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Eliane Bécache, Laurent Bourgeois, Lucas Franceschini, Jérémi Dardé

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APPLICATION OF MIXED FORMULATIONS OF QUASI-REVERSIBILITY TO SOLVE ILL-POSED PROBLEMS FOR HEAT AND WAVE EQUATIONS: THE 1D CASE

Eliane Bécache, Laurent Bourgeois and Lucas Franceschini

Laboratoire POEMS, ENSTA ParisTech
828, Boulevard des Maréchaux
91762, Palaiseau Cedex, France

Jérémi Dardé

Institut de Mathématiques, Université de Toulouse
118, Route de Narbonne
F-31062 Toulouse Cedex 9, France

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Abstract. In this paper we address some ill-posed problems involving the heat or the wave equation in one dimension, in particular the backward heat equation and the heat/wave equation with lateral Cauchy data. The main objective is to introduce some variational mixed formulations of quasi-reversibility which enable us to solve these ill-posed problems by using some classical Lagrange finite elements. The inverse obstacle problems with initial condition and lateral Cauchy data for heat/wave equation are also considered, by using an elementary level set method combined with the quasi-reversibility method. Some numerical experiments are presented to illustrate the feasibility for our strategy in all those situations.

1. **Introduction.**

The method of quasi-reversibility has now a quite long history since the pioneering book of Lattès and Lions in 1967 [1]. The original idea of these authors was, starting from an ill-posed problem which satisfies the uniqueness property, to introduce a perturbation of such problem involving a small positive parameter ε. This perturbation has essentially two effects. Firstly the perturbation transforms the initial ill-posed problem into a well-posed one for any ε, secondly the solution to such problem converges to the solution (if it exists) to the initial ill-posed problem when ε tends to 0. Generally, the ill-posedness in the initial problem is due to unsuitable boundary conditions. As typical examples of linear ill-posed problems one may think of the backward heat equation, that is the initial condition is replaced by a final condition, or the heat or wave equations with lateral Cauchy data, that is the usual Dirichlet or Neumann boundary condition on the boundary of the domain is replaced by a pair of Dirichlet and Neumann boundary conditions on the same subpart of the boundary, no data being prescribed on the complementary part of the boundary.

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In this paper we mainly focus on the numerical aspects of quasi-reversibility. From the numerical point of view, the main drawback of quasi-reversibility is the fact that the perturbations which have been proposed up to now leads to multiply the initial operator involved in the problem by its adjoint, so that the order of the operator is multiplied by 2. For example, if we aim to approximate the solution of a second order ill-posed problem with the help of the Finite Element Method, some finite elements of Hermite type have to be used instead of traditional finite elements of Lagrange type. Those Hermite finite elements are cumbersome and rarely available in codes (see for example the discussion in [2]). One way to cope with this problem consists in using some mixed formulations of quasi-reversibility: the idea is to introduce a novel unknown which enables us to replace a fourth-order problem by two coupled second-order problems, which then can be solved by Lagrange finite elements. Different choices for this additional unknown are possible and lead to different mixed formulations. The first one was introduced to solve the ill-posed Cauchy problem for the Laplace operator in [3], another one was introduced in [4] for the same problem. These two mixed formulations were extended to the case of the stationary Stokes system in [5]. The main objective of this article is to introduce similar mixed formulations as [3] to solve ill-posed time dependent problems, both for the heat and the wave equations. More precisely, we consider the backward heat equation and the heat/wave equation with lateral Cauchy data. This paper can be considered as a preliminary attempt in the sense that only the one-dimensional case is considered. Our mixed formulations are in no way limited to one dimension of space, but the 1D case enables us to conduct many computations in a short period of time. Our numerical examples have hence to be viewed as toy models which will enable us to be confident in the feasibility of our methods in more realistic 2D or 3D cases. As an application of the mixed formulation of quasi-reversibility for heat/wave equation with lateral Cauchy data we also consider the inverse obstacle problem: it consists in finding an unknown fixed Dirichlet obstacle from some lateral Cauchy data on a subpart of the boundary. In the one-dimensional case, the obstacle is a single point in some bounded interval. The method we use to solve such inverse obstacle problem, called the "exterior approach", is a coupling between the quasi-reversibility method and a level set method. In our one-dimensional context the level set method is very simple and inspired from the one introduced for the two-dimensional (but stationary) context in [2] and reused in [5]. In [2] and [5] the level set method was based on a simple Poisson equation instead of the standard eikonal equation. In one dimension, we content ourselves with computing a function given by its derivative and its initial condition. Note that in this introduction we don't list the bibliography of all existing methods to solve the various inverse problems mentioned above. We will describe them further when each of these problems will be specified.

Our paper is organized as follows. Section 2 is devoted to ill-posed problems governed by the heat equation. More precisely, we first introduce our mixed formulation of quasi-reversibility for solving the backward heat equation. This formulation is then adapted to the ill-posed problem of heat equation with lateral Cauchy data, with or without initial condition. Lastly, the inverse obstacle problem for the heat equation is considered. Section 3 is devoted to ill-posed problems governed by the wave equation. We first introduce a mixed formulation to solve the wave equation with lateral Cauchy data, with or without initial condition, and secondly consider the inverse obstacle problem. The discretization of our mixed formulations by using the Finite Element Method is described in section 4, while some numerical experiments are conducted in section 5. Lastly, we indicate a few conclusions and perspectives in section 6 while the section 7 is an appendix that explains how our mixed formulations can be derived.

2. **Some ill-posed problems related to the heat equation.**

2.1. **The backward heat equation.**

In this section we consider the backward heat equation in dimension 1. Let us consider the domain $Q = (0,1) \times (0,T)$ and for $r, s \geq 0$ the Hilbert space

$$H^{r,s}(Q) := L^2(0,T;H^r(0,1)) \cap H^s(0,T;L^2(0,1)),$$

following the notations of [6]. In the sequel we will consider solutions in space $H^{1,1}(Q)$, which happens to coincide with $H^1(Q)$. Let us define the following subparts of ∂Q:

$$S_0 = (0,1) \times \{t = 0\}, \quad S_T = (0,1) \times \{t = T\},$$

$$\Gamma_0 = \{x = 0\} \times (0,T), \quad \Gamma_1 = \{x = 1\} \times (0,T).$$

Let us denote $H^{\frac{1}{2}}_{00}(S_T)$, the subset of functions in $H^{\frac{1}{2}}(S_T)$ such that their extensions by zero on ∂Q belongs to $H^{\frac{1}{2}}(\partial Q)$. We first consider the classical backward heat equation, which for some $u_T \in H^{\frac{1}{2}}_{00}(S_T)$, consists in finding $u \in H^1(Q)$ such that

$$\left\{ \begin{array}{rcll} \partial_t u - \partial_x^2 u & = & 0 & \text{in } Q \\ u & = & 0 & \text{on } \Gamma_0 \cup \Gamma_1 \\ u & = & u_T & \text{on } S_T. \end{array} \right. \tag{1}$$

It is well-known that problem (1) satisfies the uniqueness property, that is $u_T = 0$ in S_T implies $u = 0$ in Q, but is however ill-posed: there may be data u_T for which no solution u to the problem (1) exists. All these properties may for example be found in [7]. The regularity of u may seem surprising: the typical space for solutions to problem (1) is $L^2(0,T;H^1(0,1)) \cap C^0(0,T;L^2(0,1))$ rather than $L^2(0,T;H^1(0,1)) \cap H^1(0,T;L^2(0,1))$, the second space being included in the first one. The need for this additional regularity will be clearer when we will introduce our mixed formulation of quasi-reversibility.

A slightly different (and less well-known) problem consists, for some non empty open subset $S^I_T \Subset S_T$ (i.e. $\overline{S^I_T} \subset S_T$) and some data $u_T \in H^{\frac{1}{2}}(S^I_T)$, in finding $u \in H^1(Q)$ such that

$$\left\{ \begin{array}{rcll} \partial_t u - \partial_x^2 u & = & 0 & \text{in } Q \\ u & = & 0 & \text{on } \Gamma_0 \cup \Gamma_1 \\ u & = & u_T & \text{on } S^I_T. \end{array} \right. \tag{2}$$

Problem (2) might be more interesting than problem (1) from the point of view of applications, since the data u_T corresponds to measurements which might be accessible only on a subpart of the spatial domain. For such problem the uniqueness property still holds (see proposition 2.2 in [8]), but obviously the ill-posedness of (2) is even more severe than that of (1).

The quasi-reversibility method proposed in [1] to regularize the problem (1) consists, for $\varepsilon > 0$, in solving the problem

$$\left\{\begin{array}{rcll} \partial_t u_\varepsilon - \partial_x^2 u_\varepsilon - \varepsilon\,\partial_x^4 u_\varepsilon & = & 0 & \text{in } Q \\ u_\varepsilon & = & 0 & \text{on } \Gamma_0 \cup \Gamma_1 \\ \partial_x^2 u_\varepsilon & = & 0 & \text{on } \Gamma_0 \cup \Gamma_1 \\ u_\varepsilon & = & u_T & \text{on } S_T, \end{array}\right. \tag{3}$$

and then the problem

$$\left\{\begin{array}{rcll} \partial_t \tilde{u}_\varepsilon - \partial_x^2 \tilde{u}_\varepsilon & = & 0 & \text{in } Q \\ \tilde{u}_\varepsilon & = & 0 & \text{on } \Gamma_0 \cup \Gamma_1 \\ \tilde{u}_\varepsilon & = & u_\varepsilon & \text{on } S_0, \end{array}\right.$$

where u_ε is the solution to problem (3). It is proved in [1] that in a certain sense the problem (3) is well-posed and that $\tilde{u}_\varepsilon(\cdot, T)$ converges to u_T when $\varepsilon \to 0$. Such result shows that in a certain sense the quasi-solution $\tilde{u}_\varepsilon$ is an approximation of the exact solution u. Some numerical experiments are also conducted in [1] with the help of a finite difference scheme. Since this pioneering work, a lot of alternative quasi-reversibility methods have been proposed [9, 10, 11] to regularize the problem (1). In these methods, the order of the quasi-reversibility problem is higher than the original one (compare (3) with (1)). For solving such a quasi-reversibility problem with the help of a finite element method, we have then to choose some complicated finite elements. In order to simplify the discretization with finite elements, we now present a mixed formulation of quasi-reversibility to regularize the problems (1) and (2) in the $H^1(Q)$ setting.

2.1.1. *Quasi-reversibility for problem (1).* We first consider the problem (1) and introduce the following sets:

$$U_u = \{u \in H^1(Q),\ u(0,\cdot) = u(1,\cdot) = 0,\ u(\cdot,T) = u_T\},$$

$$U_0 = \{u \in H^1(Q),\ u(0,\cdot) = u(1,\cdot) = 0,\ u(\cdot,T) = 0\},$$

$$\tilde{U}_0 = \{\lambda \in H^1(Q),\ \lambda(0,\cdot) = \lambda(1,\cdot) = 0,\ \lambda(\cdot,0) = 0\},$$

the spaces U_0 and $\tilde{U}_0$ being endowed with the same norm $||\cdot||$ given by

$$||u||^2 = \int_Q (\partial_t u)^2\,dxdt + \int_Q (\partial_x u)^2\,dxdt.$$

We consider the following quasi-reversibility problem whose definition involves the real parameters $\varepsilon, \delta > 0$: for $u_T \in H^{\frac{1}{2}}_{00}(S_T)$, find $(u_{\varepsilon,\delta}, \lambda_{\varepsilon,\delta}) \in U_u \times \tilde{U}_0$ such that for all $(v,\mu) \in U_0 \times \tilde{U}_0$,

$$\left\{\begin{array}{l} -\displaystyle\int_Q v \frac{\partial \lambda_{\varepsilon,\delta}}{\partial t}\,dxdt + \varepsilon \int_Q \frac{\partial u_{\varepsilon,\delta}}{\partial t}\frac{\partial v}{\partial t}\,dxdt \\ \quad + \displaystyle\int_Q \frac{\partial v}{\partial x}\frac{\partial \lambda_{\varepsilon,\delta}}{\partial x}\,dxdt + \varepsilon \int_Q \frac{\partial u_{\varepsilon,\delta}}{\partial x}\frac{\partial v}{\partial x}\,dxdt = 0, \\ \displaystyle\int_Q \frac{\partial u_{\varepsilon,\delta}}{\partial t}\mu\,dxdt - \delta \int_Q \frac{\partial \lambda_{\varepsilon,\delta}}{\partial t}\frac{\partial \mu}{\partial t}\,dxdt \\ \quad + \displaystyle\int_Q \frac{\partial u_{\varepsilon,\delta}}{\partial x}\frac{\partial \mu}{\partial x}\,dxdt - \delta \int_Q \frac{\partial \lambda_{\varepsilon,\delta}}{\partial x}\frac{\partial \mu}{\partial x}\,dxdt = 0. \end{array}\right. \tag{4}$$

The problem (4) is well-posed and enables one to regularize the ill-posed problem (1), since we have the following result.

Proposition 1. *For any $u_T \in H^{\frac{1}{2}}_{00}(S_T)$, the problem (4) has a unique solution $(u_{\varepsilon,\delta}, \lambda_{\varepsilon,\delta})$ in $U_u \times \tilde{U}_0$. Furthermore, if δ is a bounded function of ε such that*

$$\lim_{\varepsilon \to 0} \frac{\varepsilon}{\delta(\varepsilon)} = 0, \tag{5}$$

and if there exists a (unique) solution u to problem (1) associated with data u_T, then the solution $(u_{\varepsilon,\delta}, \lambda_{\varepsilon,\delta})$ to problem (4) associated with the same data u_T satisfies

$$\lim_{\varepsilon \to 0} u_{\varepsilon,\delta(\varepsilon)} = u \quad \text{in} \quad H^1(Q), \qquad \lim_{\varepsilon \to 0} \lambda_{\varepsilon,\delta(\varepsilon)} = 0 \quad \text{in} \quad H^1(Q).$$

In order to prove proposition 1 we need the following lemma.

