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► **To cite this version:**

Ibrahim Baydoun. Analytical formula for the roots of the general complex cubic polynomial. 2018.
hal-01237234v2

HAL Id: hal-01237234

<https://hal.science/hal-01237234v2>

Preprint submitted on 17 Jan 2018

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Analytical formula for the roots of the general complex cubic polynomial

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January 17, 2018

Abstract

We present a new method to calculate analytically the roots of the general complex polynomial of degree three. This method is based on the approach of appropriated changes of variable involving an arbitrary parameter. The advantage of this method is to calculate the roots of the cubic polynomial as uniform formula using the standard convention of the square and cubic roots. In contrast, the reference methods for this problem, as Cardan-Tartaglia and Lagrange, give the roots of the cubic polynomial as expressions with case distinctions which are incorrect using the standar convention.

keyword: cubic polynomial roots, appropriated change of variable, analytical uniform formula.

1 Introduction

The problem of solution formulas of polynomial equations is fundamental in algebra [1, 2]. It is useful, by example, for the eigenvalues perturbation [3] and the solution of a system of nonlinear equations [4]. Moreover, it has many physical applications, as the study of the singularities of the surfaces of refractive indices in crystal optics [5] and the diagonal eigenvalues perturbation [6]. Another example is the calculation of the sound velocity anisotropy in cubic crystals [7], or the diagonalization of Christoffel tensor to calculate the velocities of the three quasi-modes of the elastic waves in anisotropic medium [8]. An explicit calculation for the particular case of 3×3 real symmetric matrices has been studied in [9].

For a cubic equation, Cardano-Tartaglia and Lagrange's formulas are subject to different interpretations depending on the choices for the values of the square and cubic roots of the formulas [10, 11]. The correct interpretation is generally derived from conditions between the roots and the coefficients of the cubic polynomial. It is a fundamental question to be able to define uniform formula that yields correct solutions for all cubic equations. Ting Zhao's thesis and related work recently brought an important advance on Lagrange's formulas with uniform solution formulas for cubic and quartic equations with real coefficients. The results was obtained by introducing a new convention for the argument of the cubic root of a complex number [12]. For cubic equations, we present in this paper a uniform formula that is valid with complex coefficients. This more general result are uniquely defined by the standard convention of the square and cubic roots.

To illustrate the problem, we introduce Cardan-Tartaglia formula and afterward we give two examples. Indeed, Cardan-Tartaglia calculate the 3 solutions of the equation

$$(C) : z^3 + pz + q = 0$$

as

$$z_n := \exp\left(n\frac{2\pi}{3}\sqrt{-1}\right) \sqrt[3]{\frac{1}{2}\left(-q + \sqrt{\frac{\Delta_C}{27}}\right)} + \exp\left(-n\frac{2\pi}{3}\sqrt{-1}\right) \sqrt[3]{\frac{1}{2}\left(-q - \sqrt{\frac{\Delta_C}{27}}\right)}; n \in \{0, 1, 2\},$$

where Δ_C is defined as

$$\Delta_C := -(4p^3 + 27q^2).$$

We show by the following examples that the standard convention of the cubic roots

$$\arg\left(\sqrt[3]{X}\right) = \frac{1}{3}\arg(X); X \in \mathbb{C}$$

is not relevant in the above Cardan-Tartaglia's expression. Indeed:

Example 1.1. *The solution of (C) for the case $p = 1$ and $q = 2$ is*

$$z_n \frac{-1 - \sqrt{-3}}{2}; n \in \{0, 1, 2\},$$

instead of z_n .

Example 1.1 requires to shift the standard convention by $-2\pi/3$ in order to be suitable for the Cardan-Tartaglia's solutions. Another example requires another convention:

Example 1.2. The solution of (C) for the case $p = 1$ and $q = \sqrt{-1}$ is

$$z_n = \frac{-1 + \sqrt{-3}}{2}; \quad n \in \{0, 1, 2\},$$

instead of z_n .

Therefore, a natural question is how we can calculate the roots of (C) using uniquely the standard convention?

To overcome this problem, a mathematical software as Matlab generates the symbolic solutions of (C) in terms of one cubic root R_C as follows

$$R_C = \frac{p}{3R_C},$$

$$\frac{\sqrt{-3}}{2} \left(\frac{p}{3R_C} + R_C \right) + \frac{p}{6R_C} - \frac{R_C}{2},$$

$$- \frac{\sqrt{-3}}{2} \left(\frac{p}{3R_C} + R_C \right) + \frac{p}{6R_C} - \frac{R_C}{2},$$

where R_C is given by

$$R_C = \sqrt[3]{\sqrt{\frac{-\Delta_C}{2^2 3^3}} - \frac{q}{2}}.$$

On the one hand, this formulation hid the problem of the convention of the cubic roots. But, on the other hand, it involves another problem by dividing by R_C which leads to a singularity in the formulation if R_C is zero. So, the above question can be reworded by the following manner: how we calculate the roots of (C) using uniquely the standard convention and without any singularity?

The response to this question is the objective of this paper.

We present a uniform analytical formula for the roots of the general complex cubic polynomial. Our method is based on the approach of changes of variable involving an arbitrary parameter, which rests on two principal ideas to diagonalize the general 3×3 complex matrix:

Firstly, we introduce one change of variable involving an arbitrary parameter. So, the characteristic equation of this matrix can be replaced by another non-polynomial equation in terms of the first new variable.

Secondly, we introduce another change of variable involving another arbitrary parameter. The second variable replaces the first and the second arbitrary parameter can be chosen as required to make the form of the new equation as the sum of a cube of a certain monomial and a term independent of the unknown second variable.

Therefore, we can solve easily this new polynomial equation and consequently deduce the solution of the original equation, which is the eigenvalues of the general 3×3 complex matrix.

