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HOPF ALGEBRA STRUCTURE ON PACKED SQUARE MATRICES

HAYAT CHEBALLAH, SAMUELE GIRAUDO, AND RÉMI MAURICE

Abstract. We construct a new bigraded Hopf algebra whose bases are indexed by square matrices with entries in the alphabet \( \{0, 1, \ldots, k\} \), \( k \geq 1 \), without null rows or columns. This Hopf algebra generalizes the one of permutations of Malvenuto and Reutenauer, the one of \( k \)-colored permutations of Novelli and Thibon, and the one of uniform block permutations of Aguiar and Orellana. We study the algebraic structure of our Hopf algebra and show, by exhibiting multiplicative bases, that it is free. We moreover show that it is self-dual and admits a bidendriform bialgebra structure. Besides, as a Hopf subalgebra, we obtain a new one indexed by alternating sign matrices. We study some of its properties and algebraic quotients defined through alternating sign matrices statistics.

Contents

Introduction 1
1. Packed matrices 3
   1.1. Definitions 3
   1.2. Enumeration 5
   1.3. Hopf algebra structure 6
2. Algebraic properties 8
   2.1. Multiplicative bases and freeness 8
   2.2. Self-duality 11
   2.3. Bidendriform bialgebra structure 13
3. Related Hopf algebras 16
   3.1. Links with known algebras 16
   3.2. Equivalence relations and Hopf subalgebras 20
4. Alternating sign matrices 24
   4.1. Hopf algebra structure 25
   4.2. Alternating sign matrices statistics 27
   4.3. Algebraic interpretation of some statistics 29
References 34

Introduction

The combinatorial class of permutations is naturally endowed with two operations. One of them, called \textit{shifted shuffle product}, takes two permutations as input and put these together by blending their letters. The other one, called \textit{deconcatenation coproduct}, takes one permutation as input and
takes it apart by cutting it into prefixes and suffixes. These two operations satisfy certain compatibility relations, resulting in that the vector space spanned by the set of permutations forms a Hopf algebra [MR95], namely the Malvenuto-Reutenauer Hopf algebra, also known as $\text{FQSym}$ [DHT02].

This Hopf algebra plays a central role in algebraic combinatorics for at least two reasons. On the one hand, $\text{FQSym}$ contains, as Hopf subalgebras, several structures based on well-known combinatorial objects as e.g., standard Young tableaux [DHT02], binary trees [HNT05], and integer compositions [GKL+95]. The construction of these substructures revisits many algorithms coming from computer science and combinatorics. Indeed, the insertion of a letter into a Young tableau (following Robinson-Schensted [Sch61]) or in a binary search tree [Knu98] are algorithms which prove to be as enlightening as surprising in this algebraic context [DHT02,HNT02,HNT05]. On the other hand, the polynomial realization of $\text{FQSym}$ allows to associate a polynomial with any permutation [DHT02] providing a generalization of symmetric functions, the free quasi-symmetric functions. This generalization offers alternative ways to prove several properties of (quasi)symmetric functions.

It is thus natural to enrich this theory by proposing generalizations of $\text{FQSym}$. In the last years, several generalizations were proposed and each of these depends on the way we regard permutations. By regarding a permutation as a word and allowing repetitions of letters, Hivert introduced in [Hiv99] (see [NT06] for a detailed study) a Hopf algebra $\text{WQSym}$ on packed words. Additionally, by allowing some jumps for the values of the letters of permutations, Novelli and Thibon defined in [NT07] another Hopf algebra $\text{PQSym}$ which involves parking functions. These authors also showed in [NT10] that the $k$-colored permutations admit a Hopf algebra structure $\text{FQSym}^{(k)}$. Furthermore, by regarding a permutation $\sigma$ as a bijection associating the singleton $\{\sigma(i)\}$ with any singleton $\{i\}$, Aguiar and Orellana constructed [AO08] a Hopf algebra structure $\text{UBP}$ on uniform block permutations, i.e., bijections between set partitions of $[n]$, where each part has the same cardinality as its image. Finally, by regarding a permutation within its permutation matrix, Duchamp, Hivert and Thibon introduced in [DHT02] a Hopf algebra $\text{MQSym}$ which involves some kind of integer matrices.

In this paper we propose a new generalization of $\text{FQSym}$ by regarding permutations as permutation matrices. For this purpose, we consider the set of $1$-packed matrices that are square matrices with entries in the alphabet $\{0,1\}$ which have at least one $1$ by row and by column. By equipping these matrices with a product and a coproduct, we obtain a bigraded Hopf algebra, denoted by $\text{PM}_1$. By only considering the gradation offered by the size (resp. the number of nonzero entries) of matrices, we obtain a simply graded Hopf algebra denoted by $\text{PMN}_1$ (resp. $\text{PML}_1$). Note that since permutation matrices form a Hopf subalgebra of $\text{PMN}_1$ (and $\text{PML}_1$) isomorphic to $\text{FQSym}$, $\text{PMN}_1$ (and $\text{PML}_1$) provides a generalization of $\text{FQSym}$. Now, by allowing the entries different from $0$ of a packed matrix to belong to the alphabet $\{1,\ldots,k\}$ where $k$ is a positive integer, we obtain the notion of a $k$-packed matrix. The definition of $\text{PM}_1$ (and $\text{PMN}_1$ and $\text{PML}_1$) obviously extends to these matrices and leads to the Hopf algebra $\text{PM}_k$ (and $\text{PMN}_k$ and $\text{PML}_k$) involving $k$-packed matrices. Besides, since any $k$-packed matrix is also a $k+1$-packed matrix, $(\text{PM}_k)_{k\geq 1}$ is an increasing infinite sequence of Hopf algebras for inclusion.

Our results are presented as follows. We give in Section 1 some elementary definitions about $k$-packed matrices, enumerate them according to their size, and then define the Hopf algebra of $k$-packed matrices by describing its product and its coproduct. Section 2 is devoted to the study of the algebraic properties of $\text{PM}_k$. In order to show that $\text{PM}_k$ is free as an algebra, we define, by introducing a partial order relation on the $k$-packed matrices, two multiplicative bases: the bases of the elementary and homogeneous elements. We then describe the dual Hopf algebra $\text{PM}_k^*$ of $\text{PM}_k$ in explaining the product and the coproduct and show that $\text{PM}_k$ is self-dual. In Section 3, we show how several well-known Hopf algebras are linked with $\text{PM}_k$. In particular, we show that the Hopf
HOPF ALGEBRA ON PACKED SQUARE MATRICIES

algebra of the $k$-colored permutations $\text{FQSym}^{(k)}$ embeds into $\text{PMN}_k$ (and $\text{PML}_k$) and that the dual $\text{UBP}^*$ of the Hopf algebra of uniform block permutations embeds into $\text{PMN}_1$. We also exhibit an injective algebra morphism from $\text{PML}_1^*$ to $\text{MQSym}$. We conclude this section by providing a method to construct Hopf subalgebras of $\text{PM}_k$, analogous to the construction of Hopf subalgebras of $\text{FQSym}$ by good congruences [HN07,Gir11]. The analogs of the sylvester [HNT02,HNT05], plactic [LS81,Lot02], hypoplactic [KT97,KT99], Bell [Rey07] and Baxter [Gir12] congruences are still good congruences in our context and give rise to Hopf subalgebras of $\text{PM}_k$. We end this article by Section 4 where we show that $\text{PMN}_1$ contains a Hopf subalgebra whose bases are indexed by alternating sign matrices, denoted by $\text{ASM}$. We consider then some well-known statistics on the six-vertex model with domain wall boundary conditions [Kor82], that are combinatorial objects in bijection with alternating sign matrices [Kup96,Bre99]. We study these statistics from the algebraic point of view offered by the Hopf algebra $\text{ASM}$. This section is concluded with a complete study of quotients of $\text{ASM}$ by equivalence relations defined through these statistics.

Acknowledgements. This work is based on computer exploration and the authors used, for this purpose, the open-source mathematical software Sage [S+12] and one of its extensions, Sage-Combinat [SCc08]. The authors would like to thank the anonymous referees which, by their suggestions, greatly improved Sections 3.2, 4.2, and 4.3.

1. Packed matrices

1.1. Definitions. Let $k \geq 1$ be an integer. We denote by $\mathcal{M}_{k,n,\ell}$ the set of $n \times n$ matrices with exactly $\ell$ nonzero entries in the alphabet $A_k := \{0,1,\ldots,k\}$ and by $N_r(M)$ (resp. $N_c(M)$) the set of the indices of the zero rows (resp. columns) of $M \in \mathcal{M}_{k,n,\ell}$. For example, consider the matrix

(1.1.1) \[
M := \begin{bmatrix}
0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}.
\]

We have

(1.1.2) \[N_r(M) = \{5\} \quad \text{and} \quad N_c(M) = \{1,3\}.\]

A $k$-packed matrix $M$ of size $n$ is a matrix in $\bigcup_{\ell \geq 0} \mathcal{M}_{k,n,\ell}$ in which each row and each column contains at least one entry different from 0, that is to say if the subsets $N_r(M)$ and $N_c(M)$ are empty.

We shall denote in the sequel by $\mathcal{P}_{k,n,\ell}$ the set of $k$-packed matrices of size $n$ with exactly $\ell$ nonzero entries, by $\mathcal{P}_{k,n,-}$ the set of all $k$-packed matrices of size $n$, by $\mathcal{P}_{k,-,\ell}$ the set of all $k$-packed matrices with exactly $\ell$ nonzero entries, and by $\mathcal{P}_k$ the set of all $k$-packed matrices. The $k$-packed matrix of size 0 is denoted by $\emptyset$. For instance, the seven 1-packed matrices of size 2 are

(1.1.3) \[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}, \quad \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}, \quad \begin{bmatrix}
1 & 1 \\
0 & 1
\end{bmatrix}, \quad \begin{bmatrix}
1 & 1 \\
1 & 1
\end{bmatrix}, \quad \begin{bmatrix}
1 & 1 \\
1 & 1
\end{bmatrix}, \quad \begin{bmatrix}
0 & 1 \\
0 & 1
\end{bmatrix}, \quad \begin{bmatrix}
1 & 1 \\
1 & 1
\end{bmatrix}.
\]

Besides, the ten 1-packed matrices of $\mathcal{P}_{1,-,3}$ are

(1.1.4) \[
\begin{bmatrix}
1 & 1 \\
1 & 0
\end{bmatrix}, \quad \begin{bmatrix}
1 & 1 \\
0 & 1
\end{bmatrix}, \quad \begin{bmatrix}
1 & 1 \\
1 & 1
\end{bmatrix}, \quad \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}, \quad \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}, \quad \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}, \quad \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}, \quad \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}, \quad \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}, \quad \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}.
\]
Let us now define some operations on packed matrices. We shall denote by $Z^n_n$ the $n \times m$ null matrix. Given $M_1$ and $M_2$ two $k$-packed matrices of respective sizes $n_1$ and $n_2$, set

\begin{equation}
M_1 / M_2 := \begin{bmatrix} Z^n_{n_2} & M_1[n_2] \\ Z^n_{n_2} & M_2 \end{bmatrix} \quad \text{and} \quad M_1 \backslash M_2 := \begin{bmatrix} Z^n_{n_2} & M_1[n_1] \\ M_2[n_2] & Z^n_{n_2} \end{bmatrix}.
\end{equation}

Note that these two matrices are $k$-packed matrices of size $n_1 + n_2$. We shall respectively call $/$ and \ the over and under operators. These two operators are obviously associative.

Given a matrix $M$ whose entries are elements of the alphabet $A_k$, the compression of $M$ is the matrix $\text{cp}(M)$ obtained by deleting in $M$ all null rows and columns. Let $M$ be a $k$-packed matrix. The tuple $(M_1, \ldots, M_r)$ is a column decomposition of $M$, and we write $M = M_1 \circ \cdots \circ M_r$, if for all $i \in [r]$ the $\text{cp}(M_i)$ are square matrices (and not necessarily column matrices) and

\begin{equation}
M = \begin{bmatrix} M_1 \\ \vdots \\ M_r \end{bmatrix}.
\end{equation}

Similarly, the tuple $(M_1, \ldots, M_r)$ is a row decomposition of $M$, and we write $M = M_1 \bullet \cdots \bullet M_r$, if for all $i \in [r]$ the $\text{cp}(M_i)$ are square matrices (and not necessarily row matrices) and

\begin{equation}
M = \begin{bmatrix} M_1 \\ \vdots \\ M_r \end{bmatrix}.
\end{equation}

For instance, here are a 1-packed matrix of size 5, one of its column decompositions and one of its row decompositions:

\begin{equation}
\begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \circ \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 1 \end{bmatrix}.
\end{equation}

These two decompositions have the following property.

**Lemma 1.1.** Let $M$ be a packed square matrix and $(M_1, M_2)$ be a column (resp. row) decomposition of $M$. Then, there is no integer $i$ such that the $i$th rows (resp. columns) of $M_1$ and $M_2$ contain both a nonzero entry.

**Proof.** We prove here the lemma only when $(M_1, M_2)$ is a column decomposition of $M$. The case of a row decomposition can be proven in an analogous way.