Lemma 2.1. *The function $u \in H^1(Q)$ is the solution of problem (1) if and only if $u \in U_u$ and for all $\mu \in \tilde{U}_0$,*

$$\int_Q \frac{\partial u}{\partial t}\, \mu \, dxdt + \int_Q \frac{\partial u}{\partial x} \frac{\partial \mu}{\partial x}\, dxdt = 0. \tag{6}$$

Proof. Let $u \in H^1(Q)$ be a solution of problem (1). Then $u \in U_u$. Since u solves the heat equation $\partial_t u - \partial_x^2 u = 0$ in Q and $\partial_t u \in L^2(0,T;L^2(0,1))$, we have $\partial_x^2 u \in L^2(0,T;L^2(0,1))$ and then $u \in L^2(0,T;H^2(0,1))$. By the integration by parts formula, we obtain for $\mu \in \tilde{U}_0$

$$\int_Q \frac{\partial u}{\partial x} \frac{\partial \mu}{\partial x}\, dxdt = \int_0^T \left(\int_0^1 \frac{\partial u}{\partial x} \frac{\partial \mu}{\partial x}\, dx \right) dt$$

$$= \int_0^T \left(\frac{\partial u}{\partial x}(1,t)\mu(1,t) - \frac{\partial u}{\partial x}(0,t)\mu(0,t) \right) dt - \int_0^T \left(\int_0^1 \frac{\partial^2 u}{\partial x^2} \mu \, dx \right) dt$$

$$= -\int_Q \frac{\partial^2 u}{\partial x^2} \mu \, dxdt,$$

which implies (6). Conversely, assume that $u \in U_u$ satisfies (6). By taking $\mu \in C_0^\infty(Q)$ in (6), we obtain that the heat equation is satisfied in the distributional sense, that is u satisfies problem (1). □

Proof of proposition 1. Let us prove well-posedness of the quasi-reversibility formulation. Because $u_T \in H^{\frac{1}{2}}_{00}(S_T)$, the set U_u contains at least one element Φ. By denoting $\hat{u}_{\varepsilon,\delta} = u_{\varepsilon,\delta} - \Phi$, the problem (4) can be rewritten: find $(\hat{u}_{\varepsilon,\delta}, \lambda_{\varepsilon,\delta}) \in U_0 \times \tilde{U}_0$ such that for all $(v,\mu) \in U_0 \times \tilde{U}_0$,

$$A((\hat{u}_{\varepsilon,\delta}, \lambda_{\varepsilon,\delta}), (v,\mu)) = L(v,\mu),$$

where L is a continuous linear form on $U_0 \times \tilde{U}_0$ (which is not necessary to give explicitly as a function of Φ), and A is the continuous bilinear form on $U_0 \times \tilde{U}_0$ given by

$$\left\{ \begin{array}{l} A((\hat{u},\lambda),(v,\mu)) = -\displaystyle\int_Q v \frac{\partial \lambda}{\partial t}\, dxdt - \int_Q \frac{\partial \hat{u}}{\partial t} \mu \, dxdt \\ + \displaystyle\int_Q \frac{\partial v}{\partial x} \frac{\partial \lambda}{\partial x}\, dxdt - \int_Q \frac{\partial \hat{u}}{\partial x} \frac{\partial \mu}{\partial x}\, dxdt \\ + \varepsilon \displaystyle\int_Q \frac{\partial \hat{u}}{\partial t} \frac{\partial v}{\partial t}\, dxdt + \varepsilon \int_Q \frac{\partial \hat{u}}{\partial x} \frac{\partial v}{\partial x}\, dxdt \\ + \delta \displaystyle\int_Q \frac{\partial \lambda}{\partial t} \frac{\partial \mu}{\partial t}\, dxdt + \delta \int_Q \frac{\partial \lambda}{\partial x} \frac{\partial \mu}{\partial x}\, dxdt. \end{array} \right. \tag{7}$$

It suffices to prove that A is coercive on $U_0 \times \tilde{U}_0$. We remark that

$$\int_Q \hat{u}\,\frac{\partial \lambda}{\partial t}\,dxdt + \int_Q \frac{\partial \hat{u}}{\partial t}\,\lambda\,dxdt = \int_0^1 \left(\int_0^T \frac{\partial(\hat{u}\lambda)}{\partial t}\,dt\right)\,dx = 0,$$

by using the fact that $\hat{u}(\cdot,0)=0$ and $\lambda(\cdot,T)=0$.
Then it is clear that

$$A((\hat{u},\lambda),(\hat{u},\lambda)) = \varepsilon||\hat{u}||^2 + \delta||\lambda||^2, \tag{8}$$

which proves coercivity and completes the proof in view of Lax-Milgram's theorem.
Now let us prove the second part of the proposition. For sake of simplicity we denote $u_\varepsilon := u_{\varepsilon,\delta(\varepsilon)}$ and $\lambda_\varepsilon := \lambda_{\varepsilon,\delta(\varepsilon)}$. By the lemma 2.1, the exact solution $u \in U_u$ satisfies (6). By subtracting the equation (6) from the second equation of problem (4), we obtain that for all $(v,\mu) \in U_0 \times \tilde{U}_0$,

$$\left\{\begin{array}{l} -\displaystyle\int_Q v\frac{\partial \lambda_\varepsilon}{\partial t}\,dxdt + \varepsilon\int_Q \frac{\partial u_\varepsilon}{\partial t}\frac{\partial v}{\partial t}\,dxdt \\ +\displaystyle\int_Q \frac{\partial v}{\partial x}\frac{\partial \lambda_\varepsilon}{\partial x}\,dxdt + \varepsilon\int_Q \frac{\partial u_\varepsilon}{\partial x}\frac{\partial v}{\partial x}\,dxdt = 0, \\ \displaystyle\int_Q \frac{\partial(u_\varepsilon - u)}{\partial t}\mu\,dxdt - \delta\int_Q \frac{\partial \lambda_\varepsilon}{\partial t}\frac{\partial \mu}{\partial t}\,dxdt \\ +\displaystyle\int_Q \frac{\partial(u_\varepsilon - u)}{\partial x}\frac{\partial \mu}{\partial x}\,dxdt - \delta\int_Q \frac{\partial \lambda_\varepsilon}{\partial x}\frac{\partial \mu}{\partial x}\,dxdt = 0. \end{array}\right. \tag{9}$$

Then we choose $v = u_\varepsilon - u \in U_0$ and $\mu = \lambda_\varepsilon \in \tilde{U}_0$ in the above system (9) and subtracting the two obtained equations, we find

$$\begin{array}{l} \varepsilon\displaystyle\int_Q \frac{\partial u_\varepsilon}{\partial t}\frac{\partial(u_\varepsilon - u)}{\partial t}\,dxdt + \varepsilon\int_Q \frac{\partial u_\varepsilon}{\partial x}\frac{\partial(u_\varepsilon - u)}{\partial x}\,dxdt \\ +\delta\displaystyle\int_Q \left(\frac{\partial \lambda_\varepsilon}{\partial t}\right)^2\,dxdt + \delta\int_Q \left(\frac{\partial \lambda_\varepsilon}{\partial x}\right)^2\,dxdt = 0. \end{array} \tag{10}$$

If $((\cdot,\cdot))$ denotes the scalar product which corresponds to the norm $||\cdot||$, it follows that

$$\varepsilon||u_\varepsilon||^2 + \delta||\lambda_\varepsilon||^2 = \varepsilon((u_\varepsilon,u)) \le \varepsilon||u_\varepsilon||\,||u||,$$

which implies that

$$||u_\varepsilon|| \le ||u||, \quad ||\lambda_\varepsilon|| \le \sqrt{\frac{\varepsilon}{\delta(\varepsilon)}}||u||.$$

In particular u_ε is bounded in $H^1(Q)$ while using assumption (5) λ_ε tends to 0 when ε tends to 0. There exists a subsequence of u_ε, still denoted u_ε, that weakly converges to some $w \in H^1(Q)$. In fact $w \in U_u$ because the set U_u is weakly closed. Passing to the limit in the second equation of (4), since δ is a bounded function of ε, we obtain that for all $\mu \in \tilde{U}_0$,

$$\int_Q \frac{\partial w}{\partial t}\,\mu\,dxdt + \int_Q \frac{\partial w}{\partial x}\frac{\partial \mu}{\partial x}\,dxdt = 0,$$

that is w solves system (1) by using lemma 2.1 again, and from uniqueness $w = u$. Then u_ε weakly converges to u in $H^1(Q)$. From the identify

$$||u_\varepsilon - u||^2 = ((u_\varepsilon, u_\varepsilon - u)) - ((u, u_\varepsilon - u))$$

and the equation (10), it follows that

$$||u_\varepsilon - u||^2 \le -((u, u_\varepsilon - u))$$

which implies strong convergence. Convergence of the whole sequence u_ε (not only of the subsequence) follows from the uniqueness of the limit, which completes the proof. □

Remark 1. The reader may consider the mixed formulation (4) as unintuitive. The appendix gives some explanations on how it can be obtained starting from problem (1).

2.1.2. *Quasi-reversibility for problem (2).* In order to regularize the ill-posed problem (2), we have to replace the sets $U_u, U_0, \tilde{U}_0$ by the sets $U_u^I, U_0^I, \tilde{U}_0^I$ defined as follows:

$$U_u^I = \{u \in H^1(Q),\ u(0,\cdot) = u(1,\cdot) = 0,\ u(\cdot,T)|_I = u_T\},$$
$$U_0^I = \{u \in H^1(Q),\ u(0,\cdot) = u(1,\cdot) = 0,\ u(\cdot,T)|_I = 0\},$$
$$\tilde{U}_0^I = \{\lambda \in H^1(Q),\ \lambda(0,\cdot) = \lambda(1,\cdot) = 0,\ \lambda(\cdot,0) = 0,\ \lambda(\cdot,T)|_{\tilde{I}} = 0\},$$

where here $I \Subset (0,1)$ is implicitly defined by $S_T^I = I \times (0,T)$ and $\tilde{I} = (0,1) \setminus \overline{I}$. With these new definitions, for $u_T \in H^{\frac{1}{2}}(S_T^I)$ instead of $u_T \in H_{00}^{\frac{1}{2}}(S_T)$, the formulation of quasi-reversibility (4) is unchanged, and it is easy to prove that an analogous proposition as proposition 1 is valid.

2.2. **The heat equation with lateral Cauchy data.** Another classical ill-posed problem is the heat equation with lateral Cauchy data. To set the problem we need to correctly define the normal derivative of a solution to the heat equation on the subpart Γ_0 of the boundary. In the proof of lemma 2.1 we have remarked that if $u \in H^1(Q)$ solves the heat equation $\partial_t u - \partial_x^2 u = 0$ in Q, then $u \in L^2(0,T;H^2(0,1))$, hence $\partial_x u|_{\Gamma_0} \in L^2(\Gamma_0)$. The heat equation with lateral Cauchy data consists, for $(g_0, g_1) \in H^{\frac{1}{2}}(\Gamma_0) \times L^2(\Gamma_0)$, in finding $u \in H^1(Q)$ such that

$$\left\{\begin{array}{rcll} \partial_t u - \partial_x^2 u & = & 0 & \text{in } Q \\ u & = & g_0 & \text{on } \Gamma_0 \\ -\partial_x u & = & g_1 & \text{on } \Gamma_0. \end{array}\right. \tag{11}$$

It is well-known that the uniqueness property holds for that problem, that is $(g_0, g_1) = (0,0)$ on Γ_0 implies that $u = 0$ in Q, due to Holmgren's theorem (see theorem 5.3.3 in [12]).

Let us now describe a slightly different problem with lateral Cauchy data and initial condition. To this aim, we now define $H_{S_0}^{\frac{1}{2}}(\Gamma_0)$ as the subset of traces on Γ_0 of functions in $H^1(Q)$ that vanish on S_0. The problem is now the following: for $(g_0, g_1) \in H_{S_0}^{\frac{1}{2}}(\Gamma_0) \times L^2(\Gamma_0)$, find $u \in H^1(Q)$ such that

$$\left\{\begin{array}{rcll} \partial_t u - \partial_x^2 u & = & 0 & \text{in } Q \\ u & = & g_0 & \text{on } \Gamma_0 \\ -\partial_x u & = & g_1 & \text{on } \Gamma_0 \\ u & = & 0 & \text{on } S_0. \end{array}\right. \tag{12}$$

Both problems (11) and (12) are ill-posed (see for example [13]). Several quasi-reversibility methods are proposed in [1] to regularize the problem (11), and a simplified one is proposed in [14]. This last one consists in minimizing, for $\varepsilon > 0$ and appropriate data (g_0, g_1), the cost function

$$||\partial_t u - \partial_x^2 u||_{L^2(Q)}^2 + \varepsilon ||u||_{H^{2,1}(Q)}^2$$

among all functions u in $H^{2,1}(Q)$ which satisfy $u(0,\cdot)=g_0$ and $-\partial_x u(0,\cdot)=g_1$ on Γ_0. Again, the optimality condition associated with such a minimization problem leads to a fourth-order problem which, once discretized with the help of a finite element method, requires some complicated finite elements. This is precisely what we want to avoid by using a mixed formulation of quasi-reversibility in the $H^1(Q)$ setting.

2.2.1. *Quasi-reversibility for problem (12).* We first consider the second problem (12), which will be useful in order to tackle the inverse obstacle problem. In order to regularize such ill-posed problem, we consider new sets $V_g, V_0, \tilde{V}_0$ as follows:

$$V_g=\{u\in H^1(Q),\ u(0,\cdot)=g_0,\ u(\cdot,0)=0\},$$

$$V_0=\{u\in H^1(Q),\ u(0,\cdot)=0,\ u(\cdot,0)=0\},$$

$$\tilde{V}_0=\{\lambda\in H^1(Q),\ \lambda(1,\cdot)=0,\ \lambda(\cdot,T)=0\}.$$

We consider the following quasi-reversibility problem whose definition involves the real parameters $\varepsilon,\delta>0$: for $(g_0,g_1)\in H^{\frac{1}{2}}_{S_0}(\Gamma_0)\times L^2(\Gamma_0)$, find $(u_{\varepsilon,\delta},\lambda_{\varepsilon,\delta})\in V_g\times\tilde{V}_0$ such that for all $(v,\mu)\in V_0\times\tilde{V}_0$,

$$\left\{\begin{array}{l}
-\displaystyle\int_Q v\frac{\partial\lambda_{\varepsilon,\delta}}{\partial t}\,dxdt+\varepsilon\int_Q\frac{\partial u_{\varepsilon,\delta}}{\partial t}\frac{\partial v}{\partial t}\,dxdt\\
+\displaystyle\int_Q\frac{\partial v}{\partial x}\frac{\partial\lambda_{\varepsilon,\delta}}{\partial x}\,dxdt+\varepsilon\int_Q\frac{\partial u_{\varepsilon,\delta}}{\partial x}\frac{\partial v}{\partial x}\,dxdt=0,\\
\displaystyle\int_Q\frac{\partial u_{\varepsilon,\delta}}{\partial t}\mu\,dxdt-\delta\int_Q\frac{\partial\lambda_{\varepsilon,\delta}}{\partial t}\frac{\partial\mu}{\partial t}\,dxdt\\
+\displaystyle\int_Q\frac{\partial u_{\varepsilon,\delta}}{\partial x}\frac{\partial\mu}{\partial x}\,dxdt-\delta\int_Q\frac{\partial\lambda_{\varepsilon,\delta}}{\partial x}\frac{\partial\mu}{\partial x}\,dxdt=\int_{\Gamma_0}g_1\,\mu\,dt.
\end{array}\right.\tag{13}$$

We have the following results by using the same arguments as in the proof of proposition 1.

Proposition 2. *For any $(g_0,g_1)\in H^{\frac{1}{2}}_{S_0}(\Gamma_0)\times L^2(\Gamma_0)$, the problem (13) has a unique solution $(u_{\varepsilon,\delta},\lambda_{\varepsilon,\delta})$ in $V_g\times\tilde{V}_0$. Furthermore, if δ is a bounded function of ε such that (5) is satisfied, and if there exists a (unique) solution u to problem (12) associated with data (g_0,g_1), then the solution $(u_{\varepsilon,\delta},\lambda_{\varepsilon,\delta})$ to problem (13) associated with the same data (g_0,g_1) satisfies*

$$\lim_{\varepsilon\to 0}u_{\varepsilon,\delta(\varepsilon)}=u\quad\text{in}\quad H^1(Q),\qquad\lim_{\varepsilon\to 0}\lambda_{\varepsilon,\delta(\varepsilon)}=0\quad\text{in}\quad H^1(Q).$$

2.2.2. *Quasi-reversibility for problem (11).* In order to regularize the ill-posed problem (11), we have to replace the sets $V_g, V_0, \tilde{V}_0$ by the new sets $W_g, W_0, \tilde{W}_0$ defined as follows:

$$W_g=\{u\in H^1(Q),\ u(0,\cdot)=g_0\},$$

$$W_0=\{u\in H^1(Q),\ u(0,\cdot)=0\},$$

$$\tilde{W}_0=\{\lambda\in H^1(Q),\ \lambda(1,\cdot)=0,\ \lambda(\cdot,0)=0,\ \lambda(\cdot,T)=0\}.$$

With these new definitions, for $(g_0,g_1)\in H^{\frac{1}{2}}(\Gamma_0)\times L^2(\Gamma_0)$ instead of $H^{\frac{1}{2}}_{S_0}(\Gamma_0)\times L^2(\Gamma_0)$, the formulation of quasi-reversibility (13) is unchanged, and it is easy to prove that an analogous proposition as proposition 2 is valid.

