Existence of global Chebyshev nets on surfaces of absolute Gaussian curvature less than 2π

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Abstract. We prove the existence of a global smooth Chebyshev net on complete, simply connected surfaces when the total absolute curvature is bounded by 2π. Following Samelson and Dayawansa, we look at Chebyshev nets given by a dual curve, splitting the surface into two connected half-surfaces, and a distribution of angles along it. An analogue to the Hazzidakis' formula is used to control the angles of the net on each half-surface with the integral of the Gaussian curvature of this half-surface and the Cauchy boundary conditions. We can then prove the main result using a theorem about splitting the Gaussian curvature with a geodesic, obtained by Bonk and Lang.

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1 Introduction

In this paper we call *surface* a Riemannian 2-manifold, whose metric will be denoted g, and we consider complete, simply connected surfaces. A Chebyshev net Φ on a surface is a parameterization of the surface satisfying

$$\forall (u^1,u^2) \in \mathbb{R}^2, \quad \left|\partial_1 \Phi(u^1,u^2)\right|_{g(\Phi(u^1,u^2))} = \left|\partial_2 \Phi(u^1,u^2)\right|_{g(\Phi(u^1,u^2))} = 1, \quad (1)$$

which means that length is preserved in two directions, called the *primal coordinates*. Let $\Omega(u^1,u^2)$ be the angle between the coordinate curves at $\Phi(u^1,u^2)$.

A question of interest both for the existence of bi-Lipschitz maps [4] and for applications in textile or architectural shape design [7, 5] is the existence of a Chebyshev net on a given surface. While local existence of a Chebyshev net holds for all surfaces [2], two approaches have been explored to obtain global existence. The first approach relies on a geometric proof on more general Alexandrov surfaces. The first result on global existence of a Chebyshev net on complete and simply connected Alexandrov surfaces has been obtained by Bakelman [1]. Existence is proved on every sector Q_α delimited by two geodesics crossing at an angle α such that $\int_{Q_\alpha} K^+ < \alpha$ and $\int_{Q_\alpha} K^- < \pi - \alpha$, where K^+ and K^- denote, respectively, the positive and negative parts of the Gaussian curvature K. A global Chebyshev net is then obtained on the surface for $\alpha = \pi/2$. Burago, Ivanov and Malev [4] improved this result by showing that a complete and simply connected Alexandrov surface S admits a global Chebyshev net under the relaxed constraints $\int_S K^+ < 2\pi$ and $\int_S K^- < 2\pi$, but this net is not necessarily smooth on a Riemannian surface as the coordinate curves might present kinks at the boundaries between adjacent sectors.

A second analytical approach was developed by Samelson and Dayawansa [9] using the dual coordinates. A geodesic cutting the surface into two connected components $\mathcal{R}^+$ and $\mathcal{R}^-$ is chosen as a dual curve. In each connected component, existence and uniqueness of a smooth solution to Servant's equations (see (5) below) in the dual coordinates is proved for any smooth distribution of angles along the dual curve. Then, choosing a constant angle $\pi/2$ along the geodesic and using an estimate on the angle Ω of the net on $\mathcal{R}^\pm$, it is shown that the solution to Servant's equations is a diffeomorphism under the condition that $\int_{\mathcal{R}^\pm} K^+ < \pi/2$ and $\int_{\mathcal{R}^\pm} K^- < \pi/2$. In the present paper, we add some arguments to the proof to improve this result by relaxing the constraints on the integrals of the Gaussian curvature. Our main result is the following:

Theorem 1.1. *Let S be a complete, simply connected, C^∞ surface. Suppose that $\int_S |K| < 2\pi$. Then S admits a global Chebyshev net.*

Let us note that the Hazzidakis' formula [8] suggests that this result is optimal in the sense that a smooth global Chebyshev net does not exist on a complete, simply connected surface S such that $|\int_S K| > 2\pi$.

