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HOMOGENIZATION NEAR RESONANCES AND ARTIFICIAL MAGNETISM IN 3D DIELECTRIC METAMATERIALS

GUY BOUCHITTÉ, CHRISTOPHE BOUREL, DIDIER FELBACQ

ABSTRACT. It is now well established that the homogenization of a periodic array of parallel dielectric fibers with suitably scaled high permittivity can lead to a (possibly) negative frequency-dependent effective permeability. However this result based on a two-dimensional approach holds merely in the case of linearly polarized magnetic fields, reducing thus its applications to infinite cylindrical obstacles. In this paper we consider a dielectric structure placed in a *bounded* domain of $\mathbb{R}^3$ and perform a full 3D asymptotic analysis. The main ingredient is a new averaging method for characterizing the bulk effective magnetic field in the vanishing-period limit. We evidence a vectorial spectral problem on the periodic cell which determines micro-resonances and encodes the oscillating behavior of the magnetic field from which artificial magnetism arises. At a macroscopic level we deduce an effective permeability tensor that we can be make explicit as a function of the frequency. As far as sign-changing permeability are sought after, we may foresee that periodic bulk dielectric inclusions could be an efficient alternative to the very popular metallic split-ring structure proposed by Pendry. Part of these results have been announced in [8].

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1. INTRODUCTION AND DESCRIPTION OF THE MODEL

Physical background and recent mathematical progress. The behavior of an homogeneous material with respect to electromagnetic waves is characterized by its electric permittivity $\varepsilon(\omega)$ and its magnetic permeability $\mu(\omega)$: two physical quantities which depend on the frequency ω. For frequencies of visible light, the permittivity is a complex number $\varepsilon := \varepsilon' + i\varepsilon''$ where $\varepsilon' \in \mathbb{R}$ (can be negative for some metals) and $\varepsilon'' \geq 0$ (for passive media and if the harmonic time-dependence is assumed to be $e^{-i\omega t}$). It is different for the permeability because all natural materials present a non-magnetic behavior in the visible region of the spectrum, i.e. their relative permeability is very close to one as in vacuum.

For the past fifteen years, there have been many researches on the realization of artificial materials, generally periodically micro-structured, behaving as homogeneous media, i.e. described by effective tensors $\boldsymbol{\varepsilon}^{\text{eff}}$ and $\boldsymbol{\mu}^{\text{eff}}$. An important issue is to design structures which allows a non-trivial permeability (possibly negative), a negative permittivity or both. The later case corresponds to a "left-handed medium" presenting a *negative refractive index*.

The first metamaterial possessing a negative effective permittivity was proposed by Pendry in 1996 [28] and consists in high conductivity parallel fibers occupying a very small volume fraction. A rigorous proof of this effective behavior, based on homogenization techniques, appeared in [13, 28, 31] in the case of infinitely long fibers and under a polarization assumption. Surprisingly the same kind of behavior does not hold for finite-length fibers as demonstrated in[6, 10] where the resulting permittivity law is shown to be non-local. However by inserting such a finite structure in a larger scale structure, a reiterated homogenization procedure makes it possible mathematically to reach effective tensors $\boldsymbol{\varepsilon}^{\text{eff}}$ with negative eigenvalues (see [4]).

In a similar way it is a challenging issue to design metamaterials able to display an artificial magnetic activity i.e. $\boldsymbol{\mu}^{\text{eff}}(\omega) \neq 1$ and more specifically such that $\Re(\boldsymbol{\mu}^{\text{eff}}(\omega)) < 0$ in some range of frequencies. In photonic devices such a property is usally explained by the ability of the structure to induce a local magnetic field presenting a Fano-like resonance [18]. The first and the most famous structure illustrating this phenomenon was proposed by Pendry in [26, 28] and consists in a periodic set of *metallic split-ring resonators*. A field incident on this device induces micro-currents looping in each ring from which results a macroscopic magnetic moment. The mathematical study of this structure was made recently in [17] for the 2D case and in [7] for the 3D general case.

Another way to produce artificial magnetism from dielectric structures was proposed in [27]. Therein another kind of internal resonances is exploited, the so-called *Mie resonances*. These resonances take place inside each dielectric inclusion and generate loops of displacement current inside the obstacle. This phenomenon can be evidenced experimentally on a composite structure with a much simpler geometry than the one of the split-rings: it consists of periodically disposed micro-cavities filled with a high dielectric material [30]. Subsequent works have shown the interest of Mie resonances to tailor the properties of dielectric metamaterials. These allow for the control of the Purcell effect [16], the design of hyperbolic metamaterials and perfect reflectors [21] or the realization of zero-index metamaterials [22]. It has also been demonstrated [14, 20] that the possibility of tailoring the artificial magnetic activity was a key to the design of invisibility cloaks based on dielectric materials. This has the great advantage of limiting the losses, as compared to metallic structures were strong losses are unavoidable, apart at the price of inserting active media [15].

The first mathematical study of this kind of dielectric metamaterials was made in the particular case where the structure is invariant in one direction (see [5, 12] and [9] for a generalization to the random case). In these papers the inclusions are infinite cylinders and the incident wave is polarized with a magnetic field parallel to the axis of the cylinders. As a consequence the original 3D-problem problem can be reduced in a two-dimension setting, this allowing a quite simple and rigorous asymptotic analysis.

Our contribution. In this paper we consider a dielectric structure placed in a *bounded* domain of $\mathbb{R}^3$ and perform a full 3D asymptotic analysis. The infinitesimal parameter denoted η represents the scale factor associated with the distance between inclusions. As it was announced in the note [8], we will prove that the structure behaves, when $\eta \to 0$, as a *local* material described by a frequency-dependent permeability tensor. Although this conclusion looks qualitatively in perfect agreement with what is obtained in the 2D case, the mathematical analysis reveals several novelties that we wish to emphasize here:

- In contrast with the 2D case, the fast oscillations of the magnetic field are not anymore localized on the dielectric inclusions. It follows that the induced magnetic activity cannot result simply from the superposition of independent Helmholtz micro-resonators as depicted in [12]. Interactions between the inclusions and the substrate have to be well understood.

- The averaging procedure we need for the asymptotic analysis has to be compatible with the classical transmission conditions across the boundary of the structure, namely the continuity of the tangential components of the electromagnetic field. This issue appears to be crucial regarding the magnetic field: as it will be demonstrated, the bulk average (weak limit in $L^2_{\rm loc}$) will present tangential discontinuities. A key argument to overcome this difficulty will be the introduction of a new averaging process for periodic magnetic fields that we handle as one-forms on the complementary of the periodic inclusions.

- The new spectral problem we have evidenced for describing the resonance modes of the structure is quite interesting by its interaction with the geometry. It involves divergence free periodic vector fields on the three dimensional torus which are curl free on the complementary of the inclusions. A direct numerical approximation problem of it turns out to be very costly computationally. A lot of attention has been devoted to finding equivalent formulations making possible efficient numerical simulations for various type of shapes of dielectric inclusions. Some of these simulations are presented in Section 3.

Notations.

- $\mathbb{C}^+ := \{z \in \mathbb{C} \,:\, \Im(z) \geq 0\}$,
- $\overline{z}$ denotes the complex conjugate of complex number z,
- $Y :=]-\frac{1}{2},\frac{1}{2}[^3$,
- $\Sigma^* := Y \setminus \overline{\Sigma}$,
- $B_R = \{x \in \mathbb{R}^3 : |x| < R\}$ ($|x|$ denotes the Euclidean norm),
- $A \subset\subset B$ for subsets of $\mathbb{R}^3$ means that $\overline{A}$ is a compact subset of the interior of B,
- $|B|$ denotes the Lebesgue measure of a Borel set $B \subset \mathbb{R}^3$,
- $1_B(x)$ denotes the characteristic function of B,
- $[\boldsymbol{y}]$ is the step function defined on $\mathbb{R}^3$ by $[\boldsymbol{y}] := \boldsymbol{k}$ for all $\boldsymbol{y} \in Y + \boldsymbol{k}$,
- $\langle u \rangle := \int_Y u(y)\,dy$ denotes the mean value of a fonction $u \in L^1(Y)$,
- I_3 is the identity matrix of order 3,
- $\boldsymbol{M} : \boldsymbol{N}$ denotes the usual scalar product for 3×3 matrices ,
- $\boldsymbol{u} \wedge \boldsymbol{v}$ denotes the cross product of vectors $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^3$,
- $\boldsymbol{u} \otimes \boldsymbol{v}$ denotes the tensor product of two vectors in $\mathbb{R}^3$,
- $\fint_B f(x)\,dx := \dfrac{1}{|B|}\displaystyle\int_B f(x)\,dx$ for all $f \in L^1_{\rm loc}(\mathbb{R}^3)$ and Borel set $B \subset \mathbb{R}^3$.
- $C^\infty(\mathbb{R}^3)$ (resp.$C^\infty(K)$ if K is a compact domain of $\mathbb{R}^3$), denotes the space of functions which are C^∞ on $\mathbb{R}^3$ (resp.on K)
- $C^\infty_c(D)$ for an open subset $D \subset \mathbb{R}^3$, is the set of C^∞-functions with compact support in D,
- $C^\infty_\sharp(Y)$ the subset of Y-periodic functions in $C^\infty(\mathbb{R}^3)$,
- $L^2_\sharp(Y) := \left\{ f \in L^2_{loc}(\mathbb{R}^3) \,:\, f \ Y\text{-periodic} \right\}$,
- $\langle \boldsymbol{f} \cdot \boldsymbol{g} \rangle := \int_Y \boldsymbol{f} \cdot \overline{\boldsymbol{g}}\, dy$ the standard scalar product in $(L^2_\sharp(Y))^3$,
- $W^{1,2}_\sharp(Y) := \left\{ f \in L^2_\sharp(Y) \,:\, \partial_i f \in L^2_\sharp(Y)\,,\ i = 1,2,3 \right\}$ ($\partial_i f$ is meant in the distributional sense).
- $W^{1,2}_\sharp(\Sigma^*)$ denotes the space of restrictions to Σ^* of functions in $W^{1,2}_\sharp(Y)$.
- $\mathscr{D}'(A)$ denotes the distributions on the open subset $A \subset \mathbb{R}^3$.

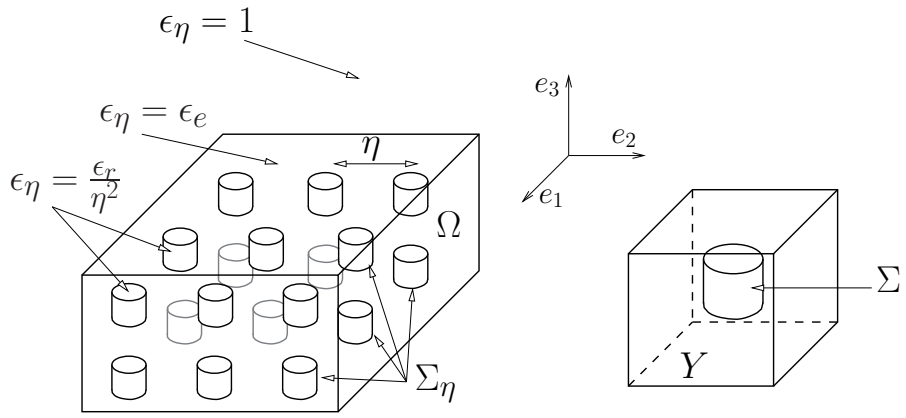


FIGURE 1. Diffracting structure and unit cell.

Geometrical assumptions. All along this paper the geometric domain of $\mathbb{R}^3$ in which the small dielectric inclusions are disposed will be denoted by Ω. This domain Ω is assumed to be bounded, simply-connected with Lipschitz boundary. For every value of the small parameter $\eta > 0$, we consider a diffracting obstacle occupying a subregion $\Sigma_\eta \subset \Omega$ which is obtained by periodization of a small inclusion of size η. More precisely Σ_η is given by

$$\Sigma_\eta := \bigcup_{k\in I_\eta} \eta(k+\Sigma) \quad , \quad I_\eta = \left\{ i \in \mathbb{Z}^3 \mid \eta(i+Y) \subset \Omega \right\} , \tag{1.1}$$

being $Y :=]-\frac{1}{2},\frac{1}{2}[^3$ the unit cell and $\Sigma \subset\subset Y$ a reference inclusion.

The complexity of the diffracting obstacle is then encoded by the fast oscillating behavior of Σ_η as η becomes infinitesimal. Let us notice that the *filling ratio* of the inclusions remains positive when $\eta \to 0$ since it converges to $|\Sigma|$ (the Lebesgue measure of Σ).

It turns out that the topology of the inclusion Σ plays an important role in the asymptotic analysis as $\eta \to 0$. In this paper we will assume that:

$$\begin{array}{ll} \text{i)} & \Sigma \text{ is a connected compact subset of } Y \text{ with Lipschitz boundary,} \\ \text{ii)} & \Sigma^* = Y \setminus \Sigma \text{ is simply connected.} \end{array} \tag{1.2}$$

Constitutive parameters and scaling. The local behavior of the medium is represented by its relative permittivity and permeability tensors at every position $x \in \mathbb{R}^3$. In order to fit with the common use in the optical domain, we will assume the overall relative permeability to be constant equal to 1. The dielectric properties of the structure under study are described by a function ε_η of the form

$$\varepsilon_\eta(x) := \frac{\varepsilon_r}{\eta^2}\, \mathbb{1}_{\Sigma_\eta}(x) + \varepsilon_e\, \mathbb{1}_{\Omega\setminus\Sigma_\eta}(x) + \mathbb{1}_{\mathbb{R}^3\setminus\Omega}(x)\,, \tag{1.3}$$

where the parameters $\varepsilon_e \in \mathbb{R}^+$ and $\frac{\varepsilon_r}{\eta^2} \in \mathbb{C}^+$ represent respectively the relative permittivity in the matrix and in the inclusions. Here ε_r is a complex parameter such that:

$$\Im(\varepsilon_r) > 0\,, \tag{1.4}$$

while the scaling factor $1/\eta^2$ is responsible of a high contrast becoming larger and larger as the period parameter η of the structure decreases to zero. The choice of this scaling is not new (see [5, 9, 12]). It ensures that the optical thickness of the inclusions remain constant and therefore the Mie resonances of each dielectric inclusion appear at frequencies which are independent of η (see [23]).

Diffraction problem. The structure is illuminated by $(\boldsymbol{E}^{\rm inc}, \boldsymbol{H}^{\rm inc})$ an incident monochromatic wave travelling from infinity. We assume the harmonic time-dependence to be $e^{-i\omega t}$ where $\omega > 0$ is a fixed frequency. The total electromagnetic field $(\boldsymbol{E}_\eta, \boldsymbol{H}_\eta)$ satisfies Maxwell equations given in

distributional sense in $\mathbb{R}^3$ by

$$\begin{cases} \operatorname{curl} \boldsymbol{E}_\eta = i\omega\mu_0 \boldsymbol{H}_\eta \,, \\ \operatorname{curl} \boldsymbol{H}_\eta = -i\omega\varepsilon_0\, \varepsilon_\eta \boldsymbol{E}_\eta \,, \end{cases} \tag{1.5}$$

where $\varepsilon_0 > 0$ and $\mu_0 > 0$ are respectively the permittivity and permeability in the vacuum. The influence of the incident wave is encoded by the fact that the diffracted field $(\boldsymbol{E}_\eta^d, \boldsymbol{H}_\eta^d) := (\boldsymbol{E}_\eta - \boldsymbol{E}^{\rm inc}, \boldsymbol{H}_\eta - \boldsymbol{H}^{\rm inc})$ satisfies the Silver-Müller's condition at infinity

$$(\boldsymbol{E}_\eta^d, \boldsymbol{H}_\eta^d) = O\left(\frac{1}{|\boldsymbol{x}|}\right), \qquad \omega\varepsilon_0 \left(\frac{\boldsymbol{x}}{|\boldsymbol{x}|} \wedge \boldsymbol{E}_\eta^d\right) - k_0 \boldsymbol{H}_\eta^d = o\left(\frac{1}{|\boldsymbol{x}|}\right). \tag{1.6}$$

Remark 1.1. In this paper we will proceed in a dimensionless framework assuming implicitly that the physical period of the composite is in fact $\eta\, d$ being d the unit of length. Thereby the adimensional wavelength of the electromagnetic waves becomes λ/d (if λ is the real wavelength). In order to simplify notations this parameter d will not appear in the following, except in the presentation of numerical simulations in Section 3.

2. Presentation of the results

The asymptotic analysis as $\eta \to 0$ of $(\boldsymbol{E}_\eta, \boldsymbol{H}_\eta)$ solving (1.5) leads to a homogenized diffraction problem of the kind

$$\begin{cases} \operatorname{curl} \boldsymbol{E} = i\omega\mu_0\, \boldsymbol{\mu}(x,\omega)\, \boldsymbol{H} \\ \operatorname{curl} \boldsymbol{H} = -i\omega\varepsilon_0\, \boldsymbol{\varepsilon}(x)\, \boldsymbol{E} \\ (\boldsymbol{E} - \boldsymbol{E}^{\rm inc}, \boldsymbol{H} - \boldsymbol{H}^{\rm inc}) \text{ satisfies (1.6)} \end{cases} \tag{2.1}$$

where

$$\boldsymbol{\varepsilon}(x) := \boldsymbol{I}_3\, 1_{\mathbb{R}^3\setminus\Omega}(x) + \boldsymbol{\varepsilon}^{\rm eff}\, 1_\Omega(x) \quad , \quad \boldsymbol{\mu}(x,\omega) := \boldsymbol{I}_3\, 1_{\mathbb{R}^3\setminus\Omega}(x) + \boldsymbol{\mu}^{\rm eff}(\omega)\, 1_\Omega(x)\,, \tag{2.2}$$

Here the effective tensors $\boldsymbol{\varepsilon}^{\rm eff}, \boldsymbol{\mu}^{\rm eff}$ describe for each frequency ω a homogeneous medium occupying the domain Ω. They are described in a precise way in the next subsection. The convergence of $(\boldsymbol{E}_\eta, \boldsymbol{H}_\eta)$ to the solution of 2.1 will be specified in our main Theorem (Theorem 2.2).

Let us point out that the two first equations in (2.1) are to be understood in the distributional sense in $\mathbb{R}^3$. In particular, under mild regularity assumptions, they imply the following transmission conditions on $\partial\Omega$:

$$[\boldsymbol{n} \wedge \boldsymbol{E}] = [\boldsymbol{n} \wedge \boldsymbol{H}] = 0\,, \qquad [\boldsymbol{n} \cdot \boldsymbol{\mu}\boldsymbol{H}] = [\boldsymbol{n} \cdot \boldsymbol{\varepsilon}\boldsymbol{E}] = 0\,, \tag{2.3}$$

with $\boldsymbol{n}$ denoting the outward unit vector and $[\cdot]$ the jump across $\partial\Omega$.

The proof of the existence and uniqueness of the solution to (1.5), (1.6) under the dissipativity condition (1.4) is classical and can be found e.g. in [11]. With regard to the uniqueness for a limit problem of the form given in (2.1), we have

Lemma 2.1. *Assume that $\boldsymbol{\varepsilon}^{\rm eff}$ is real symmetric positive and that $\boldsymbol{\mu}^{\rm eff}$ is a symmetric tensor whose imaginary part $\Im(\boldsymbol{\mu}^{\rm eff})$ is positive definite. Then the solution to* (2.1) *is unique.*

Proof. By linearity, it is enough to check that if $(\boldsymbol{E}, \boldsymbol{H})$ solves (2.1) for a vanishing $(\boldsymbol{E}^{\rm inc}, \boldsymbol{H}^{\rm inc})$, then $(\boldsymbol{E}, \boldsymbol{H}) = (0,0)$. Let $R > 0$ so large that $\Omega \subset\subset B_R$ and denote $\mathscr{P}(R) := \int_{\partial B_R} (\boldsymbol{E} \wedge \overline{\boldsymbol{H}}) \cdot \boldsymbol{n}$ the flux of the Poynting vector. As usual, the real part of $\mathscr{P}(R)$ does not depend on R. Indeed exploiting the identity $\operatorname{div} \boldsymbol{E} \wedge \overline{\boldsymbol{H}} = \operatorname{curl} \boldsymbol{E} \cdot \overline{\boldsymbol{H}} - \operatorname{curl} \overline{\boldsymbol{H}} \cdot \boldsymbol{E}$, we may integrate by parts and , by taking into account (2.1), we get

$$\mathscr{P}(R) = \int_{B_R} \left(\operatorname{curl} \boldsymbol{E} \cdot \overline{\boldsymbol{H}} - \operatorname{curl} \overline{\boldsymbol{H}} \cdot \boldsymbol{E}\right) = i\omega\Big(\mu_0 \int_{B_R} \boldsymbol{\mu}\boldsymbol{H} \cdot \overline{\boldsymbol{H}} - \varepsilon_0 \int_{B_R} \overline{\boldsymbol{\varepsilon}}\boldsymbol{E} \cdot \overline{\boldsymbol{E}}\Big)\,. \tag{2.4}$$

In particular, since $\boldsymbol{\varepsilon}$ is real and $\boldsymbol{\mu}$ agrees with the identity tensor outside Ω, by identifying the real and imaginary parts, we deduce that

$$\Re(\mathscr{P}(R)) = -\omega\mu_0 \int_\Omega \Im\Big(\boldsymbol{\mu}^{\rm eff} \boldsymbol{H} \cdot \overline{\boldsymbol{H}}\Big)\,. \tag{2.5}$$

We may now pass to the limit $R \to +\infty$ in the left-hand member of (2.5). Now by exploiting the fact that $(\boldsymbol{E}, \boldsymbol{H})$ satisfies (1.6) with $\boldsymbol{E}^{\rm inc} = \boldsymbol{H}^{\rm inc} = 0$, we find that $\lim_{R\to\infty} \mathscr{P}(R) = 0$. Thus the left-hand

member of (2.5) vanishes as well as the integral of $\Im\Big(\boldsymbol{\mu}^{\rm eff}\boldsymbol{H}\cdot\overline{\boldsymbol{H}}\Big)$ over Ω. By the positivity assumption on $\Im(\boldsymbol{\mu}^{\rm eff})$ and since $\boldsymbol{\mu}^{\rm eff}$ is symmetric, we have for a suitable constant $c>0$:

$$\Im\Big(\boldsymbol{\mu}^{\rm eff}\boldsymbol{H}\cdot\overline{\boldsymbol{H}}\Big) \;=\; \Im(\boldsymbol{\mu}^{\rm eff})\boldsymbol{H}\cdot\overline{\boldsymbol{H}} \;\geq\; c\,|\boldsymbol{H}|^2\,,$$

Thus $\boldsymbol{H}$ vanishes on Ω. By the second equation of (2.1) and the fact that $\boldsymbol{\varepsilon}$ is a real positive tensor, it is also the case of $\boldsymbol{E}$. It is then classical to deduce that $(\boldsymbol{E},\boldsymbol{H})$ vanishes in the exterior domain $\mathbb{R}^3\setminus\Omega$ as well. □

2.1. Effective laws.

The limit diffraction problem we wrote in the form (1.5) is completely determined by relations (2.2) once we know the effective permittivity tensor $\boldsymbol{\varepsilon}^{\rm eff}$ and the effective permeability tensor $\boldsymbol{\mu}^{\rm eff}$. These tensors are described in a precise way in the next two paragraphs. It turns out that $\boldsymbol{\varepsilon}^{\rm eff}$ is real positive and *does not depend* on the frequency and on the dielectric parameter ε_r. In contrast, the tensor $\boldsymbol{\mu}^{\rm eff}=\boldsymbol{\mu}^{\rm eff}(\omega)$ depends on the frequency and exhibits resonances.