2.3. **The inverse obstacle problem.** Given a pair of lateral Cauchy data (g_0, g_1) on Γ_0, the inverse obstacle problem consists in finding some real $a \in (0,1)$ (independent of time t) and some function $u \in H^1(Q_a)$ with $Q_a = (0,a) \times (0,T)$ such that

$$\left\{\begin{array}{rcll} \partial_t u - \partial_x^2 u & = & 0 & \text{in } Q_a \\ u & = & g_0 & \text{on } \Gamma_0 \\ -\partial_x u & = & g_1 & \text{on } \Gamma_0 \\ u & = & 0 & \text{on } \Gamma_a \\ u & = & 0 & \text{on } S_{0,a}, \end{array}\right. \tag{14}$$

where $\Gamma_a = \{a\} \times (0,T)$ and $S_{0,a} = (0,a) \times \{0\}$. We first state the following uniqueness result for our inverse obstacle problem.

Proposition 3. *Let $a_1, a_2 \in (0,1)$ be two reals such that the corresponding functions u_1, u_2 satisfy (14) with data (g_0, g_1). If we assume that $(g_0, g_1) \neq 0$, we have $a_1 = a_2$.*

Proof. Assume $a_1 < a_2$ and consider the function $u := u_1 - u_2$ in the domain Q_{a_1}. The function u satisfies in Q_{a_1} the system

$$\left\{\begin{array}{rcll} \partial_t u - \partial_x^2 u & = & 0 & \text{in } Q_{a_1} \\ u & = & 0 & \text{on } \Gamma_0 \\ -\partial_x u & = & 0 & \text{on } \Gamma_0. \end{array}\right.$$

By unique continuation we obtain that $u = 0$ in $(0, a_1) \times (0,T)$, and in particular $u_1(a_1, \cdot) = u_2(a_1, \cdot)$ in $(0,T)$. We hence have $u_2(a_1, \cdot) = 0$ and $u_2(a_2, \cdot) = 0$ simultaneously in $(0,T)$. By using the initial condition $u_2(\cdot, 0) = 0$ in (a_1, a_2), from uniqueness of the forward problem with Dirichlet boundary condition, we conclude that u_2 vanishes in $(a_1, a_2) \times (0,T)$, and then in $(0, a_2) \times (0,T)$ from unique continuation. This contradicts the fact that $(g_0, g_1) \neq 0$. ☐

Remark 2. The initial condition on $S_{0,a}$ in problem (14) is essential to ensure uniqueness of a in proposition 3. Indeed, let us consider the function $u(x,t) = e^{-(3\pi)^2 t} \sin(3\pi x)$ in $(0,1) \times (0,T)$. Such function satisfies the heat equation, corresponds to data $(g_0, g_1) = (0, -3\pi e^{-(3\pi)^2 t}) \neq (0,0)$ on Γ_0, and satisfies $u(1/3, t) = u(2/3, t) = 0$ in $(0,T)$, which means that the two obstacles $a_1 = 1/3$ and $a_2 = 2/3$ are compatible with data (g_0, g_1). This necessary initial condition means that, starting from a system at rest for $t < 0$, the beginning of the measurements on Γ_0 must coincide with the beginning of the perturbation of the system imposed on Γ_0.

Some resolution methods based on shape derivative tools are proposed in [15] and [16] to solve the inverse obstacle problem (14) for the heat equation, while the so-called "enclosure method" is introduced in [17]. Some 2D numerical experiments are presented in [15], while some 3D ones are presented in [16]. We now introduce an alternative method to solve the inverse obstacle problem (14), the so-called "exterior approach", which combines the quasi-reversibility method (13) and a simple level set method that we present hereafter.
For some $a \in (0,1)$ and $u \in H^1(Q_a)$, let us define in $(0,1)$ the function F defined in the interval $(0,a)$ by

$$F(x) = \left(\int_0^T u^2(x,t)\, dt \right)^{\frac{1}{2}} \tag{15}$$

and extended by 0 in the complementary interval $(a,1)$. Is is easy to check that $F \in H^1(0,1)$ due to condition $u = 0$ on the obstacle Γ_a. Such function F is intended to serve as the velocity field in our level set method. We now consider some function $f \in L^2(0,1)$ such that

$$f \geq \max(0, -F'), \tag{16}$$

where $F' \in L^2(0,1)$ is the derivative of F. One defines the real sequence (a_n) as follows. We assume that $a_0 \in (0,a)$ and for all $n \in \mathbb{N}$, if ϕ_n is the solution in $H^1(a_n,1)$ of the equation

$$\left\{ \begin{array}{rcll} -\phi_n' &=& f & \text{in } (a_n,1) \\ \phi_n &=& F & \text{for } x = a_n, \end{array} \right. \tag{17}$$

then a_{n+1} is defined by

$$a_{n+1} = \inf\{x \in [a_n,1), \quad \phi_n(x) \leq 0\}. \tag{18}$$

We have the following

Lemma 2.2. *The sequence (a_n) is well defined and $a_n \leq a$ for all $n \in \mathbb{N}$.*

Proof. Assume by induction that a_n is defined with $a_n \leq a$. The function $\psi_n := \phi_n - F$ is the solution in $H^1(a_n,1)$ of the equation

$$\left\{ \begin{array}{rcll} -\psi_n' &=& f + F' & \text{in } (a_n,1) \\ \psi_n &=& 0 & \text{for } x = a_n, \end{array} \right.$$

and hence satisfies $\psi_n \leq 0$ in $(a_n,1)$ because $f \geq -F'$. We have $\phi_n(a) = \psi_n(a) + F(a) = \psi_n(a) \leq 0$. We conclude that the set $\{x \in [a_n,1),\, \phi_n(x) \leq 0\}$ is not empty and contains a, so that a_{n+1} is well defined by (18) and $a_{n+1} \leq a$. □

We then obtain the following theorem.

Theorem 2.3. *Let us consider some $a \in (0,1)$ and $u \in H^1(Q_a)$ such that problem (14) is satisfied. Let us define F and f by (15) and (16), respectively. Then with initial condition $a_0 \in (0,a)$ the sequence (a_n) defined in $(0,1)$ by (17) and (18) converges to a.*

Proof. From lemma 2.2 we obtain that the sequence (a_n) is non decreasing and bounded by a, hence it converges to some a^* with $a^* \leq a$. It remains to prove that $a \leq a^*$. Let us assume on the contrary that $a^* < a$. By using the fact that (17) is equivalent to

$$\phi_n(x) = F(a_n) - \int_{a_n}^x f(y)\, dy$$

for $x \in [a_n,1]$, we obtain by passing to the limit that for all $x \in [a^*,1]$, given the fact that $F \in C^0([0,1])$ and $f \in L^2(0,1)$, $\phi_n(x)$ tends to $\phi(x)$ with

$$\phi(x) = F(a^*) - \int_{a^*}^x f(y)\, dy,$$

so that ϕ is the solution in $H^1(a^*,1)$ of

$$\left\{ \begin{array}{rcll} -\phi' &=& f & \text{in } (a^*,1) \\ \phi &=& F & \text{for } x = a^*. \end{array} \right.$$

By using the fact that for $x \in [a_{n+1},1]$

$$\phi_n(x) = \phi_n(a_{n+1}) - \int_{a_{n+1}}^x f(y)\, dy$$

and $f \geq 0$, we obtain that $\phi_n \leq 0$ in $[a_{n+1}, 1]$, then $\phi_n \leq 0$ in $[a^*, 1]$ and lastly $\phi \leq 0$ in $[a^*, 1]$ by passing to the limit. We conclude that in $[a^*, 1]$,

$$0 \leq F \leq F - \phi.$$

It follows that $F(a^*) = 0 = F(a)$, and lastly $u(a^*, \cdot) = 0 = u(a, \cdot)$ on $(0, T)$. By using again uniqueness for the forward problem with Dirichlet boundary condition, we obtain that $u = 0$ in $(a^*, a) \times (0, T)$, that is $u = 0$ in Q_a, which contradicts the fact that $(g_0, g_1) \neq 0$. $\square$

Remark 3. As a result of the proof of theorem 2.3, the set $\{x \in [a_n, 1),\ \phi_n(x) \leq 0\}$ involved in the definition (18) of a_{n+1} coincides with the interval $[a_{n+1}, 1)$.

The theorem 2.3 cannot be directly used to solve the inverse obstacle problem (14) since the true solution u is unknown. However, the quasi-reversibility method (13) presented in the previous section enables one to approach such solution. This is why we propose the following algorithm to approximately solve the problem (14), that is to retrieve the obstacle a from the Cauchy data (g_0, g_1) on Γ_0.

Algorithm :

1. Choose an initial guess $a_0 \in (0, 1)$ such that $a_0 < a$.
2. First step: the real a_n being given, solve the quasi-reversibility problem (13) in $Q_{a_n} = (0, a_n) \times (0, T)$ for some selected parameters $\varepsilon, \delta > 0$. The solution is denoted (u_n, λ_n).
3. Second step: the function u_n being given, solve the equation $-\phi_n' = f$ in $(a_n, 1)$ with initial condition

$$\phi_n(a_n) = \left(\int_0^T (u_n(a_n, t))^2\, dt \right)^{\frac{1}{2}}$$

for some selected $f \in L^2(0, 1)$. Define

$$a_{n+1} = \inf\{x \in [a_n, 1), \quad \phi_n(x) \leq 0\}.$$

4. Go back to the first step until the stopping criterion is reached.

Remark 4. In practice, the function f is chosen as a sufficiently large constant, the choice of which is discussed in the numerical section. In addition, the parameters ε and δ are chosen such that ε is small and $\delta(\varepsilon)$ is in accordance with the statement of proposition 2. Lastly, due to the discretization of the equation (17), the sequence (a_n) is stationary for a sufficiently large n: this provides us a very simple stopping criterion for our algorithm.

3. Some ill-posed problems related to the wave equation.

3.1. The wave equation with lateral Cauchy data.

In the case of the wave equation, since the time is reversible, the forward wave equation coincides with the backward wave equation, so that the analogous of the backward heat equation problem does not exist. However, the wave equation with lateral Cauchy data, that is the analogous problems (11) and (12) for wave equation instead of heat equation, are of interest. To introduce these two problems, once again we have to correctly define the normal derivative of a solution to the wave equation. For $u \in H^1(Q)$, let us denote $\mathbf{v} := (v_x, v_t) := (\partial_x u, -\partial_t u) \in (L^2(Q))^2$, as well as $\text{div}_2 \mathbf{v} := \partial_x v_x + \partial_t v_t$. If u solves the wave equation $\partial_t^2 u - \partial_x^2 u = 0$, then $\text{div}_2 \mathbf{v} = 0$, so that $\mathbf{v} \in H_{\text{div}}(Q)$.

This fact enables us to define $\mathbf{v}\cdot\nu_2|_{\partial Q}\in H^{-\frac{1}{2}}(\partial Q)$, which is the dual space of $H^{\frac{1}{2}}(\partial Q)$. Here ν_2 denotes the outward unit normal to the 2-dimensional domain Q. As a result, the normal derivatives $-\partial_x u|_{\Gamma_0}$ and $-\partial_t u|_{S_0}$ are well defined in $H^{-\frac{1}{2}}(\Gamma_0)$ and $H^{-\frac{1}{2}}(S_0)$, which are respectively the restrictions to Γ_0 and S_0 of the distributions in $H^{-\frac{1}{2}}(\partial Q)$. We recall that $H^{-\frac{1}{2}}(\Gamma_0)$ coincides with the dual space of $H^{\frac{1}{2}}_{00}(\Gamma_0)$, which denotes the subset of functions in $H^{\frac{1}{2}}(\Gamma_0)$ which once extended by 0 on the whole boundary ∂Q are still in $H^{\frac{1}{2}}(\partial Q)$. The wave equation with lateral Cauchy data consists, for $(g_0,g_1)\in H^{\frac{1}{2}}(\Gamma_0)\times H^{-\frac{1}{2}}(\Gamma_0)$, in finding $u\in H^1(Q)$ such that

$$\left\{\begin{array}{rcll}\partial_t^2 u-\partial_x^2 u &=& 0 & \text{in } Q\\ u &=& g_0 & \text{on } \Gamma_0\\ -\partial_x u &=& g_1 & \text{on } \Gamma_0.\end{array}\right. \tag{19}$$

Let us now introduce the wave equation with lateral Cauchy data and initial condition. We define $H^{\frac{1}{2}}_{\Gamma_1,S_T}(\Gamma_0)$ as the set of traces on Γ_0 of functions in $H^1(Q)$ that vanish on Γ_1 and on S_T, that is in other words the restrictions on Γ_0 of functions in $\tilde{V}_0$. Its dual space $H^{-\frac{1}{2}}_{S_0}(\Gamma_0)$ coincides with the set of restrictions on Γ_0 of distributions in $H^{-\frac{1}{2}}(\partial Q)$ the support of which is contained in $\overline{\Gamma_0\cup\Gamma_1\cup S_T}$. The problem is the following: for $(g_0,g_1)\in H^{\frac{1}{2}}_{S_0}(\Gamma_0)\times H^{-\frac{1}{2}}_{S_0}(\Gamma_0)$, find $u\in H^1(Q)$ such that

$$\left\{\begin{array}{rcll}\partial_t^2 u-\partial_x^2 u &=& 0 & \text{in } Q\\ u &=& g_0 & \text{on } \Gamma_0\\ -\partial_x u &=& g_1 & \text{on } \Gamma_0\\ u &=& 0 & \text{on } S_0\\ \partial_t u &=& 0 & \text{on } S_0.\end{array}\right. \tag{20}$$

Let us now discuss the uniqueness property for problems (19) and (20). By the Holmgren's theorem applied to the wave equation (see for example [1]), for $T>2$ uniqueness in problem (19) holds in the subset $R_T\subset Q$ defined by

$$R_T=\{(x,t),\quad 0<x<1,\quad x<t<T-x\}.$$

However, the values of u are arbitrary in the complementary domain $Q\setminus\overline{R_T}$. In the case of problem (20), for $T>1$ uniqueness holds in the subdomain R_T^0, with

$$R_T^0=\{(x,t),\quad 0<x<1,\quad 0<t<T-x\}.$$

Indeed, if u satisfies problem (20), due to the initial condition u can be extended by 0 for $t<0$, as well as the data (g_0,g_1), and we hence come back to the previous uniqueness problem in the time domain $(-T,T)$. However, the values of u are arbitrary in the complementary domain $Q\setminus\overline{R_T^0}$.

Remark 5. As far as existence in problems (19) and (20) are concerned, the $1D$ case is very specific. If we consider the first problem (19) for $T>2$, by inverting the roles classically played by x and t, we obtain from the d'Alembert's formula that the function defined by

$$u_A(x,t)=\frac{1}{2}(g_0(t+x)+g_0(t-x))-\frac{1}{2}\int_{t-x}^{t+x}g_1(s)\,ds \tag{21}$$

solves problem (19) in the subdomain R_T, at least if datum g_1 is a function. This shows that the problem (19) is well-posed in the subdomain R_T of Q.
If we consider now problem (20) for $T>1$, it is not difficult to see that the existence of u in the subdomain R_T^0 is not guaranteed due to redundant data on S_0.

We now consider some mixed formulations of quasi-reversibility to regularize the problems (19) and (20).