In the sequel, we apply this approach to calculate the roots of the general complex cubic polynomial. In fact, we construct a particular matrix such that its characteristic polynomial will coincide with this cubic polynomial. Consequently, the eigenvalues of this particular matrix become identical to the roots of this cubic polynomial.

The rest of the paper is structured as follows. In section 2, we derive the approach of appropriated changes of variable involving an arbitrary parameter. Consequently in section 3, we apply it on the general complex cubic polynomial.

2 Approach of appropriated changes of variable involving an arbitrary parameter

Let \mathbf{M} be the general 3×3 complex matrix:

$$\mathbf{M} := \begin{pmatrix} m_1 & m_2 & m_3 \\ m_4 & m_5 & m_6 \\ m_7 & m_8 & m_9 \end{pmatrix}.$$

In this section, we aim to calculate the spectrum of \mathbf{M} with this approach. We detail it by the following four steps. The first step consists of performing a change of variable for the eigenvalues of \mathbf{M} so that we get a non-polynomial equation. In the second step, we will deal with this equation in order to simplify it by some algebraic manipulations. The third step will introduce a second change of variable in order to replace the non-polynomial equation by another polynomial equation. Finally, in the fourth step, we will solve the latter polynomial equation to deduce the spectrum of \mathbf{M} .

Starting the first step by introducing the following definition which will be useful in the sequel:

Definition 2.1. We associate to \mathbf{M} the following expressions in terms of its components:

$$\begin{aligned} c_1 &:= m_1 - m_5 - \frac{m_3 m_8}{m_2}, & c_2 &:= c_1^2 + 4(m_2 m_4 + m_3 m_7), \\ e_1 &:= \frac{3}{2}(m_1 - c_1) - \frac{1}{2} \text{Tr}(\mathbf{M}), & e_2 &:= \frac{3}{2}(m_1 - c_1)^2 + \frac{3}{8}(c_2 - c_1)^2 - (m_1 - c_1) \text{Tr}(\mathbf{M}) + \frac{1}{4} \{[\text{Tr}(\mathbf{M})]^2 - \text{Tr}(\mathbf{M}^2)\}, \\ e_3 &:= \frac{1}{8} [(2m_1 - c_1)^3 + 3(2m_1 - c_1)c_2] - \frac{1}{4} [(2m_1 - c_1)^2 + c_2] \text{Tr}(\mathbf{M}) + \frac{1}{4} (2m_1 - c_1) \{[\text{Tr}(\mathbf{M})]^2 - \text{Tr}(\mathbf{M}^2)\} - \det(\mathbf{M}), \\ e_4 &:= \frac{1}{8} [3(2m_1 - c_1)^2 + c_2] - \frac{1}{2} (2m_1 - c_1) \text{Tr}(\mathbf{M}) + \frac{1}{4} \{[\text{Tr}(\mathbf{M})]^2 - \text{Tr}(\mathbf{M}^2)\}. \end{aligned}$$

$\text{Tr}(\mathbf{M})$ and $\det(\mathbf{M})$ stand respectively for the trace and the determinant of matrix \mathbf{M} .

Now, we introduce the first change of variable involving an arbitrary parameter so that the eigenvalues of \mathbf{M} can be rewritten in terms of this new variable \tilde{b} such that \tilde{b} verifies a non-polynomial equation like the following proposition:

Proposition 2.2. *An eigenvalue k of matrix \mathbf{M} can be written in terms of two unknowns \tilde{a} and \tilde{b} as follows:*

$$k = \frac{\tilde{a} - \tilde{b}}{2} + \frac{1}{2} \left[2m_1 - c_1 + \sqrt{\Delta_k} \right], \quad (1)$$

where \tilde{a} can be chosen arbitrary and \tilde{b} is governed by the following non-polynomial equation:

$$A + B\tilde{b} + C\tilde{b}^2 + D\sqrt{\Delta_k} + E\tilde{b}\sqrt{\Delta_k} + \frac{1}{2} \left(\tilde{b}^2\sqrt{\Delta_k} - \tilde{b}^3 \right) = 0, \quad (2)$$

where Δ_k is defined by

$$\Delta_k := \tilde{b}^2 + 2(c_1 - \tilde{a})\tilde{b} + \tilde{a}^2 - 2c_1\tilde{a} + c_2 = \left(\tilde{a} - \tilde{b} - m_1 + m_5 + \frac{m_3m_8}{m_2} \right)^2 + 4(m_2m_4 + m_3m_7). \quad (3)$$

The coefficients of Eq. (2) depend on parameter \tilde{a} as follows:

$$A = \frac{\tilde{a}^3}{2} + e_1\tilde{a}^2 + e_2\tilde{a} + e_3, \quad B = -\frac{3\tilde{a}^2}{2} - 2e_1\tilde{a} - e_2, \quad C = \frac{3\tilde{a}}{2} + e_1, \\ D = \frac{\tilde{a}^2}{2} + \left(e_1 + \frac{c_1}{2} \right) \tilde{a} + e_4, \quad E = -\tilde{a} - \left(e_1 + \frac{c_1}{2} \right), \quad (4)$$

where the coefficients of Eq. (4) are given by definition 2.1.