Effective permittivity law. It depends only on the geometry of Σ and on the permittivity in the matrix surrounding inclusions. The computation of $\boldsymbol{\varepsilon}^{\rm eff}$ looks similar as the one used in the two-dimensional case (see [5, 9]) where the classical ingredients of homogenization theory for Neumann problems with holes can be recognized. The entries of the tensor $\boldsymbol{\varepsilon}^{\rm eff}$ are given for $(k,l)\in\{1,2,3\}^2$ by

$$\boldsymbol{\varepsilon}^{\rm eff}_{kl} := \varepsilon_e\int_Y (e_k+\nabla\chi_k).(e_l+\nabla\chi_l)\,, \tag{2.6}$$

being $\chi_k\in W^{1,2}_\sharp(Y;\mathbb{R})$ the unique solutions of

$$\Delta_y\chi_k=0 \quad\text{in}\quad \Sigma^* \qquad\text{and}\qquad \chi_k=-y_k \quad\text{in}\quad \Sigma\,. \tag{2.7}$$

As ε_e is a positive real, it can be readily checked (see (5.10)) that the tensor $\boldsymbol{\varepsilon}^{\rm eff}$ is real symmetric positive.

Effective permeability law. The dependence of permeability tensor $\boldsymbol{\mu}^{\rm eff}$ with respect to ω is ruled by the internal resonances of the composite structure which are responsible for the magnetic activity. The description of the underlying spectral problem is quite involved due to the fact that strong oscillations of the microscopic magnetic field $\boldsymbol{H}_0(x,\cdot)$ are allowed not only in Σ (in a similar way as in the 2D case [5, 9, 12]), but also in the surrounding matrix. A nice way to circumvent this difficulty consists in looking at the curl of the magnetic field which accounts for the magnetic activity. It turns out that this curl vanishes outside Σ (see (5.13)). We therefore introduce the space $Z_0\subset L^2(Y;\mathbb{R}^3)$ defined by

$$Z_0 := \Big\{\boldsymbol{f}\in L^2(Y;\mathbb{R}^3)\,:\,\operatorname{div}\boldsymbol{f}=0,\quad \boldsymbol{f}=0\quad\text{in }\Sigma^*\Big\}\,. \tag{2.8}$$

Let us notice that the elements of Z_0 can be identified with divergence-free vector fields in $L^2(\Sigma;\mathbb{R}^3)$ with vanishing normal trace on $\partial\Sigma$. Next, we associate to every element $\boldsymbol{f}\in Z_0$ the unique solution $\boldsymbol{\psi}_f$ in $W^{1,2}_\sharp(Y;\mathbb{R}^3)$ of

$$-\Delta\boldsymbol{\psi}_f=\boldsymbol{f}\quad\text{in } Y\,,\qquad \int_Y\boldsymbol{\psi}_f=0\,. \tag{2.9}$$

Then, as will be discovered later, the resonance frequencies for the microscopic magnetic field are directly related to the following eigenvalue problem. Find $(\boldsymbol{f},\alpha)\in Z_0\times\mathbb{R}$ such that for all $\boldsymbol{g}\in Z_0$:

$$\int_Y\nabla\boldsymbol{\psi}_f:\nabla\boldsymbol{\psi}_g \;+\; \frac14\left(\int_\Sigma \boldsymbol{z}\wedge\boldsymbol{f}\,d\boldsymbol{z}\right)\cdot\left(\int_\Sigma \boldsymbol{z}\wedge\boldsymbol{g}\,d\boldsymbol{z}\right) = \alpha\int_\Sigma\boldsymbol{f}\cdot\boldsymbol{g} \tag{2.10}$$

The linear operator associated with the bilinear form in the left-hand side turns out to be positive, compact and self-adjoint on the Hilbert space Z_0 (embedded with the $L^2(Y;\mathbb{C}^3)$ scalar product). Therefore, it exists a sequence of eigenvalues $\alpha_0\geq\alpha_1\geq\cdots\geq\alpha_n\geq\cdots>0$ such that $\alpha_n\to0$ and an associated orthonormal basis of eigenvectors $\{\boldsymbol{f}_n,\ n\in\mathbb{N}\}$ in Z_0.

The effective permeability law we are going to establish for the limit diffraction problem is described by a symmetric tensor $\boldsymbol{\mu}^{\text{eff}}$. This tensor can be then written as the following series:

$$\boldsymbol{\mu}^{\text{eff}}(k_0) := \mathrm{I}_3 + \frac{1}{4}\sum_{n\in\mathbb{N}} \frac{\varepsilon_r k_0^2}{1-\varepsilon_r \alpha_n k_0^2}\left(\int_\Sigma \boldsymbol{y}\wedge \boldsymbol{f}_n\, dy\right)\otimes\left(\int_\Sigma \boldsymbol{y}\wedge \boldsymbol{f}_n\, dy\right). \tag{2.11}$$

In fact, it is convenient to present an alternative representation of $\boldsymbol{\mu}^{\text{eff}}$ involving periodic vector fields $\boldsymbol{u}_n$ on the unit cell which will be useful to describe the fast oscillating magnetic field $\boldsymbol{H}_\eta$. Let us define

$$\lambda_n = \frac{1}{\alpha_n} \quad , \quad \boldsymbol{u}_n = \frac{1}{\sqrt{\alpha_n}}\Big(\operatorname{curl}(\boldsymbol{\psi}_{f_n}) + \frac{1}{2}\int_\Sigma \boldsymbol{y}\wedge \boldsymbol{f}_n\Big). \tag{2.12}$$

Then the following relation holds

$$\boldsymbol{\mu}^{\text{eff}}(k_0) = \mathrm{I}_3 + \sum_{n\in\mathbb{N}} \frac{\varepsilon_r k_0^2}{\lambda_n-\varepsilon_r k_0^2}\left(\int_Y \boldsymbol{u}_n\right)\otimes\left(\int_Y \boldsymbol{u}_n\right). \tag{2.13}$$

As will be seen later in Section 6, the pair $(\boldsymbol{u}_n,\lambda_n)$ can be characterized directly as solutions of the following spectral problem. Find $(\boldsymbol{w},\lambda)\in X_0^{\text{div}}\times\mathbb{R}$ such that for all $\boldsymbol{v}\in X_0^{\text{div}}$:

$$\int_Y \operatorname{curl}\boldsymbol{u}_n\cdot\operatorname{curl}\boldsymbol{v} = \lambda_n\int_Y \boldsymbol{u}_n\cdot\boldsymbol{v}\,, \tag{2.14}$$

where X_0^{div} is a suitable subspace of $W^{1,2}_\sharp(Y;\mathbb{C}^3)$ consisting of functions which are curl-free in Σ^* and divergence-free in Y (see (5.17)).

2.2. Main convergence result.

In view of Lemma 2.1, we consider the unique solution $(\boldsymbol{E},\boldsymbol{H})$ of (2.1) and denote by E_k, H_k the k-th component of $\boldsymbol{E},\boldsymbol{H}$ respectively ($k\in\{1,2,3\}$). Next, we introduce two important vector fields $\boldsymbol{E}_0(x,y)$, $\boldsymbol{H}_0(x,y)$ in $L^2(B_R\times Y)$ (associated with the two-scale analysis performed in Section 4) where x represents the macroscopic variable and where a Y-periodic dependence with respect to the fast variable y is set in order to account for the oscillating behavior of the sequence $(\boldsymbol{E}_\eta,\boldsymbol{H}_\eta)$.

The "two-scale electric field" is defined by

$$\boldsymbol{E}_0(x,y) = \begin{cases} \displaystyle\sum_{k=1}^3 E_k(x)\big(\boldsymbol{e}_k+\nabla\chi_k(y)\big) & \text{if } x\in\Omega \\ \boldsymbol{E}(x) & \text{if } x\in B_R\setminus\Omega \end{cases} \tag{2.15}$$

where the functions χ_k are the solutions of (2.7). Similarly, with the help of the periodic vector fields $\boldsymbol{u}_n$ and positive numbers λ_n defined in (2.12), we define the "two-scale magnetic field" by

$$\boldsymbol{H}_0(x,y) = \begin{cases} \displaystyle\sum_{k=1}^3 H_k(x)\boldsymbol{H}^k(y) & \text{if } x\in\Omega \\ \boldsymbol{H}(x) & \text{if } x\in B_R\setminus\Omega \end{cases} \tag{2.16}$$

$$\boldsymbol{H}^k(y) := \boldsymbol{e}_k + \sum_{n\in\mathbb{N}}\langle \boldsymbol{e}_k,\boldsymbol{u}_n\rangle\frac{\varepsilon_r k_0^2}{\lambda_n-\varepsilon_r k_0^2}\boldsymbol{u}_n(y) \tag{2.17}$$

We are now in a position to state the main result of the paper:

Theorem 2.2.
Let us assume (1.2), (1.4) *and let* $\boldsymbol{\varepsilon},\boldsymbol{\mu}$ *be defined by* (2.2), (2.6), (2.13). *Let* $(\boldsymbol{E},\boldsymbol{H})$ *be the unique solution of* (2.1). *Then the solution* $(\boldsymbol{E}_\eta,\boldsymbol{H}_\eta)$ *of the diffraction problem* (1.5) *satisfies*

$$\int_{B_R}\left|\boldsymbol{H}_\eta(x)-\boldsymbol{H}_0\Big(x,\frac{x}{\eta}\Big)\right|^2 \mathrm{d}x\to 0\,, \qquad \int_{B_R}\left|\boldsymbol{E}_\eta(x)-\boldsymbol{E}_0\Big(x,\frac{x}{\eta}\Big)\right|^2 dx\to 0\,. \tag{2.18}$$

where R *is arbitrary large and* $\boldsymbol{E}_0,\boldsymbol{H}_0$ *are given by* (2.15) *and* (2.16) *respectively. Futhermore it holds* $(\boldsymbol{E}_\eta,\boldsymbol{H}_\eta)\to(\boldsymbol{E},\boldsymbol{H})$ *in* $C^\infty(K)$ *for every compact subset* $K\subset\subset\mathbb{R}^3\setminus\Omega$.

The proof of Theorem 2.2 is quite long and involved. It is postponed to Section 7 where the arguments are presented along two steps. The most delicate issue is the L^2 upper-bound estimate (see (5.2)) for the electromagnetic field. It is proved *a posteriori* in the last step by using a contradiction argument (in the same line as in [6, 7]). Before this proof, in Section 5, we assume *a priori* this L^2-upper-bound in order to prepare the two-scale analysis of the system.

Remark 2.3. Let us emphasize that the convergence result in Theorem 2.2 is unusual in the classical framework of homogenization theory: the effective magnetic field $\boldsymbol{H}$ that we use in order to describe the limiting diffraction problem (1.5) does not agree inside the obstacle with the weak limit of $\boldsymbol{H}_\eta$ in $L^2_{\rm loc}(\mathbb{R}^3)$. Indeed in view of (2.13), (2.16) and (2.17), it is easy to check that, for $x \in \Omega$, this weak limit $\tilde{\boldsymbol{H}}$ satisfies:

$$\tilde{\boldsymbol{H}}(x) := \int_Y \boldsymbol{H}_0(x,y)\,dy \;=\; \boldsymbol{\mu}^{\rm eff}\boldsymbol{H}(x)\,, \tag{2.19}$$

whereas $\tilde{\boldsymbol{H}} = \boldsymbol{H}$ in $\mathbb{R}^3 \setminus \Omega$. As shown later, this tensor $\boldsymbol{\mu}^{\rm eff}$ differs from the identity matrix (for most of the frequencies).

There is a major reason in not using $\tilde{\boldsymbol{H}}$ for describing the limit magnetic field: the tangential trace of $\tilde{\boldsymbol{H}}$ on $\partial\Omega$ turns out to differ from that of the field $\boldsymbol{H}$ outside. This is *a priori* not physically reasonable and suggests that taking the asymptotic bulk average of $\boldsymbol{H}_\eta$ would not be a good choice. Moreover, in view of Theorem 2.2 and of equations (2.1) solved by $(\boldsymbol{E},\boldsymbol{H})$, we find *a posteriori* that the limit system written in term of $\tilde{\boldsymbol{H}}$ leads to

$$\operatorname{curl}\boldsymbol{E} = i\omega\mu_0\,\tilde{\boldsymbol{H}}\,, \qquad \operatorname{curl}(\boldsymbol{\mu}^{-1}(x)\tilde{\boldsymbol{H}}) = -i\omega\varepsilon_0\,\boldsymbol{\varepsilon}(x)\,\boldsymbol{E}\,,$$

where we lose the curl structure of the second Maxwell and the fact that the magnetic activity is encoded through the tensor $\boldsymbol{\mu}^{\rm eff}$.

In order to obtain a proper notion of effective magnetic field, we will use a different averaging recipe in which the periodic field $\boldsymbol{H}_0(x,\cdot)$ is seen as a closed periodic differential 1-form on $\Gamma \setminus \Sigma$. This allows to define at every point $x \in \Omega$ a circulation vector $\boldsymbol{H}(x) = \oint \boldsymbol{H}_0(x,\cdot)$ (see Lemma 4.5). Adopting this alternative definition of $\boldsymbol{H}$ inside Ω, we will succeed in recovering the classical transmission conditions across $\partial\Omega$ namely (2.3). We notice that in contrast, the limit *magnetic induction* vector field $\boldsymbol{B} := \mu_0\boldsymbol{\mu}^{\rm eff}\boldsymbol{H}$ agrees with the weak limit of $\boldsymbol{B}_\eta = \mu_0\boldsymbol{H}_\eta$. Roughly speaking $\boldsymbol{B}_\eta$ can be seen as a closed 2-form and as $\eta \to 0$ the local flux it generates is represented by the bulk average $\boldsymbol{B}$.

2.3. Frequency-dependent permeability and band gaps.

According to Theorem 2.2, for infinitesimal η, the electromagnetic field $(\boldsymbol{E}_\eta,\boldsymbol{H}_\eta)$ outside the obstacle is close to the solution of a limit diffraction problem in which domain Ω is occupied by a homogeneous medium whose permeability and permittivity tensors are given in (2.6) and (2.13) respectively. In this asymptotic model, the most interesting issue with respect to applications stems from the properties of tensor $\boldsymbol{\mu}^{\rm eff}$, in particular its explicit dependence with respect to the angular frequency $\omega = k_0(\varepsilon_0\mu_0)^{-1/2}$ as well as its ability to exhibit eigenvalues with a *negative real part*. To see that, it is convenient to introduce V_λ the eigenspace associated with a eigenvalue λ of spectral problem (2.14). Then denoting by P_{V_λ} the orthogonal projector on V_λ (with respect to the scalar product of $L^2(Y)^3$), we may rewrite (2.13) as follows

$$\boldsymbol{\mu}^{\rm eff}(\omega) := 1 + \sum_{\lambda\in\sigma_0} \frac{\varepsilon_r k_0^2}{\lambda - \varepsilon_r k_0^2} M_\lambda \quad , \quad (M_\lambda)_{kl} := (P_{V_\lambda}(\boldsymbol{e}_k),\boldsymbol{e}_l)\,, \tag{2.20}$$

where

$$\sigma_0 := \big\{\lambda \,:\quad \lambda \text{ eigenvalue of (2.14)} \quad , \quad M_\lambda \neq 0\big\} \tag{2.21}$$

For $\lambda \in \sigma_0$, the real symmetric matrix satisfies $0 \le M_\lambda \le I_3$. It is of rank one if λ is simple. In order for it to be of full rank we need at least that the multiplicity of λ be not smaller than 3.

Let us assume for simplicity that ε_r is a positive real (lossless dielectric inclusions). Then tensor $\boldsymbol{\mu}^{\text{eff}}(\omega)$ is real symmetric, continuous with respect to ω except at the frequencies *(Mie resonances)* given by:

$$\omega_\lambda := \sqrt{\lambda}(\varepsilon_0\mu_0\varepsilon_r)^{-1/2} \quad , \quad \lambda \in \sigma_0 \, .$$

In the vicinity of this values, $\boldsymbol{\mu}^{\text{eff}}$ blows up and we are led to different consequences according to the rank of M_λ. Denote by $\mu^{\pm}$ the largest (resp. the smallest) of the eigenvalues of $\boldsymbol{\mu}^{\text{eff}}$. Clearly it follows from (2.20) that $\lim_{\omega\to\omega_\lambda^+}\mu^-(\omega) = -\infty$. The same holds true for μ^+ if and only if M_λ has full rank. In this case, by continuity, we obtain an interval of frequencies in which all eigenvalues of $\boldsymbol{\mu}^{\text{eff}}$ are negative. For such frequencies, since $\boldsymbol{\varepsilon}^{\text{eff}}$ is positive definite, the electromagnetic field cannot propagate in any direction inside the obstacle. We may therefore conclude to the existence of *a photonic band gap*. On the opposite side if M_λ is not of full rank, then vectors in its kernel determine propagative directions for the electromagnetic field. Such a partial band gap situation was already observed in the context of elastic waves [2].

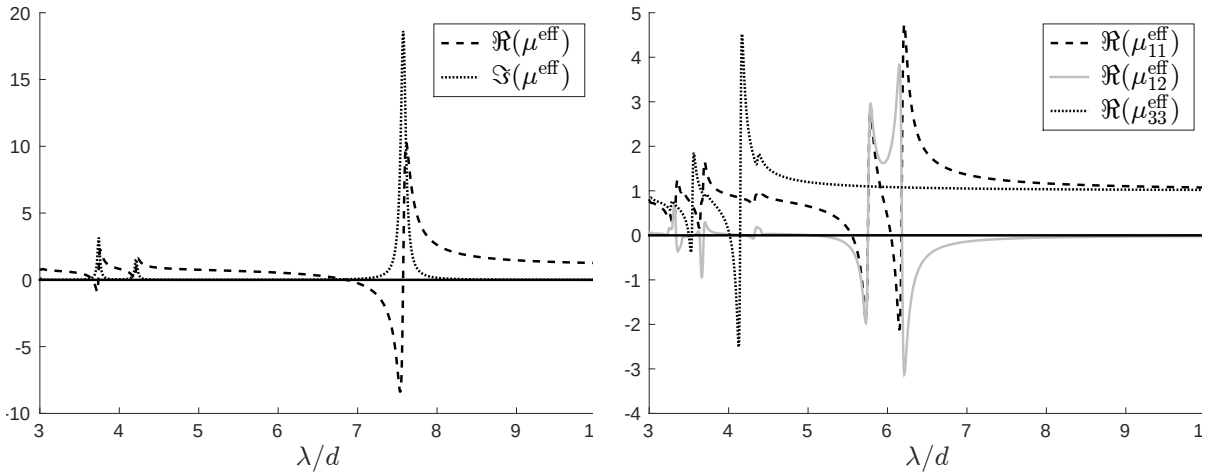


FIGURE 2. Tensor $\boldsymbol{\mu}^{\text{eff}}$ as a function of λ/d for $\varepsilon_r = 100 + i$. On the left $\Sigma = \Sigma_1$ is a cube of size 0.6 and tensor $\boldsymbol{\mu}^{\text{eff}}$ is scalar; on the right $\Sigma = \Sigma_2$ is a L-shape inclusion (and $\boldsymbol{\mu}_{22}^{\text{eff}} = \boldsymbol{\mu}_{11}^{\text{eff}}, \boldsymbol{\mu}_{13}^{\text{eff}} = \boldsymbol{\mu}_{23}^{\text{eff}} = 0$).

3. NUMERICAL SIMULATIONS.

In this Section we present a numerical approach in order to evaluate the tensor $\boldsymbol{\mu}^{\text{eff}}$ as a function of the frequency. To that aim it is convenient to use the representation (2.11) where we need to solve the three dimensional cell spectral problem (2.10) in the space Z_0 of divergence-free fields vanishing outside Σ. The advantage to work with this representation rather than with (2.13) is that the space Z_0 requires a discretization on subset Σ only. The approximation of (2.10) is performed by means of a Galerkin method making use of the piecewise affine edge-elements of Nedelec (see [25]). The main drawback of the method with respect to the computation cost is that we need to handle the non-local 3D-cell problem (2.9). For this problem we use an integral equation method with the help of the Green kernel of the inverse Laplace operator on the 3-dimensional torus (we used the explicit form given in [19]).

In order to give a nice description of $\boldsymbol{\mu}^{\text{eff}}(\omega)$ in the vicinity of the resonant frequencies, we have computed some of the local displacement currents $\boldsymbol{J}^k, 1 \le k \le 3$ appearing in the periodic cell problem (see (5.21)). Recall that $\boldsymbol{J}^k$ represents the vorticity of the shape magnetic field $\boldsymbol{H}^k$ and that it belongs to space Z_0. By solving the spectral problem (2.10), we obtain a sequence of pairs $(\boldsymbol{f}_n, \alpha_n)$ and then we recover $\boldsymbol{J}^k = \operatorname{curl}\boldsymbol{H}^k$ from the following expansion:

$$\boldsymbol{J}^k \;=\; \frac{1}{2}\sum_{n\in\mathbb{N}} \frac{\varepsilon_r\, k_0^2}{1-\alpha_n\,\varepsilon_r\, k_0^2}\langle \boldsymbol{e}_k, \boldsymbol{y}\wedge\boldsymbol{f}_n\rangle\, \boldsymbol{f}_n \, . \tag{3.1}$$

The equality above is a straightforward consequence of (2.17) taking into account that, by (2.12), one has

$$\operatorname{curl} \boldsymbol{u}_n = \frac{\boldsymbol{f}_n}{\sqrt{\alpha_n}} \quad , \quad \langle \boldsymbol{e}_k, \boldsymbol{u}_n \rangle = \frac{1}{2\sqrt{\alpha_n}} \langle \boldsymbol{e}_k, \boldsymbol{y} \wedge \boldsymbol{f}_n \rangle \, .$$

Then it follows from (2.11) and (3.1) that the entries of tensor $\boldsymbol{\mu}^{\text{eff}}(\omega)$ can be deduced from the relations

$$\boldsymbol{\mu}^{\text{eff}}_{kl} = \frac{1}{2} \langle \boldsymbol{y} \wedge \boldsymbol{J}^k, \boldsymbol{e}_l \rangle \qquad k, l \in \{1,2,3\} \, . \tag{3.2}$$

For a given frequency, the excited resonances are weighted by the strength factor $\int_\Sigma \boldsymbol{y} \wedge \boldsymbol{f}_n$ so that, in practice, only a few of the eigenvectors $\boldsymbol{f}_n$ will be contributing in the expansion (3.1). We are going to represent them showing thus the significative loops of displacement current $\boldsymbol{J}^k$ which take place in each periodic cell of the composite structure.