3.1.1. *Quasi-reversibility for problem (20).* We first consider the second problem (20), which will be useful in order to tackle the inverse obstacle problem.
The quasi-reversibility problem, for some reals $\varepsilon, \delta > 0$, is the following: for $(g_0, g_1) \in H^{\frac{1}{2}}_{S_0}(\Gamma_0) \times H^{-\frac{1}{2}}_{S_0}(\Gamma_0)$, find $(u_{\varepsilon,\delta}, \lambda_{\varepsilon,\delta}) \in V_g \times \tilde{V}_0$ such that for all $(v, \mu) \in V_0 \times \tilde{V}_0$,

$$\begin{cases} -\displaystyle\int_Q \frac{\partial v}{\partial t}\frac{\partial \lambda_{\varepsilon,\delta}}{\partial t}\,dxdt + \varepsilon \int_Q \frac{\partial u_{\varepsilon,\delta}}{\partial t}\frac{\partial v}{\partial t}\,dxdt \\ +\displaystyle\int_Q \frac{\partial v}{\partial x}\frac{\partial \lambda_{\varepsilon,\delta}}{\partial x}\,dxdt + \varepsilon \int_Q \frac{\partial u_{\varepsilon,\delta}}{\partial x}\frac{\partial v}{\partial x}\,dxdt = 0, \\ -\displaystyle\int_Q \frac{\partial u_{\varepsilon,\delta}}{\partial t}\frac{\partial \mu}{\partial t}\,dxdt - \delta \int_Q \frac{\partial \lambda_{\varepsilon,\delta}}{\partial t}\frac{\partial \mu}{\partial t}\,dxdt \\ +\displaystyle\int_Q \frac{\partial u_{\varepsilon,\delta}}{\partial x}\frac{\partial \mu}{\partial x}\,dxdt - \delta \int_Q \frac{\partial \lambda_{\varepsilon,\delta}}{\partial x}\frac{\partial \mu}{\partial x}\,dxdt = \int_{\Gamma_0} g_1 \mu\,dxdt, \end{cases} \tag{22}$$

where the last integral has the meaning of duality pairing between $H^{-\frac{1}{2}}_{S_0}(\Gamma_0)$ and $H^{\frac{1}{2}}_{\Gamma_1,S_T}(\Gamma_0)$. The mixed formulation (22) to solve problem (20) has to be compared to the fourth-order formulation used in [18]. We have the following proposition.

Proposition 4. *For any $(g_0, g_1) \in H^{\frac{1}{2}}_{S_0}(\Gamma_0) \times H^{-\frac{1}{2}}_{S_0}(\Gamma_0)$, the problem (22) has a unique solution $(u_{\varepsilon,\delta}, \lambda_{\varepsilon,\delta})$ in $V_g \times \tilde{V}_0$. Furthermore, assume that there exists a solution $u \in H^1(Q)$ to problem (20) associated with data (g_0, g_1), referred to as the exact solution, and let $\tilde{u}$ be the solution to problem (20) of minimal norm in $H^1(Q)$. If δ is a bounded function of ε such that (5) is satisfied, then the solution $(u_{\varepsilon,\delta}, \lambda_{\varepsilon,\delta})$ of problem (22) associated with the same data (g_0, g_1) satisfies*

$$\lim_{\varepsilon \to 0} u_{\varepsilon,\delta(\varepsilon)} = \tilde{u} \quad \text{in} \quad H^1(Q), \quad \lim_{\varepsilon \to 0} \lambda_{\varepsilon,\delta(\varepsilon)} = 0 \quad \text{in} \quad H^1(Q).$$

For $T > 1$, the function $\tilde{u}$ coincides with the exact solution u in the subdomain R^0_T.

In order to prove proposition 4 we need the following lemma.

Lemma 3.1. *The function u is a solution to problem (20) if and only if $u \in V_g$ and for all $\mu \in \tilde{V}_0$,*

$$-\int_Q \frac{\partial u}{\partial t}\frac{\partial \mu}{\partial t}\,dxdt + \int_Q \frac{\partial u}{\partial x}\frac{\partial \mu}{\partial x}\,dxdt = \int_{\Gamma_0} g_1\,\mu\,dt, \tag{23}$$

where the last integral means duality pairing between $H^{-\frac{1}{2}}_{S_0}(\Gamma_0)$ and $H^{\frac{1}{2}}_{\Gamma_1,S_T}(\Gamma_0)$.

Proof. We first consider some $u \in H^1(Q)$ which solves problem (20). Then $u \in V_g$. For $\mu \in H^1(Q)$, we use the following integration by parts formula, with $\mathbf{v} = (\partial_x u, -\partial_t u) \in (L^2(Q))^2$, $\mathrm{div}_2 \mathbf{v} \in L^2(Q)$ and $\nabla_2 \mu = (\partial_x \mu, \partial_t \mu) \in (L^2(Q))^2$,

$$\int_Q \mathbf{v} \cdot \nabla_2 \mu\,dxdt = -\int_Q \mu(\mathrm{div}_2 \mathbf{v})\,dxdt + \langle \mathbf{v} \cdot \nu_2, \mu \rangle_{\partial Q}, \tag{24}$$

where the bracket $\langle \cdot, \cdot \rangle_{\partial Q}$ has the meaning of duality between $H^{\frac{1}{2}}(\partial Q)$ and $H^{-\frac{1}{2}}(\partial Q)$. By considering $\mu \in \tilde{V}_0$, we use the fact that $\mathrm{div}_2 \mathbf{v} = 0$ in Q, that $\partial_t u = 0$ on S_0,

$\mu = 0$ on S_T, $-\partial_x u = g_1$ on Γ_0 and $\mu = 0$ on Γ_1 to obtain

$$\int_Q \mathbf{v} \cdot \nabla_2 \mu \, dxdt = \langle g_1, \mu \rangle_{\Gamma_0},$$

where the bracket $\langle \cdot, \cdot \rangle_{\Gamma_0}$ has the meaning of duality between $H^{-\frac{1}{2}}_{S_0}(\Gamma_0)$ and $H^{\frac{1}{2}}_{\Gamma_1, S_T}(\Gamma_0)$. The weak formulation (23) is achieved. Conversely, let $u \in V_g$ which satisfies (23). By taking $\mu \in C^\infty_0(Q)$, it follows that u solves the wave equation in the distributional sense in Q. Hence formula (24) is valid, and we obtain that for all $\mu \in \tilde{V}_0$

$$\langle \mathbf{v} \cdot \nu_2, \mu \rangle_{\partial Q} = \langle g_1, \mu \rangle_{\Gamma_0}.$$

By using the fact that $\mu = 0$ on S_T and $\mu = 0$ on Γ_1, by denoting $\tilde{g}_1$ the extension of g_1 on $\Sigma_0 = \partial Q \setminus \overline{\Gamma_1 \cup S_T}$ by 0, we obtain that

$$\langle \mathbf{v} \cdot \nu_2, \mu \rangle_{\Sigma_0} = \langle \tilde{g}_1, \mu \rangle_{\Sigma_0},$$

where the brackets $\langle \cdot, \cdot \rangle_{\Sigma_0}$ have the meaning of duality between $H^{-\frac{1}{2}}(\Sigma_0)$ and $H^{\frac{1}{2}}_{00}(\Sigma_0)$. We conclude that $\mathbf{v} \cdot \nu_2 = \tilde{g}_1$ on Σ_0, then $\mathbf{v} \cdot \nu_2 = g_1$ on Γ_0 and $\mathbf{v} \cdot \nu_2 = 0$ on S_0, that is $-\partial_x u = g_1$ on Γ_0 and $\partial_t u = 0$ on S_0. As a conclusion, problem (20) is satisfied by u. □

Proof of proposition 4. By denoting $\hat{u}_{\varepsilon,\delta} = u_{\varepsilon,\delta} - \Psi$, with $\Psi \in V_g$, the problem (22) can be rewritten: find $(\hat{u}_{\varepsilon,\delta}, \lambda_{\varepsilon,\delta}) \in V_0 \times \tilde{V}_0$ such that for all $(v, \mu) \in V_0 \times \tilde{V}_0$,

$$B((\hat{u}_{\varepsilon,\delta}, \lambda_{\varepsilon,\delta}), (v, \mu)) = M(v, \mu),$$

where M is a continuous linear form on $V_0 \times \tilde{V}_0$ (which is not necessary to give explicitly as a function of Ψ), and B is the continuous bilinear form on $V_0 \times \tilde{V}_0$ given by

$$\left\{ \begin{array}{l} B((\hat{u}, \lambda), (v, \mu)) = -\displaystyle\int_Q \frac{\partial v}{\partial t} \frac{\partial \lambda}{\partial t} \, dxdt + \int_Q \frac{\partial \hat{u}}{\partial t} \frac{\partial \mu}{\partial t} \, dxdt \\ + \displaystyle\int_Q \frac{\partial v}{\partial x} \frac{\partial \lambda}{\partial x} \, dxdt - \int_Q \frac{\partial \hat{u}}{\partial x} \frac{\partial \mu}{\partial x} \, dxdt \\ + \varepsilon \displaystyle\int_Q \frac{\partial \hat{u}}{\partial t} \frac{\partial v}{\partial t} \, dxdt + \varepsilon \int_Q \frac{\partial \hat{u}}{\partial x} \frac{\partial v}{\partial x} \, dxdt \\ + \delta \displaystyle\int_Q \frac{\partial \lambda}{\partial t} \frac{\partial \mu}{\partial t} \, dxdt + \delta \int_Q \frac{\partial \lambda}{\partial x} \frac{\partial \mu}{\partial x} \, dxdt. \end{array} \right. \tag{25}$$

We have

$$B((\hat{u}, \lambda), (\hat{u}, \lambda)) = \varepsilon ||\hat{u}||^2 + \delta ||\lambda||^2,$$

which proves that B is coercive and completes the proof in view of Lax-Milgram's theorem. We now justify the second part of the proposition. For sake of simplicity we denote $u_\varepsilon := u_{\varepsilon,\delta(\varepsilon)}$ and $\lambda_\varepsilon := \lambda_{\varepsilon,\delta(\varepsilon)}$. From lemma 3.1, the function u satisfies the problem (20) if and only if $u \in V_g$ and satisfies the weak formulation (23). As a result, the set of functions in $H^1(Q)$ that solve (20) is non empty, convex and closed, and therefore there exists a unique function $\tilde{u}$ solving (20) that minimizes the H^1 norm $|| \cdot ||$. By subtracting the equation (23) for the solution $\tilde{u}$ from the second equation of problem (22), we obtain that for all $(v,\mu)\in V_0\times\tilde{V}_0$,

$$\left\{\begin{array}{l} -\displaystyle\int_Q \frac{\partial v}{\partial t}\frac{\partial \lambda_\varepsilon}{\partial t}\,dxdt+\varepsilon\int_Q \frac{\partial u_\varepsilon}{\partial t}\frac{\partial v}{\partial t}\,dxdt \\ +\displaystyle\int_Q \frac{\partial v}{\partial x}\frac{\partial \lambda_\varepsilon}{\partial x}\,dxdt+\varepsilon\int_Q \frac{\partial u_\varepsilon}{\partial x}\frac{\partial v}{\partial x}\,dxdt=0, \\ -\displaystyle\int_Q \frac{\partial (u_\varepsilon-\tilde{u})}{\partial t}\frac{\partial \mu}{\partial t}\,dxdt-\delta\int_Q \frac{\partial \lambda_\varepsilon}{\partial t}\frac{\partial \mu}{\partial t}\,dxdt \\ +\displaystyle\int_Q \frac{\partial (u_\varepsilon-\tilde{u})}{\partial x}\frac{\partial \mu}{\partial x}\,dxdt-\delta\int_Q \frac{\partial \lambda_\varepsilon}{\partial x}\frac{\partial \mu}{\partial x}\,dxdt=0. \end{array}\right. \tag{26}$$

Then we choose $v=u_\varepsilon-\tilde{u}\in V_0$ and $\mu=\lambda_\varepsilon\in\tilde{V}_0$ in the above system and subtract the two obtained equations. We obtain a similar equation as (10), which implies

$$||u_\varepsilon||\leq||\tilde{u}||,\quad ||\lambda_\varepsilon||\leq\sqrt{\frac{\varepsilon}{\delta(\varepsilon)}}||\tilde{u}||.$$

In particular u_ε is bounded in $H^1(Q)$ while λ_ε tends to 0 when ε tends to 0 in $H^1(Q)$. There exists a subsequence of u_ε, still denoted u_ε, that weakly converges to some $w\in H^1(Q)$, with $w\in V_g$.
Passing to the limit in the second equation of (22), since δ is a bounded function of ε, we obtain that for all $\mu\in\tilde{V}_0$,

$$-\int_Q \frac{\partial w}{\partial t}\frac{\partial \mu}{\partial t}\,dxdt+\int_Q \frac{\partial w}{\partial x}\frac{\partial \mu}{\partial x}\,dxdt=\int_{\Gamma_0} g_1\mu\,dxdt,$$

and hence w solves problem (20) by using again lemma 3.1. In addition,

$$||w||\leq\liminf_{\varepsilon\to 0}||u_\varepsilon||\leq||\tilde{u}||.$$

By using the definition of $\tilde{u}$ we conclude that $w=\tilde{u}$, and lastly that the full sequence u_ε strongly converges to $\tilde{u}$ in $H^1(Q)$ like in the proof of theorem 1. From uniqueness in the subdomain R_T^0, we then obtain that $\tilde{u}$ coincides with u in R_T^0. □

3.1.2. *Quasi-reversibility for problem (19).* In order to adapt our quasi-reversibility method to the case of problem (19), we have to replace the sets $V_g, V_0, \tilde{V}_0$ by the sets $W_g, W_0, \tilde{W}_0$. For $(g_0,g_1)\in H^{\frac{1}{2}}(\Gamma_0)\times H^{-\frac{1}{2}}(\Gamma_0)$ instead of $H^{\frac{1}{2}}_{S_0}(\Gamma_0)\times H^{-\frac{1}{2}}_{S_0}(\Gamma_0)$, the formulation of quasi-reversibility (22) is unchanged, and it is easy to prove that an analogous proposition as 4 is still valid for $T>2$ with subdomain R_T instead of R_T^0.

3.2. **The inverse obstacle problem.** Given a pair of lateral Cauchy data (g_0,g_1) on Γ_0, the inverse obstacle problem consists in finding some real $a\in(0,1)$ (independent of time t) and some function $u\in H^1(Q_a)$ such that

$$\left\{\begin{array}{rcll} \partial_t^2 u-\partial_x^2 u &=& 0 & \text{in } Q_a \\ u &=& g_0 & \text{on } \Gamma_0 \\ -\partial_x u &=& g_1 & \text{on } \Gamma_0 \\ u &=& 0 & \text{on } \Gamma_a \\ u &=& 0 & \text{on } S_{0,a} \\ \partial_t u &=& 0 & \text{on } S_{0,a}. \end{array}\right. \tag{27}$$

We first recall the following uniqueness result for our inverse obstacle problem, which is strongly inspired from [19].