Proof:

Firstly, we decompose the matrix \mathbf{M} as the sum of two matrix \mathbf{M}_1 and \mathbf{M}_2 as follows:

$$\mathbf{M} = \underbrace{\begin{pmatrix} \tilde{a} & 0 & 0 \\ 0 & \tilde{b} & -m_0 \\ 0 & 0 & 0 \end{pmatrix}}_{\mathbf{M}_1} + \underbrace{\begin{pmatrix} m_1 - \tilde{a} & m_2 & m_3 \\ m_4 & m_5 - \tilde{b} & m_6 + m_0 \\ m_7 & m_8 & m_9 \end{pmatrix}}_{\mathbf{M}_2}, \quad (5)$$

where m_0 is chosen such that \mathbf{M}_2 has an eigenvector of the form $[0, y_0, z_0]$. The advantage of this choice is to construct an eigenvector of \mathbf{M} in terms of these of \mathbf{M}_2 which will be elaborated in the sequel. Indeed, if we take m_0 such that

$$m_0 = \left(m_5 - \tilde{b} + m_3 \frac{m_8}{m_2} - m_9 \right) \frac{m_3}{m_2} - m_6,$$

then \mathbf{M}_2 have the eigenvalue

$$k_0 = -m_3 \frac{m_8}{m_2} + m_9 \quad (6)$$

relative to the eigenvector

$$\mathbf{v}_0 := \left[0, -\frac{m_3}{m_2}, 1 \right].$$

Equation 6 gives one eigenvalue k_0 of \mathbf{M}_2 , then the characteristic polynomial of \mathbf{M}_2

$$\det \left[X \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \mathbf{M}_2 \right]$$

is divided in $\mathbb{C}[X]$ by the polynomial $X - k_0$. Consequently, we deduce that the two other eigenvalues of \mathbf{M}_2 verify a second order polynomial and they are given by:

$$\tilde{k} = \frac{-\tilde{a} - \tilde{b}}{2} + \frac{1}{2} \left[2m_1 - c_1 \mp \sqrt{\Delta_k} \right]. \quad (7)$$

Secondly, we search an eigenvector for \mathbf{M} of the form

$$\mathbf{v}_0 + v\mathbf{v}; \quad v \in \mathbb{C},$$

where $\hat{\mathbf{v}} := [v_1, v_2, v_3]$ is an unit eigenvector of \mathbf{M}_2 relative to \tilde{k} . Indeed, using

$$\mathbf{M}_2 \mathbf{v}_0 = k_0 \mathbf{v}_0, \\ \mathbf{M}_2 \mathbf{v} = \tilde{k} \mathbf{v},$$

we obtain from Eq. (5) that the vectorial equation

$$\mathbf{M}(\mathbf{v}_0 + v\mathbf{v}) = k(\mathbf{v}_0 + v\mathbf{v}); \quad k \in \mathbb{C},$$

is equivalent to the following system:

$$(S) \begin{cases} v_1 v (\tilde{a} + \tilde{k} - k) = 0, \\ \tilde{b} \left(v v_2 - \frac{m_3}{m_2} \right) - m_0 (v v_3 + 1) + v \tilde{k} v_2 - \frac{m_3}{m_2} k_0 - k \left(v v_2 - \frac{m_3}{m_2} \right) = 0, \\ v \tilde{k} v_3 + k_0 - k (v v_3 + 1) = 0. \end{cases}$$

The impact of the choice of m_0 appears in the fact that the first equation of (\mathcal{S}) is independent of $\mathbf{v}_0 + v\mathbf{v}$ and it can be verified by taking $k = \tilde{a} + \tilde{k}$. Thus, by taking v as

$$v = \frac{1}{v_3} \frac{k - k_0}{\tilde{k} - k},$$

the third equation of (\mathcal{S}) is verified. It remains to verify the second equation of (\mathcal{S}) . Indeed, $k = \tilde{a} + \tilde{k}$ and Eq. (7) implies Eq. (1) where k depends only on $\tilde{a} - \tilde{b}$ since Δ_k depends only on $\tilde{a} - \tilde{b}$. Therefore, by inserting Eq. (1) in the characteristic equation of \mathbf{M} , we get

$$k^3 - \text{Tr}(\mathbf{M})k^2 + \frac{1}{2}\{[\text{Tr}(\mathbf{M})]^2 - \text{Tr}(\mathbf{M}^2)\}k - \det(\mathbf{M}) = 0, \quad (8)$$

so that we can choose arbitrarily among \tilde{a} and \tilde{b} one parameter as required, while the other parameter will be the unique unknown of Eq. (8), as well as k is an eigenvalue of \mathbf{M} . We obtain that if \tilde{a} and \tilde{b} satisfy Eq. (8), then the second equation of (\mathcal{S}) is verified. To end the proof, we develop Eq. (8) to deduce that Eq. (8) and Eq. (2) are equivalent. \square

In the second step, we aim to simplify Eq. (2) of the unknown \tilde{b} , where \tilde{a} can be chosen as required. So, we derive from Eq. (2) two equations of \tilde{b} given by the following two propositions:

Proposition 2.3. For fixed \tilde{a} , \tilde{b} satisfies the following equation:

$$d_1 + d_2\tilde{b} + d_3\tilde{b}^2 + d_4\sqrt{\Delta_k} - d_3\tilde{b}\sqrt{\Delta_k} = 0, \quad (9)$$

where the coefficients of Eq. (9) are given by:

$$d_1 := d_3\tilde{a}^2 + s_1\tilde{a} + s_2, \quad d_2 := -2d_3\tilde{a} - s_1, \quad d_3 := m_2m_4 + m_3m_7, \quad d_4 := d_3\tilde{a} + s_3.$$

Here s_1 , s_2 and s_3 are given using definition 2.1 as follows:

$$s_1 := \frac{c_1c_2}{2} + c_1e_2 + c_2e_1 - 2c_1e_4 - e_3, \quad s_2 := c_1e_3 + c_2e_4, \quad s_3 := e_3 + c_1e_4. \quad (10)$$

Proof:

Eq. (9) is deduced by multiplying Eq. (2) by $\tilde{b} + \sqrt{\Delta_k} - \tilde{a} + c_1$. \square