In view of (3.1) and (3.2), numerical simulations for solving (2.10) have been performed for two kind of geometries of the dielectric inclusion Σ that we will discuss separately:

- the cubic one where $\Sigma = \Sigma_1 := (-0.3, 0.3)^3$.
- the L-shaped one where $\Sigma = \Sigma_2 := (-0.3, 0.3)^3 \setminus \big([-0.3, 0.1]^2 \times \mathbb{R}\big)$.

In both cases the permittivity parameter ε_r characterizing the dielectric inclusion is taken to be $100 + i$, meaning that the structure is slightly *dissipative*.

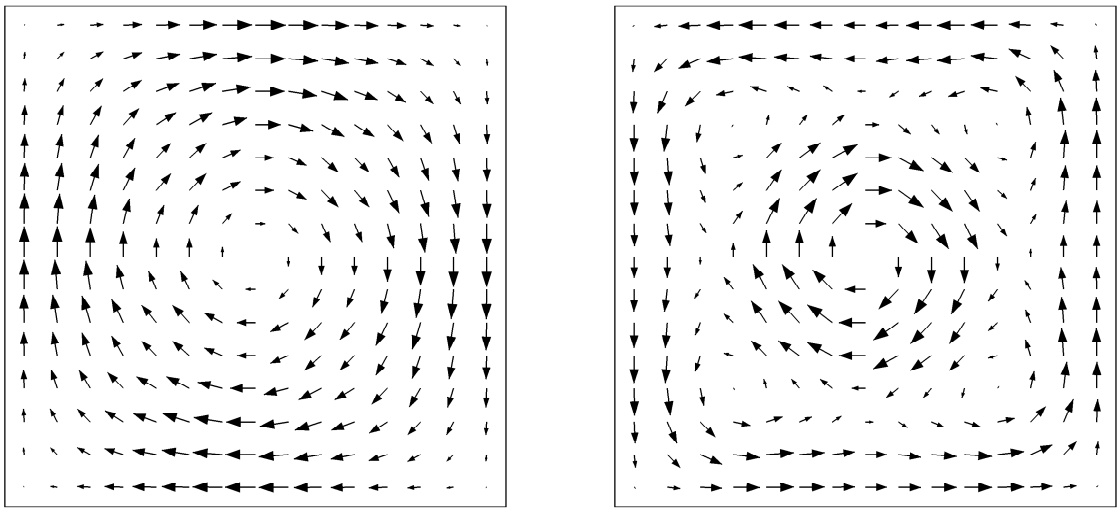


FIGURE 3. Horizontal section of $\boldsymbol{J}^3$ for the cubic geometry. Left side: $\lambda/d = 7.5$ (fundamental resonance). Right side: $\lambda/d = 3.76$. (resonance associated with the 24th eigenvalue).

Cubic geometry. As the faces of the cubic inclusion Σ_1 and of the periodic cell Y share the same orientation, many symmetry properties can be exploited, namely the invariance of the structure under all rotations of angle $\pm\pi/2$ with axis $\boldsymbol{e}_i, i \in \{1,2,3\}$. In particular it is easy to deduce that $\boldsymbol{\mu}^{\text{eff}} (= \boldsymbol{\mu}^{\text{eff}} I_3)$ is a scalar tensor. Moreover an eigenvalue α such that $\int_\Sigma \boldsymbol{y} \wedge \boldsymbol{f} \neq 0$ for any associated eigenvector $\boldsymbol{f}$ has a multiplicity ≥ 3. If it is not the case, that means that α is not contributing in expansion (2.11) (equivalently $\lambda = \alpha^{-1}$ does not belong to σ_0 defined in (2.21)). In figure 2, we represent on the left side the real and the imaginary parts of $\boldsymbol{\mu}^{\text{eff}}$ as a function of the normalized wavelength $\frac{\lambda}{d}$ (see Remark 1.1). Band gaps correspond to the frequency intervals in which the scalar permeability $\boldsymbol{\mu}^{\text{eff}}$ satisfies $\Re(\boldsymbol{\mu}^{\text{eff}}) < 0$. We represent only the part of the graph where the influence of resonant frequencies is significant namely the part corresponding to $\frac{\lambda}{d} \in [3, 10]$ in which three oscillations of $\boldsymbol{\mu}^{\text{eff}}$ can be observed. Two of them have a sufficiently large amplitude in order to reach a negative $\Re(\boldsymbol{\mu}^{\text{eff}})$: the larger one corresponds to the fundamental eigenvalue α_1 while the second one is associated with eigenvalue α_{24} (here the α_n are repeated accounting their multiplicity). The third one associated with α_{17} corresponds to a resonance whose amplitude is too small to force an

additional change of sign for $\Re(\boldsymbol{\mu}^{\text{eff}})$. In fact the numerical computations reveal that a very few of the α_n contribute to the series: among the 49 first ones only $\{\alpha_1,\alpha_2,\alpha_3,\alpha_{17},\alpha_{18},\alpha_{19},\alpha_{24},\alpha_{25},\alpha_{26}\}$ are such that $\alpha^{-1}\in\sigma_0$.

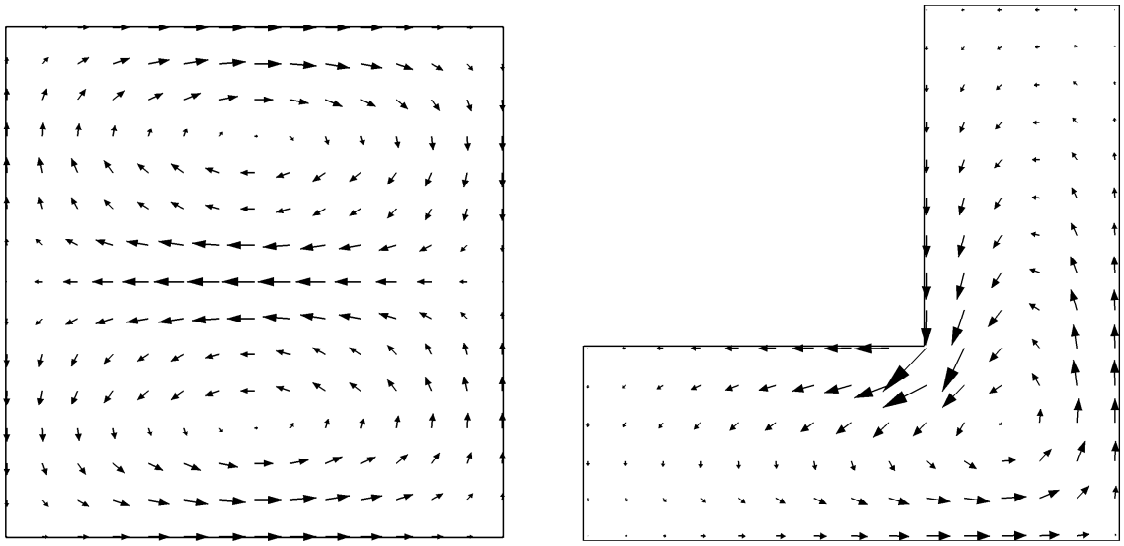


FIGURE 4. Left side: horizontal section of eigenvector $\boldsymbol{f}_4$ in the cubic geometry (mode that is not excited). Right side: horizontal section of field $\boldsymbol{J}^3$ for $\lambda/d=4.17$ in the L-shape geometry.

Now we turn to the description of the displacement current $\boldsymbol{J}^3$. In figure 3 an horizontal section of this vector field $\boldsymbol{J}^3$ is given for two particular values of λ/d (for which $\boldsymbol{J}^3$ is horizontal and very slowly varying in y_3). On the left, for $\lambda/d=7.5$, a single loop of current is obtained associated with the fundamental eigenvalue α_1. On the right, for $\lambda/d=3.76$, the second important resonance α_{24} is excited and it appears two concentric loops of current traveled in opposite directions. The induced magnetic field $\boldsymbol{H}^3$ does not vanish but has much lower amplitude than in the previous case.

Eventually in Figure 4, we draw the horizontal section of eigenvector $\boldsymbol{f}_4$ whose strength factor $\int_\Sigma \boldsymbol{y}\wedge\boldsymbol{f}_4$ vanishes. It consists of two counter rotating loops. No magnetic field is induced and the resonance α_4 is not excited from the incident wave.

L-shape geometry. In that case, the effective permeability tensor $\boldsymbol{\mu}^{\text{eff}}$ is not scalar anymore. However some symmetries are still present and it is possible to show that $\boldsymbol{\mu}^{\text{eff}}_{11}=\boldsymbol{\mu}^{\text{eff}}_{22}$ and $\boldsymbol{\mu}^{\text{eff}}_{13}=\boldsymbol{\mu}^{\text{eff}}_{23}=0$. Thus $\boldsymbol{\mu}^{\text{eff}}$ admits $\{\boldsymbol{e}_1\pm\boldsymbol{e}_2,\boldsymbol{e}_3\}$ as eigenvectors and its eigenvalues are $\left\{\boldsymbol{\mu}^{\text{eff}}_{11}\pm\boldsymbol{\mu}^{\text{eff}}_{12},\boldsymbol{\mu}^{\text{eff}}_{33}\right\}$.
In the right hand side of 2, the real parts of coefficients $\boldsymbol{\mu}^{\text{eff}}_{11},\boldsymbol{\mu}^{\text{eff}}_{12},\boldsymbol{\mu}^{\text{eff}}_{13}$ are represented in term of normalized wavelength λ/d varying in the interval $[3,10]$ where most of the significant resonances are localized. The two peaks of resonance on the right influence the response to horizontal magnetic fields. They are produced by the displacement currents $\boldsymbol{J}^1,\boldsymbol{J}^2$ each of them being deduced form the other by a rotation of angle π around axis $\boldsymbol{e}_1+\boldsymbol{e}_2$. In figure 5 we represent $\boldsymbol{J}^1$ that we draw only in the back faces of Σ_1 and in a fictitious surface element located at the junction part of the "L". On the left we take $\lambda/d=6.2$ which is close to the first fundamental mode while on the right $\lambda/d=5.7$ corresponds to the second mode. In both cases $\boldsymbol{J}^1$ exhibits two loops which rotate in the same direction in the first case (averaged induced magnetic field $\int_Y \boldsymbol{H}^1$ parallel to $\boldsymbol{e}_1+\boldsymbol{e}_2$) and in the opposite direction in the second one ($\int_Y \boldsymbol{H}^1$ parallel to $\boldsymbol{e}_1-\boldsymbol{e}_2$).

The next significative resonance is obtained for λ/d close to 4.17 and relies on displacement current $\boldsymbol{J}^3$. This vector field depicted in the right hand side of figure 4 is horizontal and independent of y_3. The averaged induced magnetic field $\int_Y \boldsymbol{H}^3$ is vertical and contributes to $\boldsymbol{\mu}^{\text{eff}}_{33}$ only, producing a change sign of $\Re(\boldsymbol{\mu}^{\text{eff}}_{33})$.

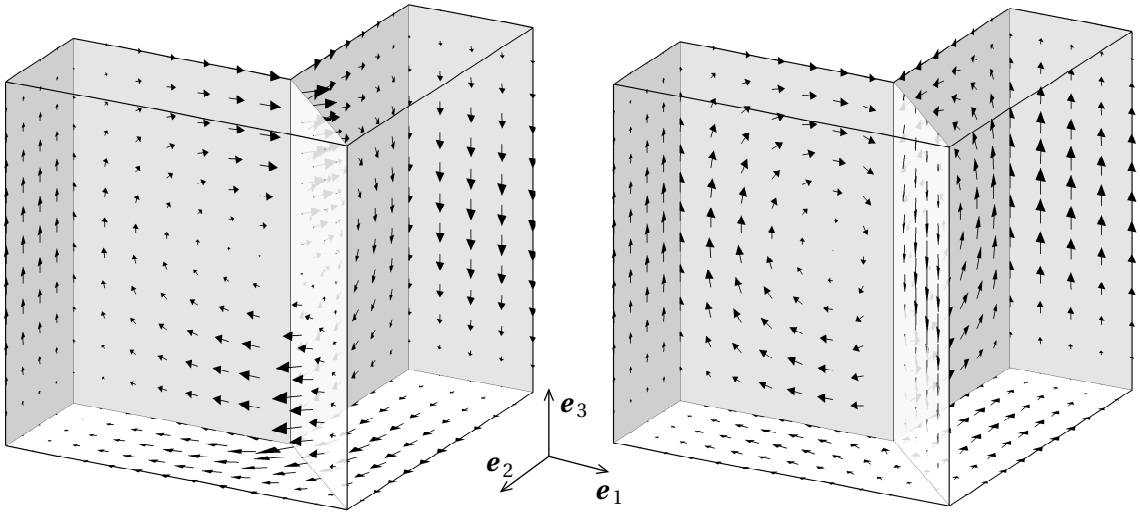


FIGURE 5. Representation of field $\boldsymbol{J}^1$ in the case of a L-shape inclusion Σ. On the left, $\lambda/d = 6.2$ (first mode is excited) and on the Right $\lambda/d = 5.7$ (second mode is excited)

4. PRELIMINARY BACKGROUD

4.1. Two-scale convergence.

Our study is based on the notion of two-scale convergence which allows to keep a precise description of the electromagnetic field in the periodic microstructure when η tends to zero. We refer to Allaire [1] for a precise presentation of the method and we recall here some principal points.

The definitions below are given on a fixed bounded domain D of $\mathbb{R}^3$ (in most cases D will be a reference ball B_R with radius R so large that $\Omega \subset\subset B_R$).

Definition 4.1 (Two-scale convergence)**.** *We say that a sequence $f_\eta \in L^2(D)$ two-scale converges to $f_0 \in L^2(D \times Y)$, and we write $f_\eta(x) \rightharpoonup f_0(x,y)$ if, for all $\varphi \in C_c^\infty(D; C_\sharp^\infty(Y))$, it holds*

$$\lim_{\eta\to 0} \int_D f_\eta(x)\,\varphi\Big(x, \frac{x}{\eta}\Big)\,\mathrm{d}x = \int_{D\times Y} f_0(x,y)\,\varphi(x,y)\,\mathrm{d}x\,\mathrm{d}y\,. \tag{4.1}$$

The sequence is said strongly two-scale convergent (denoted $f_\eta \rightarrow\!\!\!\!\rightarrow f_0$) if in addition

$$\limsup_{\eta\to 0} \int_D |f_\eta(x)|^2\,\mathrm{d}x \;=\; \int_{D\times Y} |f_0(x,y)|^2\,\mathrm{d}x\,\mathrm{d}y \tag{4.2}$$

A key justification of definition (4.1) is that any sequence f_η which is uniformly bounded in $L^2(D)$ admits a two-scale converging subsequence. For such a subsequence, the weak limit exists and is given by the bulk average $\langle f_0\rangle(x) := \int_Y f_0(x,y)\,dy$. A consequence of the strong two-scale convergence (4.2) is the following product rule:

$$f_\eta \rightharpoonup f_0\,,\; g_\eta \rightharpoonup g_0 \quad\Longrightarrow\quad \lim_{\eta\to 0} \int_D f_\eta\, g_\eta\, \varphi\, dx = \int_D \langle f_0 g_0\rangle\, \varphi\, dx\,, \tag{4.3}$$

the latter convergence holding for every φ continuous with compact support. In fact the assumption on the support of φ can be dropped once we know that f_0 satisfies a suitable admissibility criterium. It is the case in particular if $f_0(x,y) = \theta(x)\psi(y)$ with $\theta \in L^2(D)$ and $\psi \in L^2_\sharp(Y)$ (see Lemma 5.7 in [1]). Furthermore the strong convergence $f_\eta \rightarrow\!\!\!\!\rightarrow f_0$ for for such an admissible f_0 implies that

$$\lim_{\eta\to 0^+} \int_D \Big| f_\eta(x) - f_0\Big(x, \frac{x}{\eta}\Big)\Big|^2\, dx = 0.$$

In particular the strong two-scale convergence $f_\eta \rightarrow\!\!\!\!\rightarrow f_0$ with $f_0(x,y) = f(x)$ implies that $f_\eta \to f$ strongly in $L^2(D)$.

We begin by recalling a classical rule (see [1]) which applies to vector valued functions $\boldsymbol{u}_\eta : D \to \mathbb{C}^3$ which are uniformly bounded in $L^2(D)$ as well as $\eta \operatorname{div} \boldsymbol{u}_\eta$. We will use also its straightforward variant in which $\operatorname{div} \boldsymbol{u}_\eta$ is substituted with $\operatorname{curl} \boldsymbol{u}_\eta$.

Proposition 4.2. *Let $(u_\eta)_\eta \subset L^2(D;\mathbb{C}^3)$ a sequence of functions such that $u_\eta \rightharpoonup \boldsymbol{u}_0$. Then*

(i) *Assume that $\eta \operatorname{div} \boldsymbol{u}_\eta \rightharpoonup \chi_0$. Then for a.e. $x \in D$, it holds $\chi_0(x,\cdot) = \operatorname{div}_y \boldsymbol{u}_0(x,\cdot)$*

(ii) *Assume that $\eta \operatorname{curl} \boldsymbol{u}_\eta \rightharpoonup \boldsymbol{\xi}_0$. Then for a.e. $x \in D$, it holds $\boldsymbol{\xi}_0(x,\cdot) = \operatorname{curl}_y \boldsymbol{u}_0(x,\cdot)$*

Let us stress that the equalities above are meant in the sense of (Y- periodic) distributions on $\mathbb{R}^3$. On the other hand, this result can be localized as follows: let Σ be an open subset of Y and Σ_η defined as in (1.1); then the convergence $\eta \mathbb{1}_{\Sigma_\eta} \operatorname{curl} \boldsymbol{u}_\eta \rightharpoonup \boldsymbol{\xi}_0$ implies that $\boldsymbol{\xi}_0(x,\cdot) = \operatorname{curl}_y \boldsymbol{u}_0(x,\cdot)$ in $\mathscr{D}'(\Sigma)$. In the same way we have $\chi_0(x,\cdot) = \operatorname{div}_y \boldsymbol{u}_0(x,\cdot)$ in $\mathscr{D}'(\Sigma)$ provided that the convergence $\eta \mathbb{1}_{\Sigma_\eta} \operatorname{div} \boldsymbol{u}_\eta \rightharpoonup \chi_0$ holds.

Next we give an improved version of a classical result concerning a sequence of scalar-valued functions u_η which are uniformly bounded in $W^{1,2}(D)$. For every $\eta > 0$, let us set

$$Y_\eta^k = \eta(Y+k) \quad , \quad I_\eta := \{k \in \mathbb{Z}^3 \,:\, Y_\eta^k \cap D \neq \emptyset\}.$$

Then we define the step-wise approximation of u_η given by

$$[u_\eta]_\eta(x) := \sum_{k \in I_\eta} \Big(\fint_{Y_\eta^k} u_\eta \, \mathrm{d}x \Big) 1_{Y_\eta^k}(x) \,. \tag{4.4}$$

Assume that u_η converges weakly to u in $W^{1,2}(D)$. Then by Rellich's Theorem and Poincaré inequality, we easily deduce that:

$$[u_\eta]_\eta \to u \quad \text{in} \quad L^2(D) \qquad \text{and} \qquad v_\eta := \frac{u_\eta - [u_\eta]_\eta}{\eta} \quad \text{is bounded in } L^2(D) \,. \tag{4.5}$$

Possibly passing to a subsequence, we may assume that $v_\eta \rightharpoonup v_0$. In the following Proposition we make the link between v_0 and the two-scale limit of the sequence ∇u_η.

Proposition 4.3. *Let (u_η) be a bounded sequence in $W^{1,2}(D)$ such that*

$$u_\eta \to u \quad \textit{weakly in } W^{1,2}(D) \quad , \quad \frac{u_\eta - [u_\eta]_\eta}{\eta} \rightharpoonup v_0 \,.$$

Let $\psi_0(x,y) := v_0(x,y) - \nabla u(x) \cdot (y - [y])$. Then ψ_0 belongs to $L^2(D; W^{1,2}_\sharp(Y))$ and it holds

$$\nabla u_\eta \rightharpoonup \nabla_x u(x) + \nabla_y \psi_0(x,y) \,. \tag{4.6}$$

Moreover if u_η is independent of η (i.e. $u_\eta = u$), then $\psi_0 = 0$ and we have the strong two-scale convergence $\frac{u - [u]_\eta}{\eta} \twoheadrightarrow \nabla u(x) \cdot (y - [y])$.

Remark 4.4. The fact that $\psi_0(x,\cdot)$ belongs to $W^{1,2}_\sharp(Y)$ indicates that the periodic function $v_0(x,\cdot)$ shares on ∂Y the same jump as the piecewise affine function $\nabla u(x)(y - [y])$. The existence of ψ_0 satisfying (4.6) is proved in [1]. Here we give an explicit construction which will be useful in Section 5.

Proof. **Step 1**. We consider the linear map $A_\eta : W^{1,2}(D) \mapsto L^2(D)$ defined by

$$A_\eta(u) := \frac{u - [u]_\eta}{\eta} - \nabla u(x) \cdot \frac{x - [x]_\eta}{\eta} \,.$$

Since $|x - [x]_\eta| \leq \frac{\sqrt{3}}{2} \eta$, it follows from Poincaré inequality that for a suitable positive C:

$$\|A_\eta(u)\|_{L^2(D)} \leq C \, \|\nabla u\|_{L^2(D)} \qquad \forall \, u \in W^{1,2}(D) \,. \tag{4.7}$$

On the other hand, it can be easily checked by using a Taylor expansion that, for $u \in C^2(\overline{D})$, $\|A_\eta(u)\|_{L^2} \leq C \, \eta$. Therefore by (4.7) and by the density of $C^2(\overline{D})$ in $W^{1,2}(D)$, we deduce that A_η converges strongly to 0 as $\eta \to 0$. In particular, for every $u \in W^{1,2}(D)$, it holds $A_\eta u \to 0$ in $L^2(D)$.

Noticing that $\frac{x-[x]_\eta}{\eta} \rightharpoonup (y-[y])$, it follows that $\frac{u-[u]_\eta}{\eta} \rightharpoonup \nabla u(x)\cdot(y-[y])$ (which is the last statement of Proposition 4.3.

