Pseudofinite Mc-groups

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PSEUDOFINITE $\widetilde{\mathfrak{M}}_c$-GROUPS

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Abstract. A pseudofinite group satisfying the uniform chain condition on centralizers up to finite index has a big finite-by-abelian subgroup.

Introduction

We generalize the results of Elwes, Jaligot, MacPherson and Ryten [1, 2] about pseudofinite superstable groups of small rank to the pseudofinite context, possibly of infinite rankl

1. Rank

Definition 1. A *dimension* on a theory T is a function dim from the collection of all interpretable subsets of a monster model to $\mathbb{R}^{\geq 0} \cup \{\infty\}$ satisfying

- *Invariance:* If $a \equiv a'$ then $\dim(\varphi(x,a)) = \dim(\varphi(x,a'))$.
- *Algebraicity:* If X is finite, then $\dim(X) = 0$.
- *Union:* $\dim(X \cup Y) = \max\{\dim(X), \dim(Y)\}$.
- *Fibration:* If $f : X \to Y$ is a interpretable surjection whose fibres have constant dimension r, then $\dim(X) = \dim(Y) + r$.

For a partial type π we put $\dim(\pi) = \inf\{\dim(\varphi) : \pi \vdash \varphi\}$. We write $\dim(a/B)$ for $\dim(\mathrm{tp}(a/B))$.

It follows that $X \subseteq Y$ implies $\dim(X) \leq \dim(Y)$, and $\dim(X \times Y) = \dim(X) + \dim(Y)$. Moreover, any partial type π can be completed to a complete type p with $\dim(\pi) = \dim(p)$.

Definition 2. A dimension is *additive* if it satisfies

- *Additivity:* $\dim(a, b/C) = \dim(a/b, C) + \dim(b/C)$.

Remark 3. Additivity is clearly equivalent to *fibration* for type-definable maps.

Example. Examples for an additive dimension include:

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(1) coarse pseudofinite dimension (in the expansion by cardinality comparison quantifiers, in order to ensure invariance);
(2) Lascar rank, SU-rank or þ-rank, possibly localised at some $\emptyset$-invariant family of types;
(3) for any ordinal α, the coefficient of ω^α in one of the ordinal-valued ranks in (2) above.

In any of the examples, we put $\dim(X) = \infty$ if $\dim(X) > n$ for all $n < \omega$.

Note that additivity in the above examples follows from the Lascar inequalities for the corresponding rank in examples (2) and (3) and from [3] in example (1).

Definition 4. Let G be a type-definable group with an additive dimension, and suppose $0 < \dim(G) < \infty$. A partial type $\pi(x)$ implying $x \in G$ is *wide* if $\dim(\pi) = \dim(X)$. It is *broad* if $\dim(\pi) > 0$. An element $g \in G$ is *wide*/*broad* over some parameters A if $\mathrm{tp}(g/A)$ is. We say that π is *negligible* if $\dim(\pi) = 0$.

Lemma 5. *An additive dimension is invariant under definable bijections.*

Proof: Let f be an A-definable bijection, and $a \in \mathrm{dom}(f)$. Then $\dim(f(a)/A, a) = 0$ and $\dim(a/A, f(a)) = 0$, whence

$$\dim(a/A) = \dim(a, f(a)/A) = \dim(f(a)/A). \qquad \square$$

Definition 6. If $\dim(a/A) < \infty$, we say that $a \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{d}}_A B$ if $\dim(a/A) = \dim(AB)$.

It follows that $\mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{d}}$ satisfies transitivity. Moreover, for any partial type π there is a comple type $p \supseteq \pi$ (over any set of parameters containing $\mathrm{dom}(\pi)$) with $\dim(p) = \dim(\pi)$, so $\mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{d}}$ satisfies extension.

Lemma 7 (Symmetry)**.** *Let* $\dim$ *be an additive dimension. If* $\dim(a/A) < \infty$ *and* $\dim(b/A) < \infty$*, then* $a \mathop{\smile\!\!\!\!\!\!\mid}^{d}_A b$ *if and only if* $b \mathop{\smile\!\!\!\!\!\!\mid}^{d}_A a$*, if and only if* $\dim(a, b/A) = \dim(a/A) + \dim(b/A)$.