Proposition 5. *Let $a_1, a_2 \in (0,1)$ be two reals such that the corresponding functions u_1, u_2 satisfy (27) with data (g_0, g_1). If we assume that $T > 2\min(a_1, a_2)$ and that for all $t_0 > 0$, the restriction of (g_0, g_1) to time interval $(0, t_0)$ is not zero, then we have $a_1 = a_2$.*

Proof. First we extend u_i by 0 in the domain $(0, a_i) \times (-T, 0)$ for $i = 1, 2$. Without loss of generality we have $a_1 \leq a_2$. We assume that $a_1 < a_2$. The function $u := u_1 - u_2$ satisfies the system

$$\left\{\begin{array}{ccll} \partial_t^2 u - \partial_x^2 u & = & 0 & \text{in } (0, a_1) \times (-T, T) \\ u & = & 0 & \text{on } \{0\} \times (-T, T) \\ -\partial_x u & = & 0 & \text{on } \{0\} \times (-T, T). \end{array}\right.$$

By unique continuation applied to the function u we obtain that $u = 0$ in domain $\{(x,t),\ 0 < x < a_1,\ -T+x < t < T-x\}$, and it follows that $u_1(a_1, \cdot) = u_2(a_1, \cdot) = 0$ on time interval $(-T + a_1, T - a_1)$. We hence have $u_2(a_1, \cdot) = 0$ and $u_2(a_2, \cdot) = 0$ simultaneously on time interval $(-T + a_1, T - a_1)$. From uniqueness of the forward problem with Dirichlet boundary condition, we conclude that u_2 vanishes in $(a_1, a_2) \times (-T + a_1, T - a_1)$, and it follows that $u_2(a_1, \cdot)$ and $\partial_x u_2(a_1, \cdot)$ both vanish on time interval $(-T+a_1, T-a_1)$. By using unique continuation applied to the function u_2, we obtain that u_2 vanishes in domain $\{(x,t),\ 0 < x < a_1,\ -T + 2a_1 - x < t < T - 2a_1 + x\}$, and it follows that $u_2(0, \cdot)$ and $\partial_x u_2(0, \cdot)$ both vanish on time interval $(-T + 2a_1, T - 2a_1)$, in particular on $(0, T - 2a_1)$. Since $T > 2a_1$, this contradicts the assumption that (g_0, g_1) is not identically 0 on interval $(0, T - 2a_1)$. We hence have $a_1 = a_2$. □

Remark 6. Proposition 5 is valid in particular for $T > 2$.

As for the heat equation, there are already contributions to the resolution of inverse obstacle problems of type (27). Note that the specificity of problem (27) is that we use only one pair of lateral Cauchy data. In contrast, most papers addressing such kind of problem in the time domain consider much more data, like for example in the sampling methods [20]. For that reason our approach has to be compared to the other approaches that rely on a single incident wave: in this vein and in higher dimension 2 or 3, we mention [21], [22], [23] and [24]. In [21] the "enclosure method" is used to solve an inverse obstacle problem in free space. In [22], the obstacle is an infinite line that is identified with the help of the point source method. In [23] the so-called "TRAC method" is used to derive an inverse medium problem in a bounded domain from a problem originally set in free space, such inverse problem is then solved with the help of a refined gradient method. Lastly, in [24] a method based on topological sensitivity is used to solve an inverse medium problem as well (though with a few incident waves instead of only one).
Now we apply our "exterior approach" by coupling the quasi-reversibility formulation (22) and a similar level set method as the one used for the heat equation. In what follows we assume that $T > 2$ and that for all $t_0 > 0$, the restriction of (g_0, g_1) to time interval $(0, t_0)$ is not trivial, which ensures uniqueness in the inverse obstacle problem (27). The following theorem justifies our level set method.

Theorem 3.2. *Let us consider some $a \in (0,1)$ and $u \in H^1(Q_a)$ such that problem (27) is satisfied. Let us assume that $T > 2$ and that for all $t_0 > 0$, the restriction* *of (g_0, g_1) to time interval $(0, t_0)$ is not zero. Let us define $F \in H^1(0,1)$ by*

$$F(x) = \left(\int_0^{T-1} u^2(x,t)\, dt \right)^{\frac{1}{2}}$$

for $x \in (0,a)$ and extended by 0 in $(a,1)$. Let us consider $f \in L^2(0,1)$ which satisfies (16). Then with initial condition $a_0 \in (0,a)$ the sequence (a_n) defined in $(0,1)$ by (17) and (18) converges to a.

Proof. The proof is similar to that of theorem 2.3 from the beginning until $F(a^*) = 0 = F(a)$, with $a^* < a$. We conclude that $u(a^*, \cdot)$ and $u(a, \cdot)$ simultaneously vanish on $(-T+1, T-1)$. By using uniqueness for the forward problem with Dirichlet boundary condition, we obtain that $u = 0$ in $(a^*, a) \times (-T+1, T-1)$, then $u(a^*, \cdot)$ and $\partial_x u(a^*, \cdot)$ both vanish on time interval $(-T+1, T-1)$. By unique continuation, u vanishes in the domain $\{(x,t),\ 0 < x < a^*,\ -T+1+a^*-x < t < T-1-a^*+x\}$, it follows that $u(0, \cdot)$ and $\partial_x u(0, \cdot)$ both vanish on time interval $(-T+1+a^*, T-1-a^*)$, in particular on $(0, T-1-a^*)$. Since $T > 2 > 1+a^*$, this contradicts the assumption that (g_0, g_1) is not identically 0 on interval $(0, T-1-a^*)$. □

The algorithm to retrieve the obstacle a from the Cauchy data (g_0, g_1) on Γ_0 is almost the same as the algorithm for the heat equation.

Algorithm :

1. Choose an initial guess $a_0 \in (0,1)$ such that $a_0 < a$.
2. First step: the real a_n being given, solve the quasi-reversibility problem (22) in $Q_{a_n} = (0, a_n) \times (0,T)$ for some selected parameters $\varepsilon, \delta > 0$. The solution is denoted (u_n, λ_n).
3. Second step: the function u_n being given, solve the equation $-\phi_n' = f$ in $(a_n, 1)$ with initial condition

$$\phi_n(a_n) = \left(\int_0^{T-1} (u_n(a_n,t))^2\, dt \right)^{\frac{1}{2}}$$

   for some selected $f \in L^2(0,1)$. Define

$$a_{n+1} = \inf\{x \in [a_n, 1), \quad \phi_n(x) \leq 0\}.$$

4. Go back to the first step until the stopping criterion is reached.

Remark 7. In our 1D case, an alternative and simple method to solve the inverse obstacle consists in extending the lateral Cauchy data (g_0, g_1) by 0 for $t < 0$ and to use the d'Alembert's formula (21) to compute a solution u_A in the whole domain R_T^0. We can then plug the function F_A defined by

$$F_A(x) = \left(\int_0^{T-1} u_A^2(x,t)\, dt \right)^{\frac{1}{2}} \tag{28}$$

in the whole interval $(0,1)$. For exact data (g_0, g_1), the function u_A coincides with the true solution u only in the subdomain $\{(x,t),\ 0 < x < a,\ 0 < t < T-a\}$, so that the minimum of the function F_A is 0 and the smallest x such that $F_A(x) = 0$ is $x = a$, which is a practical method to compute a. Of course, such procedure is not applicable in dimension higher than 1.

4. **Discretization.** In all the mixed formulations of quasi-reversibility that we have proposed, the solution $(u_\varepsilon, \lambda_\varepsilon)$ lies in the functional space $H^1(Q) \times H^1(Q)$, which is very easy to discretize with the help of Lagrange finite elements. Since in our procedure to solve the inverse obstacle we have to compute integrals in time sections (see formula (15)), we want to separate the role of the spatial variable x and the role of the time variable t. This is why we have chosen to discretize the space $H^1(Q)$ with the help of elements of $P_1 \otimes P_1$, which denotes the tensor product between the one-dimensional P_1 elements for x and the one-dimensional P_1 elements for t. We hence consider a structured rectangular mesh of the domain $(0,1) \times (0,T)$ and it is easy to check that such finite element $P_1 \otimes P_1$ coincides with the two-dimensional quadrangular element Q_1 (see for example [25]). We denote by U_h the finite dimensional space generated by the Lagrange finite element Q_1 in a rectangular mesh of maximum diameter h, such mesh being regular in the sense of [25].

Let us introduce a discretized version of the first mixed formulation (4) based on the data $u_T \in H^{\frac{1}{2}}_{00}(S_T)$. We consider the restricted case $u_T \in H^{\frac{1}{2}}_{00}(S_T) \cap C^0(\overline{S_T})$, and we define u_{Th} as its interpolant over the space of traces on S_T of U_h-functions, that is u_{Th} has the same degrees of freedom on S_T as u_T (since such degrees of freedom are pointwise values of the function, u_T needs to be continuous on S_T). Introducing the sets

$$\begin{aligned} U_{u,h} &= \{u_h \in U_h,\ u_h(0,\cdot) = u_h(1,\cdot) = 0,\ u_h(\cdot,T) = u_{Th}\}, \\ U_{0,h} &= \{u_h \in U_h,\ u_h(0,\cdot) = u_h(1,\cdot) = 0,\ u_h(\cdot,T) = 0\}, \\ \tilde{U}_{0,h} &= \{\lambda_h \in U_h,\ \lambda_h(0,\cdot) = \lambda_h(1,\cdot) = 0,\ \lambda_h(\cdot,0) = 0\}, \end{aligned}$$

the discretized formulation associated with (4) consists for $\varepsilon, \delta, h > 0$ in solving the following problem: find $(u_{\varepsilon,\delta,h}, \lambda_{\varepsilon,\delta,h}) \in U_{u,h} \times \tilde{U}_{0,h}$ such that for all $(v_h, \mu_h) \in U_{0,h} \times \tilde{U}_{0,h}$,

$$\left\{ \begin{array}{l} -\displaystyle\int_Q v_h \frac{\partial \lambda_{\varepsilon,\delta,h}}{\partial t}\, dxdt + \varepsilon \int_Q \frac{\partial u_{\varepsilon,\delta,h}}{\partial t} \frac{\partial v_h}{\partial t}\, dxdt \\ +\displaystyle\int_Q \frac{\partial v_h}{\partial x} \frac{\partial \lambda_{\varepsilon,\delta,h}}{\partial x}\, dxdt + \varepsilon \int_Q \frac{\partial u_{\varepsilon,\delta,h}}{\partial x} \frac{\partial v_h}{\partial x}\, dxdt = 0, \\ \displaystyle\int_Q \frac{\partial u_{\varepsilon,\delta,h}}{\partial t} \mu_h\, dxdt - \delta \int_Q \frac{\partial \lambda_{\varepsilon,\delta,h}}{\partial t} \frac{\partial \mu_h}{\partial t}\, dxdt \\ +\displaystyle\int_Q \frac{\partial u_{\varepsilon,\delta,h}}{\partial x} \frac{\partial \mu_h}{\partial x}\, dxdt - \delta \int_Q \frac{\partial \lambda_{\varepsilon,\delta,h}}{\partial x} \frac{\partial \mu_h}{\partial x}\, dxdt = 0. \end{array} \right. \tag{29}$$

We have the following estimate of the discrepancy between the solution of the discretized problem (29) and the solution of the continuous problem (4).

Theorem 4.1. *For all $\varepsilon, \delta, h > 0$, the problem (29) has a unique solution $(u_{\varepsilon,\delta,h}, \lambda_{\varepsilon,\delta,h})$ in $U_{u,h} \times \tilde{U}_{0,h}$. Moreover, if $\varepsilon, \delta \le 1$ and $(u_{\varepsilon,\delta}, \lambda_{\varepsilon,\delta})$ belongs to $H^2(Q) \times H^2(Q)$, then*

$$||u_{\varepsilon,\delta,h} - u_{\varepsilon,\delta}|| + ||\lambda_{\varepsilon,\delta,h} - \lambda_{\varepsilon,\delta}|| \le C \frac{h}{\min(\varepsilon,\delta)} \left(||u_{\varepsilon,\delta}||_{H^2(Q)} + ||\lambda_{\varepsilon,\delta}||_{H^2(Q)} \right),$$

where $C > 0$ is independent of ε, δ, h.

Proof. To prove well-posedness of problem (29) we use the same arguments as in the proof of proposition 1. Let us prove the error estimate. We introduce the same function Φ and the same bilinear form A as in the proof of theorem 1, and to shorten notations we denote $X_{\varepsilon,\delta} = (u_{\varepsilon,\delta}, \lambda_{\varepsilon,\delta})$, $X = (\Phi, 0)$ and $\hat{X}_{\varepsilon,\delta} = X_{\varepsilon,\delta} - X$. Considering the interpolant $\Phi_h \in U_h$ of Φ, we also define $X_{\varepsilon,\delta,h} = (u_{\varepsilon,\delta,h}, \lambda_{\varepsilon,\delta,h})$, $X_h = (\Phi_h, 0)$ and $\hat{X}_{\varepsilon,\delta,h} = X_{\varepsilon,\delta,h} - X_h$. We now adapt the Cea's lemma to the case of non-homogeneous Dirichlet data. For all $Z_h = (v_h, \mu_h) \in U_{0,h} \times \tilde{U}_{0,h} \subset U_0 \times \tilde{U}_0$, we have from (4) and (29)

$$A_{\varepsilon,\delta}(X_{\varepsilon,\delta}; Z_h) = 0 = A_{\varepsilon,\delta}(X_{\varepsilon,\delta,h}; Z_h),$$

that is

$$A_{\varepsilon,\delta}(X_{\varepsilon,\delta} - X_{\varepsilon,\delta,h}; Z_h) = 0.$$

For any $Y_h \in U_{0,h} \times \tilde{U}_{0,h}$, we choose $Z_h = Y_h - \hat{X}_{\varepsilon,\delta,h}$. By remarking that

$$Z_h = (X_h + Y_h - X_{\varepsilon,\delta}) + (X_{\varepsilon,\delta} - X_{\varepsilon,\delta,h}),$$

we obtain

$$A_{\varepsilon,\delta}(X_{\varepsilon,\delta} - X_{\varepsilon,\delta,h}; X_{\varepsilon,\delta} - X_{\varepsilon,\delta,h}) = A_{\varepsilon,\delta}(X_{\varepsilon,\delta} - X_{\varepsilon,\delta,h}; X_{\varepsilon,\delta} - X_h - Y_h).$$

With the help of (8) and the fact that $\varepsilon, \delta \leq 1$, we obtain

$$||X_{\varepsilon,\delta} - X_{\varepsilon,\delta,h}||_\times \leq C\,\frac{1}{\min(\varepsilon,\delta)} \inf_{Y_h \in U_{0,h} \times \tilde{U}_{0,h}} ||X_{\varepsilon,\delta} - X_h - Y_h||_\times,$$

$$= C\,\frac{1}{\min(\varepsilon,\delta)} \inf_{W_h \in U_{g,h} \times \tilde{U}_{0,h}} ||X_{\varepsilon,\delta} - W_h||_\times,$$

where $||\cdot||_\times$ is the product norm of $H^1(Q) \times H^1(Q)$. The estimate follows from the classical interpolation results in space $H^1(Q)$ (see [25]). □

Remark 8. It should be noted that, in contrast to the parabolic nature of the ill-posed problem (1), the quasi-reversibility formulation (4) that enables us to regularize such problem is of elliptic nature. In parabolic problems it may happen that, depending on the numerical scheme with respect to time which is used, a stability condition be required. But in the discretized version (29) of (4), which is a global space/time finite element scheme, the stability condition amounts to impose the mesh to be regular in the sense of [25] when $h \to 0$, that is we have to prevent all the rectangles of the mesh from becoming flat. However, by giving the explicit dependence of the discretized functions with respect to time, we could interpret (29) as an implicit finite difference scheme.

Analogous convergence results as theorem 4.1 would be easily derived for the other quasi-reversibility formulations. Since they follow from the same arguments, they are not repeated in the other cases.