Let us denote by $n$ the size of $M$ and assume that $M_1$ (resp. $M_2$) has $n_1$ (resp. $n_2$) columns. The lemma follows from the fact that since $(M_1, M_2)$ is a column decomposition of $M$, there are $n_1$ nonzero rows in $M_1$, $n_2$ nonzero rows in $M_2$, and $n = n_1 + n_2$.

Lemma 1.1 provides a sufficient condition to ensure that a given pair $(M_1, M_2)$ of matrices cannot be a column (resp. row) decomposition of a matrix $M$. Nevertheless, it is not a necessary condition. Indeed, let

\begin{equation}
M := \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad (M_1, M_2) := \left( \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right).
\end{equation}

Then, even if there is no nonzero entry on the same row in $M_1$ and $M_2$, $(M_1, M_2)$ is not a column decomposition of $M$. 

\[\square\]
1.2. Enumeration. Using the sieve principle, we obtain the following enumerative result.

**Proposition 1.2.** For any $k \geq 1$, $n \geq 0$, and $\ell \geq 0$, the number $\#P_{k,n,\ell}$ of $k$-packed matrices of size $n$ with exactly $\ell$ nonzero entries is

\[
\#P_{k,n,\ell} = \sum_{0 \leq i,j \leq n} (-1)^{i+j} \binom{n}{i} \binom{n}{j} \binom{ij}{\ell} k\ell.
\]

**Proof.** For any subsets $R$ and $C$ of $[n]$ let us define the set

\[
S(R, C) := \{ M \in M_{k,n,\ell} : N_r(M) = R \text{ and } N_c(M) = C \}.
\]

Since $\#P_{k,n,\ell} = \#S(\emptyset, \emptyset)$, we shall compute $\#S(\emptyset, \emptyset)$ to prove (1.2.1).

For that, let us consider the order relation $\preceq$ defined on the set of pairs $(R, C)$ of subsets of $[n]$ by

\[
(R_1, C_1) \preceq (R_2, C_2) \text{ if and only if } R_1 \subseteq R_2 \text{ and } C_1 \subseteq C_2.
\]

We have, by setting $r := \#R$ and $c := \#C$,

\[
\sum_{(R, C) \preceq (R', C')} \#S(R', C') = \binom{(n-r)(n-c)}{\ell} k\ell
\]

since (1.2.4) is the number of matrices $M \in M_{k,n,\ell}$ such that $R \subseteq N_r(M)$ and $C \subseteq N_c(M)$. Then, by Möbius inversion on the Boolean lattice, we obtain

\[
\#S(\emptyset, \emptyset) = \sum_{(\emptyset, \emptyset) \preceq (R, C)} (-1)^{r+c} \binom{(n-r)(n-c)}{\ell} k\ell,
\]

and (1.2.1) follows. \hfill \Box

Table 1 shows the first few values of $\#P_{k,n,\ell}$. The enumeration in the case $k = 1$ is Sequence A055599 of [Slo].

<table>
<thead>
<tr>
<th>(a) Number of 1-packed matrices.</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>4</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>6</td>
<td>45</td>
<td>90</td>
<td>78</td>
<td>36</td>
<td>9</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(b) Number of 2-packed matrices.</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>8</td>
<td>32</td>
<td>16</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>48</td>
<td>720</td>
<td>2880</td>
<td>4992</td>
<td>4608</td>
<td>2304</td>
<td>512</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Table 1.** The number of $k$-packed matrices of size $n$ (vertical values) with exactly $\ell$ nonzero entries (horizontal values).
Notice that for any \( n \geq 0 \), since
\[
\mathcal{P}_{k,n,-} = \bigcup_{n \leq \ell \leq n^2} \mathcal{P}_{k,n,\ell},
\]
the set \( \mathcal{P}_{k,n,-} \) is finite. Hence, by using Proposition 1.2, we obtain
\[
\# \mathcal{P}_{k,n,-} = \sum_{0 \leq i,j \leq n} (-1)^{i+j} \binom{n}{i} \binom{n}{j} (k+1)^{ij}.
\]
Sequences \( (#\mathcal{P}_{1,-})_{n \geq 0} \) and \( (#\mathcal{P}_{2,-})_{n \geq 0} \) respectively start with
\[
1, 1, 7, 265, 45103, 24997921, 57366997447, [\text{Slo, A048291}]
\]
and
\[
1, 2, 56, 16064, 39156608, 813732073472, 14766286695991296.
\]

Similarly, since for any \( \ell \geq 0 \),
\[
\mathcal{P}_{k,-,\ell} = \bigcup_{\sqrt{\ell} \leq n \leq \ell} \mathcal{P}_{k,n,\ell},
\]
the set \( \mathcal{P}_{k,-,\ell} \) is finite. Hence, by using Proposition 1.2, we obtain
\[
\# \mathcal{P}_{k,-,\ell} = \sum_{0 \leq i,j \leq \ell} (-1)^{i+j} \binom{\ell}{i} \binom{\ell}{j} (ij)^{k\ell}.
\]
Sequences \( (#\mathcal{P}_{1,-,\ell})_{\ell \geq 0} \) and \( (#\mathcal{P}_{2,-,\ell})_{\ell \geq 0} \) respectively start with
\[
1, 1, 2, 10, 70, 642, 7246, 97052, 1503700, [\text{Slo, A104602}]
\]
and
\[
1, 2, 8, 80, 1120, 20544, 463744, 12422656, 384947200.
\]

1.3. **Hopf algebra structure.** In the sequel, all the algebraic structures have a field \( K \) of characteristic zero as ground field.

Let for any \( k \geq 1 \)
\[
\text{PM}_k := \bigoplus_{n \geq 0} \bigoplus_{\ell \geq 0} \text{Vect} (\mathcal{P}_{k,n,\ell})
\]
be the bigraded vector space spanned by the set of all \( k \)-packed matrices. The elements \( F_M \), where the \( M \) are \( k \)-packed matrices, form a basis of \( \text{PM}_k \). We shall call this basis the **fundamental basis** of \( \text{PM}_k \).

Given \( M_1 \) and \( M_2 \) two \( k \)-packed matrices of respective sizes \( n_1 \) and \( n_2 \), set
\[
M_1 \circ n_2 := \left[ \frac{M_1}{Z_{n_1}^{n_2}} \right] \quad \text{and} \quad n_1 \circ M_2 := \left[ \frac{Z_{n_2}^{n_1}}{M_2} \right].
\]
The **column shifted shuffle** \( M_1 \uplus M_2 \) of \( M_1 \) and \( M_2 \) is the set of all matrices obtained by shuffling the columns of \( M_1 \circ n_2 \) with the columns of \( n_1 \circ M_2 \).

Let us endow \( \text{PM}_k \) with a product \( \cdot \) linearly defined, for any \( k \)-packed matrices \( M_1 \) and \( M_2 \), by
\[
F_{M_1} \cdot F_{M_2} := \sum_{M \in M_1 \uplus M_2} F_M.
\]
For instance, in $\text{PM}_1$ one has
\[
\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.
\]
(1.3.4)

Moreover, we endow $\text{PM}_k$ with a coproduct $\Delta$ linearly defined, for any $k$-packed matrix $M$, by
\[
\Delta (\begin{bmatrix} F \end{bmatrix}_M) := \sum_{M = M_1 \ldots M_2} \begin{bmatrix} F \end{bmatrix}_{cp(M_1)} \otimes \begin{bmatrix} F \end{bmatrix}_{cp(M_2)}.
\]
(1.3.5)

For instance, in $\text{PM}_1$ one has
\[
\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\]
(1.3.6)

Note that by definition, the product and the coproduct of $\text{PM}_k$ are multiplicity free.

**Theorem 1.3.** The vector space $\text{PM}_k$ endowed with the product $\cdot$ and the coproduct $\Delta$ is a bigraded and connected bialgebra where homogeneous components are finite-dimensional.

**Proof.** First, it is plain that the product of $\text{PM}_k$ respects the bigradation. Moreover, Lemma 1.1 implies that it is also the case for its coproduct. Since $\emptyset$ is the only packed matrix of size 0 without nonzero entries, $\text{PM}_k$ is connected. Besides, since for all $n, \ell \geq 0$, the sets $\mathcal{P}_{k,n,\ell}$ are finite, homogeneous components of $\text{PM}_k$ are finite-dimensional.

The associativity of $\cdot$ arises from the associativity of the shifted shuffle operation on words on a totally ordered alphabet. Indeed, a packed matrix $M$ can be seen as a word $u$ where the $i$th letter of $u$ is the $i$th column of $M$. Moreover, the coassociativity of $\Delta$ comes from the fact that $(M_1 \cdot M_2) \cdot M_3$ is a column decomposition of a packed matrix $M$ if and only if $M_1 \cdot (M_2 \cdot M_3)$ also is.

It remains to show that $\Delta$ is an algebra morphism. Let $M_1$ and $M_2$ be two packed matrices. The obvious fact that $(L, R)$ is a column decomposition of a matrix $M$ appearing in the shifted shuffle of $M_1$ and $M_2$ if and only if $L$ (resp. $R$) appears in the shifted shuffle of $L_1$ and $L_2$ (resp. $R_1$ and $R_2$) where $(L_1, R_1)$ is a column decomposition of $M_1$ and $(L_2, R_2)$ is a column decomposition of $M_2$, ensures that $\Delta$ is an algebra morphism. \qed

Since $\text{PM}_k$ is, by Theorem 1.3, a bigraded and connected bialgebra, it admits an antipode and hence, is a Hopf algebra. The antipode $S$ of $\text{PM}_k$ satisfies, for any $k$-packed matrix $M$,
\[
S (\begin{bmatrix} F \end{bmatrix}_M) = \sum_{\ell \geq 1} (-1)^\ell \begin{bmatrix} F \end{bmatrix}_{cp(M_1)} \cdot \cdots \cdot \begin{bmatrix} F \end{bmatrix}_{cp(M_\ell)}.
\]
(1.3.7)
For instance, in $\text{PM}_1$ one has
\[
SF \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = -F \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} + F[1] \cdot F \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}
\] (1.3.8)
\[
= F \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} + F \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} + F \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} - F \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.
\]

Note besides that $S$ is not an involution. Indeed,
\[
S^2F \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = F \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + F \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} + F \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} - F \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}
\] (1.3.9)
\[
= F \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} - F \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} - F \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.
\]

Notice that since any $k$-packed matrix is also a $k + 1$-packed matrix, the vector space $\text{PM}_k$ is included in $\text{PM}_{k+1}$. Hence, and by Theorem 1.3,
\[
\text{PM}_1 \hookrightarrow \text{PM}_2 \hookrightarrow \cdots
\] (1.3.10)
is an increasing infinite sequence of Hopf algebras for inclusion. The first few dimensions of $\text{PM}_1$ and $\text{PM}_2$ are given by Table 1.

Let us now set
\[
\text{PMN}_k := \bigoplus_{n \geq 0} \text{Vect}(\mathcal{P}_{k,n,-}) \quad \text{and} \quad \text{PML}_k := \bigoplus_{\ell \geq 0} \text{Vect}(\mathcal{P}_{k,-,\ell})
\] (1.3.11)
the vector spaces of $k$-packed matrices respectively graded by the size and by the number of nonzero entries of matrices. By Theorem 1.3, and since each homogeneous component of these vector spaces is finite-dimensional (see Section 1.2), $\text{PMN}_k$ and $\text{PML}_k$ are Hopf algebras. Besides,
\[
\text{PMN}_1 \hookrightarrow \text{PMN}_2 \hookrightarrow \cdots \quad \text{and} \quad \text{PML}_1 \hookrightarrow \text{PML}_2 \hookrightarrow \cdots
\] (1.3.12)
are increasing infinite sequences of Hopf algebras for inclusion. The first few dimensions of $\text{PMN}_1$ and $\text{PMN}_2$ are given by (1.2.8) and (1.2.9), and the first few dimensions of $\text{PML}_1$ and $\text{PML}_2$ are given by (1.2.12) and (1.2.13). In the sequel, we shall denote by $\mathcal{H}_{k,n}(t)$ (resp. $\mathcal{H}_{k,\ell}(t)$) the Hilbert series of $\text{PMN}_k$ (resp. $\text{PML}_k$).

## 2. Algebraic properties

### 2.1. Multiplicative bases and freeness.

#### 2.1.1. Poset structure.

We endow the set $\mathcal{P}_k$ with a binary relation $\rightarrow$ defined in the following way. If $M_1$ and $M_2$ are two $k$-packed matrices of size $n$, we have $M_1 \rightarrow M_2$ if there is an index $i \in [n-1]$ such that, denoting by $s$ the number of 0 ending the $i$th column of $M_1$, and by $p$ the number of 0 starting the $(i+1)$st column of $M_1$, one has $s + p \geq n$ and $M_2$ is obtained from $M_1$ by exchanging its $i$th and $(i+1)$st columns (see Figure 1).

We now endow $\mathcal{P}_k$ with the partial order relation $\leq_k$ defined as the reflexive and transitive closure of $\rightarrow$. Figure 2 shows an interval of this partial order.

Notice that by regarding a permutation $\sigma$ of $\mathcal{S}_n$ as its permutation matrix (i.e., the 1-packed matrix $M$ of size $n$ satisfying $M_{ij} = 1$ if and only if $\sigma_j = i$), the poset $(\mathcal{P}_{k,n,-}, \leq_k)$ restricted to permutation matrices is the right weak order on permutations [GR63].