Proof: Obvious. $\square$

Lemma 8. *Let G be a type-definable group with an additive dimension, and suppose $0 < \dim(G) < \infty$. If g is wide over A and h is wide over A, g, then gh and hg are wide over A, g and over A, h.*

Proof: As dimension is invariant under definable bijections

$$\dim(gh/A, g) = \dim(h/A, g) = \dim(G).$$

The other statements follow similarly, possibly using symmetry. $\square$

2. Big abelian subgroups

Proposition 9. *Let G be a pseudofinite group, and* dim *an additive dimension on G. Then there is an element $g \in G \setminus \{1\}$ such that* $\dim(C_G(g)) \geq \frac{1}{3}\dim(G)$.

Proof: Suppose first that G has no involution. If $G \equiv \prod_I G_i/\mathcal{U}$ for some family $(G_i)_I$ of finite groups and some non-principal ultrafilter $\mathcal{U}$, then G_i has no involution for almost all $i \in I$, and is soluble by the Feit-Thompson theorem. So there is $g_i \in G_i \setminus \{1\}$ such that $\langle g_i^{G_i} \rangle$ is commutative. Put $g = [g_i]_I \in G \setminus \{1\}$. Then $\langle g^G \rangle$ is commutative and $g^G \subseteq C_G(g)$. As g^G is in definable bijection with $G/C_G(g)$, we have

$$\dim(C_G(g)) \geq \dim(g^G) = \dim(G/C_G(g)) = \dim(G) - \dim(C_G(g)).$$

In particular $\dim(C_G(g)) \geq \frac{1}{2}\dim(G)$.
Now let $i \in G$ be an involution and suppose $\dim(C_G(i)) < \frac{1}{3}\dim(G)$. Then

$$\dim(i^G) = \dim(G/C_G(i)) = \dim(G) - \dim(C_G(i)) > \frac{2}{3}\dim(G)$$

and there is $j = i^g \in i^G$ with $\dim(j) \geq \frac{2}{3}\dim(G)$. Note that

$$\dim(C_G(j)) = \dim(C_G(i)^g) = \dim(C_G(i)) < \frac{1}{3}\dim(G).$$

Then

$$\dim(j^G j) = \dim(G/C_G(j)) = \dim(G) - \dim(C_G(j)) > \frac{2}{3}\dim(G)$$

and there is $h = j^{g'}j \in j^G j$ with $\dim(h/j) \geq \frac{2}{3}\dim(G)$. Note that h is inverted by j. By additivity,

$$\dim(j/h) = \dim(j,h) - \dim(h) \geq \dim(h/j) + \dim(j) - \dim(G) > \frac{1}{3}\dim(G).$$

If $H = \{x \in G : h^x = h^{\pm 1}\}$, then H is an h-definable subgroup of G, and $C_G(h)$ has index two in H. Since $j \in H$, we have

$$\dim(C_G(h)) = \dim(H) \geq \dim(j/h) > \frac{1}{3}\dim(G). \quad \square$$

Theorem 10. *Let G be a pseudofinite $\widetilde{\mathfrak{M}}_c$-group, and* dim *an additive dimension on G. Then G has a definable broad finite-by-abelian subgroup Z. More precisely, $Z = \tilde{Z}(C)$ where C is a minimal broad centralizer (up to finite index) of a finite tuple.*

Proof: By the $\widetilde{\mathfrak{M}}_c$ condition, there is a broad centralizer C of some finite tuple, such that $C_C(g))$ is not broad for any $g \in C \setminus \tilde{Z}(C)$. Put $Z = \tilde{Z}(C)$, a definable finite-by-abelian normal subgroup of C. If Z is broad, we are done. Otherwise $\dim(Z) = 0$, and

$$\dim(C/Z) = \dim(C) - \dim(Z) = \dim(C).$$

For $g \in C \setminus Z$ we have $\dim(C_C(g)) = 0$, whence for $\bar{g} = gZ$ we have

$$\begin{aligned}\dim(\bar{g}^{C/Z}) &= \dim(g^C Z/Z) \geq \dim(g^C) - \dim(Z)\\ &= \dim(C) - \dim(C_C(g)) - \dim(Z) = \dim(C/Z).\end{aligned}$$