Step 2. In this step we prove (4.6) assuming that $u=0$. In this case ψ_0 coincides with the weak two-scale limit v_0 of $\frac{u_\eta-[u_\eta]_\eta}{\eta}$. Possibly passing to a subsequence we may assume that

$$\nabla u_\eta \rightharpoonup \xi_0(x,y) \quad , \quad \frac{u_\eta-[u_\eta]_\eta}{\eta} \rightharpoonup v_0(x,y)\,. \tag{4.8}$$

The two-scale convergence of the whole sequence ∇u_η will be deduced once we can prove that, for a.e. $x\in D$, $\xi_0(x,\cdot)$ agrees with the distributional gradient of $v_0(x,\cdot)$ on $\mathbb{R}^3$. As $v_0(x,\cdot)$ is periodic this amounts to showing that for a.e. $x\in D$ and for every test function $\boldsymbol{\theta}(y)$ in $C^\infty_\sharp(Y;\mathbb{C}^3)$, it holds

$$-\int_Y v_0(x,y)\,\mathrm{div}_y\,\boldsymbol{\theta}\,dy \;=\; \int_Y \xi_0(x,y)\cdot\boldsymbol{\theta}(y)\,dy\,. \tag{4.9}$$

Let ρ an arbitrary localizing function in $C^\infty_0(D)$. Thanks to an integration by parts, we obtain

$$\int_D \nabla u_\eta(x)\rho(x)\cdot\boldsymbol{\theta}\Big(\frac{x}{\eta}\Big)\mathrm{d}x = -\int_D u_\eta(x)\nabla\rho(x)\cdot\boldsymbol{\theta}\Big(\frac{x}{\eta}\Big)\mathrm{d}x - \frac{1}{\eta}\int_D u_\eta(x)\rho(x)\,\mathrm{div}_y\,\boldsymbol{\theta}\Big(\frac{x}{\eta}\Big)\mathrm{d}x\,. \tag{4.10}$$

Using the piecewise constant approximation operator $[\cdot]_\eta$ as defined in (4.4), we split the last integral in the right-hand member of (4.10) as follows:

$$\begin{aligned}\frac{1}{\eta}\int_D u_\eta(x)\rho(x)\,\mathrm{div}_y\,\boldsymbol{\theta}\Big(\frac{x}{\eta}\Big)\mathrm{d}x &= \int_D\left(\frac{u_\eta(x)-[u_\eta]_\eta}{\eta}\right)\rho(x)\,\mathrm{div}_y\,\boldsymbol{\theta}\Big(\frac{x}{\eta}\Big)\mathrm{d}x\\ &+\int_D [u_\eta]_\eta(x)\left(\frac{\rho(x)-[\rho]_\eta}{\eta}\right)\mathrm{div}\,\boldsymbol{\theta}\Big(\frac{x}{\eta}\Big)\mathrm{d}x + \frac{1}{\eta}\int_D [u_\eta]_\eta(x)[\rho]_\eta\,\mathrm{div}_y\,\boldsymbol{\theta}\Big(\frac{x}{\eta}\Big)\mathrm{d}x\\ &= I^1_\eta + I^2_\eta + I^3_\eta\,.\end{aligned}$$

We observe that I^3_η vanishes : indeed, the functions $[u_\eta]_\eta$ and $[\rho]_\eta$ are constant on each cell Y^k_η where $\mathrm{div}_y\,\boldsymbol{\theta}$ has a vanishing mean value. On the other hand, since $[u_\eta]_\eta$ strongly converges to $u=0$ in $L^2(D)$, we infer that $I^2_\eta\to 0$ (indeed $\frac{\rho-[\rho]_\eta}{\eta}$ is bounded in $L^2(D)$ whereas $\mathrm{div}\,\boldsymbol{\theta}\in L^\infty(D)$).

Eventually passing to the limit $\eta\to 0$ in (4.10) and taking into account the two-scale convergences (4.8) , we obtain the equality

$$\int_{D\times Y}\xi_0(x,y)\rho(x)\cdot\boldsymbol{\theta}(y)\,dx\,dy \;=\; -\int_{D\times Y} v_0(x,y)\,\mathrm{div}_y\,\boldsymbol{\theta}(y)\rho(x)\,dx\,dy\,,$$

from which follows the relation (4.9) by the arbitrariness of the test function ρ.

Step 3. We consider now a general sequence u_η converging weakly to u and we apply Step 2 to the translated sequence $\tilde u_\eta = u_\eta - u$. Then $\tilde v_\eta := \frac{\tilde u_\eta-[\tilde u_\eta]_\eta}{\eta} = v_\eta - \frac{u-[u]_\eta}{\eta}$. Thus, by Step 1, the two-scale limit $\tilde v_0$ of $\tilde v_\eta$ is given by $\tilde v_0(x,y) = v_0(x,y)-\nabla u(x)\cdot(\boldsymbol{y}-[\boldsymbol{y}]) = \psi_0(x,y)$. On the other hand, as $\tilde u_\eta\to 0$, we know by Step 2 that $\nabla\tilde u_\eta \rightharpoonup \nabla_y\tilde v_0$. It follows that

$$\nabla u_\eta = \nabla\tilde u_\eta + \nabla u \rightharpoonup \nabla_y\psi_0(x,y)+\nabla u(x).$$

The proof of Proposition 4.3 is finished. □

4.2. Geometric averaging.

Since the advent of gauge theories, it is now a classical idea in Physics to see magnetic fields as differential 1-forms in $\mathbb{R}^3$. In our case it turns out that by Lemma 5.3 the magnetic field $\boldsymbol{H}_0(x,\cdot)$ is curl-free in the simply-connected domain Σ^* (closed 1-form). It follows that the circulation of $\boldsymbol{H}_0(x,\cdot)$ along curves in Σ^* joining opposite points of ∂Y is independent of the line and of the end points we chose. It is then natural to define the "mean circulation" vector, denoted by $\oint\boldsymbol{H}_0$, and characterized by:

$$\left(\oint\boldsymbol{H}_0(x,\cdot)\right)\cdot\boldsymbol{e}_k := \int_{\Gamma_k}\boldsymbol{H}_0(x,\cdot)\cdot\boldsymbol{e}_k\,d\mathscr{H}^1\,, \tag{4.11}$$

where $\Gamma_k\subset\Sigma^*$ is a curve joining two opposite points on the faces of ∂Y orthogonal to $\boldsymbol{e}_k$.

A priori this definition makes sense only for *regular* functions. It can be extended to functions in $W^{1,2}_\sharp(Y)$ as follows:

Lemma 4.5. *Let $\boldsymbol{u} \in L^2_\sharp(Y;\mathbb{C}^3)$ such that $\operatorname{curl}\boldsymbol{u} = 0$ in Σ^*. Then there exists a unique vector $\oint \boldsymbol{u} \in \mathbb{C}^3$ and a function $\psi \in W^{1,2}_\sharp(\Sigma^*)$ (unique up to a constant) such that*

$$\boldsymbol{u} = \nabla\psi + \oint \boldsymbol{u} \quad \text{in} \quad \Sigma^* . \tag{4.12}$$

Moreover this circulation vector satisfies the following properties:

i) *For all $\boldsymbol{f} \in L^2_\sharp(Y;\mathbb{C}^3)$ such that $\operatorname{div}\boldsymbol{f} = 0$ in Y and $\boldsymbol{f} = 0$ in Σ we have*

$$\int_Y \boldsymbol{u}\cdot\boldsymbol{f}\,\mathrm{d}y = \oint \boldsymbol{u}\cdot\int_Y \boldsymbol{f}\,\mathrm{d}y . \tag{4.13}$$

ii) *If $\boldsymbol{u}$ is continuous, then for $\boldsymbol{k} \in \{1,2,3\}$*

$$\Big(\oint \boldsymbol{u}\Big)\cdot\boldsymbol{e}_k = \int_0^1 \boldsymbol{u}\big(\gamma(s)\big)\cdot\gamma'(s)\,ds , \tag{4.14}$$

for all $\gamma \in C^1([0,1];\Sigma^*)$ such that $\gamma^k(1) - \gamma^k(0) = \boldsymbol{e}_k$.

iii) *There exists a constant $C > 0$ such that*

$$\Big|\oint \boldsymbol{u}\Big| \le C\|\boldsymbol{u}\|_{L^2(Y;\mathbb{C}^3)} . \tag{4.15}$$

Remark 4.6. Note that if $\operatorname{curl}\boldsymbol{u} = 0$ in whole Y, then $\oint \boldsymbol{u} = \int_Y \boldsymbol{u}$ and we recover from (4.13) a classical property (linked to "div-curl Lemma"). On the other hand, by applying (4.13) to $\boldsymbol{f} := \boldsymbol{v}\wedge\boldsymbol{z}$ with $\boldsymbol{z}$ varying over $\mathbb{C}^3$, we get the following variant of (4.13):

$$\int (\boldsymbol{u}\wedge\boldsymbol{v})\mathrm{d}y = \oint \boldsymbol{u}\wedge\int \boldsymbol{v}\,\mathrm{d}y , \qquad \text{if} \quad \operatorname{curl}\boldsymbol{v} = 0 \ , \boldsymbol{v} = 0 \text{ in } \Sigma . \tag{4.16}$$

Proof. As Σ^* is simply-connected, the existence of the unique vector $\oint \boldsymbol{u} \in \mathbb{C}^3$ and of a function $\psi \in W^{1,2}_\sharp(\Sigma^*)$ such that (4.12) holds in the case of a smooth function $\boldsymbol{u}$ is a classical result in differential geometry (see for instance [3] p. 197). The extension to functions $\boldsymbol{u} \in L^2(Y,\mathbb{C}^3)$ such that $\operatorname{curl}\boldsymbol{u} = 0$ in Σ^* follows by using a density argument.

Now we fix $\boldsymbol{f} \in L^2_\sharp(Y;\mathbb{C}^3)$ such that $\operatorname{div}\boldsymbol{f} = 0$ in Y and $\boldsymbol{f} = 0$ in Σ. Moreover we consider $\tilde\psi \in W^{1,2}_\sharp(Y)$ to be the harmonic extension of ψ in Σ. Since $\boldsymbol{f} = 0$ in Σ, we have

$$\int_Y \boldsymbol{u}\cdot\boldsymbol{f}\,\mathrm{d}y = \int_{\Sigma^*}(\nabla\psi + \oint\boldsymbol{u})\cdot\boldsymbol{f}\,\mathrm{d}y = \int_Y \nabla\tilde\psi\cdot\boldsymbol{f} + \oint\boldsymbol{u}\cdot\int_Y\boldsymbol{f}\,\mathrm{d}y = \oint\boldsymbol{u}\cdot\int_Y\boldsymbol{f}\,dy ,$$

where the last equality holds since $\operatorname{div}\boldsymbol{f} = 0$. Relation (4.13) is proved.

For $\boldsymbol{u}$ being continuous, the associated potential ψ is Lipschitzversions dŐfinitives on Σ^* and we get

$$\int_0^1 \boldsymbol{u}\big(\gamma(s)\big)\cdot\gamma'(s)\,ds = \int_0^1 (\nabla\psi + \oint\boldsymbol{u})\big(\gamma(s)\big)\cdot\gamma'(s)\,ds = \psi\big(\gamma(1)\big) - \psi\big(\gamma(0)\big) + \oint\boldsymbol{u}\cdot\boldsymbol{e}_k .$$

The equality (4.14) follows thanks to the periodicity of ψ.

For the proof of (4.15) we consider three cylinders $C_k \subset Y$ for $k \in \{1,2,3\}$. Each of them is in direction $\boldsymbol{e}_k$, cross entirely the unit cell Y and is such that $\Sigma \cap C_k = \emptyset$. We introduce functions $\boldsymbol{f}_k \in L^2_\sharp(Y;\mathbb{C}^3)$ given by $\boldsymbol{f}_k(y) = \frac{1}{|C_k|}1_{C_k}(y)\boldsymbol{e}_k$. By construction those functions are admissible in (4.13) and satisfy $\int_Y \boldsymbol{f}_k\,\mathrm{d}y = \boldsymbol{e}_k$. Thanks to (4.13) and to the Cauchy-Schwartz inequality it comes

$$\Big|\oint\boldsymbol{u}\cdot\boldsymbol{e}_k\Big| = \Big|\oint\boldsymbol{u}\cdot\int_Y\boldsymbol{f}_k\Big| = \Big|\int_Y\boldsymbol{u}\cdot\boldsymbol{f}_k\Big| \le \frac{1}{\sqrt{|C_k|}}\|\boldsymbol{u}\|_{L^2(Y;\mathbb{C}^3)} , \qquad \text{for } k\in\{1,2,3\}.$$

The estimate (4.15) follows. □

4.3. Miscellaneous results.

For the commodity of the reader we recall a classical result related Sobolev space $W^{1,2}_\sharp(Y;\mathbb{C}^3)$

Lemma 4.7. *Let $\boldsymbol{u}\in L^2_\sharp(Y;\mathbb{C}^3)$ such that $\operatorname{curl}\boldsymbol{u}\in L^2_\sharp(Y;\mathbb{C}^3)$ and $\operatorname{div}\boldsymbol{u}\in L^2_\sharp(Y)$. Then $\boldsymbol{u}\in W^{1,2}_\sharp(Y;\mathbb{C}^3)$ and*

$$\int_Y |\nabla \boldsymbol{u}|^2 \;=\; \int_Y |\operatorname{curl}\boldsymbol{u}|^2 + \int_Y |\operatorname{div}\boldsymbol{u}|^2\,. \tag{4.17}$$

Proof. As elements of $L^2_\sharp(Y)$, the functions $\boldsymbol{u}$, $\operatorname{curl}\boldsymbol{u}$ and $\operatorname{div}\boldsymbol{u}$ can be developed in Fourier series. There exists therefore $\left\{\boldsymbol{c}_k\in\mathbb{C}^3\;:\;\boldsymbol{k}\in\mathbb{Z}^3\right\}$ such that

$$\boldsymbol{u}(\boldsymbol{y})=\sum_{\boldsymbol{k}\in\mathbb{Z}^3} e^{2i\pi\boldsymbol{k}\cdot\boldsymbol{y}}\boldsymbol{c}^k\,,\qquad \operatorname{curl}\boldsymbol{u}(\boldsymbol{y})=\sum_{\boldsymbol{k}\in\mathbb{Z}^3} 2i\pi e^{2i\pi\boldsymbol{k}\cdot\boldsymbol{y}}\;\boldsymbol{k}\wedge\boldsymbol{c}^k\,,\qquad \operatorname{div}\boldsymbol{u}(\boldsymbol{y})=\sum_{\boldsymbol{k}\in\mathbb{Z}^3} 2i\pi e^{2i\pi\boldsymbol{k}\cdot\boldsymbol{y}}\;\boldsymbol{k}\cdot\boldsymbol{c}^k\,.$$

with

$$\sum_{\boldsymbol{k}\in\mathbb{Z}^3}|\boldsymbol{c}^k|^2=\|\boldsymbol{u}\|^2_{L^2(Y;\mathbb{C}^3)}\,,\quad \sum_{\boldsymbol{k}\in\mathbb{Z}^3}|\boldsymbol{k}\wedge\boldsymbol{c}^k|^2=4\pi^2\int_Y|\operatorname{curl}\boldsymbol{u}|^2\,,\qquad \sum_{\boldsymbol{k}\in\mathbb{Z}^3}|\boldsymbol{k}\cdot\boldsymbol{c}^k|^2=4\pi^2\int_Y|\operatorname{div}\boldsymbol{u}|^2\,.$$

Since $|\boldsymbol{k}\wedge\boldsymbol{c}^k|^2+|\boldsymbol{k}\cdot\boldsymbol{c}^k|^2=|\boldsymbol{k}|^2|\boldsymbol{c}_k|^2$, we infer that $\sum_{\boldsymbol{k}}|\boldsymbol{k}|^2|\boldsymbol{c}_k|^2<+\infty$, thus $\boldsymbol{u}$ belongs to $W^{1,2}_\sharp(Y;\mathbb{C}^3)$ Moreover relation (4.17) follows obviously.

□

In order to establish the strong convergence in $L^2_{\rm loc}$ of the field $(\boldsymbol{E}_\eta,\boldsymbol{H}_\eta)$, we will use the celebrated div-curl Lemma below for which we refer to [24, 29].

Lemma 4.8. *Let B be a bounded domain of $\mathbb{R}^3$. Let $(\boldsymbol{u}_\eta)$ and $(\boldsymbol{v}_\eta)$ two sequences of $L^2(B;\mathbb{R}^3)$ converging respectively to $\boldsymbol{u}$ and $\boldsymbol{v}$ weakly in $L^2(B;\mathbb{R}^3)$. If moreover we have*

$$(\operatorname{div}\boldsymbol{u}_\eta)_\eta \quad \textit{is compact in } W^{-1,2}(B) \qquad \textit{and} \qquad (\operatorname{curl}\boldsymbol{v}_\eta)_\eta \quad \textit{is compact in } (W^{-1,2}(B))^3\,,$$

then the sequence $(\boldsymbol{u}_\eta\cdot\boldsymbol{v}_\eta)_\eta$ converges to $\boldsymbol{u}\cdot\boldsymbol{v}$ in the distributional sense in B.

For checking the strong compactness assumption in $W^{-1,2}(B)$, the following characterization will be useful:

Lemma 4.9. *Let $(T_n)_n\subset W^{-1,2}(B)$ a bounded sequence. Then (T_n) is relatively compact (for the topology of the norm) if and only if, for all sequence $(\varphi_n)_n$ such that $\varphi_n\rightharpoonup 0$ in $W^{1,2}_0(B)$, we have* $\lim_{n\to+\infty}<T_n,\varphi_n>=0$.

5. Two-scale analysis of the electromagnetic field

In this section we fix a ball B_R such that $\Omega\subset\subset B_R$. We are going to identify the two-scale limit of the electromagnetic field $(\boldsymbol{E}_\eta,\boldsymbol{H}_\eta)$ under the assumption that it is uniformly bounded in $L^2(B_R)$. In fact this analysis includes the divergence-free vector field $\boldsymbol{J}_\eta$ defined below which represents a normalized version of the so-called "displacement current":

$$\boldsymbol{J}_\eta(x)=\eta\varepsilon_\eta\boldsymbol{E}_\eta \tag{5.1}$$

Along this section, we will make the following hypothesis

$$\sup_{\eta>0}\Big(\|\boldsymbol{E}_\eta\|_{L^2(B_R)}+\|\boldsymbol{H}_\eta\|_{L^2(B_R)}+\|\boldsymbol{J}_\eta\|_{L^2(B_R)}\Big)<+\infty\,. \tag{5.2}$$

Thanks to the estimate (5.2) which will be established *a posteriori* in Section 7, we may assume, possibly after extracting subsequences, that it holds

$$\boldsymbol{E}_\eta\rightharpoonup\boldsymbol{E}_0\,,\qquad \boldsymbol{H}_\eta\rightharpoonup\boldsymbol{H}_0\quad\text{and}\quad \boldsymbol{J}_\eta\rightharpoonup\boldsymbol{J}_0\,. \tag{5.3}$$

for suitable $\boldsymbol{E}_0$, $\boldsymbol{H}_0$ and $\boldsymbol{J}_0$ belonging to $L^2(B_R\times Y;\mathbb{C}^3)$.

In view of the convergences in (5.3), we define the effective electromagnetic field $(\boldsymbol{E},\boldsymbol{H})$ to be

$$\boldsymbol{E}(x):=\int_Y\boldsymbol{E}_0(x,y)\,dy\quad,\quad \boldsymbol{H}(x):=\fint\boldsymbol{H}_0(x,\cdot)\,. \tag{5.4}$$

Recall that under (5.3), $\boldsymbol{E}(x)$ represents the weak limit of $\boldsymbol{E}_\eta$ (bulk average) whereas $\boldsymbol{H}(x)$ is associated with the new averaging procedure introduced in Subsection 4.2.

In Subsection 5.1, we identify $\boldsymbol{E}_0(x,\cdot)$ for $x\in\Omega$. The characterization of vector fields $\boldsymbol{H}_0(x,\cdot)$, $\boldsymbol{J}_0(x,\cdot)$ for $x\in\Omega$ is a quite involved issue developed along Subsection 5.2. Besides, for $x\notin\Omega$, $\boldsymbol{J}_0(x,\cdot)$ vanishes and it is a direct consequence of the uniform convergence issue developed in Subsection 7.1 that the fast oscillations of $(\boldsymbol{E}_\eta,\boldsymbol{H}_\eta)$ disappear at a positive distance from $\partial\Omega$. Accordingly, $\boldsymbol{E}_0(x,\cdot),\boldsymbol{H}_0(x,\cdot)$ are constant and agree with their respective averages:

$$\boldsymbol{E}_0(x,\cdot)=\boldsymbol{E}(x)\quad,\quad \boldsymbol{H}_0(x,\cdot)=\boldsymbol{H}(x)\qquad\text{for a.e. } x\in B_R\setminus\Omega\,. \tag{5.5}$$

5.1. Oscillating electric field and effective permittivity.

Our aim here is to identify, for $x\in\Omega$, the two-scale limit $\boldsymbol{E}_0$ given in (5.3) in term of its bulk average $\boldsymbol{E}(x)$ (see (5.4)). First we show that $\boldsymbol{E}_0(x,\cdot)$ solves an electrostatic problem in the unit cell:

Lemma 5.1. *For almost every $x\in\Omega$, the periodic function $\boldsymbol{u}:=\boldsymbol{E}_0(x,\cdot)$ satisfies in the distributional sense in Y*

$$\operatorname{curl}_y\boldsymbol{u}=0\,,\qquad \operatorname{div}_y\boldsymbol{u}=0\ \text{ in }\ \Omega\times\Sigma^*,\quad\text{and}\quad \boldsymbol{u}=0\ \text{ in }\ \Omega\times\Sigma\,. \tag{5.6}$$

In particular the restriction of $\boldsymbol{E}_0(x,\cdot)$ to Σ^* belongs to $W^{1,2}_\sharp(\Sigma^*;\mathbb{C}^3)$.

Proof. By (5.2) and the first equation in (1.5), we infer that $\eta\operatorname{curl}\boldsymbol{E}_\eta\to 0$ in $L^2(\Omega)$. Thus by applying Proposition 4.2 (ii), we deduce that $\operatorname{curl}_y\boldsymbol{E}_0(x,\cdot)$ vanishes everywhere.

From the second Maxwell equation in (1.5), we deduce that $\operatorname{div}(\varepsilon_\eta\boldsymbol{E}_\eta)=0$ in Ω. Since ε_η is constant in $\Omega\setminus\Sigma_\eta$, it follows in particular that $\eta\operatorname{div}(\boldsymbol{E}_\eta)=0$ in $\Omega\setminus\Sigma_\eta$. Passing to the limit $\eta\to 0$ with the help of assertion (i) of Proposition 4.2 (and of the localized version indicated after the statement), we derive the equality $\operatorname{div}_y\boldsymbol{E}_0=0$ holding in $\mathscr{D}'(Y\setminus\Sigma)$.