**Lemma 2.1.** Let $M$, $A$ and $B$ be three packed matrices. Then,
Figure 1. The condition for swapping the $i$th and $(i+1)$st columns of a packed matrix according to the relation $\rightarrow$. The darker regions contain any entries and the white ones, only zeros.

Figure 2. The Hasse diagram of an interval for the order $\leq_M$ on packed matrices.

(1) $A/B \leq_M M$ if and only if there are two packed matrices $A'$ and $B'$ such that $A \leq_M A'$, $B \leq_M B'$, and $M \in A' \sqcup B'$;

(2) $M \leq_M A \backslash B$ if and only if there are two packed matrices $A'$ and $B'$ such that $A' \leq_M A$, $B' \leq_M B$, and $M \in A' \setminus B'$.

Proof. Assume that $A/B \leq_M M$. By definition of the order $\leq_M$, $M$ can be obtained from $A/B$ by swapping columns coming from $A$ to obtain a matrix $A'$ satisfying $A \leq_M A'$, by swapping columns coming from $B$ to obtain a matrix $B'$ satisfying $B \leq_M B'$, and then, by swapping columns coming from $A'$ and from $B'$ together. Thereby, $M \in A' \sqcup B'$.

Conversely assume that $A \leq_M A'$, $B \leq_M B'$, and $M \in A' \sqcup B'$. Then, by definition of the shifted shuffle product and the over operator, $A'/B' \leq_M M$. This implies $A/B \leq_M M$.

By very similar arguments, (2) is established. \hfill $\square$
2.1.2. Multiplicative bases. By mimicking definitions of the bases of symmetric functions, for any \( k \)-packed matrix \( M \), the elementary elements \( E_M \) and the homogeneous elements \( H_M \) are respectively defined by

\[
E_M := \sum_{M \leq M'} F_{M'} \quad \text{and} \quad H_M := \sum_{M' \leq M} F_{M'}.
\]

By triangularity, these two families are bases of \( PM_k \). For instance, in \( PM_4 \) one has

\[
E_{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}} = F_{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}} + F_{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}} + F_{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}.
\]

and

\[
H_{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}} = F_{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}} + F_{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}} + F_{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}} + F_{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}}.
\]

**Proposition 2.2.** The elements appearing in a product of \( PM_k \) expressed in the fundamental basis form an interval for the \( \leq \)-partial order. More precisely, for any \( k \)-packed matrices \( M_1 \) and \( M_2 \),

\[
F_{M_1} \cdot F_{M_2} = \sum_{M_1/\leq M_2, M \leq M_1 \setminus M_2} F_M.
\]

**Proof.** It is plain that the left and right-hand side of (2.1.4) are multiplicity-free. Then, it is enough to show that the sets \( M_1 \sqcup M_2 \) and \( [M_1/\leq M_2, M_1 \setminus M_2] \) are equal. This is a consequence of Lemma 2.1. \( \square \)

**Proposition 2.3.** The product of \( PM_k \) satisfies, for any \( k \)-packed matrices \( M_1 \) and \( M_2 \),

\[
E_{M_1} \cdot E_{M_2} = E_{M_1/\leq M_2} \quad \text{and} \quad H_{M_1} \cdot H_{M_2} = H_{M_1/\leq M_2}.
\]

**Proof.** We shall prove the product rule for the elementary basis by expanding \( E_{M_1}, E_{M_2} \) and \( E_{M_1/\leq M_2} \) over the fundamental basis. First, since any element \( F_N \), where \( N \) is a packed matrix, appearing in \( E_{M_1} \cdot E_{M_2} \) is obtained by shifting and shuffling two matrices \( N_1 \) and \( N_2 \) such that \( M_1 \leq \leq N_1 \) and \( M_2 \leq N_2 \), \( E_{M_1} \cdot E_{M_2} \) is multiplicity-free over the fundamental basis. Moreover, by definition of the elementary basis, \( E_{M_1/\leq M_2} \) is multiplicity-free over the fundamental basis.

Therefore, it is enough to prove that the sets

\[
\{ N \in N_1 \sqcup N_2 : M_1 \leq N_1 \ \text{and} \ M_2 \leq N_2 \}
\]

and

\[
\{ N \in P_k : M_1/\leq M_2 \leq N \}
\]

are equal. This is exactly (1) of Lemma 2.1. \( \square \)

The proof for the homogeneous basis is analogous.
2.1.3. Freeness. Given a \( k \)-packed matrix \( M \neq \emptyset \), we say that \( M \) is connected (resp. anti-connected) if, for all \( k \)-packed matrices \( M_1 \) and \( M_2 \), \( M = M_1 \backslash M_2 \) (resp. \( M = M_1 \setminus M_2 \)) implies \( M_1 = M \) or \( M_2 = M \).

**Theorem 2.4.** The Hopf algebra \( PM_k \) is freely generated as an algebra by the elements \( E_M \) (resp. \( H_M \)) where the \( M \) are connected (resp. anti-connected) \( k \)-packed matrices.

**Proof.** Since any packed matrix \( M \) can be written as

\[
M = M_1 \backslash \ldots \backslash M_r,
\]

where the \( M_i \) are connected packed matrices, by Proposition 2.3, we have

\[
E_M = E_{M_1} \cdot \ldots \cdot E_{M_r},
\]

showing that the \( E_M \), where \( M \) is a connected packed matrix, generate \( PM_k \) as an algebra. Besides, the obvious unicity of the factorization (2.1.8) shows that this family is free.

The proof for the homogeneous basis is analogous. \( \square \)

Theorem 2.4 also implies that \( PMN_k \) and \( PML_k \) are freely generated by the \( E_M \) (resp. \( H_M \)) where the \( M \) are connected (resp. anti-connected) \( k \)-packed matrices. Hence, the generating series \( G_{k,n}(t) \) and \( G_{k,\ell}(t) \) of algebraic generators of \( PMN_k \) and \( PML_k \) satisfy respectively

\[
G_{k,n}(t) = 1 - \frac{1}{H_{k,n}(t)} \quad \text{and} \quad G_{k,\ell}(t) = 1 - \frac{1}{H_{k,\ell}(t)}.
\]

The first few numbers of algebraic generators of \( PMN_1 \) and \( PMN_2 \) are respectively

\[
0, 1, 6, 252, 40944, 24912120, 57316485000
\]

and

\[
0, 2, 52, 15848, 39089872, 813573857696, 147659027604370240.
\]

The first few numbers of algebraic generators of \( PML_1 \) and \( PML_2 \) are respectively

\[
0, 1, 1, 7, 51, 497, 5865, 81305, 1293333
\]

and

\[
0, 2, 4, 56, 816, 15904, 375360, 10407040, 331093248.
\]

2.2. Self-duality.

2.2.1. Dual Hopf algebra. Let us denote by \( PM_k^* \) the bigraded dual vector space of \( PM_k \), by \( F_M^* \), where the \( M \) are \( k \)-packed matrices, the adjoint basis of the fundamental basis of \( PM_k \), and by \( \langle \cdot, \cdot \rangle \) the associated duality bracket.

Let \( M_1 \) and \( M_2 \) be two \( k \)-packed matrices of respective sizes \( n_1 \) and \( n_2 \). By duality, the product in \( PM_k^* \) satisfies

\[
F_{M_1}^* \cdot F_{M_2}^* = \sum_{M \in P_k} \langle \Delta(F_M), F_{M_1}^* \otimes F_{M_2}^* \rangle F_M^*.
\]

Let us set

\[
M_1 \bullet n_2 := \left[ M_1 | Z_{n_2}^1 \right] \quad \text{and} \quad n_1 \bullet M_2 := \left[ Z_{n_2}^1 | M_2 \right].
\]
The row shifted shuffle $M_1 \ast M_2$ of $M_1$ and $M_2$ is the set of all matrices obtained by shuffling the rows of $M_1 \bullet n_2$ with the rows of $n_1 \bullet M_2$. By a routine computation, we obtain the following expression for the product of $PM_k^*$:

\[(2.2.3)\quad F_{M_1}^* \cdot F_{M_2}^* = \sum_{M \in M_1 \ast M_2} F_M^* .\]

For instance, in $PM_1^*$ one has

\[(2.2.4)\quad F_{\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}}^* \cdot F_{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}^* = F_{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}}^* + F_{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}^* + \cdots .\]

Let $M$ be a $k$-packed matrix. By duality, the coproduct in $PM_k^*$ satisfies

\[(2.2.5)\quad \Delta (F_M^*) = \sum_{M_1, M_2 \in P_k} \langle F_{M_1} \cdot F_{M_2}, F_M^* \rangle F_{M_1}^* \otimes F_{M_2}^* .\]

By a routine computation, we obtain the following expression for the coproduct of $PM_k^*$:

\[(2.2.6)\quad \Delta (F_M^*) = \sum_{M = M_1 \circ M_2} F_{\text{cp}(M_1)}^* \otimes F_{\text{cp}(M_2)}^* .\]

For instance, in $PM_1^*$ one has

\[(2.2.7)\quad \Delta F_{\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}}^* = F_{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 \end{bmatrix}}^* \otimes F_{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}^* + F_{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}^* \otimes F_{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}^* + \cdots .\]

Let us denote by $M^T$ the transpose of $M$.

**Proposition 2.5.** The map $\phi : PM_k \to PM_k^*$ linearly defined for any $k$-packed matrix $M$ by

\[(2.2.8)\quad \phi (F_M) := F_{M^T}^*\]

is a Hopf isomorphism.

**Proof.** The product and the coproduct of $PM_k$ in the fundamental basis handle the columns of the matrices while the product and the coproduct of $PM_k^*$ in the adjoint basis of the fundamental basis handle the rows. Since the transpose of a matrix swaps its rows and its columns, $\phi$ is a Hopf isomorphism. □

Since the transpose of any packed matrix of $P_{k,n,\ell}$ also belongs to $P_{k,n,\ell}$, Proposition 2.5 also implies that $PMN_k$ and $PML_k$ are self-dual for the isomorphism $\phi$. 
2.2.2. Primitive elements. For any \( k \)-packed matrix \( M \), define
\[
(2.2.9) \quad W^M := F_{M_1}^* \cdot \ldots \cdot F_{M_r}^*
\]
where the \( M_i \) are connected packed matrices (see Section 2.1.3) and \( M = M_1/\ldots/M_r \). Then, we have
\[
(2.2.10) \quad W^M = F_M^* + \sum_{M' \in R} F_M^{*'}
\]
where any matrix \( M' \) of \( R \) satisfies \( M'^T \leq_R M'^T \) since the product in \( PM_k^* \) consists in shifting and shuffling rows of matrices. Thus, by triangularity, the \( W^M \) form a basis of \( PM_k^* \). Moreover, for any \( k \)-packed matrices \( M_1 \) and \( M_2 \), the product of \( PM_k^* \) is expressed as
\[
(2.2.11) \quad W^{M_1} \cdot W^{M_2} = W^{M_1/\ldots/M_2}.
\]

Let us denote by \( V_M \), where the \( M \) are \( k \)-packed matrices, the adjoint elements of the \( W^M \).

**Proposition 2.6.** The elements \( V_M \), where \( M \) are connected \( k \)-packed matrices, form a basis of the vector space of primitive elements of \( PM_k \).

*Proof.* Since \( W^M \) is indecomposable, by duality, \( V_M \) is primitive. Moreover, let \( X \) be a primitive element of \( PM_k \). Then, \( X \) is expressed as
\[
(2.2.12) \quad X = \sum_{M \in PM_k} c_M V_M.
\]
Let \( M \) be a nonconnected \( k \)-packed matrix and \( M = M_1/\ldots/M_2 \) be a nontrivial factorization. Then, by duality, the coefficient of \( V_{M_1} \otimes V_{M_2} \) in \( \Delta(X) \) is \( c_M \). Since \( X \) is primitive, \( c_M = 0 \), showing that \( X \) is a sum of \( V_M \) where \( M \) are connected \( k \)-packed matrices. \( \square \)

By Proposition 2.6, the \( V_M \), where \( M \) are connected \( k \)-packed matrices, generate the Lie algebra of primitive elements of \( PM_k \). The first few dimensions of the Lie algebras of primitive elements of \( PMN_1, PMN_2, PML_1, PML_2 \) are respectively given by (2.1.11), (2.1.12), (2.1.13), and (2.1.14).