Hence for all $\bar{g} \in (C/Z) \setminus \{\bar{1}\}$ we have

$$\dim(C_{C/Z}(\bar{g})) = \dim(C/Z) - \dim(\bar{g}^{C/Z}) = 0.$$

As C and Z are definable, C/Z is again pseudofinite, contradicting Proposition 9. □

Theorem 10 holds in particular for any pseudofinite $\widetilde{\mathfrak{M}}_c$-group with the pseudofinite counting measure. Note that the broad finite-by-abelian subgroup is defined in the pure group, using centralizers and almost centres. Moreover, the $\widetilde{\mathfrak{M}}_c$-condition is just used in G, not in the section C/Z.

Corollary 11. *A superrosy pseudofinite group with $U^{\flat}(G) \geq \omega^\alpha$ has a definable finite-by-abelian subgroup A with $U^{\flat}(A) \geq \omega^\alpha$.*

Proof: A superrosy group is $\widetilde{\mathfrak{M}}_c$. If α is minimal with $U^{\flat}(G) < \omega^{\alpha+1}$ and we put

$$\dim(X) \geq n \quad \text{if} \quad U^{\flat}(X) \geq \omega^\alpha \cdot n,$$

then dim is an additive dimension with $0 < \dim(G) < \infty$. The assertion now follows from Theorem 10. □

Corollary 12. *For any $d, d' < \omega$ there is $n = n(d, d')$ such that if G is a finite group without elements $(g_i : i \leq d')$ such that*

$$|C_G(g_i : i < j) : C_G(g_i : i \leq j)| \geq d$$

for all $j \leq d'$, then G has a subgroup A with $|A'| \leq n$ and $n\,|A|^n \geq |G|$.

Proof: If the assertion were false, then given d, d', there were a sequence $(G_i : i < \omega)$ of finite groups satisfying the condition, such that G_i has no subgroup A_i with $|A_i'| \leq i$ and $i\,|A_i|^i \geq |G_i|$. But any non-principal ultraproduct $G = \prod G_n/\mathcal{U}$ is a pseudofinite $\widetilde{\mathfrak{M}}_c$-group, and has a definable subgroup A with A' finite and $\dim(A) \geq \frac{1}{n}\dim(G)$ for some $n < \omega$. Unravelling the definition of the pseudofinite counting measure (and possibly increasing n) we get $|A_i'| \leq n$ and $n \cdot |A_i|^n \geq |G_i|$ for almost all $i < \omega$, a contradiction for $i \geq n$. □

3. Pseudofinite $\widetilde{\mathfrak{M}}_c$-groups of dimension 2

Theorem 13. *Let G be a pseudofinite $\widetilde{\mathfrak{M}}_c$-group, and* dim *an additive integer-valued dimension on G. If $\dim(G) = 2$, then G has a broad definable finite-by-abelian subgroup N whose normalizer is wide.*

Proof: Note that since dim is integer-valued, a broad subgroup has dimension at least 1. By Corollay 11, if C is a minimal broad centralizer (up to finite index), then $A = \tilde{Z}(C)$ is broad and finite-by-abelian.
If A is commensurate with A^g for all $g \in G$, then commensurativity is uniform by compactness. So by Schlichting's Theorem there is a normal subgroup N commensurate with A. But then $\tilde{Z}(N)$ contains $A \cap N$, has finite index in N and is broad; since it is characteristic in N, it is normal in G and we are done.
Suppose $g \in \tilde{N}_G(H)$ is such that A is not commensurable with A^g. Then clearly C cannot be commensurable with C^g, and

$$\dim(A \cap A^g) \leq \dim(C \cap C^g) = 0.$$

Hence

$$\dim(AA^g) \geq \dim(AA^g/(A\cap A^g)) = \dim(A/(A\cap A^g) + \dim(A^g/(A\cap A^g)) = 2\ \dim(A).$$