Eventually, we observe that the equality $\eta\boldsymbol{J}_\eta=\varepsilon_r\boldsymbol{E}_\eta$ holds in Σ_η. By the bound (5.2), we deduce that $\boldsymbol{E}_\eta\mathbb{1}_{\Sigma_\eta}\to 0$ in $L^2(\Omega)$. Accordingly $\boldsymbol{E}_0(x,\cdot)$ vanishes in Σ. □

We are now able to characterize the set of solutions of the cell problem (5.6).

Proposition 5.2. *The set of solutions of* (5.6) *is a three dimensional vector space spanned by the fields $\boldsymbol{E}^k\in L^2(Y;\mathbb{R}^3)$ defined for $k=1,2,3$ by :*

$$\boldsymbol{E}^k=\boldsymbol{e}_k+\nabla_y\chi_k\,. \tag{5.7}$$

Functions χ_k are the unique elements of $W^{1,2}_\sharp(Y)$ satisfying (2.7)*, that is:*

$$\Delta_y\chi_k=0\quad\text{in}\quad\Sigma^*\qquad\text{and}\qquad\chi_k=-y_k\quad\text{in}\quad\Sigma\,.$$

In particular, $\boldsymbol{E}_0$ can be decomposed as

$$\boldsymbol{E}_0(x,y)=\sum_{k=1}^{3}E_k(x)\,\boldsymbol{E}^k(y)\,, \tag{5.8}$$

being $E_k(x)=\displaystyle\int\boldsymbol{E}_0(x,y)\cdot\boldsymbol{e}_k\,dy$.

Proof. By construction $\boldsymbol{E}_0$ given in (5.8) satisfies (5.6) and the average condition $\int_Y\boldsymbol{E}_0(x,y)\,\mathrm{d}x=\boldsymbol{E}(x)$. More generally, if V denotes the subspace of $W^{1,2}_\sharp(Y)$ consisting of all solutions to (5.6), we see that the linear map $\boldsymbol{u}\in V\mapsto\int_Y\boldsymbol{u}\,dy\in\mathbb{C}^3$ is surjective. The uniqueness as well as the fact that $\dim(V)=3$ will follow if we show that this map is injective. Let $\boldsymbol{u}\in V$ such that $\int_Y\boldsymbol{u}=0$. Then as $\operatorname{curl}\boldsymbol{u}=0$ in Y, there exist $\psi\in W^{1,2}_\sharp(Y)$ such that $\boldsymbol{u}=\nabla\psi$ in Y. Thanks to (5.6), it holds $\Delta\psi=0$ in Σ^* while $\nabla\psi=0$ in Σ. As Σ is connected, we may assume that ψ vanishes in Σ as well as its trace on $\partial\Sigma$. It follows that ψ vanishes everywhere, thus $\boldsymbol{u}=0$. □

Effective permittivity tensor. The functions $\boldsymbol{E}^k(y)$ defined in (5.7) that we call *shape electric fields* depend only on the geometry of Σ. They determine the oscillating behavior of the electric field as well as the effective permittivity tensor $\boldsymbol{\varepsilon}^{\rm eff}$ given in (2.6) (up to the positive real permittivity factor ε_e which represents the permittivity of the matrix). Indeed the tensor given in (2.6) can be written as follows:

$$\boldsymbol{\varepsilon}^{\rm eff}_{kl} := \varepsilon_e \int_Y \boldsymbol{E}^k \cdot \boldsymbol{E}^l \,, \tag{5.9}$$

In particular we see that this tensor is real symmetric, positive-definite and independent on the frequency. More precisely, by using Jensen's inequality, we find that

$$\boldsymbol{\varepsilon}^{\rm eff} z \cdot z \;=\; \varepsilon_e \int_Y |\sum_{k=1}^{3} z_k \, \boldsymbol{E}^k|^2 \;\geq\; \varepsilon_e \, |z|^2 \qquad \text{for all } z \in \mathbb{C}^3 \tag{5.10}$$

As a consequence no specific effect (resonances, dispersion) is expected concerning the permittivity law of the composite structure under study.

5.2. Oscillating magnetic field and displacement current.

In this subsection, we are going to identify the two-scale limits $\boldsymbol{H}_0$ and $\boldsymbol{J}_0$ given in (5.3) in terms of the effective magnetic field $\boldsymbol{H}(x)$ defined in (5.4). We start with the following result:

Lemma 5.3. *For almost all $x \in \Omega$, the periodic vector fields $\boldsymbol{H}_0(x,\cdot)$ and $\boldsymbol{J}_0(x,\cdot)$ satisfy:*

$$\operatorname{div}_y \boldsymbol{H}_0(x,\cdot) \;=\; 0 \quad , \quad \operatorname{curl}_y \boldsymbol{H}_0(x,\cdot) = -i\,\omega\,\varepsilon_0\,\boldsymbol{J}_0(x,\cdot) \quad \text{in } \mathscr{D}'(\mathbb{R}^3)\,, \tag{5.11}$$

$$\boldsymbol{J}_0(x,\cdot) \;=\; 0 \quad \text{a.e. in } \Sigma \quad , \quad \operatorname{curl}_y \boldsymbol{J}_0(x,\cdot) \;=\; i\omega\mu_0\varepsilon_r\,\boldsymbol{H}_0(x,\cdot) \quad \text{in } \mathscr{D}'(\Sigma)\,. \tag{5.12}$$

In particular $\boldsymbol{H}_0(x,\cdot)$ is an element of $W^{1,2}_\sharp(Y,\mathbb{C}^3)$.

Proof. Recalling that $\boldsymbol{H}_\eta$ is divergence-free, the first relation in (5.11) follows from the assertion (i) of Proposition 4.2. Next by the second equation in (1.5) and (5.1), we have

$$\eta \operatorname{curl} \boldsymbol{H}_\eta = -i\omega\varepsilon_0\eta\varepsilon_\eta \boldsymbol{E}_\eta = -i\omega\varepsilon_0 \boldsymbol{J}_\eta\,.$$

By taking the two-scale limit of the left-hand member exploiting the assertion (ii) of Proposition 4.2, we are led to the second equation in (5.11). In particular we find that $\operatorname{curl}\boldsymbol{H}_0(x,\cdot), \operatorname{div}\boldsymbol{H}_0(x,\cdot)$ are elements $L^2_\sharp(Y)$. Therefore, by Lemma 4.7, we conclude that $\boldsymbol{H}_0(x,\cdot)$ belongs to $W^{1,2}_\sharp(Y,\mathbb{C}^3)$.

Next we observe that, by construction, the vector field $\boldsymbol{J}_\eta(x)$ vanishes outside Σ_η, thus obviously $\boldsymbol{J}_0(x,\cdot) = 0$ a.e. in Σ^*. On the other hand, by (1.3) and (5.1), we have $\eta\,\boldsymbol{J}_\eta = \varepsilon_r\,\boldsymbol{E}_\eta$ in Σ_η so that by the first equation in (1.5), it holds

$$\eta \operatorname{curl} \boldsymbol{J}_\eta \;=\; i\,\omega\,\mu_0\,\varepsilon_r\,\boldsymbol{H}_\eta \quad \text{in } \Sigma_\eta\,.$$

Passing to the limit $\eta \to 0$ with the help of assertion (ii) of Proposition 4.2 (see the localization argument after the statement), we derive the equality $\operatorname{curl}_y \boldsymbol{J}_0(x,\cdot) \;=\; i\omega\mu_0\varepsilon_r\,\boldsymbol{H}_0(x,\cdot)$ holding in $\mathscr{D}'(\Sigma)$. The proof of Lemma 5.3 is complete. □

We notice that, by eliminating $\boldsymbol{J}_0$ in relations (5.11) (5.12), we obtain for $\boldsymbol{H}_0$ a system of equations similar to that obtained in (5.6) for $\boldsymbol{E}_0(x,\cdot)$:

$$\operatorname{div}_y \boldsymbol{H}_0 = 0\,, \quad \text{in } Y \qquad \operatorname{curl}_y \boldsymbol{H}_0 = 0 \quad \text{in } \Sigma^* \quad , \quad \Delta_y \boldsymbol{H}_0 + \varepsilon_r\, k_0^2\, \boldsymbol{H}_0 = 0 \quad \text{in } \Sigma \tag{5.13}$$

However, as we have no information on the possible tangential jump of $\boldsymbol{J}_0(x,\cdot)$ across $\partial\Sigma$, it is non straightforward in this case to see that the set of $W^{1,2}_\sharp(Y,\mathbb{C}^3)$ of solutions to (5.13) is still of dimension three or equivalently that the solution to the system is unique for a given average $\boldsymbol{H}(x)$ (see (5.4)). This apparent difficulty is overcome once we show that $\boldsymbol{H}_0(x,\cdot)$ satisfies a suitable variational principle. This is done in the next crucial Lemma.

Lemma 5.4. *Let $\boldsymbol{v} \in W^{1,2}_\sharp(Y,\mathbb{C}^3)$ such that $\operatorname{curl}\boldsymbol{v} = 0$ in Σ^* and $\oint \boldsymbol{v} = 0$. Then for almost all $x \in \Omega$, the periodic field $\boldsymbol{w} = \boldsymbol{H}_0(x,\cdot)$ satisfies the equation*

$$\int_\Sigma \operatorname{curl}\boldsymbol{w} \cdot \operatorname{curl}\boldsymbol{v} - \varepsilon_r\, k_0^2 \int_Y \boldsymbol{w}\cdot\boldsymbol{v} = 0\,. \tag{5.14}$$

Proof. From the Maxwell system (1.5) and taking into account the definition of ε_η in (1.3) and $k_0^2 = \varepsilon_0\mu_0\omega^2$, it is straightforward to deduce that the magnetic field $\boldsymbol{H}_\eta$ satisfies the variational equality

$$\frac{1}{\varepsilon_e}\int_{\Omega\setminus\Sigma_\eta} \operatorname{curl}\boldsymbol{H}_\eta\cdot\operatorname{curl}\boldsymbol{\varphi}_\eta + \frac{1}{\varepsilon_r}\int_{\Sigma_\eta}\eta^2\operatorname{curl}\boldsymbol{H}_\eta\cdot\operatorname{curl}\boldsymbol{\varphi}_\eta \;=\; k_0^2\int_\Omega \boldsymbol{H}_\eta\cdot\boldsymbol{\varphi}_\eta\,. \tag{5.15}$$

holding for every smooth vector field $\boldsymbol{\varphi}_\eta$ compactly supported in Ω.

Let $\rho(x)\in C_c^\infty(\Omega)$ and plug $\boldsymbol{\varphi}_\eta(x) := \rho(x)\boldsymbol{v}(x/\eta)$ as a test function in (5.15) with $\boldsymbol{v}$ satisfying the assumptions of the Lemma. Then, as $\boldsymbol{v}$ is curl-free in Σ^*, it holds

$$\operatorname{curl}(\boldsymbol{\varphi}_\eta) = \nabla\rho(x)\wedge\boldsymbol{v}\Big(\frac{x}{\eta}\Big) + \frac{\rho(x)}{\eta}\,\big(\operatorname{curl}_y\boldsymbol{v}\big)\Big(\frac{x}{\eta}\Big)\,.$$

It follows that we have the following strong two-scale convergences

$$\boldsymbol{\varphi}_\eta \rightarrow\!\!\!\!\rightarrow \rho(x)\,\boldsymbol{v}(y) \quad,\quad \eta\operatorname{curl}(\boldsymbol{\varphi}_\eta) \rightarrow\!\!\!\!\rightarrow \rho(x)\operatorname{curl}_y\boldsymbol{v}\,.$$

Then, by applying (4.3) and recalling that $\boldsymbol{H}_\eta \rightharpoonup\!\!\!\!\rightarrow \boldsymbol{H}_0$ and $\eta\operatorname{curl}\boldsymbol{H}_\eta \rightharpoonup\!\!\!\!\rightarrow \operatorname{curl}_y\boldsymbol{H}_0$ (see Proposition 4.2), we have

$$\lim_{\eta\to0}\int_{\Sigma_\eta}\eta^2\operatorname{curl}\boldsymbol{H}_\eta\cdot\operatorname{curl}\boldsymbol{\varphi}_\eta = \int_{\Omega\times\Sigma}\rho(x)\operatorname{curl}_y\boldsymbol{H}_0\cdot\operatorname{curl}_y\boldsymbol{v} \quad,\quad \lim_{\eta\to0}\int_\Omega \boldsymbol{H}_\eta\cdot\boldsymbol{\varphi}_\eta = \int_{\Omega\times\Sigma}\rho(x)\boldsymbol{H}_0\cdot\boldsymbol{v}\,.$$

In view of (5.15), it is then enough to prove the following claim

$$\lim_{\eta\to0}\int_{\Omega\setminus\Sigma_\eta}\operatorname{curl}\boldsymbol{H}_\eta\cdot\operatorname{curl}\boldsymbol{\varphi}_\eta \;=\; 0\,. \tag{5.16}$$

Indeed thanks to the convergences above, by passing to the limit in (5.15) , we infer that

$$\int_{\Omega\times\Sigma}\rho(x)\operatorname{curl}_y\boldsymbol{H}_0(x,y)\cdot\operatorname{curl}\boldsymbol{v}(y)\,dx\,dy = k_0^2\varepsilon_r\int_{\Omega\times Y}\rho(x)\boldsymbol{H}_0(x,y)\cdot\boldsymbol{v}(y)\,dx\,dy\,,$$

and (5.14) follows by the arbitrariness of the test function $\rho(x)$.

To prove claim (5.16), it is convenient to go back to the electric field $\boldsymbol{e}_\eta$ through the second Maxwell equation in (1.5). As $\varepsilon_\eta = \varepsilon_e$ in $\Omega\setminus\Sigma_\eta$, we have:

$$\begin{aligned}\lim_{\eta\to0}\int_{\Omega\setminus\Sigma_\eta}\operatorname{curl}\boldsymbol{H}_\eta\cdot\operatorname{curl}\boldsymbol{\varphi}_\eta &= \lim_{\eta\to0} -i\omega\varepsilon_0\int_{\Omega\setminus\Sigma_\eta}\boldsymbol{E}_\eta(x)\cdot\nabla\rho(x)\wedge\boldsymbol{v}\Big(\frac{x}{\eta}\Big)\\ &= -i\omega\varepsilon_0\int_{\Omega\times\Sigma^*}\boldsymbol{E}_0(x,y)\wedge\nabla\rho(x)\cdot\boldsymbol{v}(y)\,,\end{aligned}$$

where in the last line we used the weak two-scale convergence $\boldsymbol{E}_\eta \rightharpoonup\!\!\!\!\rightarrow \boldsymbol{E}_0$. Recall that by (5.6), we have $\boldsymbol{E}_0 = 0$ in Σ and $\operatorname{curl}_y\boldsymbol{E}_0 = 0$ in Y. In particular, for a.e. $x\in\Omega$, the periodic vector field $\boldsymbol{f}(y) = \boldsymbol{E}_0(x,\cdot)\wedge\nabla\rho(x)$ is divergence-free and vanishes in Σ. Therefore by applying (4.13), we obtain $\int_Y\boldsymbol{f}(y)\cdot\boldsymbol{v}(y) = (\int_Y\boldsymbol{f})\cdot(\fint\boldsymbol{v}) = 0$. The claim (5.16) follows and the proof of Lemma 5.4 is complete. □

Variational characterization of $\boldsymbol{H}_0$. For fixed $x\in\Omega$, we look for a solution $\boldsymbol{H}_0(x,\cdot)$ to (5.14). Let $\boldsymbol{H}(x) = \fint\boldsymbol{H}_0(x,\cdot)$. By exploiting the two first equations in (5.13), we notice that the vector field $\boldsymbol{u} := \boldsymbol{H}_0(x,\cdot) - \boldsymbol{H}(x)$ belongs to the following subspace of $W^{1,2}_\sharp(Y;\mathbb{C}^3)$

$$X_0^{\mathrm{div}} := \Big\{\boldsymbol{u}\in W^{1,2}_\sharp(Y;\mathbb{C}^3)\,:\,\operatorname{curl}\boldsymbol{u} = 0\text{ in }\Sigma^*\,,\ \operatorname{div}\boldsymbol{u} = 0\text{ in }Y\,,\ \fint\boldsymbol{u} = 0\Big\}\,. \tag{5.17}$$

Accordingly $\boldsymbol{H}_0(x,\cdot)$ has to be searched in $\mathbb{C}^3\oplus X_0^{\mathrm{div}}$. Notice that the previous sum is direct since non-zero constant functions have a non-vanishing circulation vector (see Remark 4.5).

Now we rewrite the equality (5.14) which we intentionally restrict to those elements $\boldsymbol{v}\in X_0^{\mathrm{div}}$ which are divergence-free (see Remark 5.8). With $\boldsymbol{H}_0(x,\cdot) = \boldsymbol{z}+\boldsymbol{u}(y)$, $\boldsymbol{z} = \fint\boldsymbol{H}_0(x,\cdot)$, we find that $\boldsymbol{u}$ solves in X_0^{div} the variational problem

$$b_0(\boldsymbol{u},\boldsymbol{v}) - \varepsilon_r k_0^2\int_Y\boldsymbol{u}\cdot\overline{\boldsymbol{v}} = \varepsilon_r k_0^2\int_Y\boldsymbol{z}\cdot\overline{\boldsymbol{v}} \quad,\quad \forall\boldsymbol{v}\in X_0^{\mathrm{div}}\,. \tag{5.18}$$

where b_0 denotes the Hermitian product

$$b_0(\boldsymbol{u},\boldsymbol{v}) := \int_\Sigma \operatorname{curl}\boldsymbol{u}\cdot\operatorname{curl}\overline{\boldsymbol{v}} \quad \left(= \int_Y \nabla\boldsymbol{u} : \nabla\overline{\boldsymbol{v}}\right). \tag{5.19}$$

It is easy to check that $X_0^{\rm div}$ is a closed subspace of the Hilbert space $W_\sharp^{1,2}(Y;\mathbb{C}^3)$ (thanks to (4.15)) and that b_0 defined above is a scalar product on $X_0^{\rm div}$ which induces an equivalent norm to that $W_\sharp^{1,2}(Y;\mathbb{C}^3)$. Indeed we have

Lemma 5.5. *There exists a constant $c>0$ such that*

$$c\,\|\boldsymbol{u}\|^2_{W_\sharp^{1,2}(Y;\mathbb{C}^3)} \le b_0(\boldsymbol{u},\boldsymbol{u}) \le \|\boldsymbol{u}\|^2_{W_\sharp^{1,2}(Y;\mathbb{C}^3)} \qquad \text{for all } \boldsymbol{u}\in X_0^{\rm div}$$

Proof. In view of (4.17), it is enough to show the existence of a constant $k>0$ such that

$$\forall \boldsymbol{v}\in X_0^{\rm div} \quad , \quad b_0(\boldsymbol{v},\boldsymbol{v}) = \int_Y |\nabla\boldsymbol{v}|^2 \ge k\int_Y |\boldsymbol{v}|^2 . \tag{5.20}$$

By contradiction, assume that (5.20) does not hold. Then we can find a sequence $(\boldsymbol{v}_n)\subset X_0^{\rm div}$ such that $\|\boldsymbol{v}_n\|_{L^2}=1$, $\nabla\boldsymbol{v}_n\to 0$ strongly in $L^2(Y;\mathbb{C}^3)$. Then, by Rellich's Theorem and possibly after extracting a subsequence, such a sequence would converge strongly to some constant function $\boldsymbol{v}$ in $(W_\sharp^{1,2}(Y))^3$ such that $\|\boldsymbol{v}\|_{L^2}=1$. As $X_0^{\rm div}$ is a closed subspace of $W_\sharp^{1,2}(Y;\mathbb{R}^3)$, we need also that this constant function $\boldsymbol{v}$ satisfies $\fint \boldsymbol{v}=0$. This is impossible unless $\boldsymbol{v}=0$. We get a contradiction with the requirement that $\|\boldsymbol{v}\|_{L^2}=1$. □

Lemma 5.6. *Assume that $\Im(\varepsilon_r)>0$. Then, for every $\boldsymbol{z}\in\mathbb{C}^3$, equation* (5.18) *admits a unique solution in $X_0^{\rm div}$.*

Proof. Let β a positive real such that $\beta\,\Im(\varepsilon_r)-\Re(\varepsilon_r)\ge 1$ and let

$$b(\boldsymbol{u},\boldsymbol{v}) := (1+i\beta)\left[b_0(\boldsymbol{u},\boldsymbol{v}) - \varepsilon_r k_0^2\int_Y \boldsymbol{u}\cdot\overline{\boldsymbol{v}}\right].$$

Then (5.18) is equivalent to solving $b(\boldsymbol{u},\boldsymbol{v}) = L_z(v)$ for all $\boldsymbol{v}\in X_0^{\rm div}$, being L_z the linear form on $X_0^{\rm div}$ defined by $L_z(v) = (1+i\beta)\varepsilon_r k_0^2\int_Y \boldsymbol{z}\cdot\overline{\boldsymbol{v}}$. Clearly L_z is continuous as well as $b(\cdot,\cdot)$ as a sequilinear form. On the other hand, b is coercive since, for every $\boldsymbol{u}\in X_0^{\rm div}$, it holds

$$\Re\Big(b(\boldsymbol{u},\boldsymbol{u})\Big) = \int_Y \Big(|\nabla\boldsymbol{u}|^2 + k_0^2\big(\beta\,\Im(\varepsilon_r)-\Re(\varepsilon_r)\big)|\boldsymbol{u}|^2\Big) \ge b_0(\boldsymbol{u},\boldsymbol{u}).$$

The existence and uniqueness of the solution of (5.18) follows from Lax-Milgram Lemma. □

Let us notice that the previous existence and uniqueness still holds if $\Im(\varepsilon_r)=0$ provided $\varepsilon_r k_0^2$ does not belong to the discrete set $\{\lambda_n\}$ defined in (2.12). This a consequence of Fredholm's alternative since the resolvent associated with b_0 as an operator on $L^2_\sharp(Y)$ turns out to be compact (see Section 6).

Shape magnetic fields. By applying Lemma 5.6 to $\boldsymbol{z}=\boldsymbol{e}_k$ for $k=1,2,3$, we obtain three vector fields $\boldsymbol{u}^1,\boldsymbol{u}^2,\boldsymbol{u}^3$ in $X_0^{\rm div}$. We associate the following periodic fields

$$\boldsymbol{H}^k(y) := \boldsymbol{e}_k + \boldsymbol{u}^k(y) \quad , \quad \boldsymbol{J}^k := \operatorname{curl}\boldsymbol{H}^k \qquad k\in\{1,2,3\}. \tag{5.21}$$

By linearity, $\boldsymbol{H}^k$ is characterized as the unique vector field $\boldsymbol{w}\in\mathbb{C}^3\oplus X_0^{\rm div}$ satisfying the variational equation (5.14) such that $\fint\boldsymbol{w}=\boldsymbol{e}_k$. It is now straightforward to deduce

Proposition 5.7. *The family $\{\boldsymbol{H}^1,\boldsymbol{H}^2,\boldsymbol{H}^3\}$ is a basis of solutions for equation* (5.14). *Accordingly the two-scale limits $\boldsymbol{H}_0,\boldsymbol{J}_0$ defined in* (5.3) *are uniquely determined in term of $\boldsymbol{H}(x)$ given in* (5.4) *as follows:*

$$\boldsymbol{H}_0(x,y) = \sum_{k=1}^3 H_k(x)\boldsymbol{H}^k(y) \quad , \quad \boldsymbol{J}_0(x,y) = \sum_{k=1}^3 J_k(x)\boldsymbol{J}^k(y) \qquad \text{a.e. } (x,y)\in\Omega\times Y, \tag{5.22}$$

being $H_k(x)=\boldsymbol{H}(x)\cdot\boldsymbol{e}_k$ and $J_k := -\frac{1}{i\omega\varepsilon_0}H_k$.