2.3. Bidendriform bialgebra structure.

2.3.1. Dendriform algebra structure. An algebra \( (A, \cdot) \) admits a dendriform algebra structure [Lod01] if its product can be split into two operations
\[
(2.3.1) \quad \cdot = \prec + \succ,
\]
where \( \prec, \succ : A \otimes A \to A \) are non-degenerated linear maps such that, by denoting by \( A^+ \) the augmentation ideal of \( A \), for all \( x, y, z \in A^+ \), the following relations hold
\[
(2.3.2a) \quad (x \prec y) \prec z = x \prec (y \cdot z),
\]
\[
(2.3.2b) \quad (x \succ y) \prec z = x \succ (y \prec z),
\]
\[
(2.3.2c) \quad (x \cdot y) \succ z = x \succ (y \succ z).
\]

For any nonempty matrix \( M \), we shall denote by \( \text{last}_c(M) \) its last column. Let us endow \( PM_k^+ \) with two products \( \prec \) and \( \succ \) linearly defined, for any nonempty \( k \)-packed matrices \( M_1 \) and \( M_2 \) of respective sizes \( n_1 \) and \( n_2 \), by
\[
(2.3.3) \quad F_{M_1} \prec F_{M_2} := \sum_{M \in M_1 \sqcup M_2, \text{last}_c(M) = \text{last}_c(M_1 \sqcup n_2)} F_M
\]
and
\[(2.3.4) \quad F_{M_1} \succ F_{M_2} := \sum_{M \in M_1 \shuffle M_2 \atop \text{last}_c(M) = \text{last}_c(n_1 \circ M_2)} F_M.\]

In other words, the matrices appearing in a $\prec$-product (resp. $\succ$-product) in the fundamental basis involving $M_1$ and $M_2$ are the matrices $M$ obtained by shifting and shuffling the columns of $M_1$ and $M_2$ such that the last column of $M$ comes from $M_1$ (resp. $M_2$). For example,
\[(2.3.5) \quad F_{\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}} \prec F_{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} = F_{\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}} + F_{\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}} + F_{\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}},\]
\[(2.3.6) \quad F_{\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}} \succ F_{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} = F_{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}} + F_{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}} + F_{\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}}.\]

Since the last column of any matrix appearing in the shifted shuffle of two matrices comes from one of the two operands, for any nonempty packed matrices $M_1$ and $M_2$, one obviously has
\[(2.3.7) \quad F_{M_1} \cdot F_{M_2} = F_{M_1} \prec F_{M_2} + F_{M_1} \succ F_{M_2}.\]

**Proposition 2.7.** The Hopf algebra $PM_k$ admits a dendriform algebra structure for the products $\prec$ and $\succ$.

**Proof.** We have to prove that (2.3.2a), (2.3.2b), and (2.3.2c) hold. Let $M_1$, $M_2$, and $M_3$ be three packed matrices of respective sizes $n_1$, $n_2$, and $n_3$.

By definition of $\prec$ and $\succ$, and since $\shuffle$ is associative, the set $S$ of matrices indexing the support of $(F_{M_1} \succ F_{M_2}) \prec F_{M_3}$ satisfies
\[(2.3.8) \quad S = \{ M \in (M_1 \shuffle M_2) \shuffle M_3 : \text{last}_c(M) = \text{last}_c(n_1 \circ M_2 \circ n_3) \} = \{ M \in M_1 \shuffle (M_2 \shuffle M_3) : \text{last}_c(M) = \text{last}_c(n_1 \circ M_2 \circ n_3) \}.
\]

Hence, $S$ also is the set of matrices indexing the support of $F_{M_1} \succ (F_{M_2} \prec F_{M_3})$. Since the shifted shuffle of packed matrices is multiplicity-free, (2.3.2b) holds.

By definition of $\prec$ and $\succ$, and since $\shuffle$ is associative, the set $T$ of matrices indexing the support of $(F_{M_1} \prec F_{M_2}) \prec F_{M_3}$ satisfies
\[(2.3.9) \quad T = \{ M \in (M_1 \shuffle M_2) \shuffle M_3 : \text{last}_c(M) = \text{last}_c(M_1 \circ (n_2 + n_3)) \} = \{ M \in M_1 \shuffle (M_2 \shuffle M_3) : \text{last}_c(M) = \text{last}_c(M_1 \circ (n_2 + n_3)) \}.
\]

Hence, by (2.3.7), $T$ also is the set of matrices indexing the support of $F_{M_1} \prec (F_{M_2} \cdot F_{M_3})$. Since the shifted shuffle of packed matrices is multiplicity-free, (2.3.2a) holds. By a very similar argument, (2.3.2c) also holds. \hfill \Box

2.3.2. Codendriform coalgebra structure. By dualizing the notion of dendriform algebra structure, one obtains the notion of codendriform coalgebra structure [Foi07]. A coalgebra $(C, \Delta)$ admits a codendriform coalgebra structure if its coproduct can be split into two operations
\[(2.3.10) \quad \Delta = 1 \otimes I + \Delta_\prec + \Delta_\succ + I \otimes 1,\]
where $\Delta_\prec, \Delta_\succ : \mathcal{C} \rightarrow \mathcal{C} \otimes \mathcal{C}$ are non-degenerated linear maps such that following relations hold
\begin{align*}
(2.3.11a) \quad & (\Delta_\prec \otimes I) \circ \Delta_\prec = (I \otimes \bar{\Delta}) \circ \Delta_\prec, \\
(2.3.11b) \quad & (\Delta_\succ \otimes I) \circ \Delta_\prec = (I \otimes \Delta_\prec) \circ \Delta_\succ, \\
(2.3.11c) \quad & (\bar{\Delta} \otimes I) \circ \Delta_\prec = (I \otimes \Delta_\prec) \circ \Delta_\succ,
\end{align*}
where $\bar{\Delta} := \Delta_\prec + \Delta_\succ$.

For any nonempty matrix $M$, we shall denote by $\text{last}_{r}(M)$ its last row. Let us endow $\text{PM}_k$ with two coproducts $\Delta_\prec$ and $\Delta_\succ$ linearly defined, for any nonempty $k$-packed matrix $M$, by
\begin{align*}
(2.3.12) \quad \Delta_\prec (F_M) & := \sum_{M = L \bullet R \atop \text{last}_{r}(L \bullet R) = \text{last}_{r}(M)} F_{\text{cp}(L)} \otimes F_{\text{cp}(R)} \\
& \text{and} \\
(2.3.13) \quad \Delta_\succ (F_M) & := \sum_{M = L \bullet R \atop \text{last}_{r}(L \bullet R) = \text{last}_{r}(M)} F_{\text{cp}(L)} \otimes F_{\text{cp}(R)},
\end{align*}
where $r$ (resp. $\ell$) is the number of columns of $R$ (resp. $L$). In other words, the pairs of matrices appearing in a $\Delta_\prec$-coproduct (resp. $\Delta_\succ$-coproduct) in the fundamental basis are the pairs $(L, R)$ of packed matrices such that the last row of $L$ (resp. $R$) comes from the last row of $M$. For example,
\begin{align*}
(2.3.14) \quad & \Delta_\prec F \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0
\end{bmatrix} = F \begin{bmatrix} 1 & 0 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix} \otimes F \begin{bmatrix} 0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0 \\
1 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 1 & 1 \\
0 & 1 & 0
\end{bmatrix} + F \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0
\end{bmatrix} \otimes F \begin{bmatrix} 1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}, \\
\end{align*}
\begin{align*}
(2.3.15) \quad & \Delta_\succ F \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix} = F \begin{bmatrix} 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} \otimes F \begin{bmatrix} 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}.
\end{align*}

Since by Lemma 1.1, one cannot vertically split a packed matrix by separating two nonzero entries on a same row, for any nonempty packed matrix $M$, one has
\begin{align*}
(2.3.16) \quad \Delta(F_M) = 1 \otimes F_M + \Delta_\prec (F_M) + \Delta_\succ (F_M) + F_M \otimes 1.
\end{align*}

**Proposition 2.8.** The Hopf algebra $\text{PM}_k$ admits a codendriform coalgebra structure for the co-products $\Delta_\prec$ and $\Delta_\succ$.

Since the proof of this statement is similar to that of Proposition 2.7 it has been omitted.

2.3.3. **Bidendriform bialgebra structure.** A bialgebra $(B, \cdot, \Delta)$ admits a bidendriform bialgebra structure [Foi07] if $B$ admits both a dendriform algebra $(B, \prec, \succ)$ and a codendriform coalgebra $(B, \Delta_\prec, \Delta_\succ)$ structure with some extra compatibility relations between $(\prec, \succ)$ and $(\Delta_\prec, \Delta_\succ)$.

**Theorem 2.9.** The Hopf algebra $\text{PM}_k$ admits a bidendriform bialgebra structure for the products $\prec$, $\succ$ and the co-products $\Delta_\prec$, $\Delta_\succ$.

**Proof.** By Propositions 2.7 and 2.8, $\text{PM}_k$ admits a dendriform algebra and a codendriform coalgebra structure.

The required extra compatibility relations (see [Foi07]) between $(\prec, \succ)$ and $(\Delta_\prec, \Delta_\succ)$ are established by arguments similar to the ones used in the proofs of Propositions 2.7 and 2.8. \qed
Theorem 2.9 also implies that $\text{PMN}_k$ and $\text{PML}_k$ admit a bidendriform bialgebra structure. Recall that an element $x$ of a Hopf algebra admitting a bidendriform bialgebra structure is \textit{totally primitive} if $\Delta(x) = 0 = \Delta(x)$. Following [Foi07], the generating series $T_{k,n}(t)$ and $T_{k,\ell}(t)$ of totally primitive elements of $\text{PMN}_k$ and $\text{PML}_k$ satisfy respectively

\begin{equation}
T_{k,n}(t) = \frac{H_{k,n}(t) - 1}{H_{k,n}(t)^2} \quad \text{and} \quad T_{k,\ell}(t) = \frac{H_{k,\ell}(t) - 1}{H_{k,\ell}(t)^2}.
\end{equation}

The first few dimensions of totally primitive elements of $\text{PMN}_1$ and $\text{PMN}_2$ are respectively

\begin{equation}
0, 1, 5, 240, 40404, 2482708, 57266105928
\end{equation}

and

\begin{equation}
0, 2, 48, 15640, 39023776, 813415850016, 14765576899243364.
\end{equation}

The first few dimensions of totally primitive elements of $\text{PML}_1$ and $\text{PML}_2$ are respectively

\begin{equation}
0, 1, 0, 5, 36, 381, 4720, 67867, 1109434
\end{equation}

and

\begin{equation}
0, 2, 0, 40, 576, 12192, 302080, 8686976, 284015104.
\end{equation}

\section{Related Hopf algebras}

In this section, we list some already known Hopf algebras and describe their links with $\text{PM}_k$. Next, we provide a method to construct Hopf subalgebras of $\text{PM}_k$.

\subsection{Links with known algebras}

\subsubsection{Hopf algebra of colored permutations}

Recall that a \textit{k-colored permutation} is a pair $(\sigma, c)$ where $\sigma$ is a permutation of size $n$ and $c$ is a word of length $n$ on the alphabet $A_k \setminus \{0\}$.

In [NT10], the authors endowed the vector spaces $\mathbf{FQSym}^{(k)}$ spanned by the set of all $k$-colored permutations with a Hopf algebra structure. The elements $\mathbf{F}_{(\sigma,c)}$, where the $(\sigma, c)$ are $k$-colored permutations, form the fundamental basis of $\mathbf{FQSym}^{(k)}$. These Hopf algebras provide a generalization of $\mathbf{FQSym}$ since $\mathbf{FQSym} = \mathbf{FQSym}^{(1)}$.

\begin{proposition}
The map $\alpha_k : \mathbf{FQSym}^{(k)} \to \text{PMN}_k$ linearly defined, for any $k$-colored permutation $(\sigma, c)$ by

\begin{equation}
\alpha_k (\mathbf{F}_{(\sigma,c)}) := \mathbf{F}_{M^{(\sigma,c)}}
\end{equation}

where $M^{(\sigma,c)}$ is the $k$-packed matrix satisfying $M^{(\sigma,c)}_{i,j} = c_j \delta_{i,\sigma_j}$ is an injective Hopf morphism.

In particular, Proposition 3.1 shows that $\text{PMN}_1$ contains $\mathbf{FQSym}$. Notice that the map $\alpha_k$ is still well-defined on the codomain $\text{PML}_k$ instead of $\text{PMN}_k$.\end{proposition}
3.1.2. Hopf algebra of uniform block permutations. Recall that a uniform block permutation, or a UBP for short, of size \( n \) is a bijection \( \pi : \pi^d \to \pi^c \) where \( \pi^d \) and \( \pi^c \) are set partitions of \([n]\), and, for any \( e \in \pi^d \), \( e \) and \( \pi(e) \) have same cardinality.

For instance, the map \( \pi \) defined by
\[
\begin{align*}
\pi(\{1, 4, 5\}) & := \{2, 5, 6\}, \\
\pi(\{2\}) & := \{1\}, \\
\pi(\{3, 6\}) & := \{3, 4\}
\end{align*}
\]
is a UBP of size 6.

In [AO08], the authors endowed the vector space \( \text{UBP} \) spanned by the set of all UBPs with a Hopf algebra structure. The elements \( F_{\pi} \), where the \( \pi \) are UBPs, form the fundamental basis of \( \text{UBP} \). The dimensions of \( \text{UBP} \) form Sequence A023998 of [Slo] and the first few terms are
\[
1, 1, 3, 16, 131, 1496, 22482, 426833, 9934563, 277006192, 9085194458.
\]

Proposition 3.2. The map \( \beta : \text{UBP}^* \to \text{PMN}_1 \) linearly defined, for any UBP \( \pi \) by
\[
\beta(F_{\pi}^*) := F_{M^\pi}
\]
where \( M^\pi \) is the 1-packed matrix satisfying
\[
M^\pi_{ij} :=
\begin{cases} 
1 & \text{if there is } e \in \pi^d \text{ such that } j \in e \text{ and } i \in \pi(e), \\
0 & \text{otherwise}
\end{cases}
\]
is an injective Hopf morphism.