5. Numerical experiments.

5.1. The backward heat equation.

We first begin with some numerical experiments for the backward heat equation. To produce the artificial data u_T, we solve the forward problem

$$\begin{cases} \partial_t u - \partial_x^2 u &= 0 & \text{in } Q \\ u &= 0 & \text{on } \Gamma_0 \cup \Gamma_1 \\ u &= u_0 & \text{on } S_0 \end{cases}$$

for an initial data u_0 and with the help of an implicit finite difference scheme. We study two kinds of initial data u_0:

1. $u_0^1(x) = \sin(\pi x)$

2. $u_0^2(x) = \chi_{[1/3,2/3]}(x)$,

where $\chi_{[a,b]} = 1$ on interval $[a, b]$ and $\chi_{[a,b]} = 0$ elsewhere. The first case is simple because u_0^1 corresponds to the first eigenvalue of the one-dimensional Laplacian operator with Dirichlet boundary condition. The second one is difficult because u_0^2 is non-smooth while the heat equation has a smoothing effect. We now introduce some noisy data in order to test the robustness of our method. Precisely, we impose a pointwise random noise to data u_T. The relative error in L^2 norm for such noisy data is denoted by σ. Due to the severe ill-posedness of the problem we consider a short interval of time, namely $T = 0.1$. We first consider the problem (1) and we test the discretized version (29) of the quasi-reversibility method (4) with time discretization $\Delta t = T/20$ and space dicretization $\Delta x = 1/200$, with regularization parameters $\varepsilon = 10^{-4}$ and $\delta = \sqrt{\varepsilon}$. In figures 1 and 2, we have represented the exact solution u associated with u_0^1 and u_0^2, respectively, as well as the discrepancy between the quasi-reversibility solution $u_{\varepsilon,\delta}$ and the exact solution u, with the relative amplitude of noise $\sigma = 0$ (exact data), $\sigma = 0.05$ and $\sigma = 0.1$. We observe that the quasi-reversibility method acts as if it projected the data on the first eigenfunctions of the one-dimensional Laplacian operator with Dirichlet boundary condition. This explains why the result is rather good in the first case (data u_0^1) and very bad in the second one (data u_0^2). We note that the problem is so ill-posed in the second case that the amplitude of noise has almost no effect on the reconstruction. We now consider the problem (2) and we test the corresponding

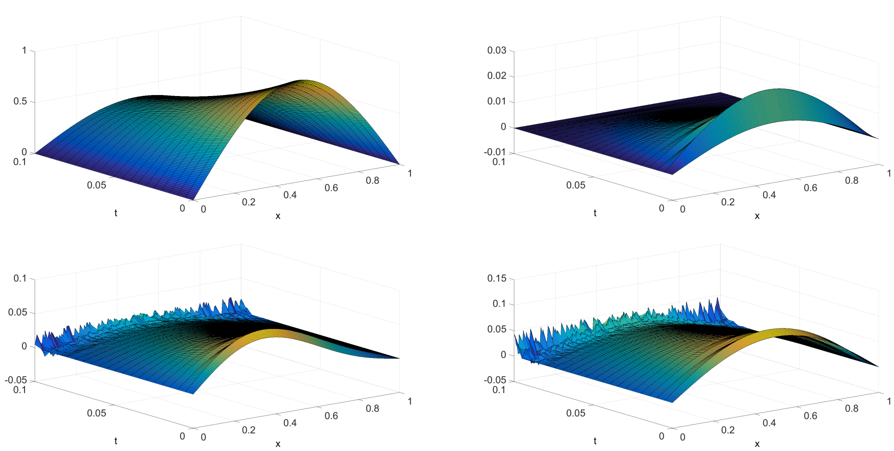

FIGURE 1. Top left: exact solution u for $u_0 = u_0^1$. Top right: discrepancy $u_{\varepsilon,\delta} - u$ for $\sigma = 0$. Bottom left: discrepancy $u_{\varepsilon,\delta} - u$ for $\sigma = 0.05$. Bottom right: discrepancy $u_{\varepsilon,\delta} - u$ for $\sigma = 0.1$.

quasi-reversibility method with the same parameters, only in the most favorable case $u_0 = u_0^1$, and for the subdomain $I = (0.1, 0.6)$. The results are represented on figure 3 and has to be compared to those represented on figure 1. The quality of the results is approximately the same.

5.2. **The heat equation with lateral Cauchy data and inverse obstacle problem.** We continue with some numerical experiments for the heat equation with lateral Cauchy data. First, we consider the solution $u(x, t) = e^{-t}\cos(x)$ to

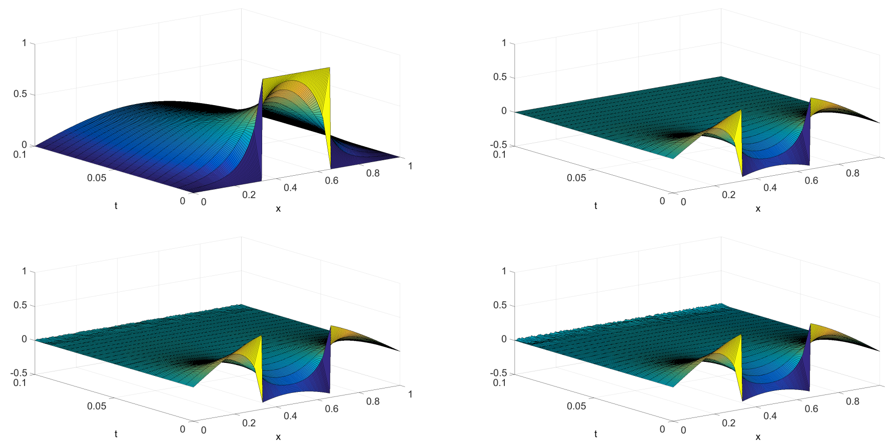

FIGURE 2. Top left: exact solution u for $u_0 = u_0^2$. Top right: discrepancy $u_{\varepsilon,\delta} - u$ for $\sigma = 0$. Bottom left: discrepancy $u_{\varepsilon,\delta} - u$ for $\sigma = 0.05$. Bottom right: discrepancy $u_{\varepsilon,\delta} - u$ for $\sigma = 0.1$.

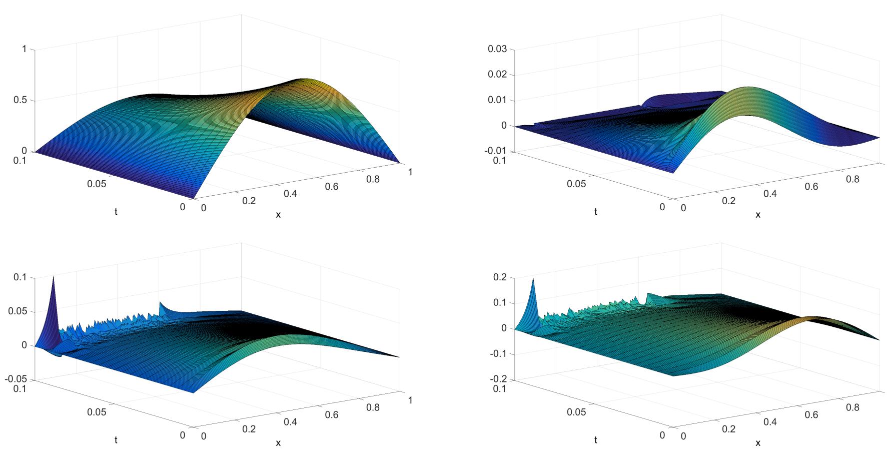

FIGURE 3. Top left: exact solution u for $u_0 = u_0^1$. Top right: discrepancy $u_{\varepsilon,\delta} - u$ for $\sigma = 0$ (partial data). Bottom left: discrepancy $u_{\varepsilon,\delta} - u$ for $\sigma = 0.05$ (partial data). Bottom right: discrepancy $u_{\varepsilon,\delta} - u$ for $\sigma = 0.1$ (partial data).

the heat equation, which enables us to obtain some lateral Cauchy data (g_0, g_1) on Γ_0, and then address problem (11) with such data. We impose a pointwise random noise to the Dirichlet data g_0 only (g_1 is noise free), the relative error in L^2 norm for such noisy data being again denoted by σ. The reason why we do not perturb the Neumann data g_1 is the following: in practice, the Neumann data is considered as the solicitation that we impose to the system while the Dirichlet data is considered as the measurement of the response of the system to such solicitation. In this sense, we have a better knowledge of the solicitation than of the response. We use the discretized version of the quasi-reversibility method (13) for sets $W_g, W_0, \tilde{W}_0$ instead of $V_g, V_0, \tilde{V}_0$, with time discretization $\Delta t = T/50$ and space discretization $\Delta x = 1/100$, the regularization parameters $\varepsilon = 10^{-4}$ and $\delta = \sqrt{\varepsilon}$ being unchanged. The impact of noise is analyzed on figure 4 for $T = 1$, where we have represented a comparison between the exact solution u (continuous line) and the quasi-reversibility solution $u_{\varepsilon,\delta}$ (dashed line) on Γ_1, for $\sigma = 0$ (exact data), $\sigma = 0.01$, $\sigma = 0.02$ and $\sigma = 0.05$. The impact of the final time T is analyzed on figure 5, where we have represented the discrepancy between the exact solution u and the quasi-reversibility solution $u_{\varepsilon,\delta}$ on Γ_1, for $\sigma = 0.01$, with $T = 0.5$, $T = 1$ and $T = 2$. We observe that the quality of the reconstruction is strongly altered when the amplitude of noise increases and is strongly improved when the final time T increases. We complete this study on the heat equation with lateral Cauchy data by a short sensitivity analysis of the reconstruction with respect to the parameters ε and δ. On figure 6 we can see a comparison between the exact solution u (continuous line) and the quasi-reversibility solution $u_{\varepsilon,\delta}$ (dashed line) on Γ_1, for $T = 1$ and noise free data, with $\varepsilon = 10^{-5}, 10^{-4}, 10^{-3}$ and with $\delta = \sqrt{\varepsilon}, 1$. The figure 7 is the same as figure 6 with $\sigma = 0.02$ instead of noise free data. We observe that the quality of the reconstruction depends on ε and δ, which should be adapted to the amplitude of noise σ. In our example, it seems that both ε and δ have to be increased with respect to the amplitude of noise, but a rigorous method to choose them, which is a delicate question, is not proposed in the present paper. We now

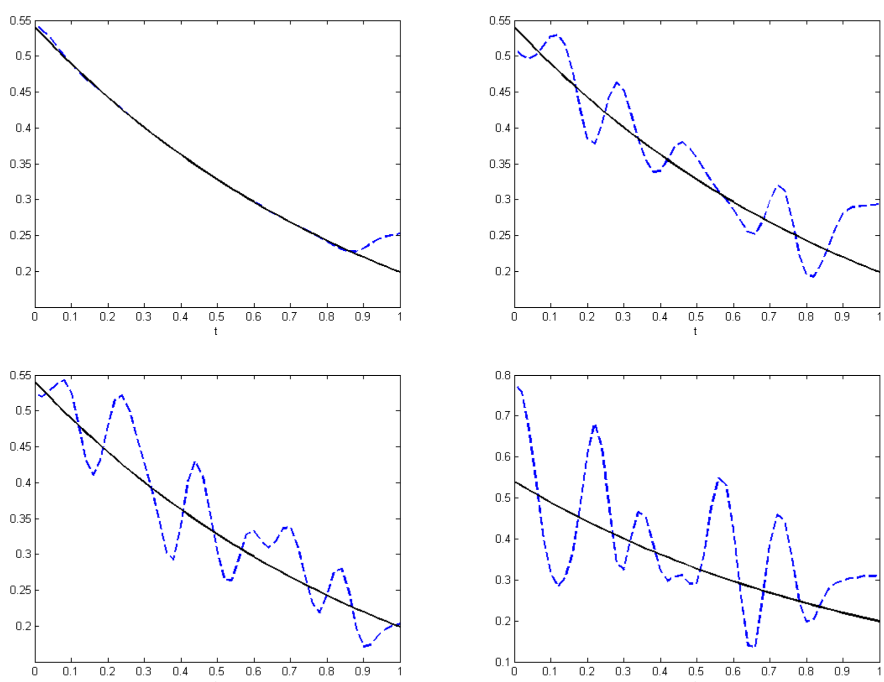

FIGURE 4. Quasi-reversibility solution $u_{\varepsilon,\delta}$ and exact solution u for $x = 1$, impact of the amplitude of noise. Top left: $\sigma = 0$. Top right: $\sigma = 0, 01$. Bottom left: $\sigma = 0.02$. Bottom right: $\sigma = 0.05$.

solve the inverse obstacle problem (14) with the help of the algorithm presented

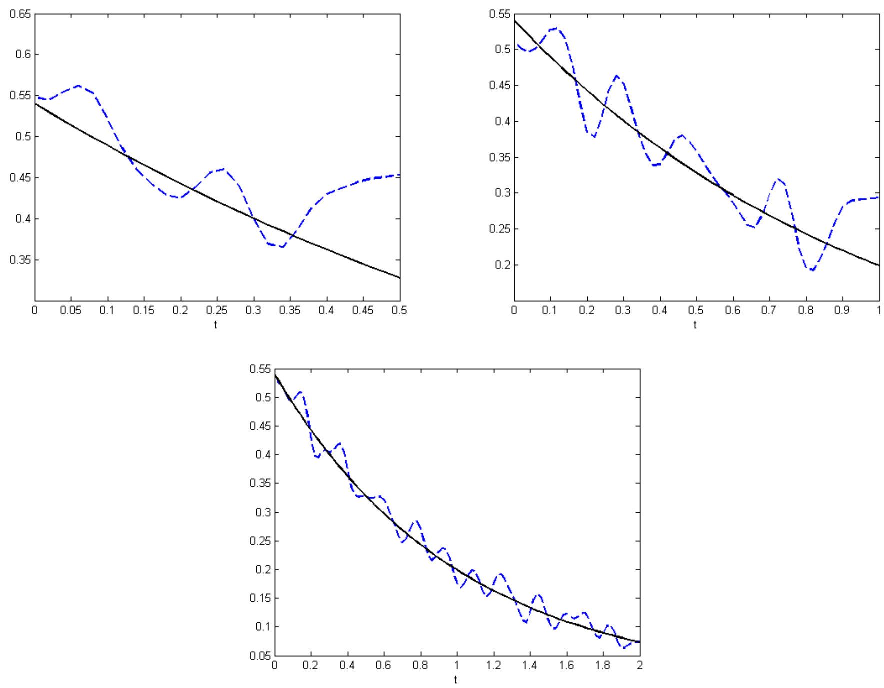

FIGURE 5. Quasi-reversibility solution $u_{\varepsilon,\delta}$ and exact solution u for $x = 1$, impact of the final time for $\sigma = 0.01$. Top left: T=0.5. Top right: T=1. Bottom: T=2.

at the end of section 2. To obtain some artificial data (g_0, g_1) on Γ_0 we solve the forward problem

$$\left\{\begin{array}{ccll} \partial_t u - \partial_x^2 u & = & 0 & \text{in } Q_a \\ -\partial_x u & = & g_1 & \text{on } \Gamma_0 \\ u & = & 0 & \text{on } \Gamma_a \\ u & = & 0 & \text{on } S_{0,a} \end{array}\right.$$

with the help of an implicit finite difference scheme, for $a = 0.8$ and $g_1(t) = \sin(\pi t/T)$ for $t \in (0, T)$. All the parameters used in the discretized version of the quasi-reversibility method (13) are the same as previously, as well as the noise which contaminates the data g_0. Concerning the resolution of the equation $-\phi_n' = f$ to obtain the sequence of reals a_n from a_0 (see the third step of the algorithm), since f is chosen as a sufficiently large positive constant, it is trivially done by hand. The figure 8 emphasizes the importance of the choice of f. It represents the sequence (a_n) for $a_0 = 0.2$, $T = 1$ and $\sigma = 0.01$. Starting from a low value of f, which does not provide convergence, we obtain such convergence above a certain threshold according to condition (16). When f is too large, the sequence reaches a stationary value with slow convergence, and such value may be below the true value a because the spatial discretization Δx is too large with respect to f. The impact of the amplitude of noise σ and of the final time T on the behaviour of the sequence (a_n) are analyzed on figure 9 and 10, respectively. The value of f is adjusted to each case following the previous remark. We observe that when the amplitude of noise increases from $\sigma = 0$ to $\sigma = 0.02$ the exact value is obtained with a lower speed