Proof. Let V be the subspace of $\mathbb{C}^3 \oplus X_0^{\rm div}$ consisting of all solutions to (5.14). Let $L : V \mapsto \mathbb{C}^3$ the linear map defined by $L(\boldsymbol{w}) := \oint \boldsymbol{w}$. We are reduced to check that this map is bijective. The surjectivity is trivial since, as noticed before, it holds $L(\boldsymbol{H}^k) = \boldsymbol{e}_k$. On the other hand, if $L(\boldsymbol{w}) = 0$, then it means that $\boldsymbol{w} \in X_0^{\rm div}$ and that it satisfies (5.18) for $\boldsymbol{z} = 0$. Thanks to the uniqueness result in Lemma 5.6, we conclude that $\boldsymbol{w} = 0$. □

Remark 5.8. If $\boldsymbol{u} \in X_0^{\rm div}$ solves (5.18), then the variational equality (5.14) is satisfied by $\boldsymbol{w} = \boldsymbol{z} + \boldsymbol{u}$ even if the test function $\boldsymbol{v}$ is not divergence-free. Indeed any admissible $\boldsymbol{v}$ for (5.14) can be decomposed as $\boldsymbol{v} = \tilde{\boldsymbol{v}} + \nabla p$ with $\tilde{\boldsymbol{v}} \in X_0^{\rm div}$ and $p \in W^{1,2}_\sharp(Y)$ solving $\Delta p = \operatorname{div} \boldsymbol{v}$ (as $\int_Y \operatorname{div} \boldsymbol{v} = 0$ such a p exists and it satsifies $\oint \nabla p = 0$). Therefore, by applying (5.18) with $\boldsymbol{u}$ and $\tilde{\boldsymbol{v}}$, we see that $(\boldsymbol{w}, \tilde{\boldsymbol{v}})$ satisfies (5.14). In fact this be also the case for $\boldsymbol{v}$ since $\operatorname{curl} \boldsymbol{v} = \operatorname{curl} \tilde{\boldsymbol{v}}$ and $\int_Y \boldsymbol{w} \cdot \nabla p = 0$ ($\boldsymbol{w}$ is divergence-free like is $\boldsymbol{u}$).

Remark 5.9. Let us stress that it is necessary that Σ^* is *simply-connected* in order to obtain a decomposition of the magnetic field $\boldsymbol{H}_0$ with three independent shape functions like in (5.22). In fact the dimension d of the space of solutions to (5.18), which is $d = 3$ in our case, can increase with the topological complexity of the inclusion Σ. If for instance Σ is a torus, then $d = 4$ as it is shown in [7] where an extra shape function $\boldsymbol{H}^4$ is needed.

5.3. Integral representation and properties of the effective permeability tensor.

As will be checked in Section 6, the shape functions $\boldsymbol{H}^k$ defined in (5.21) coincide with that introduced in (2.17). Therefore, in view of (2.19), the symmetric effective permittivity tensor $\boldsymbol{\mu}^{\rm eff}$ given by (2.13) can be recast componentwise through the following integrals

$$\boldsymbol{\mu}^{\rm eff}_{kl} = \int_Y \boldsymbol{H}^k \cdot \boldsymbol{e}_l \quad , \quad k,l \in \{1,2,3\}\,. \tag{5.23}$$

The dissipativity property of $\boldsymbol{\mu}^{\rm eff}$ (needed for the uniqueness issue of Lemma 2.1) and some other useful relations are given in Lemmas 5.10 and 5.11 below.

Lemma 5.10. *Under* (1.4)*, the real symmetric tensor* $\Im(\boldsymbol{\mu}^{\rm eff}(\omega))$ *is positive definite.*

Proof. Let $\boldsymbol{z} = (z_k) \in \mathbb{R}^3$ and $\boldsymbol{u}^z := \sum_k z_k \boldsymbol{u}^k$ be the unique solution of problem (5.18). By using (5.21) and (5.23), we get

$$\boldsymbol{\mu}^{\rm eff}\boldsymbol{z} \cdot \boldsymbol{z} = \sum_{k,l=1}^{3} z_k z_l \int_Y \boldsymbol{H}^k \cdot \boldsymbol{e}_l \;=\; |\boldsymbol{z}|^2 + \int_Y \boldsymbol{u}^z \cdot \boldsymbol{z}. \tag{5.24}$$

By taking $\boldsymbol{v} = \boldsymbol{u}^z$ in (5.18) and dividing by $\varepsilon_r\, k_0^2$, we infer that

$$\frac{1}{\varepsilon_r\, k_0^2}\, b_0(\boldsymbol{u}^z, \boldsymbol{u}^z) - \int_Y |\boldsymbol{u}^z|^2 \;=\; \int_Y \boldsymbol{z} \cdot \overline{\boldsymbol{u}}^z\,.$$

Passing to the conjugate in the equality above and plugging in (5.24), we are led to:

$$\Im(\boldsymbol{\mu}^{\rm eff})\boldsymbol{z} \cdot \boldsymbol{z} \;=\; \frac{\Im(\varepsilon_r)}{|\varepsilon_r|^2 k_0^2}\, b_0(\boldsymbol{u}^z, \boldsymbol{u}^z) \tag{5.25}$$

holding for every $z \in \mathbb{R}^3$. As b_0 is a scalar product (see Lemma 5.5) and $\Im(\varepsilon_r) > 0$, we conclude that $\Im(\boldsymbol{\mu}^{\rm eff})\boldsymbol{z} \cdot \boldsymbol{z} > 0$ unless $\boldsymbol{u}^z = 0$ which, by looking at equation (5.18), is clearly equivalent to $z = 0$. □

Lemma 5.11. *With the notations of Proposition 5.7, it holds for a.e.* $x \in \Omega$*:*

$$\boldsymbol{\mu}^{\rm eff}\boldsymbol{H}(x) \cdot \overline{\boldsymbol{H}}(x) = \int_Y |\boldsymbol{H}_0(x,\cdot)|^2 - \frac{1}{\varepsilon_r}\frac{\varepsilon_0}{\mu_0} \int_Y |\boldsymbol{J}_0(x,\cdot)|^2\,, \tag{5.26}$$

Proof. We apply Lemma 5.4 with $\boldsymbol{w} = \boldsymbol{H}_0(x,\cdot)$ and $\boldsymbol{v} = \tilde{\boldsymbol{H}}_0(x,\cdot) - \overline{\boldsymbol{H}}(x)$ which by construction has a vanishing circulation vector. Then after dividing by $\varepsilon_r\, k_0^2$ and taking into account (2.19), equation (5.14) becomes:

$$\frac{1}{\varepsilon_r\, k_0^2} \int_Y |\operatorname{curl}_y \boldsymbol{H}_0(x,\cdot)|^2 \;=\; \int_Y \big(|\boldsymbol{H}_0(x,\cdot)|^2 - \boldsymbol{H}_0(x,\cdot) \cdot \overline{\boldsymbol{H}}(x)\big) \;=\; \int_Y |\boldsymbol{H}_0(x,\cdot)|^2 - \boldsymbol{\mu}^{\rm eff}\boldsymbol{H}(x) \cdot \overline{\boldsymbol{H}}(x)\,.$$

Then relation (5.26) follows by taking into account the second equation in (5.11) and relation $k_0^2 = \varepsilon_0 \mu_0 \omega^2$. □

6. Spectral description of the effective permeability tensor

The effective permeability tensor $\boldsymbol{\mu}^{\text{eff}}$ was introduced in Section 2 with two different expressions namely (2.11) and (2.13). The first one is related to a spectral problem (2.10) set on the Hilbert space Z_0 appearing in (2.8) that we recall here:

$$Z_0 := \left\{ \boldsymbol{f} \in L^2(Y;\mathbb{R}^3) \,:\, \operatorname{div} \boldsymbol{f} = 0, \quad \boldsymbol{f} = 0 \text{ in } \Sigma^* \right\}.$$

It turns out that the spectral problem (2.10) is well suited to numerical approximations (see Section 3). The second expression (2.13) is related to another spectral problem (2.14) where the underlying Hilbert space X_0^{div} was constructed in the previous Section (see (5.17))

$$X_0^{\text{div}} := \left\{ \boldsymbol{v} \in W_\sharp^{1,2}(Y;\mathbb{C}^3) \,:\, \operatorname{curl} \boldsymbol{v} = 0 \text{ in } \Sigma^*,\ \operatorname{div} \boldsymbol{v} = 0 \text{ in } Y,\ \oint \boldsymbol{v} = 0 \right\}.$$

The aim of this construction was to represent the periodic magnetic shape functions $\boldsymbol{H}^k$ which intervene in our main Theorem 2.2. It turns out that the entries of tensor $\boldsymbol{\mu}^{\text{eff}}$ can be also recovered from functions $\boldsymbol{H}^k$ by means of relations (5.23).

In this Section we show that the spectral problems (2.10) and (2.14) are well-posed and linked together. Then we deduce that all definitions given for $\boldsymbol{\mu}^{\text{eff}}$ that is (2.11) , (2.13) and (5.23) are in agreement.

6.1. Spectral equivalence.

Keeping the notations of section 2, we associate to every element $\boldsymbol{f} \in Z_0$ the function $\Gamma \boldsymbol{f}$ defined by

$$\Gamma \boldsymbol{f} := \operatorname{curl} \boldsymbol{\psi}_f + \frac{1}{2} \int_\Sigma \boldsymbol{y} \wedge \boldsymbol{f}\,, \tag{6.1}$$

being $\boldsymbol{\psi}_f \in (W_\sharp^{1,2}(Y))^3$ the unique solution of (2.9), that is: $-\Delta \boldsymbol{\psi}_f = \boldsymbol{f}$ in Y and $\int_Y \boldsymbol{\psi}_f = 0$.

Lemma 6.1. *The linear map $\boldsymbol{f} \mapsto \Gamma \boldsymbol{f}$ is a bijective from Z_0 to X_0^{div} and satisfies*

$$\Gamma(\operatorname{curl} \boldsymbol{u}) = \boldsymbol{u} \quad , \quad \operatorname{curl}(\Gamma \boldsymbol{f}) = \boldsymbol{f} \qquad \textit{for all } (\boldsymbol{f}, \boldsymbol{u}) \in Z_0 \times X_0^{\text{div}}\,.$$

In particular, for all pairs $(\boldsymbol{f}, \boldsymbol{g}) \in Z_0^2$ it holds: $b_0(\Gamma \boldsymbol{f}, \Gamma \boldsymbol{g}) = \int_\Sigma \boldsymbol{f}\overline{\boldsymbol{g}}$. Therefore Γ is an isometry between Hilbert space Z_0 endowed with the scalar product of $L^2(Y)$ and Hilbert space X_0^{div} endowed with b_0.

Proof. Let $\boldsymbol{f} \in Z_0$ and let us show that $\Gamma \boldsymbol{f}$ belongs to X_0^{div}. Obviously it is divergence-free. On the other hand $\rho := \operatorname{div} \boldsymbol{\psi}_f$ vanishes as the unique periodic solution of $-\Delta \rho = \operatorname{div} \boldsymbol{f} = 0$ with $\int_Y \rho = 0$. It follows that

$$\operatorname{curl}(\Gamma \boldsymbol{f}) = \operatorname{curl}(\operatorname{curl}(\boldsymbol{\psi}_f) = -\Delta \boldsymbol{\psi}_f = \boldsymbol{f}\,. \tag{6.2}$$

form which follows in particular that $\operatorname{curl} \Gamma \boldsymbol{f} = 0$ in Σ^*. It remains to check the more tricky part, i.e. that $\oint \Gamma \boldsymbol{f} = 0$. Let $\boldsymbol{z} \in \mathbb{C}^3$ be arbitrary and denote by $\boldsymbol{\varphi}_z$ a function in $(W_\sharp^{1,2}(Y))^3$ such that $\boldsymbol{\varphi}_z(\boldsymbol{y}) = -\frac{1}{2}\boldsymbol{z} \wedge \boldsymbol{y}$ in Σ. Then by construction the periodic vector field $\boldsymbol{g} := \operatorname{curl} \boldsymbol{\varphi}_z + \boldsymbol{z}$ is divergence-free, vanishes in Σ and satisfies $\int_Y \boldsymbol{g} = \boldsymbol{z}$. In view of the characterization (4.13), one has therefore

$$\int_Y \Gamma \boldsymbol{f} \cdot \boldsymbol{g} \;=\; \boldsymbol{z} \cdot \oint \Gamma \boldsymbol{f}\,. \tag{6.3}$$

By integrating by parts and taking into account that $\operatorname{curl}(\operatorname{curl} \boldsymbol{\psi}_f) = \boldsymbol{f}$, one gets

$$\int_Y \operatorname{curl} \boldsymbol{\psi}_f \cdot \boldsymbol{g} = \int_Y \operatorname{curl} \boldsymbol{\psi}_f \cdot \operatorname{curl} \boldsymbol{\varphi}_z = \int_Y \boldsymbol{f} \cdot \boldsymbol{\varphi}_z = -\frac{1}{2} \boldsymbol{z} \cdot \int_\Sigma \boldsymbol{y} \wedge \boldsymbol{f}\,.$$

Then recalling definition (6.1), we deduce that $\int_Y \Gamma \boldsymbol{f} \cdot \boldsymbol{g} = 0$. Thus, by applying (6.3), we are led to $\oint \Gamma \boldsymbol{f} = 0$ and $\Gamma \boldsymbol{f} \in X_0^{\text{div}}$.

In view of (6.2), the Lemma is proved once we have checked that the relation $\Gamma(\mathrm{curl}\,\boldsymbol{u})=\boldsymbol{u}$ holds for every $\boldsymbol{u}\in X_0^{\mathrm{div}}$. We observe that, for such $\boldsymbol{u}$, the function $\mathrm{curl}\,\boldsymbol{u}$ is divergence-free and vanishes in Σ^*, thus belongs to Z_0. Let $\boldsymbol{v}:=\Gamma(\mathrm{curl}\,\boldsymbol{u})$. By applying (6.2) to $\boldsymbol{f}=\mathrm{curl}\,\boldsymbol{u}$, we get the equality $\mathrm{curl}\,\boldsymbol{u}=\mathrm{curl}\,\boldsymbol{v}$ from which follows $\boldsymbol{u}=\boldsymbol{v}$ (since $b_0(\boldsymbol{u}-\boldsymbol{v},\boldsymbol{u}-\boldsymbol{v})=0$). □

Next we intoduce the sesquilinear form on Z_0 defined by

$$a_0(\boldsymbol{f},\boldsymbol{g}):=\int_Y \Gamma\boldsymbol{f}\cdot\overline{\Gamma\boldsymbol{g}}=\int_Y \nabla\boldsymbol{\psi}_f:\overline{\nabla\boldsymbol{\psi}_g}+\frac{1}{4}\left(\int_\Sigma \boldsymbol{y}\wedge\boldsymbol{f}\right)\cdot\left(\int_\Sigma \boldsymbol{y}\wedge\overline{\boldsymbol{g}}\right), \tag{6.4}$$

where in the second equality we used definition (6.1), the fact that the periodic fields $\boldsymbol{\psi}_f,\boldsymbol{\psi}_g$ are divergence-free (see the proof above) and that their curl have vanishing bulk average over Y. It is straightforward from Lemma 6.1 that a_0 is a scalar product on Z_0. Moreover it holds

$$a_0(\boldsymbol{f},\boldsymbol{f})\le\frac{1}{k}\|\boldsymbol{f}\|^2_{L^2(\Sigma)}, \tag{6.5}$$

where k is the positive constant appearing in (5.20). The spectral problem (2.10) amounts to finding $(\boldsymbol{f},\alpha)\in Z_0\times\mathbb{R}_+^*$ such that:

$$a_0(\boldsymbol{f},\boldsymbol{g})=\alpha\int_Y \boldsymbol{f}\cdot\overline{\boldsymbol{g}},\quad\forall\boldsymbol{g}\in Z_0\,. \tag{6.6}$$

Then, in relation with the spectral problem (2.14), we obtain the following equivalence principle holding for every $(\boldsymbol{u},\lambda)\in X_0^{\mathrm{div}}\times\mathbb{R}^+$:

$$b_0(\boldsymbol{u},\boldsymbol{v})=\lambda\int_Y \boldsymbol{u}\cdot\overline{\boldsymbol{v}},\quad\forall\boldsymbol{v}\in X_0^{\mathrm{div}}\quad\Longleftrightarrow\quad(\boldsymbol{f},\alpha)=(\mathrm{curl}\,\boldsymbol{u},\frac{1}{\lambda})\quad\text{solves}\quad(6.6) \tag{6.7}$$

6.2. Reduction to the diagonal form.
In view of the equivalence (6.7), we may limit ourselves to the study of the spectral problem (6.6). Since by (6.5) the hermitian product a_0 is continuous, we may consider the bounded linear operator $K:Z_0\mapsto Z_0$ such that

$$\langle K\boldsymbol{f},\boldsymbol{g}\rangle_{L^2(Y)}=a_0(\boldsymbol{f},\boldsymbol{g}),\qquad\text{for all }(\boldsymbol{f},\boldsymbol{g})\in Z_0\,. \tag{6.8}$$

We also introduce $P_0:L^2(Y;\mathbb{R}^3)\mapsto Z_0$ the orthogonal projector on Z_0 and the finite-rank operator $M:Z_0\mapsto L^2(Y;\mathbb{R}^3)$ defined for every $\boldsymbol{f}\in Z_0$ by setting

$$M\boldsymbol{f}(z)=\frac{1_\Sigma(z)}{4}\left(\int_\Sigma \boldsymbol{y}\wedge\boldsymbol{f}\right)\wedge\boldsymbol{z}\quad\forall z\in Y\,. \tag{6.9}$$

Lemma 6.2. *With the notations above the linear operator $K:Z_0\to Z_0$ is compact positive and self-adjoint. For all $\boldsymbol{f}\in Z_0$, it holds*

$$K\boldsymbol{f}=P_0\left(\boldsymbol{\psi}_f+M\boldsymbol{f}\right), \tag{6.10}$$

Proof. Since a_0 is a continuous scalar product, K is a bounded positive self-adjoint operator on Z_0. Let us take $\boldsymbol{f}\in Z_0$ and establish the relation (6.10). In view of (6.8), It is enough to show that for every element $\boldsymbol{g}\in Z_0$, it holds:

$$a_0(\boldsymbol{f},\boldsymbol{g})=\int_Y(\boldsymbol{\psi}_f+M\boldsymbol{f})\cdot\overline{\boldsymbol{g}}\,. \tag{6.11}$$

By (6.9) one has $\int_Y M\boldsymbol{f}\cdot\overline{\boldsymbol{g}}=\frac{1}{4}\left(\int_\Sigma\boldsymbol{y}\wedge\boldsymbol{f}\right)\cdot\left(\int_\Sigma\boldsymbol{y}\wedge\overline{\boldsymbol{g}}\right)$. On the other hand, by integrating by parts and taking into account that $\boldsymbol{\psi}_g$ satisfies $-\Delta\boldsymbol{\psi}_g=\boldsymbol{g}$, we obtain $\int_Y\boldsymbol{\psi}_f\cdot\overline{\boldsymbol{g}}=\int_Y\nabla\boldsymbol{\psi}_f:\overline{\nabla\boldsymbol{\psi}_g}$. Recalling the expression on the right-hand side appearing in the definition (6.4) of a_0, we are led to (6.11). Thus (6.10) is established. The compactness property of operator K is then straightforward. Indeed, let $(\boldsymbol{f}_n)$ be a weakly convergent sequence of Z_0. Then $\boldsymbol{\psi}_{f_n}$ is bounded $(W^{1,2}_\sharp(Y))^3$ hence strongly convergent in $L^2(Y)^3$. It is also the case of $M\boldsymbol{f}_n$ since M is finite rank. The conclusion follows by the continuity of P_0 as an operator from $L^2(Y)^3$ to Z_0. □

As a consequence of Lemma 6.2 and of the equivalent principle (6.7), we have the following result:

Lemma 6.3. *There exists a sequence of positive numbers such that $\alpha_0 \geq \alpha_1 \geq \cdots \geq \alpha_n$ with $\alpha_n \to 0$ and an orthonormal basis $\{\boldsymbol{f}_n\}_{n\in\mathbb{N}}$ of Z_0 consisting of real valued functions such that*

$$a_0(\boldsymbol{f}_n, \boldsymbol{g}) = \alpha_n \int_Y \boldsymbol{f}_n \cdot \overline{\boldsymbol{g}}\,, \qquad \forall n \in \mathbb{N} \quad , \quad \forall \boldsymbol{g} \in Z_0\,. \tag{6.12}$$

Moreover, by setting for every $n \in \mathbb{N}$

$$\boldsymbol{u}_n = \frac{1}{\sqrt{\alpha_n}}\,\Gamma \boldsymbol{f}_n\,, \qquad \lambda_n = \frac{1}{\alpha_n}\,, \tag{6.13}$$

we obtain an orthogonal basis $\{\boldsymbol{u}_n\}$ of real valued functions in $X_0^{\rm div}$ endowed with the scalar product b_0 such that for all $n \in \mathbb{N}$:

$$b_0(\boldsymbol{u}_n, \boldsymbol{v}) = \lambda_n \int_Y \boldsymbol{u}_n \cdot \overline{\boldsymbol{v}} \qquad \forall \boldsymbol{v} \in X_0^{\rm div} \quad , \quad \int_Y \boldsymbol{u}_n \cdot \boldsymbol{u}_m = \delta_{nm}. \tag{6.14}$$

Proof. In view of the compactness property obtained in Lemma 6.2, the existence of the pairs $(\boldsymbol{f}_n, \alpha_n)$ satisfying (6.12) is obvious. As $\overline{K}\,\boldsymbol{f} = K\overline{\boldsymbol{f}}$ for every $\boldsymbol{f} \in Z_0$, we can chose the eigenvectors $\boldsymbol{f}_n$ to be real. Then we deduce (6.14) by applying the equivalence (6.7) to $\boldsymbol{u}_n$ given in (6.13). On the other hand, by the isometry property in Lemma 6.1, we know that $\{\Gamma \boldsymbol{f}_n\}$ is orthonormal basis of $(X_0^{\rm div}, b_0)$. Thus $\{\boldsymbol{u}_n\}_{n\in\mathbb{N}}$ is an orthogonal basis of eigenvectors normalized so that $b_0(\boldsymbol{u}_n, \boldsymbol{u}_n) = \lambda_n$ and $\int_Y |\boldsymbol{u}_n|^2 = 1$. □

6.3. Power series representation of the effective permeability.

We are now in a position to express the effective permeability tensor $\boldsymbol{\mu}^{\rm eff}$ given in (5.23) in terms of a power series expansion. This is done by projecting the solution of the variational equation (5.18) on the orthogonal basis $\{\boldsymbol{u}_n\}_{n\in\mathbb{N}}$. As a byproduct we recover representation formulas (2.11) , (2.13) making explicit the dependence of $\boldsymbol{\mu}^{\rm eff}$ with respect to the incident wave number $k_0 = \omega\sqrt{\varepsilon_0\mu_0}$.