For example, with the UBP \( \pi \) defined in (3.1.2), we have
\[
\beta(F_{\pi}^*) = F_{\begin{bmatrix} 
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 
\end{bmatrix}}.
\]

Corollary 3.3. The Hopf algebra \( \text{UBP}^* \) is a free, cofree, and self-dual Hopf algebra which admits a bidendriform bialgebra structure.

Proof. By Proposition 3.2 and the definition of the product on the fundamental basis of \( \text{UBP}^* \) (see [AO08]), we can see \( \text{UBP}^* \) as a Hopf subalgebra of \( \text{PMN}_1 \) restricted on the elements \( F_M \) where the \( M \) are 1-packed matrices such that there are UBPs \( \pi \) satisfying \( M^\pi = M \). This shows that \( \text{UBP}^* \) inherits from the bidendriform bialgebra structure of \( \text{PMN}_1 \) (see Theorem 2.9). Now, since \( \text{UBP}^* \) admits a bidendriform bialgebra structure, by [Foi07], it is free, cofree, and self-dual. □

By using the same arguments as those used in Section 2.1, one can build multiplicative bases of \( \text{UBP}^* \) by setting, for any UBP \( \pi \),
\[
E_{M^\pi} := \sum_{M^\pi \leq \pi M^\pi'} F_{M^\pi'} \quad \text{and} \quad H_{M^\pi}^* := \sum_{M^\pi' \preceq M^\pi} F_{M^\pi'}.
\]

This gives another way to prove the freeness of \( \text{UBP}^* \) by using the same arguments as those of Theorem 2.4. Hence, \( \text{UBP}^* \) is freely generated by the elements \( E_{M^\pi} \) (resp. \( H_{M^\pi}^* \)) where the \( \pi \) are UBPs such that the \( M^\pi \) are connected (resp. anti-connected) 1-packed matrices. The first few numbers of algebraic generators of \( \text{UBP}^* \) are
\[
0, 1, 2, 11, 98, 1202, 375692, 8981392, 255253291, 8488918198.
\]
and the first few dimensions of totally primitive elements are

\[(3.1.9) \quad 0, 1, 7, 72, 962, 16135, 330624, 8117752, 235133003, 7929041828.\]

Moreover, since for any UBP \(\pi\), there exists a UBP \(\pi^{-1}\) such that the transpose of \(M^\pi\) is \(M^{\pi^{-1}}\), by Proposition 2.5, the map \(\phi : \text{UBP}^* \to \text{UBP}\) linearly defined for any UBP \(\pi\) by

\[(3.1.10) \quad \phi(F^*_M) := F^*_M + T\]

is an isomorphism.

### 3.1.3. Algebra of matrix quasi-symmetric functions.

In [DHT02] (see also [Hiv99]), the authors defined the vector space \(\text{MQSym}\) spanned by the set of the (not necessarily square) matrices with entries in \(\mathbb{N}\), and such that each row and each column contains at least one nonzero entry. In this section, we simply call matrices such sort of matrices. The elements \(\text{MS}\) such that \(M\) is a matrix form the quasi-multiword basis of \(\text{MQSym}\). The degree of a \(\text{MS}\) is given by the sum of the entries of \(M\).

This vector space is endowed with an algebra structure where the product of two basis elements is provided by the augmented shuffle \(\uplus\). Let \(M_1\) and \(M_2\) be two matrices. Any matrix \(M\) of \(M_1 \uplus M_2\) is obtained by concatenating \(N_1\) and \(N_2\) where \(N_1\) (resp. \(N_2\)) is obtained from \(M_1\) (resp. \(M_2\)) by inserting some null rows and so that \(N_1\) and \(N_2\) have both a same number of rows and each row of \(M\) has at least one nonzero entry. For example,

\[(3.1.11) \quad \text{MS} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \cdot \text{MS} \begin{bmatrix} 1 & 3 \end{bmatrix} = \text{MS} \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \end{bmatrix} + \text{MS} \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \end{bmatrix} + \text{MS} \begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 0 \end{bmatrix} + \text{MS} \begin{bmatrix} 0 & 0 & 1 & 3 \\ 2 & 1 & 0 & 0 \end{bmatrix}.\]

Let us endow the set of matrices indexing \(\text{MQSym}\) with a binary relation \(\rightarrow\) defined in the following way. If \(M_1\) and \(M_2\) are two matrices such that \(M_1\) has \(n\) rows and \(m\) columns, we have \(M_1 \rightarrow M_2\) if there is an index \(i \in [n-1]\) such that, denoting by \(s\) the number of 0 which end the \(i\)th row of \(M_1\), and by \(p\) the number of 0 which start the \((i+1)\)st row of \(M_1\), one has \(s + p \geq m\) and \(M_2\) is obtained from \(M_1\) by overlaying its \(i\)th and \((i+1)\)st rows (see Figure 3).

\[\begin{array}{c}
\text{Figure 3. The condition for overlaying the } i\text{th and } (i + 1)\text{st rows of a (not necessarily square) packed matrix according to the relation } \rightarrow. \text{ The darker regions contain any entries and the white ones, only zeros.}
\end{array}\]

We now endow the set of matrices that index \(\text{MQSym}\) with the partial order relation \(\leq\) defined as the reflexive and transitive closure of \(\rightarrow\). Figure 4 shows an interval of this partial order.

**Lemma 3.4.** Let \(A\) and \(B\) be two \(k\)-packed matrices. Then,

\[(3.1.12) \quad \{C' : C \leq\text{MQ} C', C \in A \ast B\} = \{C' \in A' \uplus B' : A \leq\text{MQ} A', B \leq\text{MQ} B'\},\]

where \(\ast\) is the row shifted shuffle of \(k\)-packed matrices and \(\uplus\) is the augmented shuffle of matrices.
Proof. Let \( C' \) be a matrix such that \( C \leq_{MQ} C' \) and \( C \in A \ast B \). By definition of the order \( \leq_{MQ} \) and the product \( \ast \), \( C' \) can be obtained from \( C \) by overlaying rows coming from \( A \), rows coming from \( B \), or rows coming from \( A \) and \( B \). Let us denote by \( A' \) (resp. \( B' \)) the matrix obtained from \( A \) (resp. \( B \)) by overlaying some of its rows. Then, we have \( A \leq_{MQ} A' \) and \( B \leq_{MQ} B' \), and, by definition of the augmented shuffle, \( C' \in A' \amalg B' \).

Conversely, let \( C' \) be a matrix such that \( C' \in A' \amalg B' \) where \( A' \) and \( B' \) are matrices satisfying \( A \leq_{MQ} A' \) and \( B \leq_{MQ} B' \). Then, by definition of the augmented shuffle of matrices, \( C' \) can be obtained from a matrix \( C \) of \( A \ast B \) by overlaying rows coming from \( A \), rows coming from \( B \), or rows coming from \( A \) and \( B \). Hence, \( C \leq_{MQ} C' \). \( \square \)

**Proposition 3.5.** The map \( \gamma : PML_1^* \to MQSym \) linearly defined, for any 1-packed matrix \( M \) by

\[
\gamma(F_M^*) := \sum_{M \leq_{MQ} M'} MS_{M'},
\]

is an injective algebra morphism.

Proof. Let \( M_1 \) and \( M_2 \) be two 1-packed matrices. By definition of \( \gamma \), \( \gamma(F_{M_1}^* \cdot F_{M_2}^*) \) is multiplicity-free over the quasi-multiword basis of \( MQSym \). Moreover, since the augmented shuffle is multiplicity-free, \( \gamma(F_{M_1}^*) \cdot \gamma(F_{M_2}^*) \) also is. Lemma 3.4 implies that these two elements are equal and then, that \( \gamma \) is an algebra morphism. The injectivity of \( \gamma \) follows by triangularity. \( \square \)

For instance, one has

\[
\gamma(F_1^*) = MS \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} + MS \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + MS \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} + MS \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.
\]

Notice that \( \gamma \) is not a Hopf morphism since it is not a coalgebra morphism. Indeed, we have

\[
\Delta \gamma (F_1^*) = 1 \otimes MS \begin{bmatrix} 1 \\ 1 \end{bmatrix} + MS \begin{bmatrix} 1 \\ 1 \end{bmatrix} \otimes 1,
\]

but

\[
(\gamma \otimes \gamma) \Delta F_1^* = 1 \otimes MS \begin{bmatrix} 1 \\ 1 \end{bmatrix} + MS \begin{bmatrix} 1 \end{bmatrix} \otimes MS \begin{bmatrix} 1 \end{bmatrix} + MS \begin{bmatrix} 1 \end{bmatrix} \otimes 1.
\]
3.1.4. Diagram of embeddings. The following diagram summarizes the relations between known Hopf algebras related to \( \text{PM}_k \) and, more specifically, to its simple gradations \( \text{PMN}_k \) and \( \text{PML}_k \).

Plain arrows are Hopf algebra morphisms and the dotted arrow is an algebra morphism. The Hopf algebra \( \text{ASM} \) is the subject of Section 4.

\[
\begin{array}{c}
\text{PMN}_k \\
\downarrow \alpha_k \\
\text{PMN}_2 \\
\downarrow \text{FQSym}^{(k)} \\
\text{PMN}_1 \\
\downarrow \beta \\
\text{UBP}^* \\
\downarrow \alpha_1 \\
\text{ASM} \\
\downarrow \alpha_1 \\
\text{FQSym} \\
\end{array}
\]

\[
\begin{array}{c}
\text{PML}_k \\
\downarrow \alpha_k \\
\text{PML}_2 \\
\downarrow \text{FQSym}^{(2)} \\
\text{PML}_1 \\
\downarrow \gamma \\
\text{PML}_1^* \\
\end{array}
\]

3.2. Equivalence relations and Hopf subalgebras. Several Hopf algebras can be constructed as Hopf subalgebras of the Malvenuto-Reutenauer Hopf algebra \( \text{FQSym} \) [MR95, DHT02]. The main examples are the Hopf algebra \( \text{PBT} \) based on planar binary trees, first defined by Loday and Ronco [LR98] and reconstructed by Hivert, Novelli, and Thibon [HNT05], and \( \text{FSym} \) based on standard Young tableaux, first discovered by Poirier and Reutenauer [PR95] and reconstructed by Duchamp, Hivert, and Thibon [DHT02].

The starting point of these constructions is to define a congruence \( \equiv \) on the free monoid \( A^* \) where \( A \) is a totally ordered infinite alphabet. Then, when \( \equiv \) satisfies some properties [HN07, Gir11], the elements

\[
P_{[\sigma]_\equiv} := \sum_{\sigma \in [\sigma]_\equiv} F_\sigma
\]

span a Hopf subalgebra of \( \text{FQSym} \) indexed by the \( \equiv \)-equivalence classes restricted to permutations. We shall show in this section that an analogous construction works to construct Hopf subalgebras of \( \text{PM}_k \).

3.2.1. The sylvester and the plactic congruences. Recall that the congruence allowing to reconstruct \( \text{PBT} \) is the sylvester congruence (see [HNT02, HNT05]). It is denoted by \( \equiv_\mathcal{S} \) and is the reflexive and transitive closure of the sylvester adjacency relation \( \leftrightarrow_\mathcal{S} \) defined for \( u \in A^* \) and \( a, b, c \in A \) by

\[
a c u b \leftrightarrow_\mathcal{S} c a u b \quad \text{where} \quad a \leq b < c.
\]

For example, the \( \equiv_\mathcal{S} \)-equivalence class of the permutation 15423 (see Figure 5) is

\[
\{12543, 15243, 15423, 51243, 51423, 54123\}.
\]
HOPF ALGEBRA ON PACKED SQUARE MATRICES

Figure 5. The sylvester equivalence class of the permutation 15423. Edges represent sylvester adjacency relations.

Besides, recall that the congruence allowing to reconstruct $\mathbf{FSym}$ is the plactic congruence (see [LS81, Lot02]). It is denoted by $\equiv_{P}$ and is the reflexive and transitive closure of the plactic adjacency relation $←→_{P}$ defined for $a, b, c ∈ A$ by

\begin{align}
\text{(3.2.4a)} & \quad abc ←→_{P} cab \quad \text{where} \quad a \leq b < c, \\
\text{(3.2.4b)} & \quad bac ←→_{P} bca \quad \text{where} \quad a < b \leq c.
\end{align}

3.2.2. The monoid of words of columns. Let $C_{k}^{∗}$ be the free monoid generated by the set $C_{k}$ of all $n \times 1$-matrices with entries in $A_{k}$, for all $n \geq 1$. In other words, the elements of $C_{k}^{∗}$ are words whose letters are columns and its product $\bullet$ is the concatenation of such words. When all the letters of an element $M ∈ C_{k}^{∗}$ have, as columns, a same number of rows, $M$ is a matrix and we shall denote it as such in the sequel.

The alphabet $C_{k}$ is naturally equipped with the total order $\leq$, where, for any $c_{1}, c_{2} ∈ C_{k}$, $c_{1} \leq c_{2}$ if and only if the bottom to top reading of the column $c_{1}$ is lexicographically smaller than the bottom to top reading of $c_{2}$. For instance,

\begin{align}
\text{(3.2.5)} & \quad \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \leq \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \leq \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix} \leq \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \leq \begin{bmatrix} 1 \\ 2 \end{bmatrix}.
\end{align}

Since $C_{k}$ is then totally ordered and $C_{k}^{∗}$ is a free monoid, one can consider the previous two congruences on $C_{k}^{∗}$ instead on $A^{∗}$. For instance, Figure 6 represents a $≡_{S}$-equivalence class and a $≡_{P}$-equivalence class of packed matrices.