Proposition 2.4. For fixed \tilde{a} , \tilde{b} satisfies the following equation:

$$f_1 + f_2\tilde{b} + f_3\tilde{b}^2 + f_4\sqrt{\Delta_k} + f_3\tilde{b}\sqrt{\Delta_k} = 0, \quad (11)$$

where the coefficients of Eq. (11) are given by:

$$\begin{aligned} f_1 &:= f_3\tilde{a}^2 + s_4\tilde{a} + s_5, & f_2 &:= -2f_3\tilde{a} - s_4, \\ f_3 &:= (m_5 - m_1)m_3m_7 + (m_9 - m_1)m_2m_4 - m_2m_6m_7 - m_3m_4m_8, & f_4 &:= -f_3\tilde{a} + s_6. \end{aligned}$$

Here s_4 , s_5 and s_6 are given using definition 2.1 as follows:

$$s_4 := c_2d_3 - 2c_1^2e_4 - 2c_1e_3 - 2d_3e_2, \quad s_5 := c_2e_3 - 2d_3e_3 + c_1c_2e_4, \quad s_6 := c_1e_3 + c_2e_4 - 2d_3e_4. \quad (12)$$

Proof:

Similarly to the proof of proposition 2.3, on one hand we multiply Eq. (2) by $2d_3$ and on the other hand we multiply Eq. (9) by $\sqrt{\Delta_k}$. Consequently, we subtract the obtained equations to deduce Eq. (11). \square

Now, we show that equations (9) and (11) imply the two equations (13) and (14) given by the following corollary:

Corollary 2.5. For fixed \tilde{a} , \tilde{b} satisfies the following equations:

$$r_1 + r_2\tilde{b} + r_3\sqrt{\Delta_k} - 2\tilde{b}\sqrt{\Delta_k} = 0, \quad (13)$$

$$r_4 + r_5\tilde{b} + r_6\sqrt{\Delta_k} + 2\tilde{b}^2 = 0, \quad (14)$$

where the coefficients of Eq. (13) and Eq. (14) are given by:

$$\begin{aligned} r_1 &:= -r_2\tilde{a} + s_9, & r_2 &:= -\frac{s_1}{d_3} + \frac{s_4}{f_3}, & r_3 &:= 2\tilde{a} + s_{10}, \\ r_4 &:= 2\tilde{a}^2 + s_7\tilde{a} + s_8, & r_5 &:= -4\tilde{a} - s_7, & r_6 &:= \frac{s_3}{d_3} + \frac{s_6}{f_3} \end{aligned} \quad (15)$$

Here, the coefficients of Eq. (15) are defined using definition 2.1, Eq. (10) and Eq. (12) as follows:

$$s_7 := \frac{s_1}{d_3} + \frac{s_4}{d_3}, \quad s_8 := \frac{s_2}{d_3} + \frac{s_5}{f_3}, \quad s_9 := \frac{s_2}{d_3} - \frac{s_5}{f_3}, \quad s_{10} := \frac{s_3}{d_3} - \frac{s_2 - 2d_3e_4}{f_3}.$$

Proof:

We divide Eq. (9) and Eq. (11) respectively by d_3 and f_3 . Then, the subtraction and the addition of the obtained equations imply Eq. (13) and Eq. (14). \square

In the third step, we introduce the second change of variable

$$l := -\left(\tilde{b} + o\sqrt{\Delta_k}\right) \quad (16)$$

in order to deduce from the intractable equations of \tilde{b} another polynomial equation of the new variable l , where o is an arbitrary parameter can be chosen as required:

Proposition 2.6. Let $o \in \mathbb{C}$. Then l verifies the following equation:

$$2l^3 + B_l l^2 + C_l l + D_l = 0, \quad (17)$$

where the coefficients of Eq. (17) are given by:

$$\begin{aligned} B_l &:= -l_1 o + 6\tilde{a} - l_2, & C_l &:= l_3 o^2 - 2(l_1 \tilde{a} + l_4) o + 6\tilde{a}^2 - 2l_2 \tilde{a} + l_5, \\ D_l &:= -l_6 o^3 + (l_3 \tilde{a} - l_7) o^2 - (l_1 \tilde{a}^2 + 2l_4 \tilde{a} + l_8) o + 2\tilde{a}^3 - l_2 \tilde{a}^2 + l_5 \tilde{a} + l_9. \end{aligned}$$

Here B_l , C_l and D_l are defined using definition 2.1, Eq. (10), Eq. (12) and Eq. (15) as follows:

$$\begin{aligned} l_1 &:= 2\frac{s_1}{d_3} - s_7 + r_6, & l_2 &:= r_6 - s_7 - 2\frac{s_3}{d_3}, & l_3 &:= s_8 - 2c_2 + 2c_1 r_6 + \frac{r_6 s_1 - (s_7 + 4c_1) s_3}{d_3}, \\ l_4 &:= 2\frac{s_2}{d_3} - s_8 - c_1 r_6, & l_5 &:= s_8 + \frac{s_7 s_3 - r_6 s_1}{d_3}, & l_6 &:= 2c_1 s_8 + c_2 s_7 + \frac{s_1 (s_8 - 2c_2) - (4c_1 + s_7) s_2}{d_3}, \\ l_7 &:= 2c_1 s_8 + c_2 (s_7 + r_6) + \frac{(s_8 - 2c_2) s_3 - r_6 s_2}{d_3}, & l_8 &:= c_2 r_6 + \frac{s_7 s_2 - s_1 s_8}{d_3}, & l_9 &:= \frac{s_8 s_3 - r_6 s_2}{d_3}. \end{aligned}$$

Proof:

The first part of the proof consists in getting expressions for \tilde{b}^2 and $\sqrt{\Delta_k}$ just in terms of l and o . Indeed, Eq. (16) implies:

$$\tilde{b} = -\left(o\sqrt{\Delta_k} + l\right). \quad (18)$$

We take the square of Eq. (18) and we replace Δ_k by its value given in Eq. (3). Then, we insert Eq. (18) in the obtained equation to deduce an expression of \tilde{b}^2 . This chain of operations is illustrated as follows:

$$\begin{aligned} \tilde{b}^2 &= o^2 \underbrace{\Delta_k}_{\tilde{b}^2 + 2(c_1 - \tilde{a})\tilde{b} + \tilde{a}^2 - 2c_1 \tilde{a} + c_2} + l^2 + 2ol\sqrt{\Delta_k} \Rightarrow (1 - o^2) \tilde{b}^2 = 2o^2 (c_1 - \tilde{a}) \underbrace{\tilde{b}}_{-(o\sqrt{\Delta_k} + l)} + o^2 (\tilde{a}^2 - 2c_1 \tilde{a} + c_2) + l^2 + 2ol\sqrt{\Delta_k} \\ &\Rightarrow \tilde{b}^2 = \frac{2o [l - (c_1 - \tilde{a}) o^2] \sqrt{\Delta_k} + (\tilde{a}^2 - 2c_1 \tilde{a} + c_2) o^2 + l^2 - 2(c_1 - \tilde{a}) o^2 l}{1 - o^2}. \quad (19) \end{aligned}$$

So, inserting Eq. (18) and Eq. (19) in Eq. (14) in order to substitute \tilde{b} and \tilde{b}^2 and consequently to get the expression of $\sqrt{\Delta_k}$ as follows:

$$\begin{aligned} r_4 + r_5 \underbrace{\tilde{b}}_{-(o\sqrt{\Delta_k} + l)} + r_6 \sqrt{\Delta_k} + 2 \underbrace{\tilde{b}^2}_{(1-o^2)^{-1} \{2o[l - (c_1 - \tilde{a}) o^2] \sqrt{\Delta_k} + (\tilde{a}^2 - 2c_1 \tilde{a} + c_2) o^2 + l^2 - 2(c_1 - \tilde{a}) o^2 l\}} &= 0 \\ \Rightarrow \sqrt{\Delta_k} &= \frac{(1 - o^2) (r_4 - r_5 l) + 2 [(\tilde{a}^2 - 2c_1 \tilde{a} + c_2) o^2 - 2(c_1 - \tilde{a}) o^2 l + l^2]}{(1 - o^2) (r_5 o - r_6) + 2 [2(c_1 - \tilde{a}) o^3 - 2ol]}. \quad (20) \end{aligned}$$

The second part of the proof consists to use the previous expressions of \tilde{b}^2 and $\sqrt{\Delta_k}$ in order to deduce Eq. (17). Indeed, we take the square of the equation $\tilde{b} + o\sqrt{\Delta_k} + l = 0$ deduced from Eq. (16). Then, we insert Eq. (3) and Eq. (9) in the obtained equation to substitute Δ_k and $\tilde{b}\sqrt{\Delta_k}$. Afterwards we insert Eq. (18) to substitute \tilde{b} . This chain of operations is illustrated as follows:

$$\begin{aligned} \tilde{b}^2 + o^2 \underbrace{\Delta_k}_{\tilde{b}^2 + 2(c_1 - \tilde{a})\tilde{b} + \tilde{a}^2 - 2c_1 \tilde{a} + c_2} + l^2 + 2o \underbrace{\tilde{b}\sqrt{\Delta_k}}_{d_3^{-1} (d_1 + d_2 \tilde{b} + d_3 \tilde{b}^2 + d_4 \sqrt{\Delta_k})} + 2lo\sqrt{\Delta_k} + 2l\tilde{b} &= 0 \Rightarrow \\ (1+o)^2 \tilde{b}^2 + 2 \left[(c_1 - \tilde{a}) o^2 + l + \frac{d_2}{d_3} o \right] \underbrace{\tilde{b}}_{-(o\sqrt{\Delta_k} + l)} + 2 \left(l + \frac{d_4}{d_3} \right) o \sqrt{\Delta_k} + (\tilde{a}^2 - 2c_1 \tilde{a} + c_2) o^2 + l^2 + 2\frac{d_1}{d_3} o &= 0 \Rightarrow \\ (1+o)^2 \tilde{b}^2 + 2 \left[\frac{d_4}{d_3} - (c_1 - \tilde{a}) o^2 - \frac{d_2}{d_3} o \right] o \sqrt{\Delta_k} - 2 \left[(c_1 - \tilde{a}) o^2 + l + \frac{d_2}{d_3} o \right] l + (\tilde{a}^2 - 2c_1 \tilde{a} + c_2) o^2 + l^2 + 2\frac{d_1}{d_3} o &= 0. \end{aligned}$$

Finally, to complete the proof, it is enough to insert Eq. (19) and Eq. (20) in the above equation in order to substitute \tilde{b}^2 and $\sqrt{\Delta_k}$ and consequently to deduce Eq. (17). \square

Since o is an arbitrary parameter, then we can choose o as required to solve equation (17) of the unknown l :

Lemma 2.7. If o verifies the following polynomial equation of degree two:

$$(l_1^2 - 6l_3) o^2 + 2(l_1 l_2 + 6l_4) o + l_2^2 - 6l_5 = 0, \quad (21)$$

then $l = -\tilde{a} + r_m$, where

$$r_m = \frac{l_1 o + l_2}{6} + m s_l; \quad m \in \left\{ -1, \frac{1 - \sqrt{-3}}{2}, \frac{1 + \sqrt{-3}}{2} \right\}.$$

Here s_l is given by:

$$s_l := \sqrt[3]{\left(\frac{l_1 o + l_2}{6}\right)^3 - \frac{1}{2}(l_6 o^3 + l_7 o^2 + l_8 o - l_9)}.$$

Proof:

Equation (17) can be rewritten as follows

$$l^3 + 3\frac{B_l}{6}l^2 + 3\frac{C_l}{6}l + \left(\frac{B_l}{6}\right)^3 + s_l^3 - \frac{(l_1^2 - 6l_3)o^2 + 2(l_1l_2 + 6l_4)o + l_2^2 - 6l_5}{12}\tilde{a} = 0. \quad (22)$$

The condition $B_l^2 = 6C_l$, which is equivalent to Eq. (21), allows us to factorise Eq. (22) as follows:

$$\left(l + \frac{B_l}{6}\right)^3 + s_l^3 = \left(l + \frac{B_l}{6} + s_l\right) \left[\left(l + \frac{B_l}{6}\right)^2 - s_l\left(l + \frac{B_l}{6}\right) + s_l^2\right] = 0. \quad (23)$$

So, Eq. (23) can be solved easily, which ends the proofs. \square

Lemma 2.7 gives the expressions for the set of solutions of l . Then, we deduce from the change of variable (18) the following polynomial equation of \tilde{b} , which will be useful to give an explicit formula for the set of solutions of \tilde{b} :

Corollary 2.8. *If o verifies Eq. (21), then \tilde{b} verifies the following equation*

$$(1 - o^2)\tilde{b}^2 + 2[r_m - \tilde{a} + o^2(\tilde{a} - c_1)]\tilde{b} + (r_m - \tilde{a})^2 - o^2(\tilde{a}^2 - 2c_1\tilde{a} + c_2) = 0, \quad (24)$$

where

$$r_m \in \mathcal{S}_l := \left\{ \frac{l_1o + l_2}{6} - s_l; \frac{l_1o + l_2}{6} + \frac{1 - \sqrt{-3}}{2}s_l; \frac{l_1o + l_2}{6} + \frac{1 + \sqrt{-3}}{2}s_l \right\}.$$

Proof:

We have from Eq. (16) that

$$\sqrt{\Delta_k} = -\frac{1}{o}(\tilde{b} + l). \quad (25)$$

The proof ends by taking the square of Eq. (25) and afterwards replacing Δ_k by its value given by Eq. (3) in the obtained equation, where l is replacing by $r_m - \tilde{a}$ from lemma 2.7. \square

Finally, in the fourth step, we can determine the set of solutions of \tilde{b} by solving the nonlinear system constructed by Eq. (24) and one of the following equations (9), (11), (13) or (14) of \tilde{b} . Thus, we can solve this nonlinear system by choosing \tilde{a} as required to simplify the calculation. Consequently, we deduce from Eq. (1) the spectrum of \mathbf{M} :

Corollary 2.9. *The spectrum of matrix \mathbf{M} is given by:*

$$k = \frac{1}{2} \left[\left(1 + \frac{1}{o}\right) \frac{d_3r_m^2 + s_3(1-o)r_m - s_2o(1-o) - d_3c_2o^2}{(1-o)(s_3 + s_1o) - 2d_3c_1o^2 + d_3(1+o)r_m} + 2m_1 - c_1 - \frac{r_m}{o} \right], \quad (26)$$

where $r_m \in \mathcal{S}_l$ and o verifies Eq. (21).

Proof:

Inserting Eq. (25) in Eq. (9), we get an equation of \tilde{b} and \tilde{b}^2 . Then, we substitute \tilde{b}^2 using Eq. (24) to obtain:

$$\tilde{b} = \tilde{a} - \frac{d_3r_m^2 + (1-o)s_3r_m - s_2o(1-o) - d_3c_2o^2}{(1-o)(s_3 + s_1o) - 2d_3c_1o^2 + d_3(1+o)r_m}. \quad (27)$$

Finally, to deduce Eq. (26), we insert Eq. (25) in Eq. (1) in order to substitute $\sqrt{\Delta_k}$ and afterwards we insert Eq. (27) in the obtained equation in order to substitute \tilde{b} . \square

The denominator of the fraction in Eq. (26) contains r_m which involves a cubic root. So, we calculate another expression for spectrum of \mathbf{M} in order to rationalize this denominator. The two expressions of spectrum of \mathbf{M} will be useful in the proof of theorem 3.3.

Corollary 2.10. *The spectrum of matrix \mathbf{M} is given by:*

$$k = \frac{1}{2} \left[\left(1 + \frac{1}{o}\right) \frac{s_9o^2 - s_8o + (r_6 - s_{10}o)r_m}{(s_7 - s_{10})o + r_2o^2 + r_6 + 2or_m} + 2m_1 - c_1 - \frac{r_m}{o} \right], \quad (28)$$

where $r_m \in \mathcal{S}_l$ and o verifies Eq. (21).

Proof:

We insert Eq. (25) in both Eq. (9) and Eq. (11) to replace $\sqrt{\Delta_k}$ by its value in terms of \tilde{b} , l and o . Then, by substituting \tilde{b}^2 from one obtained equation in the other, we deduce that:

$$\tilde{b} = \tilde{a} - \frac{s_9o^2 - s_8o + (r_6 - s_{10}o)r_m}{(s_7 - s_{10})o + r_2o^2 + r_6 + 2or_m}. \quad (29)$$

Finally, to deduce Eq. (28), we insert Eq. (25) in Eq. (1) to substitute $\sqrt{\Delta_k}$ and afterwards we insert Eq. (29) in the obtained equation to substitute \tilde{b} . \square

3 Application on the cubic polynomial

Let (\mathcal{P}) be the general complex polynomial of degree three in $\mathbb{C}[x]$:

$$(\mathcal{P}) : x^3 + bx^2 + cx + d = 0; \quad b, c, d \in \mathbb{C}.$$

We aim to calculate analytically the roots of (\mathcal{P}) by applying the results of section 2. So, we introduce the following definitions which will be useful to find the expressions for the roots of (\mathcal{P}) :

