash \neg \neg A$, but sequent (14) is valid in IR only because $B \vdash B$ is an axiom (of course $\neg \neg A \land A \vdash B$ is claimed to be unprovable in IR).

2. At least since Leibniz, this law of substitution of logical equivalents is regarded as a basic logical law. To quote Leibniz’ example [2, pp. 85-87], if $A$ and $B$ have the same meaning, for example “trilateral plane figure” and “plane figure triangular”, $A$ can be replaced with $B$ *salva veritate* in all statements (provided a context as referentially transparent as Euclidean geometry). Conversely, it is generally accepted in mathematical logic that, if $A$ cannot be replaced by $B$ in system $S$, it is because $A$ and $B$ have different meanings i.e. are not logically equivalent in $S$. For example $\neg \neg A$ cannot be replaced by $A$ *salva veritate* in intuitionistic logic, and $\bot \lor B$ cannot be replaced by $B$ in minimal logic. This law is essential to our understanding of the rules of logical systems in general. By proving that Leibniz’ law fails in IR, section 3 helps explain why this so-called “Core logic” is so difficult to understand. Of course, Tennant could reply that the virtue of IR is to prevent the transitivity of deductions from sequents $\Gamma \vdash \bot \lor B$ to $\Gamma \vdash B$ when $\Gamma$ is itself $\bot$. But the trouble is that the logical equivalence between $\bot \lor B$ and $B$ is *syntactically* provable in IR regardless of context, and therefore should be *semantically* valid in IR also regardless of context; proof has been given in this paper that it is not the case in IR, and it is a serious problem if harmony between syntax and semantics is expected in Core logic.
3. To my knowledge, nowhere has Tennant clearly explained the difference between a provable inference “at the level of the turnstile” and a provable conditional, i.e. an implication. After all, why (ECQ) should be unprovable, while the conditional \((\neg A \land A) \rightarrow B\) is a theorem of IR? This lack of harmony between provable sequents and provable conditionals is shocking. The proof given in section 4 shows that Tennant’s slogan is not even in harmony with the theorem of IR according to which any contradiction \((\neg A \land A)\) is equivalent to another one, say \((\neg B \land B)\). From this point of view, minimal logic is a better candidate to be a Core logic. With minimal logic, by contrast with IR, you are not embarrassed to recognize that if you assume that any contradiction is equivalent to any other one, it is easy to show, at the level of the turnstile, that from this assumption and the assumption of \(\neg A \land A\), you can infer \(B\) (i.e. (9) is provable in minimal logic). In my opinion, it is contrary to the meaning of “Core logic” to contradict minimal logic, and that is probably the main reason why Tennant’s logic cannot be a Core logic.

Notes

1. Tennant calls this rule “Dilution”. I use the most usual name for this rule.

2. In [5, p. 189] Tennant repeats that IR “requires non-vacuous discharge of assumptions”, but he adds in footnote 6 at the same page “except, crucially, in one half of the rule of implication introduction!” . It means that it is only for the other half of implication introduction, i.e. for rule \(\rightarrow I_o\), that vacuous discharge i.e. Weakening is forbidden.

References

5.1 Acknowledgments I thank Neil Tennant for his friendly correspondence and his patience in explaining the intricacies of IR.
I also had very useful discussions on this subject with Sara Negri, Jan von Plato, Konstantine Arkoudas, Roy Dyckhoff, Dominique Larchey-Wendling and Richard Zach. I thank each of them, especially Roy Dyckhoff, because I am indebted to him for theorem 3.1.
Finally, my warmest thanks to my good old friends, Henri Astier and Christian Desmier, who have both revised and improved my English in this paper.

Joseph Vidal-Rosset
Université de Lorraine - Département de philosophie
Archives Poincaré - CNRS - UMR 7117
91 Avenue de la Libération
BP 454 - 54001 NANCY CEDEX
FRANCE
Email: joseph.vidal-rosset@univ-lorraine.fr