The paper is organized as follows. In Section 2, we restate two existing results in a form that is useful for our purpose. The first one is that the estimates obtained on half the surface in [9] can be doubled using a theorem proved by Bonk and Lang [3] stating that the surface can be split by a geodesic into two connected components (called half-surfaces) such that each component contains half of the integral of K^+ and K^- on the surface. The second result concerns the existence (and uniqueness) of a smooth solution to Servant's equations proved in [9] which we restate by specifying the boundary conditions from the choice of a dual curve and an arbitrary smooth angle distribution on that curve. Then in Section 3, we prove the existence of a Chebyshev net on each half-surface. To that end, we first observe that it is sufficient to keep the angles Ω of the net uniformly away from 0 and π for Φ to be a global diffeomorphism. This means that no nonlocal self-intersection of the parameterization can occur when Φ is locally injective everywhere. Proving an analogue to the Hazzidakis' formula in the dual coordinates, we are able to choose an adequate (constant) angle distribution on the dual geodesic curve to control the angles in each half-surface. Finally, we conclude the proof in Section 4 by assembling the intermediate results.

2 Preliminaries

In this section, we restate two existing results on splitting the surface into two components and on the solution to Servant's equations.

2.1 Splitting the surface into two components

We give a restatement of [3, prop. 6.1] applied to the continuous non-negative function K^-.

Proposition 2.1. *Let S be a smooth, complete surface homeomorphic to the plane. Suppose that $\int_S K^+ < 2\pi$ and $\int_S K^- < \infty$. Then there exists a properly embedded, complete geodesic γ, splitting S into two connected components S_1 and S_2 such that*

$$\int_{S_1} K^+ = \int_{S_2} K^+, \qquad \int_{S_1} K^- = \int_{S_2} K^-. \tag{2}$$

2.2 Solution to Servant's equations

Following Samelson and Dayawansa [9], we introduce the change of variables $D : (u^1, u^2) \mapsto (x^1, x^2) = (u^1 - u^2, u^1 + u^2)$ to the *dual coordinates.* We use upper case letters Φ and Ω for the primal parameterization and angle between primal coordinate curves, lower case letters $\varphi = \Phi \circ D^{-1}$ and $\omega = \Omega \circ D^{-1}$ for their dual counterparts. The first fundamental form in a Chebyshev net is then

$$g = (du^1)^2 + 2\cos(\Omega(u^1, u^2))du^1du^2 + (du^2)^2 \tag{3}$$

$$= \sin^2\left(\frac{\omega(x^1, x^2)}{2}\right)(dx^1)^2 + \cos^2\left(\frac{\omega(x^1, x^2)}{2}\right)(dx^2)^2. \tag{4}$$

A derivation of (1) along coordinates (u^1, u^2) leads to Servant's equations, expressed here in the dual coordinates (x^1, x^2),

$$\partial^2_{22}\varphi^i - \partial^2_{11}\varphi^i + \sum_{k,l=1}^{2} \Gamma^i_{k,l}(\varphi)\left(\partial_2\varphi^k\partial_2\varphi^l - \partial_1\varphi^k\partial_1\varphi^l\right) = 0, \quad \text{for } i = 1, 2, \tag{5}$$

where $\Gamma^i_{k,l}$ denote the Christoffel symbols, together with the Cauchy boundary conditions

$$\begin{cases} \varphi(x^1, 0) = \gamma(x^1), \\ \partial_2\varphi(x^1, 0) = \psi(x^1), \end{cases} \qquad \forall x^1 \in \mathbb{R}, \tag{6}$$

for given functions $\gamma, \psi : \mathbb{R} \to S$. The following result on smooth solutions with smooth boundary conditions is proved in [9].

Proposition 2.2. *Assume that γ, ψ are C^∞ functions. Then there exists a unique C^∞ solution φ to* (5)-(6).

The primal parameterization $\Phi = \varphi \circ D$, where φ is the solution to (5)-(6), is then a Chebyshev net if and only if (1) is satisfied by Φ on the boundary. This is expressed, on the dual coordinates, by

$$\begin{cases} \left\langle \partial_1\varphi(x^1,0), \partial_2\varphi(x^1,0)\right\rangle_{g(\varphi(x^1,0))} = 0, \\ \left|\partial_1\varphi(x^1,0)\right|^2_{g(\varphi(x^1,0))} + \left|\partial_2\varphi(x^1,0)\right|^2_{g(\varphi(x^1,0))} = 1, \end{cases} \quad \forall x^1 \in \mathbb{R}. \tag{7}$$