Proposition 6.4. *Let $(\lambda_n, \boldsymbol{u}_n)$ as in Lemma 6.3. Then the magnetic shape functions $\boldsymbol{H}^k$ introduced in* (5.21) *are given by*

$$\boldsymbol{H}^k \;=\; \boldsymbol{e}_k + \sum_{n\in\mathbb{N}} \langle \boldsymbol{e}_k, \boldsymbol{u}_n\rangle \frac{\varepsilon_r k_0^2}{\lambda_n - \varepsilon_r k_0^2}\,\boldsymbol{u}_n \quad , \quad k \in \{1,2,3\}\,. \tag{6.15}$$

Accordingly the tensor $\boldsymbol{\mu}^{\rm eff}$ defined in (5.23) *admits the following representation:*

$$\boldsymbol{\mu}^{\rm eff}_{ij}(k_0) \;=\; \delta_{ij} + \sum_n \frac{\varepsilon_r k_0^2}{\lambda_n - \varepsilon_r k_0^2} \left(\int_Y \boldsymbol{u}_n \cdot \boldsymbol{e}_i\right)\left(\int_Y \boldsymbol{u}_n \cdot \boldsymbol{e}_j\right). \tag{6.16}$$

or alternatively in terms of $(\alpha_n, \boldsymbol{f}_n)$ by

$$\boldsymbol{\mu}^{\rm eff}_{ij}(k_0) \;=\; \delta_{ij} + \frac{1}{4}\sum_n \frac{\varepsilon_r k_0^2}{1 - \varepsilon_r \alpha_n k_0^2} \left(\int_\Sigma \boldsymbol{y}\wedge \boldsymbol{f}_n\right)_i \left(\int_\Sigma \boldsymbol{y}\wedge \boldsymbol{f}_n\right)_j . \tag{6.17}$$

Proof. In view of (5.23) and (6.13), the two series expansions for $\boldsymbol{\mu}^{\rm eff}$ follow straightforward from (6.15). Following (5.21), for $k \in \{1,2,3\}$, we write $\boldsymbol{H}^k = \boldsymbol{e}_k + \boldsymbol{u}^k$ where $\boldsymbol{u}^k$ is the unique element of $X_0^{\rm div}$ solving (5.18) for $z = \boldsymbol{e}_k$ that is

$$b_0(\boldsymbol{u}^k, \boldsymbol{v}) - \varepsilon_r k_0^2 \int_Y \boldsymbol{u}^k \cdot \overline{\boldsymbol{v}} \;=\; \varepsilon_r k_0^2 \int_Y \boldsymbol{e}_k \cdot \overline{\boldsymbol{v}} \quad , \quad \forall \boldsymbol{v} \in X_0^{\rm div}\,. \tag{6.18}$$

As $\{\boldsymbol{u}_n\}$ is an orthogonal basis of $X_0^{\rm div}$, we may expand $\boldsymbol{u}^k$ as

$$\boldsymbol{u}^k = \sum_{n\in\mathbb{N}} c_n^k\,\boldsymbol{u}_n \quad , \quad \text{where} \quad c_n^k = \int_Y \boldsymbol{u}^k \cdot \boldsymbol{u}_n = \frac{1}{\lambda_n}\, b_0(\boldsymbol{u}^k, \boldsymbol{u}_n)\,. \tag{6.19}$$

Note that the convergence of the series above holds with respect to the Sobolev norm of $W^{1,2}_\sharp(Y)$, thus also in $L^2(Y)$. Plugging (6.19) in (6.18) where we take $\boldsymbol{v} = \boldsymbol{u}_n$, we obtain the relations

$$(\lambda_n - \varepsilon_r k_0^2)\, c_n^k \;=\; \varepsilon_r k_0^2 \int_Y \boldsymbol{e}_k \cdot \boldsymbol{u}_n \quad , \quad \forall n \in \mathbb{N}\,,$$

from which follows (6.15). □

7. PROOF OF THE MAIN RESULT

This section is devoted to the proof of Theorem 2.2. We will proceed according to the following steps:

- In a first step, we work under the following energy bound which will be proven *a posteriori* in a second step.
$$\sup_{\eta>0} \int_{B_R} |\boldsymbol{E}_\eta|^2 + |\boldsymbol{H}_\eta|^2 < +\infty \,, \tag{7.1}$$
where B_R is an open ball containing $\overline{\Omega}$.
  - In Proposition 7.1 we start by establishing the uniform convergence of a subsequence of $(\boldsymbol{E}_\eta, \boldsymbol{H}_\eta)$ in every compact subset of $\mathbb{R}^3 \setminus \Omega$ from which follows the convergence of the flux of Poynting vectors and a uniform L^2 estimate for the rescaled displacement current $\boldsymbol{J}_\eta$ (see Lemma 7.3). This estimate together with hypothesis (7.1) legitimates, up to extracting a subsequence, the two-scale analysis of the triple $(\boldsymbol{E}_\eta, \boldsymbol{H}_\eta, \boldsymbol{J}_\eta)$ performed in Section 5.
  - In Proposition 7.3, we establish that the *effective* electromagnetic field $(\boldsymbol{E}, \boldsymbol{H})$ deduced from the analysis of Subsections 5.1 and 5.2 and of Proposition 7.1 satisfies the limit diffraction problem given in (2.1). Note that, thanks to the results of Section 6, the tensor $\boldsymbol{\mu}^{\text{eff}}$ defined therein (see (2.11) or (2.13)) agrees with the expression obtained in (5.23). On the other hand the uniqueness of the solution to (2.1) (see Lemma 2.1) ensures that the whole sequence $(\boldsymbol{E}_\eta, \boldsymbol{H}_\eta, \boldsymbol{J}_\eta)$ is weakly two-scale convergent.
  - In the crucial Subsection 7.3, we improve the convergence above and establish the *strong two-scale* convergence of the triple $(\boldsymbol{E}_\eta, \boldsymbol{H}_\eta, \boldsymbol{J}_\eta)$.
- In a second Step (Lemma 7.7), we establish the bound (7.1) by means of a contradiction argument in which we use once again the *uniqueness* of the solution of the limit problem and the *strong two-scale* convergence established in the first step.

7.1. Behavior far from the obstacle and energy estimate.

This first step is related to the convergence of $(\boldsymbol{E}_\eta, \boldsymbol{H}_\eta)$ at a positive distance from Ω. We use the following result whose proof relies on the hypo-ellipticity of the operator $\Delta + k_0$ (ses for instance [6, Lemma 2.1].

Proposition 7.1. *Let $(\boldsymbol{E}_\eta, \boldsymbol{H}_\eta)$ be the solution of problem* (1.5) *associated to an incident wave $(\boldsymbol{E}_\eta^{\text{inc}}, \boldsymbol{H}_\eta^{\text{inc}})$ such that $(\boldsymbol{E}_\eta^{\text{inc}}, \boldsymbol{H}_\eta^{\text{inc}}) \to (\boldsymbol{E}^{\text{inc}}, \boldsymbol{H}^{\text{inc}})$ uniformly as $\eta \to 0$. If $(\boldsymbol{E}_\eta, \boldsymbol{H}_\eta) \rightharpoonup (\boldsymbol{E}, \boldsymbol{H})$ weakly in $(L^2(B_R \setminus \Omega))^3$, then we can extend $\boldsymbol{E}$ and $\boldsymbol{H}$ to $\mathbb{R}^3 \setminus \Omega$ so that both satisfy the Helmholtz equation $\Delta \boldsymbol{u} + k_0^2 \boldsymbol{u} = 0$ in $\mathbb{R}^3 \setminus \overline{\Omega}$. Moreover the convergence $(\boldsymbol{E}_\eta, \boldsymbol{H}_\eta) \to (\boldsymbol{E}, \boldsymbol{H})$ holds in $C^\infty(K)$ for all compact $K \subset\subset \mathbb{R}^3 \setminus \overline{\Omega}$ while $(\boldsymbol{E} - \boldsymbol{E}^{\text{inc}}, \boldsymbol{H} - \boldsymbol{H}^{\text{inc}})$ satisfies Silver Müler condition* (1.6) *with respect to $(\boldsymbol{E}^{\text{inc}}, \boldsymbol{H}^{\text{inc}})$.*

A first consequence of the previous result is the boundedness (see Lemma below) of the flux of the Poynting vector on ∂B_R given by

$$\mathscr{P}_\eta := \int_{\partial B_R} \boldsymbol{E}_\eta \wedge \overline{\boldsymbol{H}}_\eta \cdot \boldsymbol{n} \, d\sigma \,. \tag{7.2}$$

It is useful to rewrite $\mathscr{P}_\eta$ as a bulk integral. Integrating by parts with the help of the identity

$$\operatorname{div}\big(\boldsymbol{E}_\eta \wedge \overline{\boldsymbol{H}}_\eta\big) = \operatorname{curl} \boldsymbol{E}_\eta \cdot \overline{\boldsymbol{H}}_\eta - \operatorname{curl} \overline{\boldsymbol{H}}_\eta \cdot \boldsymbol{E}_\eta$$

and taking into account equations (1.5), we obtain:

$$\mathscr{P}_\eta = \int_{B_R} (\operatorname{curl} \boldsymbol{E}_\eta \cdot \overline{\boldsymbol{H}}_\eta - \operatorname{curl} \overline{\boldsymbol{H}}_\eta \cdot \boldsymbol{E}_\eta) = i\omega \int_{B_R} \left(\mu_0 |\boldsymbol{H}_\eta|^2 - \varepsilon_0 \overline{\varepsilon}_\eta |\boldsymbol{E}_\eta|^2\right) . \tag{7.3}$$

The real part of $\mathscr{P}_\eta$ is independent of the radius R and represents the energy dissipated by the obstacle. Taking into account the definitions (1.3) (5.1), this dissipation can be recast in terms of

the displacement current $\boldsymbol{J}_\eta$ as follows

$$\Re\big(\mathscr{P}_\eta\big) = -\omega\,\varepsilon_0\,\frac{\Im(\varepsilon_r)}{|\varepsilon_r|^2}\int_{\Sigma_\eta}|\boldsymbol{J}_\eta|^2\,. \tag{7.4}$$

Lemma 7.2. *Let $(\boldsymbol{E}_\eta,\boldsymbol{H}_\eta)$ satisfy the uniform upper bound* (7.1) *and solve problem* (1.5)*. Then with $\mathscr{P}_\eta$ defined by* (7.2) *and $\boldsymbol{J}_\eta$ defined in* (5.1)*, it holds*

$$\sup_\eta\big|\mathscr{P}_\eta\big|<+\infty\quad,\quad\sup_{\eta>0}\|\boldsymbol{J}_\eta\|_{L^2(B_R)}<+\infty\,. \tag{7.5}$$

Proof. Up to passing to a subsequence, we can assume that $\limsup_\eta\big|\mathscr{P}_\eta\big|=\lim_\eta\big|\mathscr{P}_\eta\big|$ and, by the uniform upper bound (7.1), that $(\boldsymbol{E}_\eta,\boldsymbol{H}_\eta)$ does converge weakly in $L^2(B_R)$. We may therefore apply the convergence in $C^\infty(K)$ obtained in Proposition 7.1 with K containing a neighbourhood of ∂B_R. This shows that $\limsup_\eta\big|\mathscr{P}_\eta\big|<+\infty$. Then under (1.4), the second upper bound in (7.5) follows directly from (7.4) (notice that in view of (7.1) and (7.3), we can reach the same conclusion assuming merely that $\varepsilon_r\neq 0$). □

7.2. Weak convergence to the solution of the homogenized problem.

Let $\boldsymbol{\varepsilon},\boldsymbol{\mu}$ be defined by (2.2), (2.6), (2.13). Let $(\boldsymbol{E},\boldsymbol{H})$ be the unique solution of (2.1) and let B_R be any ball such that $\Omega\subset\subset B_R$. We associate the fields $\boldsymbol{E}_0,\boldsymbol{H}_0$ defined on $B_R\times Y$ by (2.15)(2.16) as well as the field $\boldsymbol{J}_0$ determined in Lemma 5.3. The flux of the Poynting vector through ∂B_R (already used in the proof of Lemma 2.1) will be denoted

$$\mathscr{P}(=\mathscr{P}(R)):=\int_{\partial B_R}\boldsymbol{E}\wedge\overline{\boldsymbol{H}}\cdot\boldsymbol{n}\,d\sigma\,. \tag{7.6}$$

Proposition 7.3.
Let us assume that the upper bound (7.1) *holds and let $(\boldsymbol{E}_\eta,\boldsymbol{H}_\eta)$ be the solution of the diffraction problem* (1.5) *and $\boldsymbol{J}_\eta$ given by* (5.1)*. Then we have the weak two-scale convergence of the triple $(\boldsymbol{E}_\eta,\boldsymbol{H}_\eta,\boldsymbol{J}_\eta)\rightharpoonup(\boldsymbol{E}_0,\boldsymbol{H}_0,\boldsymbol{J}_0)$ in B_R whereas $(\boldsymbol{E}_\eta,\boldsymbol{H}_\eta)\to(\boldsymbol{E},\boldsymbol{H})$ in $C^\infty(K)$ for every compact subset $K\subset\subset\mathbb{R}^3\setminus\Omega$. In particular, it holds $\mathscr{P}_\eta\to\mathscr{P}$ as $\eta\to 0$.*

Proof. Under the assumption (7.1) and thanks to Lemma 7.2 we may, up to passing to a subsequence, assume that the convergences (5.3) hold for suitable fields $\hat{\boldsymbol{E}}_0,\hat{\boldsymbol{H}}_0,\hat{\boldsymbol{J}}_0$ belonging to $L^2(B_R\times Y)$, namely

$$\boldsymbol{E}_\eta\rightharpoonup\hat{\boldsymbol{E}}_0\,,\qquad\boldsymbol{H}_\eta\rightharpoonup\hat{\boldsymbol{H}}_0\quad\text{and}\quad\boldsymbol{J}_\eta\rightharpoonup\hat{\boldsymbol{J}}_0\,. \tag{7.7}$$

We are done if we show that the triple $(\hat{\boldsymbol{E}}_0,\hat{\boldsymbol{H}}_0,\hat{\boldsymbol{J}}_0)$ agrees with the triple $(\boldsymbol{E}_0,\boldsymbol{H}_0,\boldsymbol{J}_0)$ appearing in our statement (i.e. (2.15) , (2.16) and (5.11) for $\boldsymbol{J}_0$). Let us define:

$$\hat{\boldsymbol{E}}(x)=\int_Y\hat{\boldsymbol{E}}_0(x,\cdot)\,,\qquad\hat{\boldsymbol{H}}(x)=\fint_Y\hat{\boldsymbol{H}}_0(x,\cdot)\quad\text{for } x\in\Omega\,. \tag{7.8}$$

By applying Propositions 5.2 and 5.7, we infer that

$$\hat{\boldsymbol{E}}_0(x,y)=\begin{cases}\hat{\boldsymbol{E}}(x) & \text{in } (B_R\setminus\Omega)\times Y\\ \displaystyle\sum_{k=1}^3(\hat{\boldsymbol{E}}(x)\cdot\boldsymbol{e}_k)\boldsymbol{E}^k(y) & \text{in } \Omega\times Y\end{cases}\quad,\quad\hat{\boldsymbol{H}}_0(x,y)=\begin{cases}\hat{\boldsymbol{H}}(x) & \text{in } (B_R\setminus\Omega)\times Y\\ \displaystyle\sum_{k=1}^3(\hat{\boldsymbol{H}}(x)\cdot\boldsymbol{e}_k)\boldsymbol{H}^k(y) & \text{in } \Omega\times Y\end{cases} \tag{7.9}$$

being $\boldsymbol{E}^k,\boldsymbol{H}^k$ the periodic vector fields introduced in (5.7) and (5.21) respectively. Accordingly we have that $(\boldsymbol{E}_\eta,\boldsymbol{H}_\eta)\to(\hat{\boldsymbol{E}},\hat{\boldsymbol{H}})$ weakly in $L^2(B_R\setminus\Omega)$. Then, by applying Proposition 7.1 with a constant incident wave $(\boldsymbol{E}^{\rm inc},\boldsymbol{H}^{\rm inc})$, we may extend $(\hat{\boldsymbol{E}},\hat{\boldsymbol{H}})$ to all $\mathbb{R}^3\setminus B_R$ so that it satisfies radiation condition (1.6) and Maxwell system in the vacuum outside Ω. Furthermore $(\boldsymbol{E}_\eta,\boldsymbol{H}_\eta)\to(\hat{\boldsymbol{E}},\hat{\boldsymbol{H}})$ in $C^\infty(K)$ for every compact subset of $\mathbb{R}^3\setminus\Omega$. In particular it follows that $\mathscr{P}_\eta\to\hat{\mathscr{P}}:=\int_{\partial B_R}\hat{\boldsymbol{E}}\wedge\overline{\hat{\boldsymbol{H}}}\cdot\boldsymbol{n}\,d\sigma$.

Summarizing we see that all conclusions of Proposition 7.3 hold, provided we can establish that $(\hat{\boldsymbol{E}},\hat{\boldsymbol{H}})$ agrees with the solution $(\boldsymbol{E},\boldsymbol{H})$ to (2.1). As by Lemma 2.1 this solution is unique, we are thus reduced to check that $(\hat{\boldsymbol{E}},\hat{\boldsymbol{H}})$ satisfies the two first equations in (2.1). In fact, as we know already that $(\hat{\boldsymbol{E}},\hat{\boldsymbol{H}})$ satisfies these equations outside Ω, we may restrict ourselves to B_R.

The first equation is readily derived by passing to the distributional limit in the first equation of (1.5). Indeed the weak limits in $L^2(B_R)$ of $\boldsymbol{E}_\eta, \boldsymbol{H}_\eta$ agree with their bulk averages. In view of (7.8) and (5.23), there are given respectively by:

$$\int_Y \hat{\boldsymbol{E}}_0(\cdot, y)\, dy = \hat{\boldsymbol{E}}(x) \quad , \quad \int_Y \hat{\boldsymbol{H}}_0(\cdot, y)\, dy \;=\; \boldsymbol{\mu}\hat{\boldsymbol{H}}\,.$$

Thus we are led to

$$\operatorname{curl} \hat{\boldsymbol{E}} \;=\; i\omega\mu_0\, \boldsymbol{\mu}\hat{\boldsymbol{H}} \quad \text{in } \mathscr{D}'(B_R)\,.$$

In order to check the second equation, we consider $\varphi \in C_c^\infty(B_R)$ and, for fixed $k \in \{1,2,3\}$, the shape function $\boldsymbol{E}^k$ defined in (5.7). Multiplying the second equation of (1.5) by $\varphi(x)\boldsymbol{E}^k(x/\eta)$ and integrating by parts the left-hand side (recall that $\operatorname{curl}_y \boldsymbol{E}^k = 0$), we get

$$\int_{B_R} \boldsymbol{H}_\eta(x)\cdot\left[\nabla\phi(x)\wedge \boldsymbol{E}^k\left(\frac{x}{\eta}\right)\right] = -i\omega\varepsilon_0 \int_{B_R} \varphi(x)\boldsymbol{E}_\eta(x)\cdot \boldsymbol{E}^k\left(\frac{x}{\eta}\right).$$

By exploiting (7.7), the strong two-scale convergence of $\boldsymbol{E}^k(x/\eta)$ and the product rule (4.3) (and comments below), we may pass to the limit in the equality above:

$$\int_{B_R\times Y} \hat{\boldsymbol{H}}_0(x,y)\cdot[\nabla\varphi(x)\wedge \boldsymbol{E}^k(y)] \;=\; -i\omega\varepsilon_0\varepsilon_e \int_{B_R\times Y} \varphi(x)\hat{\boldsymbol{E}}_0(x,y)\cdot \boldsymbol{E}^k(y)\,. \tag{7.10}$$

Next we observe that $\hat{\boldsymbol{H}}_0(x,\cdot)$ is curl-free in Σ^* and $\nabla\varphi(x)\wedge \boldsymbol{E}^k$ is divergence-free and vanishes in Σ. Therefore, thanks to (4.13) and taking into account (7.8), we are led to

$$\int_Y \hat{\boldsymbol{H}}_0(x,y)\cdot[\nabla\varphi(x)\wedge \boldsymbol{E}^k(y)]\, dy = \hat{\boldsymbol{H}}(x)\cdot[\nabla\varphi(x)\wedge \boldsymbol{e}_k].$$

On the other hand, in view of definition (2.2) and (5.9), we have for a.e. $x \in B_R$:

$$\varepsilon_e \int_Y \hat{\boldsymbol{E}}_0(x,\cdot)\cdot \boldsymbol{E}^k = \boldsymbol{\varepsilon}(x)\hat{\boldsymbol{E}}(x)\cdot \boldsymbol{e}_k\,.$$

Then by applying Fubini's formula to the left and right sides of (7.10), we are led to

$$\int_{B_R} \hat{\boldsymbol{H}}(x)\cdot[\nabla\varphi(x)\wedge \boldsymbol{e}_k] \;=\; -i\omega\varepsilon_0 \int_{B_R} \varphi(x)\boldsymbol{\varepsilon}(x)\hat{\boldsymbol{E}}(x)\cdot \boldsymbol{e}_k\,. \tag{7.11}$$

Eventually, thanks to the identity $\operatorname{div}(e_k\wedge\hat{\boldsymbol{H}}) = -\operatorname{curl}\hat{\boldsymbol{H}}\cdot e_k$ and by integrating by parts the left-hand side of (7.11), we infer the following equality for all $k \in \{1,2,3\}$ and every test function $\varphi \in C_c^\infty(B_R)$:

$$\int_{B_R} \operatorname{curl}\hat{\boldsymbol{H}}\cdot \boldsymbol{e}_k\,\varphi \;=\; -i\omega\varepsilon_0 \int_{B_R} \boldsymbol{\varepsilon}\hat{\boldsymbol{E}}\cdot \boldsymbol{e}_k\,\varphi(x)$$

Thus we can conclude that $\operatorname{curl}\hat{\boldsymbol{H}} = -i\omega\varepsilon_0\boldsymbol{\varepsilon}\hat{\boldsymbol{E}}$ holds in $\mathscr{D}'(B_R)$. The proof of Proposition 7.3 is finished. □

7.3. Strong two-scale convergence.

In the following, we keep all assumptions and notations of Subsection 7.2. We are going to improve the convergence result of Proposition 7.3 by showing that the two-scale convergence of the triple $(\boldsymbol{E}_\eta, \boldsymbol{H}_\eta, \boldsymbol{J}_\eta)$ holds strongly. This is done within the three next Lemmas. We begin with the field $\boldsymbol{J}_\eta$ whose strong convergence is a straightforward consequence of the fact that $\mathscr{P}_\eta \to \mathscr{P}$ (see Proposition 7.3).