As A is broad and $\dim(G) = 2$, we have $\dim(A) = 1$.
Choose some d-independent wide a, b_0, c_0 in A over g. Then $\dim(a^g b_0/g, c_0) = 2$, and

$$2 \geq \dim(c_0 a^g b_0/g) \geq \dim(c_0 a^g b_0/g, c_0) = \dim(a^g b_0/g, c_0) = \dim(a^g b_0/g) = 2$$

and $c_0 \mathop{\smile\!\!\!\!\!\!\,|}^{\mathrm{d}}_g c_0 a^g b_0$. Thus c_0 is wide in A over $g, c_0 a^g b c_0$. Similarly, b_0 is wide in A over $g, c_0 a^g b c_0$.
Choose $d, b_1, c_1 \equiv_{c_0 a^g b_0, g} a, b_0, c_0$ with $d, b_1, c_1 \mathop{\smile\!\!\!\!\!\!\,|}^{\mathrm{d}}_{c_0 a^g b_0, g} a, b_0, c_0$. Then $c_0 a^g b_0 = c_1 d^g b_1$, whence

$$a^g b = c d^g,$$

where $c = c_0^{-1} c_1$ and $b = b_0 b_1^{-1}$. Moreover,

$$\dim(b/a, g) \geq \dim(b_0 b_1^{-1}/a, b_0, c_0, g) \geq \dim(b_1/a, b_0, c_0, g) = \dim(b_1/c_0 a^g b_0, g) = 1,$$

so b is wide in A over g, a. Similarly, c is wide in A over g, d.
Let x and y be two d-independent wide elements of $C_A(a, b, c, d)$ over a, b, c, d, g. Then they are d-independent wide in A, and for $z = xgy$ we have

$$\dim(z/a, b, c, d, g) = 2,$$

so z is wide in G over g, a, b, c, d. Moreover

$$a^z b = a^{xgy} b = a^{gy} b^y = (a^g b)^y = (cd^g)^y = c^y d^{gy} = cd^{xgy} = cd^z.$$

Choose $z' \in G$ with $z' \equiv_{a,b,c,d} z$ and $z' \mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{d}}_{a,b,c,d} z$, and put $r = z'^{-1}z$. Then r is wide in G over g, a, b, c, d, z, and

$$a^z b^r = a^{z'r} b^r = (a^{z'} b)^r = (cd^{z'})^r = c^r d^{z'r} = c^r d^z.$$

Hence

$$c^{-1} a^z b = d^z = c^{-r} a^z b^r.$$

Putting $b' = bb^{-r}$ and $c' = cc^{-r}$, we obtain

$$a^z b' = c' a^z,$$

where a is wide in A. As $r \mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{d}}_{g,a,b,c,d} z$ we have

$$\dim(z/a, b', c') \geq \dim(z/a, b, c, d, r) = \dim(z/a, b, c, d) = 2,$$

and z is wide in G over a, b', c'. If $z'' \equiv_{a,b',c'} z$ with $z'' \mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{d}}_{a,b',c'} z$, then $a^{z''} b' = c' a^{z''}$, whence

$$b'^{a^z} = c' = b'^{a^{z''}}$$

and $a^z a^{-z''} = a'^z$ commutes with b', where $a' = aa^{-z''z^{-1}}$.

Claim. $\dim(b') \geq 1$, $\dim(a'/b') \geq 1$, $\dim(z/a', b') \geq 2$ *and* $\dim(a'^z/b') \geq 1$.