But (7) is equivalent to the existence of an arc length parameterized curve $\tilde{\gamma}$ and an angle distribution $\bar{\omega} : \mathbb{R} \to (0;\pi)$ along $\tilde{\gamma}$ such that

$$\begin{cases} \varphi(x^1,0) = \tilde{\gamma}(\alpha(x^1)), \\ \partial_2\varphi(x^1,0) = \cos\left(\dfrac{\bar{\omega}(x^1)}{2}\right) n(\alpha(x^1)), \end{cases} \quad \forall x^1 \in \mathbb{R}, \tag{8}$$

where $t(x^1)$ is the tangential vector to $\tilde{\gamma}$ at x^1, $n(x^1)$ is the normal vector to this curve at x^1 such that $(t(x^1), n(x^1))$ is positively oriented, and

$$\alpha(x^1) = \int_0^{x^1} \sin\left(\frac{\bar{\omega}(s)}{2}\right) ds.$$

Moreover, we have $\bar{\omega}(x^1) = \omega(x^1,0)$ where ω is the angle between the primal coordinate curves. The Cauchy boundary conditions for Servant's equations (5) satisfying the Chebyshev property (1) are now prescribed by specifying

- an arc length parametrized curve $\tilde{\gamma} : \mathbb{R} \to S$;
- an angle distribution $\bar{\omega} : \mathbb{R} \to (0;\pi)$ uniformly bounded away from 0 and π.

We will say that a map $\varphi : \mathbb{R}^2 \to S$ is a local Chebyshev net on S if it satisfies Servant's equations (5) with Cauchy boundary conditions (8).

3 Existence of a Chebyshev net on each half-surface

In this section we prove the existence of a Chebyshev net on each half-surface. The key ingredient is a new Hazzidakis-type formula on the dual (see Lemma 3.2) which allows us to specify a suitable angle distribution along the geodesic (see Lemma 3.4).

3.1 Global injectivity of the map φ

Lemma 3.1. *Let S be a complete, simply connected, C^∞ surface. A mapping $\varphi : \mathbb{R}^2 \to S$ satisfying (1) is a global Chebyshev net if it satisfies*

$$\exists \varepsilon > 0 \quad s.t. \quad \varepsilon < \omega(x^1,x^2) < \pi - \varepsilon, \quad \forall (x^1,x^2) \in \mathbb{R}^2 \tag{9}$$

where ω is the angle of the map defined in (3).

Proof. Let ds^2 be the metric (3) associated with φ. Since ω satisfies (9), $(\mathbb{R}^2, ds^2)$ is a geodesically complete, simply connected, C^∞ surface. Moreover, $\varphi : (\mathbb{R}^2, ds^2) \to S$ is a local isometry. That φ is a global isometry follows from [6, prop. 2.106]. Hence $\varphi : \mathbb{R}^2 \to S$ is a global Chebyshev net. ☐

3.2 Hazzidakis' formula on the dual

Lemma 3.1 shows that the map φ can be proved to be a diffeomorphism by deriving a uniform estimate on the angle ω of the net. To this purpose, we derive an equivalent to the Hazzidakis' formula in the dual coordinates. We derive the result from the classical relation (see for instance [10]):

$$\partial^2_{12}\Omega(u^1,u^2) = -K\left(\Phi(u^1,u^2)\right)\sin(\Omega(u^1,u^2)). \tag{10}$$

Lemma 3.2. *Let φ be a local Chebyshev net of S. Define the three points $A = \varphi(x_0, y_0)$, $B = \varphi(x_0 + h, y_0)$, and $C = \varphi(x_0 + \frac{h}{2}, y_0 + \frac{h}{2})$. Consider the dual line $\gamma = \varphi(\cdot, y_0)$ (containing the points A and B) and let ABC be the triangle delimited by the two primal lines AC and BC and the dual line γ, depicted on Figure 1. Then the following holds:*

$$\forall (x_0,y_0)\in\mathbb{R}^2, \quad \forall h>0,$$
$$\omega(x_0+\tfrac{h}{2}, y_0+\tfrac{h}{2}) = \frac{\omega(x_0,y_0)+\omega(x_0+h,y_0)}{2} + \int_{AB} k_\gamma - \int_{ABC} K \tag{11}$$