Lemma 7.4. *The following convergence holds*

$$\lim_{\eta\to 0} \left\| \boldsymbol{J}_\eta(x) - \boldsymbol{J}_0\Big(x, \frac{x}{\eta}\Big)\right\|_{L^2(B_R)} = 0\,. \tag{7.12}$$

Proof. As $\Re(\mathscr{P}_\eta)\to\Re(\mathscr{P})$, by (7.4) and (2.5) and recalling that $\boldsymbol{J}_\eta=0$ outside Σ_η , one has

$$\lim_\eta\int_\Omega|\boldsymbol{J}_\eta|^2=-\frac{|\varepsilon_r|^2}{\omega\,\varepsilon_0\Im(\varepsilon_r)}\lim_\eta\Re(\mathscr{P}_\eta)=-\frac{|\varepsilon_r|^2}{\omega\,\varepsilon_0\Im(\varepsilon_r)}\Re(\mathscr{P})=\frac{\mu_0}{\varepsilon_0}\frac{|\varepsilon_r|^2}{\Im(\varepsilon_r)}\int_\Omega\Im\Big(\boldsymbol{\mu}^{\rm eff}\boldsymbol{H}\cdot\overline{\boldsymbol{H}}\Big)$$
$$=\int_{\Omega\times Y}|\boldsymbol{J}_0|^2\,dx\,dy\,,$$

where in the last equality we used (5.26).The strong two-scale convergence $\boldsymbol{J}_\eta\twoheadrightarrow\boldsymbol{J}_0$ follows. On the other hand, in view of the decomposition (5.22), it is easy to check that $\boldsymbol{J}_0(x,x/\eta)\twoheadrightarrow\boldsymbol{J}_0(x,y)$ (J_0 is admissible) and we are led to (7.12). □

Proposition 7.5. *The following convergence holds*

$$\lim_{\eta\to0}\left\|\boldsymbol{H}_\eta(x)-\boldsymbol{H}_0\Big(x,\frac{x}{\eta}\Big)\right\|_{L^2(B_R)}=0\,. \tag{7.13}$$

Proof. Our strategy is based on the argument of compensated compactness in Lemma 4.8, that we apply on the open set B_R to the pair $(\boldsymbol{u}_\eta,\boldsymbol{v}_\eta)$ defined by

$$\boldsymbol{u}_\eta=\boldsymbol{v}_\eta:=\boldsymbol{H}_\eta(x)-\boldsymbol{H}_0\Big(x,\frac{x}{\eta}\Big)\,.$$

By (2.19), it is clear that $\boldsymbol{u}_\eta\rightharpoonup0$ weakly in $(L^2(B_R))^3$. We need only to prove the following claims:

$$a_\eta:=\operatorname{div}[\boldsymbol{H}_\eta-\boldsymbol{H}_0(x,x/\eta)]\to0\qquad\text{strongly in } W^{-1,2}(B_R) \tag{7.14}$$

$$\boldsymbol{b}_\eta:=\operatorname{curl}[\boldsymbol{H}_\eta-\boldsymbol{H}_0(x,x/\eta)]\to0\qquad\text{strongly in } W^{-1,2}(B_R) \tag{7.15}$$

Indeed, under (7.14)(7.15), we infer from Lemma 4.8 that the sequence of non-negative functions $|\boldsymbol{u}_\eta|^2$ converges to zero in $\mathscr{D}'(B_R)$. On the other hand, we know from Lemma 7.1 that $\boldsymbol{u}_\eta\to0$ uniformly at a positive distance from Ω. By using a trivial localization argument, we may readily conclude that $|\boldsymbol{u}_\eta|^2\to0$ in $L^1(B_R)$ that is (7.13). Notice that, by the bound (7.1), a_η and $\boldsymbol{b}_\eta$ are bounded thus weakly convergent to 0 in $W^{-1,2}(B_R)$.

Proof of Claim (7.14)**.** Let $\{\varphi_\eta\}$ be a sequence such that $\varphi_\eta\rightharpoonup0$ weakly in $W_0^{1,2}(B_R)$. In view of Lemma 4.9, we need only to show the convergence to zero of the duality bracket $\langle a_\eta,\varphi_\eta\rangle$. Since $\operatorname{div}\boldsymbol{H}_\eta=0$, we have

$$\langle a_\eta,\varphi_\eta\rangle=-\int_{B_R}\boldsymbol{H}_0\Big(x,\frac{x}{\eta}\Big)\cdot\nabla\varphi_\eta(x)\,\mathrm{d}x\,. \tag{7.16}$$

Up to passing to a subsequence, we can assume that $\nabla\varphi_\eta\rightharpoonup\xi_0$ for a suitable element of $(L^2(\Omega\times Y))^3$. Furthermore, by Proposition 4.3 (see (4.6)), there exists $\psi_0\in L^2(B_R;W^{1,2}_\sharp(Y))$ such that $\xi_0=\nabla_y\psi_0$. As $\boldsymbol{H}^0=\sum_k H_k(x)\boldsymbol{H}^k(y)$ is a finite sum of admissible functions (see the comments after (4.3)), we may pass to the limit in (7.16) applying (4.3)) for $\varphi=1$. We obtain:

$$\lim_{\eta\to0}\langle a_\eta,\varphi_\eta\rangle=-\int_{B_R}\left(\int_Y\boldsymbol{H}_0(x,\cdot)\cdot\nabla_y\psi_0(x,\cdot)\,dy\right)dy\,.$$

Recalling that $\boldsymbol{H}_0(x,\cdot)$ is divergence-free (see (5.11)), we see that the right-hand side vanishes by integrating by parts over Y and by taking into account the periodicity of $\boldsymbol{H}_0(x,\cdot)$ and $\psi_0(x,\cdot)$.

Proof of Claim (7.15)**.** This is the most delicate part. In a similar way as before, we consider a sequence $\{\boldsymbol{\varphi}_\eta\}$ such that $\boldsymbol{\varphi}_\eta\rightharpoonup0$ weakly in $(W_0^{1,2}(B_R))^3$. We need to show that $\langle\boldsymbol{b}_\eta,\boldsymbol{\varphi}_\eta\rangle\to0$ where

$$\begin{aligned}\langle\boldsymbol{b}_\eta,\boldsymbol{\varphi}_\eta\rangle&=\int_{B_R}\operatorname{curl}\boldsymbol{H}_\eta\cdot\boldsymbol{\varphi}_\eta-\int_{B_R}\boldsymbol{H}_0\Big(x,\frac{x}{\eta}\Big)\cdot\operatorname{curl}\boldsymbol{\varphi}_\eta\\&=-i\,\omega\varepsilon_0\frac{1}{\eta}\int_{B_R}\boldsymbol{J}_\eta\cdot\boldsymbol{\varphi}_\eta-\int_{B_R}\boldsymbol{H}_0\Big(x,\frac{x}{\eta}\Big)\cdot\operatorname{curl}\boldsymbol{\varphi}_\eta\,.\end{aligned} \tag{7.17}$$

Here, in the second equality, we used equation (1.5) and the definition (5.1). It is now useful to consider the piecewise constant function

$$[\boldsymbol{\varphi}_\eta]_\eta(x) := \frac{1}{\eta^3}\sum_{k\in I_\eta}\left(\int_{Y_\eta^k}\boldsymbol{\varphi}_\eta\right)1_{Y_\eta^k}(x) \quad , \quad Y_\eta^k = \eta(Y+\boldsymbol{k})\,,\ I_\eta := \{\boldsymbol{k}\in\mathbb{Z}^3\ :\ Y_\eta^k\subset B_R\}\,.$$

By applying Proposition 4.3 to each component of $\boldsymbol{\varphi}_\eta$, we are ensured of the existence of an element $\boldsymbol{\psi}_0\in W^{1,2}(B_R;W^{1,2}_\sharp(Y;\mathbb{C}^3))$ such that

$$\frac{\boldsymbol{\varphi}_\eta-[\boldsymbol{\varphi}_\eta]_\eta}{\eta}\rightharpoonup\boldsymbol{\psi}_0 \quad , \quad \mathrm{curl}_y\,\boldsymbol{\varphi}_\eta\rightharpoonup\mathrm{curl}_y\,\boldsymbol{\psi}_0\,.$$

In view of (7.17), let us write $\langle\boldsymbol{b}_\eta,\boldsymbol{\varphi}_\eta\rangle = r_\eta+s_\eta$ where

$$r_\eta := -i\omega\varepsilon_0\int_{B_R}\boldsymbol{J}_\eta\cdot\frac{\boldsymbol{\varphi}_\eta-[\boldsymbol{\varphi}_\eta]_\eta}{\eta}-\int_{B_R}\boldsymbol{H}_0\Big(x,\frac{x}{\eta}\Big)\cdot\mathrm{curl}\,\boldsymbol{\varphi}_\eta\,, \tag{7.18}$$

$$s_\eta := -i\omega\varepsilon_0\int_{B_R}\frac{\boldsymbol{J}_\eta}{\eta}\cdot[\boldsymbol{\varphi}_\eta]_\eta\,. \tag{7.19}$$

Thanks to the strong two-scale convergences $\boldsymbol{H}_0\Big(x,\frac{x}{\eta}\Big)\twoheadrightarrow\boldsymbol{H}_0$ and $\boldsymbol{J}_\eta\rightharpoonup\boldsymbol{J}_0$ (see Lemma 7.4), we may apply twice the multiplication rule (4.3) (and subsequent comments). We are led to

$$\lim_{\eta\to0}r_\eta = -\int_{B_R\times Y}\big[i\omega\varepsilon_0\boldsymbol{J}_0\cdot\boldsymbol{\psi}_0+\boldsymbol{H}_0\cdot\mathrm{curl}_y\,\boldsymbol{\psi}_0\big]\,\mathrm{d}x\,\mathrm{d}y\,.$$

Here we used the fact that $\boldsymbol{H}^0$ is an admissible function and also that $\boldsymbol{J}_\eta$ is compactly supported in B_R (so that (4.3) can be applied with a localizing test function φ such that $\varphi=1$ in Ω). Eventually, as $\mathrm{curl}_y\,\boldsymbol{H}_0(x,\cdot)=-i\omega\varepsilon_0\,\boldsymbol{J}_0(x,\cdot)$ (see (5.11)), by integrating by parts on Y and taking into account the periodicity of $\boldsymbol{\psi}_0(x,\cdot)$, we readily deduce that the right-hand member in the equality above vanishes, thus $r_\eta\to0$.

In order to conclude, it remains to show that s_η defined in (7.19) vanishes as $\eta\to0$. To that aim we consider, for $k\in\{1,2,3\}$, a function $\theta_k\in C_c^\infty(Y)$ verifying $\nabla\theta_k=\boldsymbol{e}_k$ in Σ and extended by periodicity in $\mathbb{R}^3$. Then we consider the potential w_k defined by

$$w_\eta(x) := \eta\sum_{k=1}^{k=3}\big([\boldsymbol{\varphi}_\eta]_\eta(x)\cdot\boldsymbol{e}_k\big)\,\theta_k\Big(\frac{x}{\eta}\Big)\,.$$

By construction and recalling that $[\boldsymbol{\varphi}_\eta]_\eta$ is piecewise constant, it holds

$$\nabla w_\eta = [\boldsymbol{\varphi}_\eta]_\eta \qquad \text{on } \Sigma_\eta\,. \tag{7.20}$$

On the other hand, since the θ_k's are Lipschitzian , one checks easily that w_η belongs to $W^{1,2}_0(B_R)$ and satisfies $|\nabla w_\eta|\le C\,\Big|[\boldsymbol{\varphi}_\eta]_\eta\Big|$ for a suitable constant $C>0$. In particular as $[\boldsymbol{\varphi}_\eta]_\eta\to0$ strongly in $L^2(B_R)$, we have that

$$\lim_{\eta\to0}\|\nabla w_\eta-[\boldsymbol{\varphi}_\eta]_\eta\|_{L^2(B_R)} = 0\,. \tag{7.21}$$

Thanks to (7.20) and noticing that $\frac{\boldsymbol{J}_\eta}{\eta}$ agrees with $\varepsilon_e\,\boldsymbol{E}_\eta$ in $B_R\setminus\Sigma_\eta$, we have

$$s_\eta = \frac{1}{\eta}\int_{B_R}\boldsymbol{J}_\eta\cdot\nabla w_\eta+\varepsilon_e\int_{B_R}\boldsymbol{E}_\eta\cdot\big([\boldsymbol{\varphi}_\eta]_\eta-\nabla w_\eta\big)\,.$$

Clearly the first integral above vanishes (since $\boldsymbol{J}_\eta$ is divergence-free) while the second one converges to zero by Cauchy-Schwarz inequality and by taking into account (7.21) and the upper bound (7.1). Summarizing, we have proved that $\langle\boldsymbol{b}_\eta,\boldsymbol{\varphi}_\eta\rangle = r_\eta+s_\eta\to0$ and our claim (7.15) follows. The proof of Proposition 7.5 is finished. □

Proposition 7.6. *The following convergence holds*

$$\lim_{\eta\to 0}\left\| \boldsymbol{E}_\eta(x)-\boldsymbol{E}_0\Big(x,\frac{x}{\eta}\Big)\right\|_{L^2(B_R)}=0\,. \tag{7.22}$$

Proof. First we observe that $|\boldsymbol{E}_\eta|^2=\frac{\eta^2}{|\varepsilon_r|^2}|\boldsymbol{J}_\eta|^2$ holds in Σ_η and then, thanks to the strong two-scale convergence of $\boldsymbol{J}_\eta$ obtained in Lemma 7.4, we have

$$\lim_{\eta\to 0}\frac{1}{\eta^2}\int_{\Sigma_\eta}|\boldsymbol{E}_\eta|^2\;=\;\frac{1}{|\varepsilon_r|^2}\int_{\Omega\times Y}|\boldsymbol{J}_0|^2\,. \tag{7.23}$$

In particular, it holds $\int_{\Sigma_\eta}|\boldsymbol{E}_\eta|^2\to 0$. Thus , recalling that $\boldsymbol{E}_0(x,\cdot)=\boldsymbol{E}(x)$ for $x\in B_R\setminus\Omega$ and that $\boldsymbol{E}_0=0$ in $\Omega\times\Sigma$, the convergence (7.22) is proved once we have shown that

$$\limsup_\eta\int_{B_R\setminus\Omega}|\boldsymbol{E}_\eta|^2\;\le\;\int_{B_R\setminus\Omega}|\boldsymbol{E}|^2\quad,\quad\limsup_\eta\int_{\Omega\setminus\Sigma_\eta}|\boldsymbol{E}_\eta|^2\;\le\;\int_{\Omega\times Y}|\boldsymbol{E}_0|^2\,. \tag{7.24}$$

As in Lemma 7.4, the main argument is the convergence $\mathscr{P}_\eta\to\mathscr{P}$ established in Proposition 7.3. It is convenient to rewrite Claim (7.24) in the following equivalent form

$$\alpha:=\limsup_\eta\left(\int_{B_R\setminus\Omega}|\boldsymbol{E}_\eta|^2+\varepsilon_e\int_{\Omega\setminus\Sigma_\eta}|\boldsymbol{E}_\eta|^2\right)\le\int_{B_R\setminus\Omega}|\boldsymbol{E}|^2+\varepsilon_e\int_{\Omega\times Y}|\boldsymbol{E}_0|^2\;=\;\int_{B_R}\boldsymbol{\varepsilon}\,\boldsymbol{E}\cdot\overline{\boldsymbol{E}}\,, \tag{7.25}$$

where in the last equality we used (2.2) and (5.9). It is trivial that (7.24) implies (7.25). The converse implication follows from the fact that one has

$$\liminf_\eta\int_{B_R\setminus\Omega}|\boldsymbol{E}_\eta|^2\ge\int_{B_R\setminus\Omega}|\boldsymbol{E}|^2\quad,\quad\liminf_\eta\int_{\Omega\setminus\Sigma_\eta}|\boldsymbol{E}_\eta|^2\ge\int_{\Omega\times Y}|\boldsymbol{E}_0|^2.$$

Now, by taking the imaginary parts in (7.3) and after dividing by $\omega\varepsilon_0$, we obtain the relation

$$\int_{B_R\setminus\Omega}|\boldsymbol{E}_\eta|^2+\varepsilon_e\int_{\Omega\setminus\Sigma_\eta}|\boldsymbol{E}_\eta|^2\;=\;-\frac{\Im(\mathscr{P}_\eta)}{\omega\varepsilon_0}+\frac{\mu_0}{\varepsilon_0}\int_{B_R}|\boldsymbol{H}_\eta|^2-\frac{\Re(\varepsilon_r)}{\eta^2}\int_{\Sigma_\eta}|\boldsymbol{E}_\eta|^2\,.$$

Letting $\eta\to 0$, with the convergence of $\mathscr{P}_\eta$ to $\mathscr{P}$ and thanks to (7.23) and the strong two-scale convergence $\boldsymbol{H}_\eta\twoheadrightarrow\boldsymbol{H}$, we see that the left-hand member α of (7.25) is actually a limit and it holds

$$\begin{aligned}\alpha\;&=\;-\frac{\Im(\mathscr{P})}{\omega\varepsilon_0}+\frac{\mu_0}{\varepsilon_0}\int_{B_R\times Y}|\boldsymbol{H}_0|^2-\frac{\Re(\varepsilon_r)}{|\varepsilon_r|^2}\int_{\Omega\times Y}|\boldsymbol{J}_0|^2\\&=\;-\frac{\Im(\mathscr{P})}{\omega\varepsilon_0}+\frac{\mu_0}{\varepsilon_0}\int_{B_R}\Re\Big(\boldsymbol{\mu}(x)\boldsymbol{H}\cdot\overline{\boldsymbol{H}}\Big)\,,\end{aligned} \tag{7.26}$$

where $\boldsymbol{\mu}$is define in (2.2) and in the last equality we took the integral over Ω of the real parts of the terms appearing in the equality (5.26) while for $x\in B_R\setminus\Omega$ we used the fact that $\boldsymbol{H}_0(x,\cdot)=\boldsymbol{H}(x)$ and $\boldsymbol{J}_0(x,\cdot)=0$. Now it remains to evaluate $\Im(\mathscr{P})$. In view of the second equality in (2.4) in which we take the imaginary parts and where $\boldsymbol{\varepsilon}$ is real, we get

$$\Im(\mathscr{P})=\omega\left(\mu_0\int_{B_R}\Re\Big(\boldsymbol{\mu}(x)\boldsymbol{H}\cdot\overline{\boldsymbol{H}}\Big)-\varepsilon_0\int_{B_R}\boldsymbol{\varepsilon}(x)\boldsymbol{E}\cdot\overline{\boldsymbol{E}}\right). \tag{7.27}$$

Putting (7.26) and (7.27) together, we can conclude that $\alpha=\int_{B_R}\boldsymbol{\varepsilon}\,\boldsymbol{E}\cdot\overline{\boldsymbol{E}}$. Thus the claim (7.24) holds and the proof of Proposition 7.6 is finished. □

7.4. End of the proof.

Collecting the results of Proposition 7.3, Lemma 7.4, Proposition 7.5 and Proposition 7.6, we see that the proof of our main Theorem 2.2 is achieved once the energy bound (7.1) has been established. The following final Lemma allows us to conclude.

Lemma 7.7. *Under the assumptions of Theorem 2.2, the energy bound given in* (7.1) *holds.*

Proof. Let us assume by contradiction that $(\boldsymbol{E}_\eta, \boldsymbol{H}_\eta)$ solving (1.5) is not uniformly bounded in $L^2(B_R)^3$. Then it exists a subsequence (still denoted η) such that $\lim_\eta(\int_{B_R}|\boldsymbol{E}_\eta|^2+\int_{B_R}|\boldsymbol{H}_\eta|^2)=+\infty$. For such a subsequence we denote

$$t_\eta := \left(\int_{B_R}|\boldsymbol{E}_\eta|^2+\int_{B_R}|\boldsymbol{H}_\eta|^2\right)^{\frac{1}{2}} \to +\infty \quad , \quad \hat{\boldsymbol{E}}_\eta := \frac{\boldsymbol{E}_\eta}{t_\eta} \quad \text{and} \quad \hat{\boldsymbol{H}}_\eta := \frac{\boldsymbol{H}_\eta}{t_\eta}\,. \tag{7.28}$$

By construction $(\hat{\boldsymbol{E}}_\eta, \hat{\boldsymbol{H}}_\eta)$ satisfies (7.1) whereas, by linearity, it is the solution of (1.5) when the incident wave is given by $(\frac{\boldsymbol{E}^{\rm inc}}{t_\eta}, \frac{\boldsymbol{H}^{\rm inc}}{t_\eta})$. We can then apply Proposition 7.3: the limit of $(\hat{\boldsymbol{E}}_\eta, \hat{\boldsymbol{H}}_\eta)$ is characterized in term of the unique solution $(\hat{\boldsymbol{E}}, \hat{\boldsymbol{H}})$ to the diffraction problem (2.1) in which the incident wave $(\hat{\boldsymbol{E}^{\rm inc}}, \hat{\boldsymbol{H}^{\rm inc}}) = \lim_{\eta\to 0}(\frac{\boldsymbol{E}^{\rm inc}}{t_\eta}, \frac{\boldsymbol{H}^{\rm inc}}{t_\eta}) = (0,0)$. Therefore, by the uniqueness Lemma 2.1, it holds $(\hat{\boldsymbol{E}}, \hat{\boldsymbol{H}}) = (0,0)$ and the two-scale limit $(\hat{\boldsymbol{E}}_0, \hat{\boldsymbol{H}}_0)$ of $(\hat{\boldsymbol{E}}_\eta, \hat{\boldsymbol{H}}_\eta)$ vanishes as well on $B_{\mathbb{R}} \times Y$. Then, by applying the strong convergence results (7.13) and (7.22), we are led to

$$\lim_{\eta\to 0}\int_{B_R}|\hat{\boldsymbol{E}}_\eta|^2 \;=\; \lim_{\eta\to 0}\int_{B_R}|\hat{\boldsymbol{H}}_\eta|^2 = 0\,.$$

This is impossible since, by (7.28), we have $\int_{B_R}|\hat{\boldsymbol{E}}_\eta|^2+\int_{B_R}|\hat{\boldsymbol{H}}_\eta|^2 = 1$. □

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