Proof of Claim: If $\dim(b'/b) = 0$, then $\dim(r/b, b') = \dim(r/b) = 2$. Choose $r' \equiv_{b,b'} r$ with $r' \mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{d}}_{b,b'} r$. So $b^r = b'^{-1}b = b^{r'}$ and $r'r^{-1} \in C_G(b)$. Since $r'r^{-1}$ is wide in G over b, so is $C_G(b)$, and it has finite index in G by minimality. Thus $b \in \tilde{Z}(G)$ and $\dim \tilde{Z}(G) \geq 1$, so we can take $N = \tilde{Z}(G)$. Thus we may assume $\dim(b') \geq \dim(b'/b) \geq 1$.
Next, note that z and $z''z^{-1}$ are wide and d-independent over a, b', c' by Lemma 8. An argument similar to the first paragraph yields $\dim(a'/b') \geq \dim(a'/a, b', z''z^{-1}) \geq 1$ and $\dim(a'^z/b') \geq 1$. □
To finish, note that $\dim(C_G(b')) \geq \dim(a'^z/b') \geq 1$. If $\dim(C_G(b')) = 2$, then $b' \in \tilde{Z}(G)$ and $\dim(\tilde{Z}(G)) \geq \dim(b') \geq 1$, so we are done again. Otherwise $\dim(C_G(b')) = 1$. By the $\widetilde{\mathfrak{M}}_c$-condition there is a broad centralizer $D \leq C_G(b')$ of some finite tuple, minimal up to finite index, and $\dim(D) = \dim(C_G(b')) = 1$, whence $\dim(C_G(b')/D) = 0$. Choose $z^* \in G$ with $z^* \equiv_{a',b'} z$ and $z^* \mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{d}}_{a',b'} z$, and put $h = z^{*-1}z$. Then $a'^{z^*} \in C_G(b')$, so $a'^z \in C_G(b', b'^h)$. Moreover, h is wide in G over a', b' and d-independent of z, so

$$\dim(C_G(b', b'^h)) \geq \dim(a'^z/b', h) = \dim(a'^z/b') \geq 1$$

and $\dim(C_G(b')/C_G(b', b'^h)) = 0$. It follows that

$$\begin{aligned}\dim(D/D \cap D^h) &= \dim(D/(D \cap C_G(b'^h))) + \dim((D \cap C_G(b'^h))/(D \cap D^h)) \\ &\leq \dim(C_G(b')/C_G(b', b'^h)) + \dim(C_G(b'^h)/D^h) = 0,\end{aligned}$$

whence $\dim(D \cap D^h) = 1$, and D^h is commensurable with D by minimality. Thus

$$\dim(\tilde{N}_G(D)) \geq \dim(h/b') = 2$$

and $\tilde{N}_G(D)$ is wide in G. Since D is finite-by-abelian by Corollary 11, we finish as before, using Schlichting's Theorem. □

Corollary 14. *Let G be a pseudofinite group whose definable sections are $\widetilde{\mathfrak{M}}_c$, and* dim *an additive integer-valued dimension on G. If* $\dim(G) = 2$*, then G has a definable wide soluble subgroup.*

Proof: By Theorem 13, there is a definable finite-by-abelian group N such that $N_G(N)$ is wide. Replacing N by $C_N(N')$, we may assume that N is (finite central)-by-abelian. If $\dim(N) = 2$ we are done. Otherwise $\dim(N_G(N)/N) = 1$; by Corollary 11 there is a definable finite-by-abelian subgroup S/N with $\dim(S/N) = 1$. As above we may assume that S/N is (finite central)-by-abelian, so S is soluble. Moreover,

$$\dim(S) = \dim(N) + \dim(S/N) = 1 + 1 = 2,$$

so S is wide in G. □

Corollary 15. *A pseudofinite superrosy group G with $\omega^\alpha \cdot 2 \leq U^\flat(G) < \omega^\alpha \cdot 3$ has a definable soluble subgroup S with $U^\flat(S) \geq \omega^\alpha \cdot 2$.*

Proof: Superrosiness implies that all definabel sections of G are $\widetilde{\mathfrak{M}}_c$. We put

$$\dim(X) = n \quad \Leftrightarrow \quad \omega^\alpha \cdot n \leq U^\flat(X) < \omega^\alpha \cdot (n+1).$$

This defines an additive dimension with $\dim(G) = 2$. The result now follows from Corollary 14. □

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