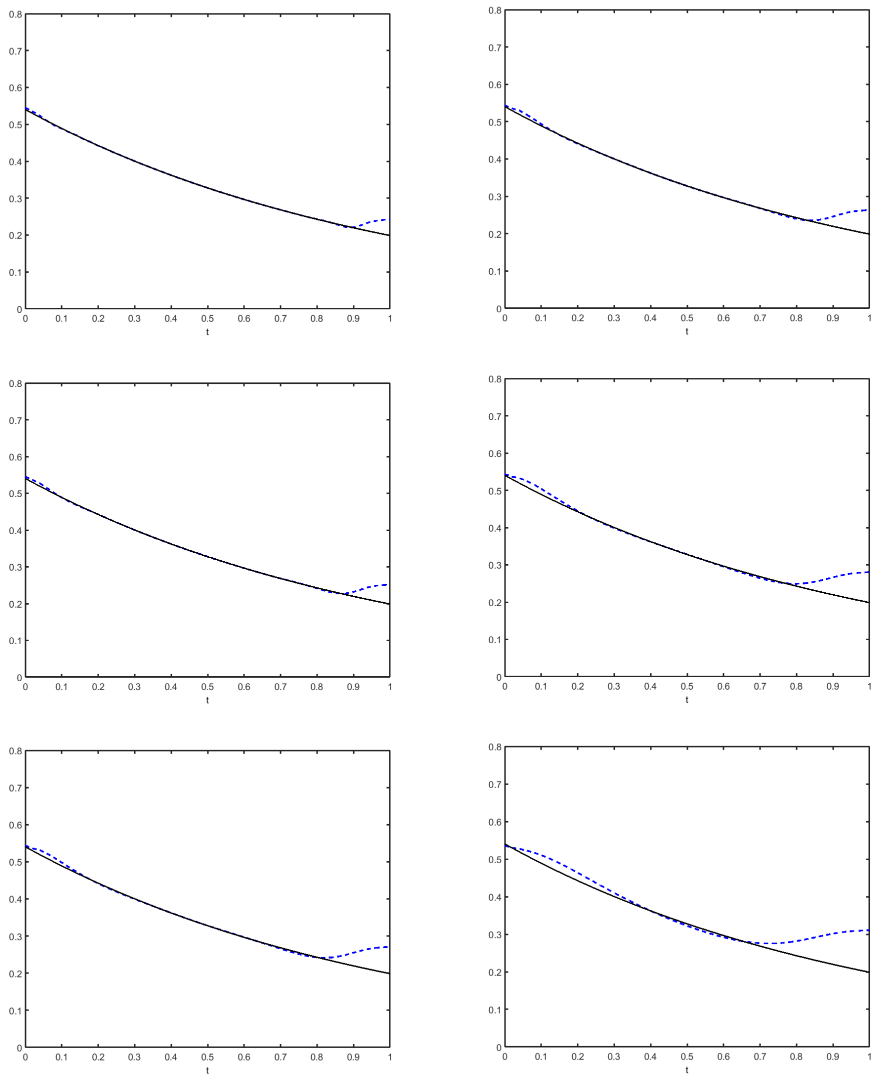

FIGURE 6. Quasi-reversibility solution $u_{\varepsilon,\delta}$ and exact solution u for $x = 1$, impact of the regularization parameters for $\sigma = 0$. Top left: $\varepsilon = 10^{-5}$ and $\delta = \sqrt{\varepsilon}$. Top right: $\varepsilon = 10^{-5}$ and $\delta = 1$. Middle left: $\varepsilon = 10^{-4}$ and $\delta = \sqrt{\varepsilon}$. Middle right: $\varepsilon = 10^{-4}$ and $\delta = 1$. Bottom left: $\varepsilon = 10^{-3}$ and $\delta = \sqrt{\varepsilon}$. Bottom right: $\varepsilon = 10^{-3}$ and $\delta = 1$.

of convergence, and when $\sigma = 0.05$ a wrong value of 0.77 is obtained instead of 0.8. Similarly, for $\sigma = 0.01$, we observe that when the final time T increases from $T = 0.5$ to $T = 2$, the exact value a is obtained with a higher speed of convergence.

5.3. **The wave equation with lateral Cauchy data and inverse obstacle problem.** We complete this numerical section with some numerical experiments

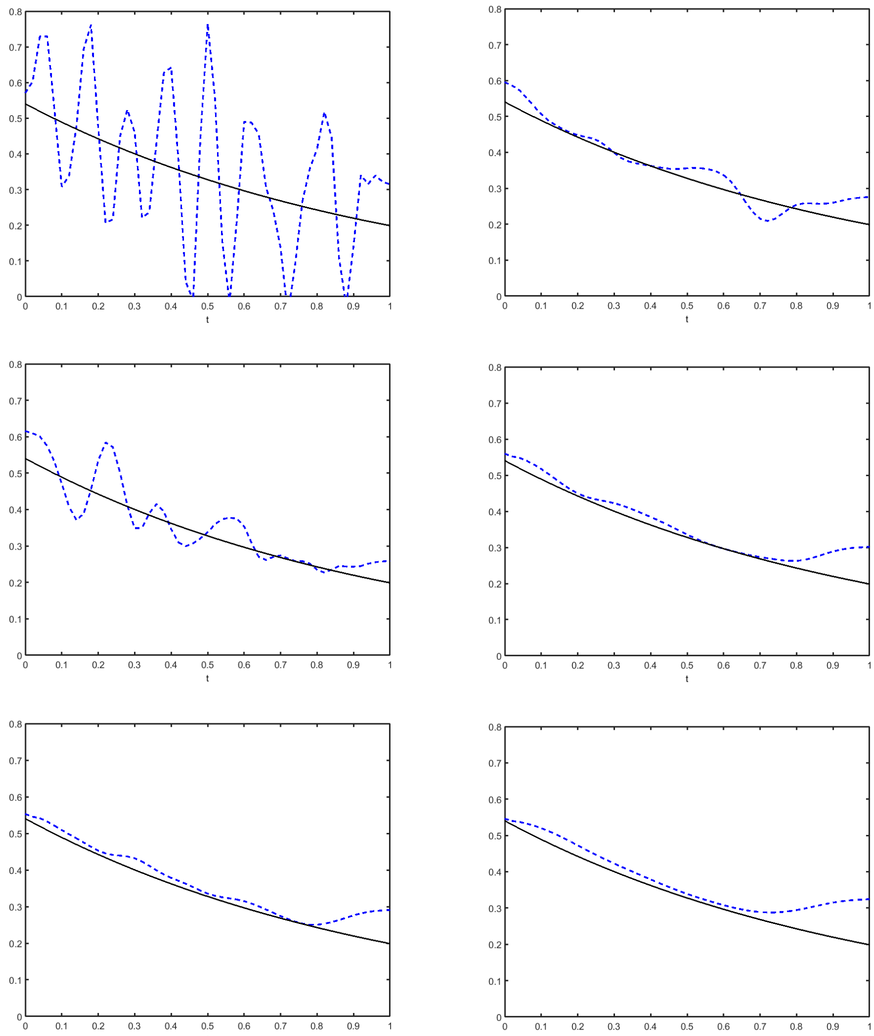

FIGURE 7. Quasi-reversibility solution $u_{\varepsilon,\delta}$ and exact solution u for $x = 1$, impact of the regularization parameters for $\sigma = 0.02$. Top left: $\varepsilon = 10^{-5}$ and $\delta = \sqrt{\varepsilon}$. Top right: $\varepsilon = 10^{-5}$ and $\delta = 1$. Middle left: $\varepsilon = 10^{-4}$ and $\delta = \sqrt{\varepsilon}$. Middle right: $\varepsilon = 10^{-4}$ and $\delta = 1$. Bottom left: $\varepsilon = 10^{-3}$ and $\delta = \sqrt{\varepsilon}$. Bottom right: $\varepsilon = 10^{-3}$ and $\delta = 1$.

for the wave equation with lateral Cauchy data. First, we consider the solution $u(x,t) = x^2 + t^2$ to the wave equation for $T = 3$, which enables us to obtain some lateral Cauchy data (g_0, g_1) on Γ_0, and then to address problem (19) with such data. Like for the heat equation, we impose a pointwise random noise to the Dirichlet data g_0 only (g_1 is noise free), the relative error in L^2 norm for such noisy data being again denoted by σ. In order to approximate the solution to problem (11) in the subdomain R_T, we use the discretized version of the quasi-reversibility method (22)

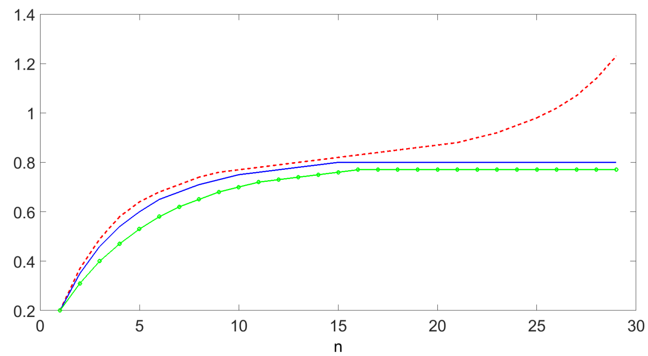

FIGURE 8. Impact of the constant f for $\sigma = 0.01$. Dashed line: f=2. Continuous line: f=2.3. Line with circles: f=3.

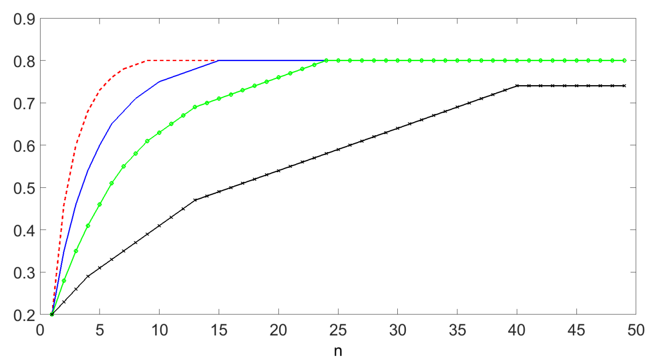

FIGURE 9. Impact of the amplitude of noise. Dashed line: $\sigma = 0$. Continuous line: $\sigma = 0.01$. Line with circles: $\sigma = 0.02$. Line with crosses: $\sigma = 0.05$.

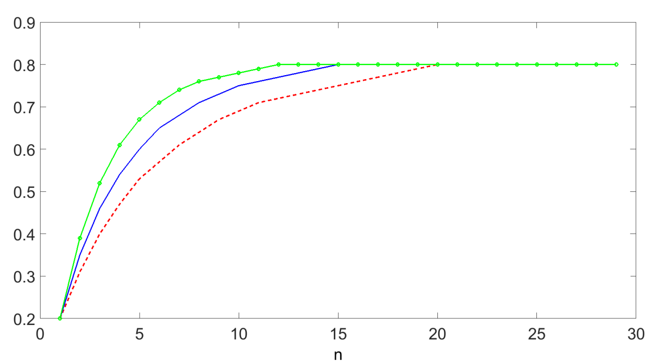

FIGURE 10. Impact of the final time for $\sigma = 0.01$. Dashed line: $T = 0.5$. Continuous line: $T = 1$. Line with circles: $T = 2$.

for sets $W_g, W_0, \tilde{W}_0$ instead of $V_g, V_0, \tilde{V}_0$. The time and space discretizations are $\Delta t = \Delta x = T/50$, the regularization parameters are $\varepsilon = 10^{-4}$ and $\delta = \sqrt{\varepsilon}$. We compare the solution obtained by such method with that obtained directly by using the d'Alembert's formula (21). Such comparison is presented in figure 11, in the case of exact data ($\sigma = 0$) and in the case of noisy data ($\sigma = 0.05$). We remark that the discrepancy between the exact solution and the approximated solution in R_T, for each method, is of the same order as the amplitude of noise, which reflects the fact that the problem (19) is well-posed in R_T. Our aim is now to solve the inverse obstacle problem (27). To obtain some artificial data (g_0, g_1) on Γ_0 we solve the forward problem

$$\begin{cases} \partial_t^2 u - \partial_x^2 u &= 0 & \text{in } Q_a \\ -\partial_x u &= g_1 & \text{on } \Gamma_0 \\ u &= 0 & \text{on } \Gamma_a \\ u &= 0 & \text{on } S_{0,a} \\ \partial_t u &= 0 & \text{on } S_{0,a} \end{cases}$$

with the help of an implicit finite difference scheme, for $a = 0.8$ and $g_1(t) = \sin(\pi t/T)$ for $t \in (0, T)$. We compare on figure 12 the result obtained with the help of the algorithm presented at the end of section 3 to the result obtained with the direct method presented in remark 7. All the parameters used in the discretized version of the quasi-reversibility method (22) are the same as previously, as well as the noise which contaminates the data g_0. For various amplitudes of noise, that is $\sigma = 0$, $\sigma = 0.05$ and $\sigma = 0.1$, the function F_A defined by (28) involved in the direct method is represented in the left column of figure 12, while the sequence (a_n) involved in the quasi-reversibility/level set algorithm is plotted in the right column of figure 12. Unsurprisingly, we observe that the identification of the obstacle is better for the wave equation than for the heat equation, due to the fact that in 1D the wave equation with lateral Cauchy data is well-posed in the light cone.