Definition 3.1. We associate to (\mathcal{P}) the following expressions in terms of its coefficients:

$$\begin{aligned} \Delta_l &:= 2c^3(8b^6 + 132b^3d + 36d^2 + c^3 + 33b^2c^2 - 66bcd) + 12b^4c(d^2 - 7c^3) - b^2c^2d(24b^3 + 291d) + d^3(144bc - 2b^3 - 27d), \\ \Delta_o &:= -4b^3d + b^2c^2 + 18bcd - 4c^3 - 27d^2, \end{aligned}$$

and

$$d_o := 4b^4c^2 - 4b^3cd - 14b^2c^3 + b^2d^2 + 28bc^2d + c^4 - 12cd^2.$$

Definition 3.2. We associate to (\mathcal{P}) the following expressions in terms of these given by definition 3.1:

$$\delta_l := (d - bc)\sqrt{\Delta_o}(4b^2c^2 - 4bcd + 2c^3 + d^2) + \frac{\sqrt{-3}}{9}\Delta_l,$$

and

$$\begin{aligned} A_1 &:= -\frac{2\sqrt{-3}}{3}(4b^3c - 2db^2 - 13bc^2 + 15dc) + 2c\sqrt{\Delta_o}, \\ A_2 &:= 8b^5c^2 - 8b^4cd - 40b^3c^3 + 2b^3d^2 + 116b^2c^2d + 23bc^4 - 99bcd^2 - 21c^3d + 27d^3 - \sqrt{-3}(8b^2c^2 - 10bcd + c^3 + 3d^2)\sqrt{\Delta_o}. \end{aligned}$$

Now, we present the principal result of the paper which is the formula of the roots of (\mathcal{P}) . This formula is in fact a Lagrange-type [11], where the square and cubic are given by the standard convention, contrary to Cardan-Tartaglia and Lagrange formulas which are incorrect with the standard convention. Indeed, the difference with the previous works is the two terms α_1 and α_2 of Eq. (30) which were missed in Cardan-Tartaglia and Lagrange formulas:

Theorem 3.3. Let the two cubic roots \mathcal{R}_1 and \mathcal{R}_2 :

$$\begin{aligned} \mathcal{R}_1 &:= \sqrt[3]{\frac{\sqrt{-3}}{9}\sqrt{\Delta_o} + \frac{2b^3 - 9cb + 27d}{27}}, \\ \mathcal{R}_2 &:= \sqrt[3]{\frac{\sqrt{-3}}{9}\sqrt{\Delta_o} - \frac{2b^3 - 9cb + 27d}{27}}. \end{aligned}$$

The set of solutions of (\mathcal{P}) is given in terms of \mathcal{R}_1 and \mathcal{R}_2 as follows:

$$x_m = m\alpha_1\mathcal{R}_1 + m^2\alpha_2\mathcal{R}_2 - \frac{b}{3}; \quad m \in \left\{-1, \frac{1 - \sqrt{-3}}{2}, \frac{1 + \sqrt{-3}}{2}\right\}, \quad (30)$$

where α_1 and α_2 are defined as follows:

$$\begin{aligned} \alpha_1 &:= \frac{\sqrt[3]{4}}{2} \exp \left\{ \sqrt{-1} \left[\arg \left(A_1 \sqrt[3]{\delta_l} \right) - \arg \left(-d_o \mathcal{R}_1 \right) \right] \right\}, \\ \alpha_2 &:= \frac{\sqrt[3]{4}}{2} \exp \left\{ \sqrt{-1} \left[\arg \left(A_2 \sqrt[3]{\delta_l^2} \right) - \arg \left(d_o^2 \mathcal{R}_2 \right) \right] \right\}. \end{aligned}$$

Here, Δ_o , d_o , δ_l , A_1 and A_2 are given by definitions 3.1 and 3.2.

Proof:

Firstly, we construct the following matrix

$$\mathbf{A} := \begin{pmatrix} -b & 1 & d^{-1}c \\ 0 & 0 & 1 \\ -d & 0 & 0 \end{pmatrix},$$

which has (\mathcal{P}) as its characteristic polynomial. Consequently, the roots of (\mathcal{P}) are identical to the spectrum of \mathbf{A} , which can be calculated from corollaries 2.9 and 2.10. But, these corollaries require firstly to make explicitly o and r_m relative to matrix \mathbf{A} . Indeed, lemma 2.7 gives Eq. (21) of o and the expression of l in terms of the components of \mathbf{M} . So, we identify the components of \mathbf{M} with these of \mathbf{A} so that $\mathbf{M} = \mathbf{A}$. In other words, we replace in expressions of section 2 the components of \mathbf{M} by their corresponding values in terms of b , c and d . Thus, we deduce that

$$o = \frac{b^2c^3 - c^4 - 3cd^2 + b^2d^2 - 2b^3cd \pm \sqrt{-3}c(d - bc)\sqrt{\Delta_o}}{d_o}, \quad (31)$$

since Eq. (21) is a quadratic polynomial equation for o . In the sequel, we take the positive sign before $\sqrt{-3}c(d - bc)\sqrt{\Delta_o}$ in Eq. (31). Also, the applying of lemma 2.7 on matrix \mathbf{A} implies that

$$r_m = l_o - m\sqrt[3]{4}\frac{c}{d_o}\sqrt{\Delta_o}\sqrt[3]{\delta_l}; \quad m \in \left\{-1, \frac{1 - \sqrt{-3}}{2}, \frac{1 + \sqrt{-3}}{2}\right\}.$$

Here, l_o is defined as follows:

$$l_o := d_o^{-1} \left[2b^4 cd + b^3 c^3 - b^3 d^2 - 5b^2 c^2 d - 4bc^4 + 6bcd^2 + 5c^3 d + \frac{\sqrt{-3}}{3} c (b^2 c - 2bd - c^2) \sqrt{\Delta_o} \right].$$