The order relation $\leq$ on $C_{k}$ is compatible with the shifted shuffle of packed matrices in the following sense. Let $M_{1}$ and $M_{2}$ be two nonempty packed matrices and $M$ be a matrix appearing in $M_{1} \sqcup M_{2}$. Then, if $c_{1}$ (resp. $c_{2}$) is a column of $M$ coming from $M_{1}$ (resp. $M_{2}$), we necessarily have $c_{1} \leq c_{2}$ and $c_{1} \neq c_{2}$. The obvious analogous property holds for words of $A^{∗}$ and the shifted shuffle of words.

3.2.3. Properties of equivalence relations. An equivalence relation $≡$ on $C_{k}^{∗}$ is a monoid congruence if for all $u, v, u', v' ∈ C_{k}^{∗}$,

\begin{align}
\text{(3.2.6)} & \quad u ≡ u' \quad \text{and} \quad v ≡ v' \quad \text{imply} \quad u \bullet v ≡ u' \bullet v'.
\end{align}
Besides, we say that $\equiv$ is compatible with the restriction to alphabet intervals if for any interval $I$ of $C_k$ and for all $u, v \in C_k^*$,

$$u \equiv v \implies u|_I \equiv v|_I,$$

where $u|_I$ denotes the word obtained by erasing in $u$ the letters that are not in $I$.

Finally, we say that $\equiv$ is compatible with the decompression process if for all $u, v \in C_k^*$ such that $u$ and $v$ are matrices,

$$u \equiv v \iff \text{cp}(u) \equiv \text{cp}(v) \text{ and ev}(u) = \text{ev}(v),$$

where ev($u$) denotes the commutative image of $u$.

3.2.4. Construction of Hopf subalgebras. Given an equivalence relation $\equiv$ on the words of $C_k^*$ and a $\equiv$-equivalence class $[M]_\equiv$ of packed matrices of $C_k^*$, we consider the elements

$$P_{[M]} := \sum_{M' \in [M]} F_{M'}$$

of $PM_k$.

One has for instance

$$P_{\begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}} = F_{\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}} + F_{\begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}} + F_{\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}} + F_{\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}} + F_{\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}}.$$
In particular, if \( \equiv \) is compatible with the decomposition process, any \( \equiv \)-equivalence class of a packed matrix only contains packed matrices. The family \( P_{[M]} \), where the \([M]\) are \( \equiv \)-equivalence classes of packed matrices, forms then a basis of a vector subspace of \( PM_k \) denoted by \( PM_k^\equiv \).

**Theorem 3.6.** Let \( \equiv \) be an equivalence relation on the words of \( C_k^* \) such that \( \equiv \)

1. is a monoid congruence on \( C_k^* \);
2. is compatible with the restriction to alphabet intervals;
3. is compatible with the decompression process.

Then, \( PM_k^\equiv \) is a Hopf subalgebra of \( PM_k \).

**Proof.** Let us show that the product is well-defined on \( PM_k^\equiv \). Let \([M_1]\) and \([M_2]\) be two \( \equiv \)-equivalence classes of \( k \)-packed matrices. We have

\[
(3.2.11) \quad P_{[M_1]} \cdot P_{[M_2]} = \sum_{M_1 \equiv [M_1]} \sum_{M \equiv [M_2]} F_M.
\]

Let \( M \) be a \( k \)-packed matrix such that \( F_M \) appears in \((3.2.11)\) and \( M' \) be a \( k \)-packed matrix such that \( M' \equiv M \). Then, there is a pair of \( k \)-packed matrices \((M_1, M_2)\) such that \( M_1 \in [M_1] \), \( M_2 \in [M_2] \), and \( M \in M_1 \sqcup M_2 \). By definition of the shifted shuffle, this pair is unique. Let \( m_1 \) (resp. \( m_2 \)) be the size of \( M_1 \) (resp. \( M_2 \)). Let \( c_1 \) (resp. \( d_1 \)) be the smallest (resp. greatest) column of \( M_1 \circ m_2 \) and \( c_2 \) (resp. \( d_2 \)) be the smallest (resp. greatest) column of \( m_1 \circ M_2 \). Then, since all columns of \( M_1 \circ m_2 \) are strictly smaller than the ones of \( m_1 \circ M_2 \), the intervals \([c_1, d_1]\) and \([c_2, d_2]\) are disjoint. By \((2)\), \( M \equiv M' \) implies \( M_{[c_1, d_1]} \equiv M'_{[c_1, d_1]} \) and \( M_{[c_2, d_2]} \equiv M'_{[c_2, d_2]} \). Moreover, by \((3)\) and by definition of \( \circ \), we have

\[
(3.2.12) \quad M_1 = \text{cp} (M_{[c_1, d_1]}) \equiv \text{cp} (M'_{[c_1, d_1]}) =: M'_1
\]

and

\[
(3.2.13) \quad M_2 = \text{cp} (M_{[c_2, d_2]}) \equiv \text{cp} (M'_{[c_2, d_2]}) =: M'_2.
\]

Thus, we have \( M' \in M'_1 \sqcup M'_2 \), showing that \( F_{M'} \) also appears in \((3.2.11)\) and that the product is well-defined on \( PM_k^\equiv \).

Let us now show that the coproduct is well-defined on \( PM_k^\equiv \). Let \([M]\) be a \( \equiv \)-equivalence class of \( k \)-packed matrices. We have

\[
(3.2.14) \quad \Delta (P_{[M]} \equiv) = \sum_{M \equiv [M]} \sum_{M = L \bullet R} F_{\text{cp}(L)} \otimes F_{\text{cp}(R)}.
\]

Let \( M_1 \) and \( M_2 \) be two \( k \)-packed matrices such that \( F_{M_1} \otimes F_{M_2} \) appears in \( (3.2.14) \) and \( M'_1 \) and \( M'_2 \) two \( k \)-packed matrices such that \( M'_1 \equiv M_1 \) and \( M'_2 \equiv M_2 \). Then, there is a \( k \)-packed matrix \( M \in [M] \) such that \( M = L \bullet R \), \( \text{cp}(L) = M_1 \), and \( \text{cp}(R) = M_2 \). By \((3)\), \( M'_1 \) (resp. \( M'_2 \)) is a permutation of \( M_1 \) (resp. \( M_2 \)). Thus, there exist two elements \( L' \) and \( R' \) of \( C_k^* \) that are respectively permutations of \( L \) and \( R \) which satisfy \( \text{cp}(L') = M'_1 \) and \( \text{cp}(R') = M'_2 \). Again by \((3)\), we have \( L' \equiv L \) and \( R' \equiv R \). Now, by \((1)\),

\[
(3.2.15) \quad M = L \bullet R \equiv L' \bullet R' =: M'.
\]

Hence, \( M' \equiv M \) and \( F_{M'_1} \otimes F_{M'_2} \) also appears in \( (3.2.14) \).

We have shown that the product and the coproduct of \( PM_k \) are still well-defined on \( PM_k^\equiv \). Hence, \( PM_k^\equiv \) is a Hopf subalgebra of \( PM_k \). \( \square \)
We say that an equivalence relation $\equiv$ on $C^*_k$ is a good congruence if it satisfies (1), (2) and (3) of Theorem 3.6. Let $\equiv$ be a good congruence. Note that since $\equiv$ is compatible with the decompression process, any matrix contained in a $\equiv$-equivalence class $[M]_{\equiv}$ is obtained by switching columns of $M$. Then, any $\equiv$-equivalence class $[M]_{\equiv}$ of $k$-packed matrices only contains matrices whose size and number of nonzero entries are the same as in $M$. Hence, Theorem 3.6 also implies that the family (3.2.9) forms a basis of Hopf subalgebras of both $\text{PMN}_k$ and $\text{PML}_k$. We respectively denote these by $\text{PMN}_k^{\equiv}$ and $\text{PML}_k^{\equiv}$.

3.2.5. Computer experiments. Let us recall here the definitions of some well-known good congruences.

The Baxter congruence (see [Gir12]), denoted by $\equiv_{\text{Bx}}$, is the reflexive and transitive closure of the Baxter adjacency relation $\leftrightarrow_{\text{Bx}}$ defined for $u, v \in A^*$ and $a, b, c, d \in A$ by

$$(3.2.16a)\quad c \ u \ a \ d \ v \ b \leftrightarrow_{\text{Bx}} c \ u \ d \ a \ v \ b \quad \text{where} \quad a \leq b < c \leq d,$$

$$(3.2.16b)\quad b \ u \ d \ a \ v \ c \leftrightarrow_{\text{Bx}} b \ u \ a \ d \ v \ c \quad \text{where} \quad a < b \leq c < d.$$

The Bell congruence (see [Rey07]), denoted by $\equiv_{\text{Bl}}$, is the reflexive and transitive closure of the Bell adjacency relation $\leftrightarrow_{\text{Bl}}$ defined for $u \in A^*$ and $a, b, c \in A$ by

$$(3.2.17)\quad a c \ u \ b \leftrightarrow_{\text{Bl}} c a \ u \ b \quad \text{where} \quad a \leq b < c \quad \text{and for all} \quad d \in u, d \geq c.$$

The hypoplactic congruence (see [KT97,KT99]), denoted by $\equiv_{\text{H}}$, is the reflexive and transitive closure of the hypoplactic adjacency relation $\leftrightarrow_{\text{H}}$ defined for $u \in A^*$ and $a, b, c \in A$ by

$$(3.2.18a)\quad a c \ u \ b \leftrightarrow_{\text{H}} c a \ u \ b \quad \text{where} \quad a \leq b < c,$$

$$(3.2.18b)\quad b u \ c a \leftrightarrow_{\text{H}} b u \ a c \quad \text{where} \quad a < b \leq c.$$

The total congruence equivalence relation, denoted by $\equiv_{\text{T}}$, is the reflexive and transitive closure of the total adjacency relation $\leftrightarrow_{\text{T}}$ defined by $u \equiv_{\text{T}} v$ for any $u, v \in A^*$ such that $\text{ev}(u) = \text{ev}(v)$.

By Theorem 3.6, all these congruences lead to bigraded Hopf subalgebras of $\text{PM}_k$. Table 2 shows first few dimensions of the Hopf subalgebras of $\text{PMN}_1$ and $\text{PML}_1$ obtained from these congruences, computed by computer exploration.

4. Alternating sign matrices

Recall that an alternating sign matrix [MRR83], or an ASM for short, of size $n$ is a square matrix of order $n$ with entries in the alphabet $\{0, +, -\}$ such that every row and column starts and ends by $+$ and in every row and column, the $+$ and the $-$ alternate. For instance,

$$(4.0.19)\quad \delta := \begin{bmatrix} 0 & + & 0 & 0 & 0 \\ 0 & 0 & + & 0 & 0 \\ + & 0 & 0 & + & 0 \\ 0 & + & - & + & 0 \\ 0 & 0 & + & 0 & 0 \end{bmatrix}$$

is an ASM of size 5.
Hopf algebra structure. Let $\delta$ be an ASM. We denote by $M^\delta$ the matrix satisfying

$$M^\delta_{ij} := \begin{cases} 1 & \text{if } \delta_{ij} \in \{+,-\}, \\ 0 & \text{otherwise.} \end{cases}$$

For instance, with the ASM $\delta$ defined above, we obtain

$$M^\delta = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$
Theorem 4.1. The vector space $\text{ASM}$, endowed with the product and coproduct of $\text{PM}_1$, forms a free, cofree, and self-dual bigraded Hopf algebra which admits a bidendriform bialgebra structure.

Proof. Let $\delta_1$ and $\delta_2$ be two ASMs of respective sizes $n_1$ and $n_2$ and let $M \in M^{\delta_1} \sqcup M^{\delta_2}$. Let us denote by $M_1$ (resp. $M_2$) the matrix consisting in the first $n_1$ (resp. last $n_2$) rows of $M$. By construction, $M_1$ contains columns coming from $\delta_1$ and some null columns. The relative order of columns of $M_1$ is the same as in $M_1$, i.e., the $i$th column of $M_1$ is the $i$th nonzero column of $M$. Hence, the rows of $M_1$ start and end with $+$ and then $+$ and $-$ alternate. Similarly, the same property is satisfied in $M_2$. Furthermore, the nonzero column of $M_1$ are followed by null columns of $M_2$ and the nonzero column of $M_2$ are preceded by null columns of $M_1$. Hence, the columns of $M$ start and end with $+$ and $+$ and $-$ alternate. Thus $M$ is an ASM so that $\text{ASM}$ is stable for the product of $\text{PM}_1$.

Let $\delta$ be an ASM and $L \bullet R$ be a column decomposition of $M_\delta$. By Lemma 1.1, a column decomposition never splits a matrix by separating two nonzero entries on a same row. Then, the nonzero rows of $L$ and $R$ start and end with $+$ and $+$ and $-$ alternate. Thus, $\text{cp}(L)$ and $\text{cp}(R)$ are ASMs and $\text{ASM}$ is stable for the coproduct of $\text{PM}_1$.