6. **Conclusion and perspectives.** The numerical results that we have obtained in the one-dimensional case show the feasibility of our mixed formulations of quasi-reversibility to solve ill-posed problems for heat and wave equations, in particular the inverse obstacle problems from lateral Cauchy data on some subpart of the boundary. This is the reason why we intend to address the same kind of problems in the 2D case in a close future. The extension of our mixed formulation of quasi-reversibility to the 2D case is straightforward, and concerning the discretization with finite elements, it is again convenient to use a tensor product between some 2D Lagrange triangular finite elements for space variables and some 1D Lagrange finite elements for the time variable, which amounts to use some Lagrange prismatic finite elements in 3D. However, since we then obtain a global 3D problem, computations will be much heavier, the more so as the inverse obstacle problem requires to repeat such computations as many times as the number of iterations in the level set method. Concerning the level set method itself, the method of order one introduced in the present paper is specific to the one-dimensional case and can no longer be used in the 2D case. It has for instance to be replaced by the method of order two (based on a Poisson partial differential equation instead of an ordinary differential equation) introduced in [2]. Another practical issue is to find a systematic way to choose the regularization parameters ε and δ in the quasi-reversibility methods as a function of the amplitude of noise that contaminates the data, by adapting for instance the Morozov's discrepancy principle, like in [27]. Lastly, from a theoretical point of view, an open question is to establish some (probably logarithmic) convergence rates with respect to ε and δ for the discrepancy between the solutions of our

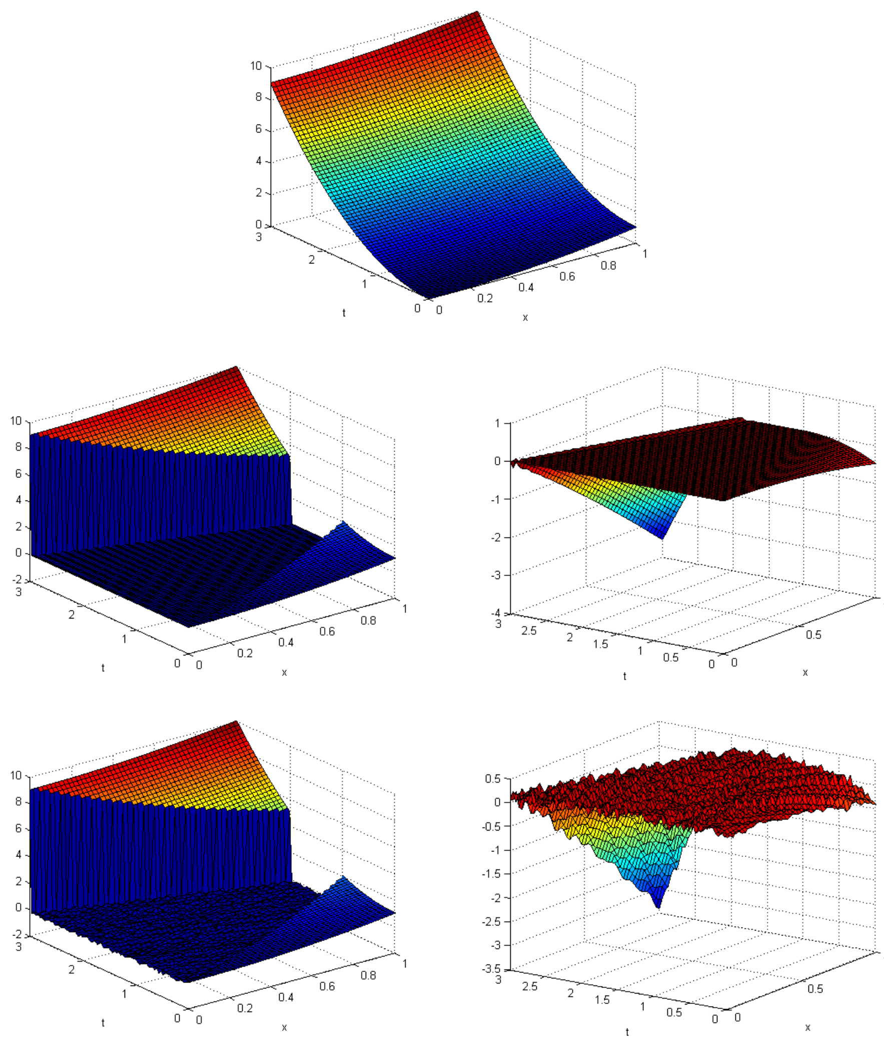

FIGURE 11. Impact of the amplitude of noise. Top: exact solution u. Middle left: discrepancy $u_A - u$ for $\sigma = 0$. Middle right: discrepancy $u_{\varepsilon,\delta} - u$ for $\sigma = 0$. Bottom left: discrepancy $u_A - u$ for $\sigma = 0.05$. Bottom right: discrepancy $u_{\varepsilon,\delta} - u$ for $\sigma = 0.05$.

mixed formulations and the corresponding exact solutions (if they exist) in the whole space/time domain for any dimension of space. In particular, in comparison with [28], which tackles such kind of problem for the stationary case, there are two additional challenges: the case of heat or wave equation equation is more difficult than the case of Laplace equation, and the case of mixed formulations is more difficult than the case of classical ones.

7. **Appendix : derivation of the mixed formulations.** The aim of the appendix is to give an idea of how the mixed formulation (4) can be obtained from the original ill-posed problem (1). A similar approach can be used to derive the

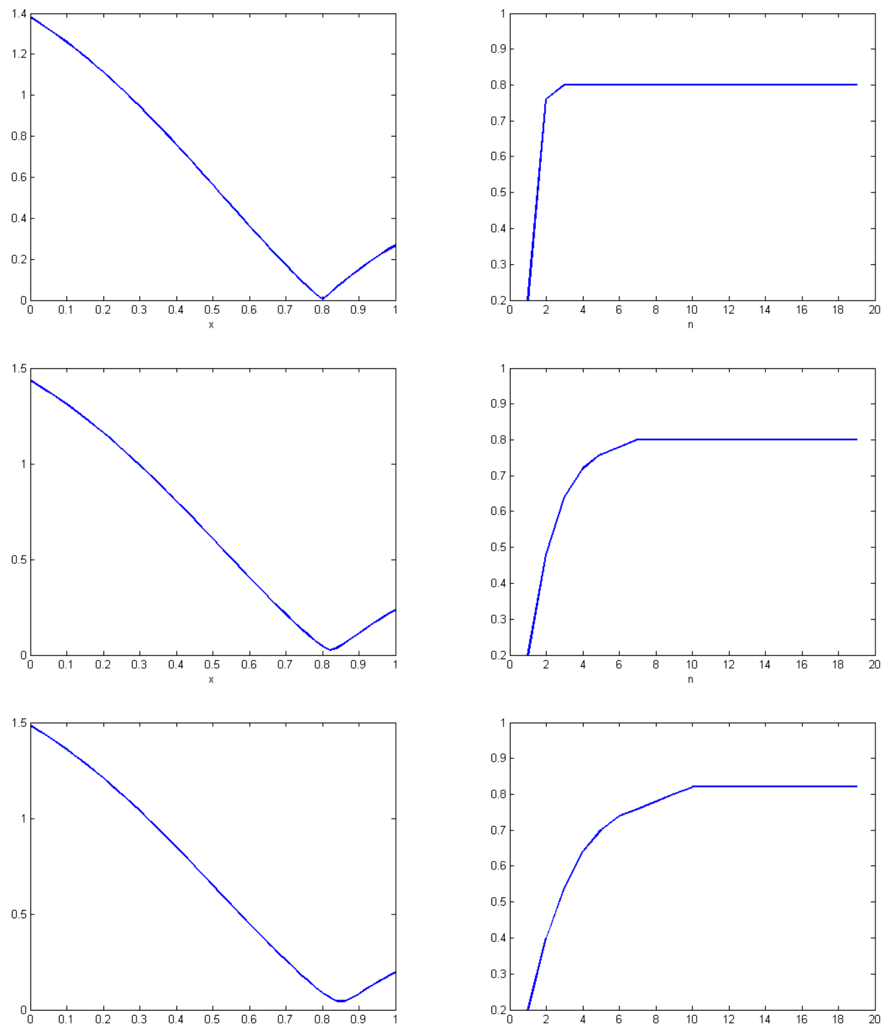

FIGURE 12. Impact of the amplitude of noise. Top left: function F_A for $\sigma = 0$. Top right: sequence (a_n) for $\sigma = 0$. Middle left: function F_A for $\sigma = 0.05$. Middle right: sequence (a_n) for $\sigma = 0.05$. Bottom left: function F_A for $\sigma = 0.1$. Bottom right: sequence (a_n) for $\sigma = 0.1$.

other mixed formulations introduced in the present paper. We first introduce, for $u_T \in H^{1/2}_{00}(S_T)$, the set

$$H_u = \{u \in H^1(Q),\ \partial_t u - \partial_x^2 u \in L^2(Q),\ u(0,\cdot) = u(1,\cdot) = 0,\ u(\cdot,T) = u_T\},$$

which is assumed to be not empty. By virtue of a standard result of optimization, for a small parameter $\varepsilon > 0$ the minimization problem

$$(P_\varepsilon) \qquad \inf_{u \in H_u} \left(||\partial_t u - \partial_x^2 u||^2_{L^2(Q)} + \varepsilon ||u||^2_{H^1(Q)} \right)$$

has a unique solution. Such problem (P_ε) corresponds to a classical quasi-reversibility formulation associated with the ill-posed problem (1) in the spirit of [1]. In particular, its solutions converges when ε tends to 0 to the solution of (1) in $H^1(Q)$, if it exists.

Let us deduce the mixed formulation (4) from problem (P_ε). Clearly,

$$(P_\varepsilon) \Leftrightarrow \inf_{(\phi,u)\in\mathcal{U}_u} \left(||\phi||^2_{L^2(Q)} + \varepsilon||u||^2_{H^1(Q)}\right)$$

where

$$\mathcal{U}_u = \{(\phi,u) \in L^2(Q)\times U_u,\ \phi = -\partial_t u + \partial_x^2 u\}.$$

It is easy to see that

$$(P_\varepsilon) \Leftrightarrow \inf_{(\phi,u)\in\mathcal{U}_u} a_\varepsilon((\phi,u);(\phi,u)),$$

with

$$a_\varepsilon((\phi,u);(\psi,v)) = (\phi,\psi)_{L^2(Q)} + \varepsilon(u,v)_{H^1(Q)},$$

and that

$$\mathcal{U}_u = \{(\phi,u) \in L^2(Q)\times U_u,\ b((\phi,u);\mu) = 0,\ \forall \mu \in \tilde{U}_0\},$$

with

$$b((\phi,u);\mu) = \int_Q (\phi\,\mu - u\,\partial_t\mu + \partial_x u\,\partial_x\mu)\,dxdt.$$

If the bilinear form b on the space $(L^2(Q)\times U_0)\times\tilde{U}_0$ satisfied the inf-sup condition, that is

$$\exists\,\beta>0,\quad \inf_{(\phi,u)\in L^2(Q)\times U_0}\ \sup_{\mu\in\tilde{U}_0} \frac{b((\phi,u);\mu)}{\left(||\phi||_{L^2(Q)} + ||u||_{H^1(Q)}\right)||\mu||_{H^1(Q)}} \geq \beta, \tag{30}$$

then by virtue of a standard result on mixed formulations (see for example [29]) the problem (P_ε) would be equivalent to the following mixed formulation: find $(\phi_\varepsilon,u_\varepsilon,\lambda_\varepsilon) \in L^2(Q)\times U_u\times\tilde{U}_0$ such that for all $(\psi,v,\mu)\in L^2(Q)\times U_0\times\tilde{U}_0$,

$$\left\{\begin{array}{lcccl} a_\varepsilon((\phi_\varepsilon,u_\varepsilon);(\psi,v)) & + & b((\psi,v);\lambda_\varepsilon) & = & 0 \\ b((\phi_\varepsilon,u_\varepsilon);\mu) & & & = & 0. \end{array}\right. \tag{31}$$

However, let us prove by a contradiction argument that the bilinear form b does not satisfy the inf-sup condition. Indeed, if it did, we would infer from (30) by setting $\phi = 0$ that

$$\inf_{u\in U_0}\ \sup_{\mu\in\tilde{U}_0} \frac{b((0,u);\mu)}{||u||_{H^1(Q)}||\mu||_{H^1(Q)}} \geq \beta,$$

which would imply that the simpler bilinear form b_0 defined on $U_0\times\tilde{U}_0$ by

$$b_0(u,\mu) = b((0,u);\mu) = \int_Q (-u\,\partial_t\mu + \partial_x u\,\partial_x\mu)\,dxdt$$

would satisfy the inf-sup condition as well. We remark in addition that b_0 satisfies the property

$$\forall\mu\in\tilde{U}_0,\ (\forall u\in U_0,\ b_0(u,\mu)=0) \Rightarrow (\mu=0), \tag{32}$$

which is equivalent to the uniqueness property for the backward heat equation. If b_0 satisfied the inf-sup condition, such condition together with condition (32) would imply from the Banach-Necas-Babuska theorem (see for example theorem 2.6 in [30]) that for any f in the dual space of $\tilde{U}_0$ the problem: find $u\in U_0$ such that for all $\mu\in\tilde{U}_0$

$$b_0(u,\mu) = f(\mu),$$

would be well-posed. By choosing $f(\mu) = -\int_{S_T} u_T\,\mu\,dx$, such problem exactly coincides with the ill-posed problem (1). In conclusion, neither the bilinear form b_0, nor the bilinear form b, do satisfy the inf-sup condition. Hence it is not possible to transform the problem (P_ε) into the mixed formulation (31).

Following for instance [29], the idea to cope with this issue is to add a penalization term in the system (31) in order to form a well-posed problem for some small parameters $\varepsilon, \delta > 0$, that is: find $(\phi_{\varepsilon,\delta}, u_{\varepsilon,\delta}, \lambda_{\varepsilon,\delta}) \in L^2(Q) \times U_u \times \tilde{U}_0$ such that for all $(\psi, v, \mu) \in L^2(Q) \times U_0 \times \tilde{U}_0$,

$$\left\{ \begin{array}{lcccl} a_\varepsilon((\phi_{\varepsilon,\delta}, u_{\varepsilon,\delta}); (\psi, v)) & + & b((\psi, v); \lambda_{\varepsilon,\delta}) & = & 0 \\ b((\phi_{\varepsilon,\delta}, u_{\varepsilon,\delta}); \mu) & - & c_\delta(\lambda_{\varepsilon,\delta}, \mu) & = & 0, \end{array} \right. \tag{33}$$

with for $\lambda, \mu \in \tilde{U}_0$,

$$c_\delta(\lambda, \mu) = \delta(\lambda, \mu)_{H^1(Q)}.$$

Well-posedness in problem (33) results from the coercivity of both bilinear forms a_ε and c_δ. In view of the definitions of $a_\varepsilon, b, c_\delta$, we obtain that $\phi_{\varepsilon,\delta} = -\lambda_{\varepsilon,\delta}$ and that $(u_{\varepsilon,\delta}, \lambda_{\varepsilon,\delta})$ is the solution to the system: find $(u_{\varepsilon,\delta}, \lambda_{\varepsilon,\delta}) \in U_u \times \tilde{U}_0$ such that for all $(v, \mu) \in U_0 \times \tilde{U}_0$,

$$\left\{ \begin{array}{l} -\displaystyle\int_Q v\,\partial_t \lambda_{\varepsilon,\delta}\,dxdt + \int_Q \partial_x v\,\partial_x \lambda_{\varepsilon,\delta}\,dxdt \\ +\varepsilon \displaystyle\int_Q \partial_t u_{\varepsilon,\delta}\,\partial_t v\,dxdt + \varepsilon \int_Q \partial_x u_{\varepsilon,\delta}\,\partial_x v\,dxdt = 0 \\ \displaystyle\int_Q \partial_t u_{\varepsilon,\delta}\,\mu\,dxdt + \int_Q \partial_x u_{\varepsilon,\delta}\,\partial_x \mu\,dxdt - \int_Q \lambda_{\varepsilon,\delta}\,\mu\,dxdt \\ -\delta \displaystyle\int_Q \partial_t \lambda_{\varepsilon,\delta}\,\partial_t \mu\,dxdt - \delta \int_Q \partial_x \lambda_{\varepsilon,\delta}\,\partial_x \mu\,dxdt = 0. \end{array} \right. \tag{34}$$

A comparison between the formulation (34) and the formulation (4) shows that they coincide up to the L^2 term $\int_Q \lambda_{\varepsilon,\delta}\,\mu\,dxdt$ in the second equation of (34). It is readily seen that such term can be removed from (34) by preserving the well-posed character of the formulation. In addition, as specified in the statement of proposition 1, in contrast with parameter ε the parameter δ has not necessarily to be small to ensure the convergence result. Lastly, the above analysis shows that $\lambda_{\varepsilon,\delta}$ is expected to be close to $\partial_t u - \partial_x^2 u = 0$, where u is the solution to (1).

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E-mail address: laurent.bourgeois@ensta.fr
E-mail address: eliane.becache@ensta-paristech.fr
E-mail address: lucas.franceschini@ensta-paristech.fr
E-mail address: Jeremi.Darde@math.univ-toulouse.fr