where ω is the angle of the net introduced in (3) and k_γ is the geodesic curvature of γ defined as

$$k_\gamma = -\frac{1}{|\partial_1\varphi|_{g(\varphi)}}\left\langle \partial_1\left(\frac{\partial_1\varphi}{|\partial_1\varphi|_{g(\varphi)}}\right), \frac{\partial_2\varphi}{|\partial_2\varphi|_{g(\varphi)}}\right\rangle_{g(\varphi)}$$

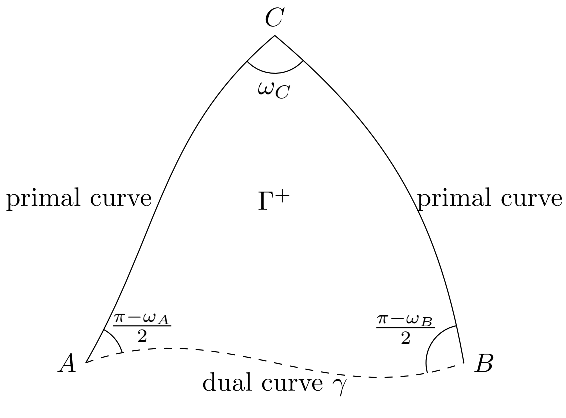

FIGURE 1. Scheme of the Hazzidakis' formula on triangle *ABC*

The proof of Lemma 3.2 hinges on the following lemma.

Lemma 3.3. *Let $\varphi : \mathbb{R}^2 \to S$ be a local Chebyshev net of S. The following holds for all $y_0 \in \mathbb{R}$:*

$$\partial_2\omega(\cdot, y_0) = 2k_\gamma \sin\left(\frac{\omega(\cdot,y_0)}{2}\right). \tag{12}$$

Proof. Define $\partial_{x^1} := \partial_1\varphi$ and $\partial_{x^2} := \partial_2\varphi$. Using the commutation of ∂_{x^1} and ∂_{x^2}, we obtain

$$\begin{aligned}\partial_1 \left\langle \frac{\partial_{x^1}}{|\partial_{x^1}|_{g(\varphi)}}, \partial_{x^2} \right\rangle_{g(\varphi)} &= \left\langle \partial_1 \left(\frac{\partial_{x^1}}{|\partial_{x^1}|_{g(\varphi)}} \right), \partial_{x^2} \right\rangle_{g(\varphi)} + \frac{1}{2\,|\partial_{x^1}|_{g(\varphi)}} \partial_2 \left(|\partial_{x^1}|^2_{g(\varphi)} \right) \\ &= -\,|\partial_{x^2}|_{g(\varphi)}\,|\partial_{x^1}|_{g(\varphi)}\,k_\gamma + \frac{1}{2\,|\partial_{x^1}|_{g(\varphi)}} \partial_2 \left(\sin^2\left(\frac{\omega}{2}\right)\right).\end{aligned}$$

Moreover, $\langle \partial_{x^1}, \partial_{x^2}\rangle_{g(\varphi)} = 0$, $|\partial_{x^1}|_{g(\varphi)} = \sin(\frac{\omega}{2})$ and $|\partial_{x^2}|_{g(\varphi)} = \cos(\frac{\omega}{2})$ due to (4), so that (12) holds. □

Proof. (Hazzidakis' formula): Let $(u_0, v_0) = D^{-1}(x_0, y_0)$. Let Γ^+ be the triangle delimited by the dual curve joining A to B and the two primal curves joining B to C and C to A (see Figure 1):

$$\begin{aligned}\Gamma^+ &= \varphi\left(\left\{(x,y) \in \mathbb{R}^2 |\, x_0 \le x \le x_0 + h,\ y_0 \le y \le y_0 + \tfrac{h}{2} - \left|x - x_0 - \tfrac{h}{2}\right|\right\}\right) \\ &= \Phi\left(\left\{(u,v) \in \mathbb{R}^2 |\, u \le u_0 + \tfrac{h}{2},\ v \le v_0,\ u + v \ge u_0 + v_0\right\}\right).\end{aligned}$$