This shows that $\text{ASM}$ is a Hopf subalgebra of $\text{PM}_1$ and also that $\text{ASM}$ inherits from the bidendriform bialgebra structure of $\text{PM}_1$ (see Theorem 2.9). Finally, since $\text{ASM}$ admits a bidendriform bialgebra structure, by [Foi07], it is free, cofree, and self-dual.

From now, we shall see $\text{ASM}$ as a simply graded Hopf algebra so that the degree of any $F_\delta$, where $\delta$ is an ASM, is the size of $\delta$. The dimensions of $\text{ASM}$ form Sequence A005130 of [Slo] and the first few terms are

\[ (4.1.5) \quad 1, 1, 2, 7, 42, 429, 7436, 218348, 10850216, 911835460, 129534272700. \]

By using same arguments as those used in Section 2.1, one can build multiplicative bases of $\text{ASM}$ by setting, for any ASM $\delta$,

\[ (4.1.6) \quad E_\delta := \sum_{M^\delta \subseteq \delta M^\delta'} F_{\delta'} \quad \text{and} \quad H_\delta := \sum_{M^\delta' \subseteq \delta M^\delta} F_{\delta'}. \]

This gives another way to prove the freeness of $\text{ASM}$ by using same arguments as those of Theorem 2.4. Hence, $\text{ASM}$ is freely generated by the elements $E_\delta$ (resp. $H_\delta$) where the $\delta$ are ASMs such that the $M^\delta$ are connected (resp. anti-connected) 1-packed matrices. The first few numbers of algebraic generators of $\text{ASM}$ are

\[ (4.1.7) \quad 0, 1, 1, 4, 29, 343, 6536, 202890, 10403135, 889855638, 127697994191 \]

and the first few dimensions of totally primitive elements are

\[ (4.1.8) \quad 0, 1, 0, 2, 20, 277, 5776, 188900, 9980698, 868571406, 125895356788. \]

Moreover, since the transpose of an ASM is also an ASM, by Proposition 2.5, the map $\phi : \text{ASM} \to \text{ASM}^*$ linearly defined for any ASM $\delta$ by

\[ (4.1.9) \quad \phi (F_\delta) := F_{\delta^T}^* \]

is an isomorphism.
4.2. Alternating sign matrices statistics. We recall here the definitions of some statistics on ASMs. Their description passes through six-vertex configurations and osculating paths, combinatorial objects in bijection with ASMs.

The statistics discussed in this article have been already exploited in the literature. For instance, in [EKLP92], the authors focused on these statistics to understand the relationship between domino tilings of Aztec diamonds and ASMs.

4.2.1. Six-vertex configurations. A six-vertex configuration (see for example [Bre99,Bax08] for further information and references) of size \(n\) is a \(n \times n\) square grid with oriented edges so that each vertex has two incoming and two outgoing edges. There are six possible configurations for each vertex. We consider here the six-vertex model with domain wall boundary conditions [Kor82] i.e., all horizontal (resp. vertical) edges on the boundary of this model are oriented inwardly (resp. outwardly) (see Figure 8(c)).

The bijection [Kup96] between ASMs of size \(n\) and six-vertex configurations of the same size consists in replacing each vertex configuration by \(0\), +, or \(-\) according to the rules described in Figure 7. Reciprocally, to recover a six-vertex model from an ASM \(\delta\), we first replace each nonzero entry of \(\delta\) by the corresponding vertex configuration (see the last two configurations of Figure 7). Then, for each zero entry of \(\delta\), we look at the sum \(\ell\) (resp. \(a\)) of the entries to the left (resp. above) of it and in the same row (resp. column). By the alternating property of the ASMs, \(\ell\) and \(a\) belong to \(\{0, 1\}\). Now, set in \(\delta\) the configuration \(\longleftrightarrow\) (resp. \(\uparrow\downarrow\)) if \(\ell = 1\) (resp. \(\ell = 0\)) together with the configuration \(\uparrow\downarrow\) (resp. \(\longleftrightarrow\)) if \(a = 1\) (resp. \(a = 0\)). Figures 8(a) and 8(c) form an example.

4.2.2. Statistics on six-vertex configurations and ASMs. Let us denote by \(\text{ne}(\delta)\) (resp. \(\text{sw}(\delta), \text{se}(\delta), \text{nw}(\delta), \text{oi}(\delta), \text{io}(\delta)\)) the number of vertices \(\text{ne}\) (resp. \(\text{sw}, \text{se}, \text{nw}, \text{oi}, \text{io}\)) in the six-vertex configuration in bijection with the ASM \(\delta\). Let \(\mathcal{Z} := \{\text{se}, \text{nw}, \text{se}\}\) be the set of the statistics counting the four configurations of 0 and \(\mathcal{R} := \{\text{io}, \text{oi}\}\) be the set of the statistics counting the two nonzero configurations.

4.2.3. Sets of osculating paths. These statistics share some symmetries that are naturally interpreted on sets of osculating paths (see [BMH95,Belh08]). Let \(\Pi\) be a \(n \times n\) square of lattice points with rows (resp. columns) labelled from 1 to \(n\) from top to bottom (resp. from left to right). A lattice path on \(\Pi\) is a sequence \((v_0, v_1, \ldots, v_r)\) of vertices of \(\Pi\) such that \(v_i - v_{i-1} \in \{(1, 0), (0, -1)\}\) for all \(i \in [r]\). A set of osculating paths on \(\Pi\) is a set of lattice paths on \(\Pi\) in which different paths do not cross each other but can share some vertices.

We can associate a set of osculating paths with any six-vertex configuration according to the rules described in Figure 9. Domain boundary conditions ensure that each path starts at \((i, 1)\) and ends at \((1, j)\) for some \(i\) and \(j\). Figures 8(c) and 8(d) form an example.
The direct interpretation of ASMs in terms of sets of osculating paths is directly based upon the corner sum matrix introduced in [RR86]: given an $n \times n$ matrix $M$, the corner sum matrix $\bar{M}$ of $M$ is defined by

$$\bar{M}_{i,j} := \sum_{i \leq k \leq n, j \leq \ell \leq n} M_{k,\ell}.$$  

Figures 8(a) and 8(b) form an example. We associate with any ASM $\delta$ of size $n$ the set of osculating paths described as follows. By regarding $\delta$ as a $(n+1) \times (n+1)$ square of lattice points, we draw on it the south and the east boundaries of the areas consisting in a same value greater than zero. This produces a set of $n$ osculating paths. Figures 8(b) and 8(d) form an example. Since the steps of the paths in the first row (resp. column) are, by construction, always vertical (resp. horizontal), this set of osculating paths can be seen without loss of information on the $n \times n$ square of lattice points.

The different $2 \times 2$ submatrices configurations in a corner sum matrix of an ASM are exactly

$$(4.2.2)$$

They obviously describe the correspondence given in Figure 9.

---

**Figure 8.** Four objects in correspondence: ASMs, six-vertex configurations, corner sum matrices, and sets of osculating paths.

**Figure 9.** Correspondence between vertices of six-vertex configurations and osculating paths.
4.2.4. Symmetries between ASMs statistics.

Proposition 4.2. Let $\delta$ be an ASM of size $n$. Then,

\[
\text{se}(\delta) = \text{nw}(\delta), \quad \text{ne}(\delta) = \text{sw}(\delta), \quad \text{oi}(\delta) = \text{io}(\delta) + n.
\]

Proof. Consider the set of osculating paths $P$ associated with $\delta$ and the correspondence between the vertices of six-vertex configurations and osculating paths (see Figure 9). The first identity of (4.2.3) is equivalent to say that there are in $P$ as many horizontal steps as vertical steps. Since in $P$, any osculating path connects the $i$th vertex of the first column with the $i$th vertex of the first row of the grid, for any $i \in [n]$, this property holds. Whence the first identity.

Consider now the ASM $\delta'$ where, for any $i \in [n]$, the $i$th row of $\delta'$ is the $(n - i + 1)$st row of $\delta$. Then, in the six-vertex configuration in bijection with $\delta'$, all ne (resp. sw) configurations come from se (resp. nw) configurations of the six-vertex configuration in bijection with $\delta$. Then, the second identity follows from the first one.

The last identity follows immediately from the fact that any row and column of $\delta$ starts and ends by $+$, and the $+$ and the $-$ alternate. $\square$

4.3. Algebraic interpretation of some statistics. We provide algebraic interpretations of the statistics on ASMs recalled in the previous section by using the Hopf algebra $\text{ASM}$. To be more precise, we study the algebraic quotients of $\text{ASM}$ by equivalence relations defined via ASMs statistics.

4.3.1. Maps from $\text{ASM}$ to $q$-rational functions. Let us recall the following notations in the algebra $\mathbb{K}(q)$ of $q$-rational functions:

\[
\begin{align*}
[n]_q &:= 1 + q + \cdots + q^{n-1}, \quad n \geq 1, \\
[0]_q! &:= 1, \quad [n]_q! := [1]_q[2]_q \cdots [n]_q, \quad n \geq 1, \\
[n_1+n_2]_{n_1,n_2} &:= \frac{[n_1+n_2]_q!}{[n_1]_q ![n_2]_q !}, \quad n_1, n_2 \geq 0.
\end{align*}
\]

Lemma 4.3. Let $\delta, \delta_1$, and $\delta_2$ be three ASMs such that $M^\delta \in M^{\delta_1} \sqcup M^{\delta_2}$. Then, for any $s \in \mathcal{R}$,

\[
s(\delta) = s(\delta_1) + s(\delta_2).
\]

Proof. The two statistics $\text{oi}$ and $\text{io}$ of $\mathcal{R}$, respectively count the number of entries $+$ and $-$ in ASMs. This result follows from the fact that the shifted shuffle of packed matrices does not add nor remove nonzero entries and the fact that any nonzero entry encoding a $+$ (resp. $-$) in the operands $M^{\delta_1}$ and $M^{\delta_2}$ gives rise to a $+$ (resp. $-$) in $M^\delta$. $\square$

Here is the product (4.1.3) in $\text{ASM}$, seen on six-vertex configurations, where boldfaced vertices are of kind $\text{io}$.
Proposition 4.4. The map $\phi_s : \text{ASM} \to \mathbb{K}(q)$ linearly defined, for any $s \in \mathcal{R}$ and any ASM $\delta$ of size $n$ by

$$(4.3.6) \quad \phi_s (F_\delta) := \frac{q^{s(\delta)}}{n!}$$

is an algebra morphism.

Proof. This result follows immediately from Lemma 4.3 and the fact that the product of two matrices of sizes $n_1$ and $n_2$ in $\text{ASM}$ over the fundamental basis contains $\binom{n_1 + n_2}{n_1}$ terms.

Lemma 4.5. Let $\delta, \delta_1,$ and $\delta_2$ be three ASMs such that $M^\delta \in M^{\delta_1} \sqcup M^{\delta_2}$. Let $m$ be the size of $\delta_2$ (resp. $\delta_1$) and $\{k_1 < k_2 < \cdots < k_m\}$ be the set of the indices of the columns of $M^\delta$ coming from $M^{\delta_2}$ (resp. $M^{\delta_1}$). Then, for any $s \in \{\text{nw, se}\}$ (resp. $s \in \{\text{sw, ne}\}$),

$$(4.3.7) \quad s(\delta) = s(\delta_1) + s(\delta_2) + \sum_{1 \leq j \leq m} (k_j - j).$$

Proof. Let us prove the statement for the nw statistic. Let us denote by $n_1$ the size of $\delta_1$ and by $M_1$ (resp. $M_2$) the first $n_1$ (resp. the last $m$) rows of $\delta$.

Notice that the zero columns of $M_2$ have no nw configuration and that the nw configurations lying in the nonzero columns of $M_1$ (resp. $M_2$) are those of $\delta_1$ (resp. $\delta_2$). It remains to count, for all $j \in [m]$, the number of nw configurations in the $k_j$th column of $M_1$. Observe that the sums of the entries above any zero of the $k_j$th column are 0. Besides, there are exactly $k_j - j$ zeros in the $k_j$th column such that the sums of the entries to their left are 1. These zeros are, by definition, nw configurations, whence (4.3.7).

This is also valid for the statistic sw since the symmetry consisting in swapping the $i$th and $(n - i + 1)$st row of ASMs of size $n$ exchanges the nw configurations into sw configurations. By Proposition 4.2, this also proves the statement of the se and ne statistics.

Here is the product (4.1.3) in ASM, seen on six-vertex configurations, where boldfaced vertices are of kind nw.

$$(4.3.8) \quad \text{F} \cdot \text{F} = \text{F} + \text{F} + \text{F} + \text{F}$$

Proposition 4.6. The map $\phi'_s : \text{ASM} \to \mathbb{K}(q)$ linearly defined, for any $s \in \mathcal{S}$ and any ASM $\delta$ of size $n$ by

$$(4.3.9) \quad \phi'_s (F_\delta) := \frac{q^{s(\delta)}}{[n]_q!}$$

is an algebra morphism.
Proposition 4.7. The maps $\psi_s: \text{ASM}^* \to \mathbb{K}(q)$ and $\psi_t^*: \text{ASM}^* \to \mathbb{K}(q)$ linearly defined, for any $s \in \mathcal{S}$, $t \in \mathcal{S}$, and any ASM $\delta$ of size $n$ by

$$
\psi_s(F^*_\delta) := \frac{q^{s(\delta)}}{n!} \quad \text{and} \quad \psi_t^*(F^*_\delta) := \frac{q^{t(\delta)}}{n! q!}
$$

are algebra morphisms.