Secondly, since we have the expressions of o and r_m relative to matrix \mathbf{A} , then by applying corollaries 2.9 and 2.10 on matrix \mathbf{A} , we deduce that the roots of (\mathcal{P}) are given by

$$x_m = \frac{1}{2} \left[\left(1 + \frac{1}{o} \right) \frac{-cr_m^2 + d(1-o)r_m + bdo(1-o) + c(b^2 - 4c)o^2}{(1-o)[d + (bc-d)o] - 2bco^2 - c(1+o)r_m} - b - \frac{r_m}{o} \right] = \frac{1}{2} \left[\left(1 + \frac{1}{o} \right) \frac{bd(bc-d) + cd(b^2 - 2c) - d(2c^2 - bd)o^{-1} + do^{-1}(do^{-1} - d + 2bc)r_m}{2b^2c^2 - 4bcd - 2c^3 + d^2 - 2(c^3 - bcd + d^2)o^{-1} + d^2o^{-2} - 2c(d-bc)o^{-1}r_m} - b - \frac{r_m}{o} \right]. \quad (32)$$

Then, we aim to simplify the expression of x_m in Eq. (32), indeed Eq. (32) itself implies that

$$\frac{-cr_m^2 + d(1-o)r_m + bdo(1-o) + c(b^2 - 4c)o^2}{(1-o)[d + (bc-d)o] - 2bco^2 - c(1+o)r_m} = \frac{bd(bc-d) + cd(b^2 - 2c) - d(2c^2 - bd)o^{-1} + do^{-1}(do^{-1} - d + 2bc)r_m}{2b^2c^2 - 4bcd - 2c^3 + d^2 - 2(c^3 - bcd + d^2)o^{-1} + d^2o^{-2} - 2c(d-bc)o^{-1}r_m}. \quad (33)$$

So, the following property allows us to eliminate, in the denominator of the fractions of Eq. (33), the cubic root concealed in r_m :

$$\frac{\mathcal{A}}{\mathcal{B}} = \frac{\mathcal{C}}{\mathcal{D}} \Rightarrow \frac{\mathcal{A}}{\mathcal{B}} = \frac{\mathcal{C}}{\mathcal{D}} = \frac{\mathcal{M}\mathcal{A} + \mathcal{N}\mathcal{C}}{\mathcal{M}\mathcal{B} + \mathcal{N}\mathcal{D}}; \quad \forall \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{M}, \mathcal{N} \in \mathbb{C}^*. \quad (34)$$

By applying property (34) on Eq. (33) for $\mathcal{M} = 2o^{-1}c(d-bc)$ and $\mathcal{N} = -c(1+o)$, we just obtain a square root in the denominator as follows:

$$x_m = -\frac{b}{3} + \frac{m^2 \sqrt[3]{4^2 \delta_l^2}}{12\delta_l} \left[\frac{\sqrt{-3}}{3} (4b^3c - 2db^2 - 13bc^2 + 15dc) + c\sqrt{\Delta_o} \right] - m \sqrt[3]{4\delta_l} \times \frac{8b^5c^2 - 8b^4cd - 40b^3c^3 + 2b^3d^2 + 116b^2c^2d + 23bc^4 - 99bcd^2 - 21c^3d + 27d^3 + \sqrt{-3}(8b^2c^2 - 10bcd + c^3 + 3d^2)\sqrt{\Delta_o}}{18\delta_l}. \quad (35)$$

Therefore, we rationalize Eq. (35) using the following identities:

$$8b^5c^2 - 8b^4cd - 40b^3c^3 + 2b^3d^2 + 116b^2c^2d + 23bc^4 - 99bcd^2 - 21c^3d + 27d^3 + \sqrt{-3}(8b^2c^2 - 10bcd + c^3 + 3d^2)\sqrt{\Delta_o} = \frac{9A_1\delta_l}{4d_o},$$

$$\frac{2\sqrt{-3}}{3} (4b^3c - 2db^2 - 13bc^2 + 15dc) + 2c\sqrt{\Delta_o} = \frac{3A_2\delta_l}{d_o^2}$$

and consequently we deduce that

$$x_m = -\frac{\sqrt[3]{4}}{8} \frac{md_o A_1 \sqrt[3]{\delta_l} - m^2 \sqrt[3]{4} A_2 \sqrt[3]{\delta_l^2}}{d_o^2} - \frac{b}{3}. \quad (36)$$

Thirdly, in order to simplify d_o^2 in the denominator of Eq. (36), we prove that:

$$\left(A_1 \sqrt[3]{\delta_l} \right)^3 = (-4d_o \mathcal{R}_1)^3 \Rightarrow A_1 \sqrt[3]{\delta_l} = \exp \left\{ \sqrt{-1} \left[\arg \left(A_1 \sqrt[3]{\delta_l} \right) - \arg \left(-4d_o \mathcal{R}_1 \right) \right] \right\} (-4d_o \mathcal{R}_1),$$

$$\left(A_2 \sqrt[3]{\delta_l^2} \right)^3 = \left(\sqrt[3]{4^2 d_o^2 \mathcal{R}_2} \right)^3 \Rightarrow A_2 \sqrt[3]{\delta_l^2} = \exp \left\{ \sqrt{-1} \left[\arg \left(A_2 \sqrt[3]{\delta_l^2} \right) - \arg \left(\sqrt[3]{4^2 d_o^2 \mathcal{R}_2} \right) \right] \right\} \sqrt[3]{4^2 d_o^2 \mathcal{R}_2}. \quad (37)$$

Then, by inserting Eq. (37) in Eq. (36), we deduce Eq. (30). \square

Acknowledgments

I would like to thank, in particularly, Dalia Ibrahim for her support during the difficult moment to work this paper. Also, I wish to thank the reviewers of the first submission for their useful comments.

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