Integrating (10) on Γ^+ leads to

$$\begin{aligned}-\int_{\Gamma^+} K &= \int_{u_0}^{u_0+\frac{h}{2}} \int_{u_0+v_0-u}^{v_0} \partial^2_{12}\Omega(u,v)\mathrm{d}v\mathrm{d}u \\ &= \int_{u_0}^{u_0+\frac{h}{2}} \left(\partial_1\Omega(u, v_0) - \partial_1\Omega(u, u_0 + v_0 - u)\right)\mathrm{d}u \\ &= \Omega(u_0 + \tfrac{h}{2}, v_0) - \Omega(u_0, v_0) \\ &\qquad - \frac{1}{2}\int_{u_0}^{u_0+\frac{h}{2}} (\partial_1 - \partial_2)\Omega(u, u_0 + v_0 - u)\mathrm{d}u \\ &\qquad - \frac{1}{2}\int_{u_0}^{u_0+\frac{h}{2}} (\partial_1 + \partial_2)\Omega(u, u_0 + v_0 - u)\mathrm{d}u \\ &= \Omega(u_0 + \tfrac{h}{2}, v_0) - \Omega(u_0, v_0) - \frac{1}{2}\left[\Omega(u_0 + \tfrac{h}{2}, v_0 - \tfrac{h}{2}) - \Omega(u_0, v_0)\right] \\ &\qquad - \frac{1}{2}\int_{x_0+y_0}^{x_0+y_0+h} \partial_2\omega(-y_0 + x, y_0)\mathrm{d}x \\ &= \Omega(u_0 + \tfrac{h}{2}, v_0) - \frac{1}{2}\Omega(u_0, v_0) - \frac{1}{2}\Omega(u_0 + \tfrac{h}{2}, v_0 - \tfrac{h}{2}) - \int_{AB} \frac{\partial_2\omega}{2\sin(\frac{\omega}{2})} \\ &= \omega(x_0 + \tfrac{h}{2}, y_0 + \tfrac{h}{2}) - \frac{1}{2}\omega(x_0, y_0) - \frac{1}{2}\omega(x_0 + h, y_0) - \int_{AB} k_\gamma.\end{aligned}$$

This completes the proof. □

3.3 Angle distribution along the dual curve

Lemma 3.4. *Let γ be a geodesic of S which splits this surface into two connected components S_1 and S_2. If*

$$\max_{i=1,2} \int_{S_i} K^+ + \max_{i=1,2} \int_{S_i} K^- < \pi, \tag{13}$$

then there exists a distribution $\bar{\omega} : \mathbb{R} \to (0;\pi)$ such that the solution of Servant's equations (5) *given by the Cauchy boundary conditions γ and $\bar{\omega}$ in* (8) *is a C^∞-diffeomorphism.*

Proof. Lemma 3.2 applied with $x_0 = x^1 - x^2$, $y_0 = 0$ and $h = 2x^2$ leads to the following estimate:

$$\forall (x^1, x^2) \in \mathbb{R}^2, \quad \inf_{s\in\mathbb{R}} \bar{\omega}(s) - \max_{i=1,2} \int_{S_i} K^+ \leq \omega(x^1, x^2) \leq \max_{i=1,2} \int_{S_i} K^- + \sup_{s\in\mathbb{R}} \bar{\omega}(s).$$

Hence, choosing for instance the constant distribution of angles along γ such that

$$\bar{\omega}(x) = \frac{\pi}{2} + \frac{1}{2} \max_{i=1,2} \int_{S_i} K^+ - \frac{1}{2} \max_{i=1,2} \int_{S_i} K^-$$

gives a global Chebyshev net owing to Lemma 3.1. □

4 Proof of the main theorem

Let γ be a geodesic splitting S into two connected components S_1 and S_2, as resulting from Proposition 2.1. Then the hypotheses of Lemma 3.4 are satisfied upon choosing the geodesic γ and the constant distribution of angles $\bar{\omega} = \frac{\pi}{2} + \frac{1}{4} \int_S K$ as Cauchy boundary conditions in (8). This proves the main theorem.

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