Here is the product (4.1.3) in ASM* seen on six-vertex configurations, where the vertices represented by squares are of kind io while those represented by circles are of kind nw.

### 4.3.2. Equivalence relations on ASMs and associated subspaces of ASM

Let $S \subseteq \mathcal{S} \cup \mathcal{N}$ be a set of statistics and $\sim_S$ be the equivalence relation on the set of ASMs defined, for any ASMs $\delta_1$ and $\delta_2$ of the same size, by

$$
\delta_1 \sim_S \delta_2 \quad \text{if and only if} \quad s(\delta_1) = s(\delta_2) \quad \text{for all} \quad s \in S.
$$

We denote by $I_S$ the associated vector space spanned by

$$
\{F_{\delta_1} - F_{\delta_2}, \, \delta_1 \sim_S \delta_2\}.
$$

### 4.3.3. The algebra $\text{ASM}/I_{io}$

Let us first study the statistic $\text{io} \in \mathcal{N}$.

Proposition 4.8. The quotient $\text{ASM}/I_{io}$ is a commutative algebra.

Proof. The subspace $I_{io}$ of ASM is a two-sided ideal of ASM. Indeed, let $\delta, \delta_1$, and $\delta_2$ be three ASMs such that $\delta_1 \sim_{io} \delta_2$. Since the products $F_\delta \cdot F_{\delta_1}$ and $F_\delta \cdot F_{\delta_2}$ for $i \in \{1, 2\}$ have the same number of terms, Lemma 4.3 implies that the products $F_\delta \cdot (F_{\delta_1} - F_{\delta_2})$ and $(F_{\delta_1} - F_{\delta_2}) \cdot F_\delta$ are in $I_{io}$. Hence, $\text{ASM}/I_{io}$ is an algebra.

Besides, the ideal $I_{io}$ contains the commutators. Indeed, let $\delta_1$ and $\delta_2$ be two ASMs. Since the products $F_{\delta_1} \cdot F_{\delta_2}$ and $F_{\delta_2} \cdot F_{\delta_1}$ have the same number of terms, Lemma 4.3 implies that $F_{\delta_1} \cdot F_{\delta_2} - F_{\delta_2} \cdot F_{\delta_1}$ is in $I_{io}$. Thus, $\text{ASM}/I_{io}$ is commutative as an algebra. \qed
Note however that $\text{ASM}/\text{io}$ does not inherit the structure of a coalgebra of $\text{ASM}$ because even if

\begin{equation}
(4.3.14) \quad x := F \begin{bmatrix} 0 & 0 & 0 \\ + & + & 0 \\ 0 & 0 & 0 \end{bmatrix} - F \begin{bmatrix} 0 & 0 & 0 \\ + & + & 0 \\ 0 & 0 & 0 \end{bmatrix}
\end{equation}

is an element of $I_{\text{io}}$, the element

\begin{equation}
(4.3.15) \quad \Delta(x) = 1 \otimes x + F \begin{bmatrix} 0 & 0 \\ + & + \\ 0 & 0 \end{bmatrix} \otimes F[+] + x \otimes 1
\end{equation}

is not in $\text{ASM} \otimes I_{\text{io}} + I_{\text{io}} \otimes \text{ASM}$. Hence, $I_{\text{io}}$ is not a coideal.

**Proposition 4.9.** The dimension $A_{n}^{\text{io}}$ of the $n$th graded component of $\text{ASM}/\text{io}$ is $\left\lfloor \frac{n^2}{2} \right\rfloor + 1$.

**Proof.** Let $\delta$ be an ASM of size $n$ with a maximal number of no configurations (i.e., a maximal number of $-$). Then, it is easy to see that

\begin{equation}
(4.3.16) \quad \text{io}(\delta) = \sum_{i=0}^{\left\lfloor \frac{n}{2} \right\rfloor - 1} i + \sum_{i=1}^{\left\lfloor \frac{n-1}{2} \right\rfloor} i.
\end{equation}

Indeed, the first and last row of an ASM can contain only one $+$ and no $-$. Let $i \geq 2$ and let $A_{i-1}$ be the matrix consisting in the first $i-1$ rows of $A$. The $-$ in row $i$ can only be in those columns for which the corresponding column sum of the submatrix $A_{i-1}$ is 1. Since the row sums of $A_{i-1}$ are 1 and the column sums of $A_{i-1}$ are 0 or 1, exactly $i-1$ of the column sums of $A_{i-1}$ are 1. We conclude that there are at most $(i-1)$ $-$s in row $i$. The same argument applies to the column sums taken from bottom to top. Hence, the rows $i$ and $n-i+1$ are at most $(i-1)$ $-$s. If $n$ is odd, then the row $(n+1)/2$ has only nonzero entries, alternating between $+$ and $-$, and the row $(n+1)/2$ has $\left\lfloor \frac{n}{2} \right\rfloor - 8$.

Now, since for any $0 \leq k \leq \text{io}(\delta)$, there exists an ASM $\delta'$ such that $\text{io}(\delta') = k$, we obtain, by a simple computation, the statement of the proposition. \hfill $\Box$

The dimensions of $\text{ASM}/\text{io}$ form Sequence A033638 of [Slo] and the first few terms are

\begin{equation}
(4.3.17) \quad 1, 1, 1, 2, 3, 5, 7, 10, 13, 17, 21.
\end{equation}

A basic argument on generating series implies that these dimensions cannot be the ones of a free commutative algebra and hence, $\text{ASM}/\text{io}$ is not free as a commutative algebra.

Using the symmetry between the statistics $\text{io}$ and $\text{oi}$ provided by Proposition 4.2, we immediately have $\sim_{\text{oi}} = \sim_{\text{io}}$ and then, $\text{ASM}/\text{io} = \text{ASM}/\text{oi}$.

4.3.4. The algebra $\text{ASM}/\text{inw}$. Let us now study the statistic $\text{nw} \in \mathcal{Z}$.

**Proposition 4.10.** The quotient $\text{ASM}/\text{inw}$ is a commutative algebra.

**Proof.** The subspace $I_{\text{inw}}$ of $\text{ASM}$ is a two-sided ideal of $\text{ASM}$. Indeed, let $\delta, \delta_1$ and $\delta_2$ be three ASMs of respective sizes $n, n_1$ and $n_2$ such that $\delta_1 \sim_{\text{inw}} \delta_2$. Lemma 4.5 implies that the number of $\text{nw}$ configurations of an ASM $\delta'$ such that $F_{\delta'}$ appears in $F_{\delta} \cdot F_{\delta_1}$ (resp. $F_{\delta} \cdot F_{\delta_2}$) depends only on the number of $\text{nw}$ configurations in $\delta$ and $\delta_1$ (resp. $\delta_2$) and a subset of $[n + n_1]$ (resp. $[n + n_2]$) of size $n_1$ (resp. $n_2$) corresponding to the positions in $\delta'$ of the columns coming from $\delta_1$ (resp. $\delta_2$).
Each entry of an ASM matrices are in different iteration obviously increases by one the number of considering a process that associates with a permutation matrix \( M \).

**Proposition 4.11.** The dimension \( A^{nw}_n \) of the \( n \)th graded component of \( \text{ASM}/I_{nw} \) is \( \binom{n}{2} + 1 \).

**Proof.** Let us first show that there are at least \( \binom{n}{2} + 1 \) \( \sim_{nw} \)-equivalence classes of ASMs of size \( n \) by considering a process that associates with a permutation matrix \( M_1 \) of size \( n \) a permutation matrix \( M_2 \) such that \( w(M_2) = w(M_1) + 1 \). If \( M_1 \) is not the permutation matrix \( I_n \) of the identity, there is a greatest integer \( k \geq 0 \) such that \( M_1 = I_k / M'_1 \) and \( M'_1 \) is not empty. Consider now the matrix \( M_2 := I_k / M'_2 \) where \( M'_2 \) is the matrix obtained by swapping the \( (i-1) \)st and \( i \)th columns of \( M'_1 \) so that \( i \) is the index of the column of \( M'_1 \) containing its uppermost 1. Starting with the matrix \( M_1 \) of size \( n \) of the form \( 1 \, \cdots \, \underbrace{1}_{1} \), we can iteratively apply the previous process \( \binom{n}{2} \) times. Since each iteration obviously increases by one the number of \( nw \) configurations, all the \( \binom{n}{2} + 1 \) permutation matrices are in different \( \sim_{nw} \)-equivalence classes.

Let us then show that there are no more than \( \binom{n}{2} + 1 \) \( \sim_{nw} \)-equivalence classes of ASMs of size \( n \). Each entry of an ASM \( \delta \) of size \( n \) gives rise to a configuration among the six possible. Then,

\[
 n^2 = w(n) + nw(n) + sw(n) + io(n) + 2io(n) + n \]

By using the symmetries provided by Proposition 4.2, \( (4.3.20) \) becomes

\[
 n^2 = 2sw(n) + 2nw(n) + 2io(n) + n
\]

and we deduce that \( w(n) \leq \frac{n^2-n}{2} = \binom{n}{2} \). 

The dimensions of \( \text{ASM}/I_{nw} \) form Sequence \text{A152947} of [Slo] and the first few terms are

\[
 1, 1, 2, 4, 7, 11, 16, 22, 29, 37, 46, 56.
\]

A basic argument on generating series implies that these dimensions cannot be the ones of a free commutative algebra and hence, \( \text{ASM}/I_{nw} \) is not free as a commutative algebra.

Using the symmetry between the statistics \( nw \) and \( se \) provided by Proposition 4.2, we immediately have \( \sim_{se} = \sim_{nw} \) and then, \( \text{ASM}/I_{nw} = \text{ASM}/I_{se} \). Moreover, by using the same arguments as before, \( \text{ASM}/I_{nw} \) and \( \text{ASM}/I_{se} \) are the same commutative algebras.
Note that the map $\theta: \text{ASM}/I_{\text{nw}} \to \text{ASM}/I_{\text{sw}}$ linearly defined for any ASM $\delta$ by
\begin{equation}
\theta(\pi_{\text{nw}}(F_{\delta})) := \pi_{\text{sw}}(F_{\overrightarrow{\delta}}),
\end{equation}
where $\pi_{\text{nw}}$ (resp. $\pi_{\text{sw}}$) is the canonical projection from $\text{ASM}$ to $\text{ASM}/I_{\text{nw}}$ (resp. $\text{ASM}/I_{\text{sw}}$) and $\overrightarrow{\delta}$ is the ASM where, for any $i \in [n]$, the $i$th column of $\overrightarrow{\delta}$ is the $(n-i+1)$st column of $\delta$, is an isomorphism between $\text{ASM}/I_{\text{nw}}$ and $\text{ASM}/I_{\text{sw}}$.

### 4.3.5. The algebra $\text{ASM}/I_{\text{io}, \text{nw}}$.

Let us finally study the set of statistics $\{\text{io, nw}\}$.

**Proposition 4.12.** The quotient $\text{ASM}/I_{\text{io}, \text{nw}}$ is a commutative algebra.

**Proof.** This follows directly from Propositions 4.8 and 4.10. \qed

Note however that $\text{ASM}/I_{\text{io}, \text{nw}}$ does not inherit the structure of a coalgebra of $\text{ASM}$ because even if
\begin{equation}
x := F \begin{bmatrix}
0 & + & 0 & 0 \\
0 & + & 0 & 0 \\
0 & 0 & 0 & + \\
0 & + & 0 & 0
\end{bmatrix} - F \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\end{equation}
is an element of $I_{\text{io}, \text{nw}}$, the element
\begin{equation}
\Delta(x) = 1 \otimes x + F \begin{bmatrix}
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0
\end{bmatrix} \otimes F \begin{bmatrix} + \\
+ \\
+ \\
+
\end{bmatrix} + x \otimes 1
\end{equation}
is not in $\text{ASM} \otimes I_{\text{io}, \text{nw}} + I_{\text{io}, \text{nw}} \otimes \text{ASM}$. Hence, $I_{\text{io}, \text{nw}}$ is not a coideal.

By computer exploration, the first few dimensions of $\text{ASM}/I_{\text{io}, \text{nw}}$ are
\begin{equation}
1, 1, 2, 5, 13, 31, 66, 127, 225,
\end{equation}
and seems to be Sequence A116701 of [Slo].

A basic argument on generating series implies that these dimensions cannot be the ones of a free commutative algebra and hence, $\text{ASM}/I_{\text{io}, \text{nw}}$ is not free as a commutative algebra.

### 4.3.6. Others quotients of ASM.

Using the symmetries provided by Proposition 4.2, all the algebras $\text{ASM}/I_S$, where $S$ contains two nonsymmetric statistics, are equal to $\text{ASM}/I_{\text{io}, \text{nw}}$. Moreover, note that by using the same arguments as before, one can prove that for any $S \in 3 \cup \mathcal{N}$, $\text{ASM}/I_S$ is a commutative algebra isomorphic to $\text{ASM}/I_{\text{io}}$, $\text{ASM}/I_{\text{nw}}$, or $\text{ASM}/I_{\text{io}, \text{nw}}$.

### References

HOPF ALGEBRA ON PACKED SQUARE MATRICES


