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Abstract

The present article illustrates a credibility approach to mortality. Interest from life insurers to assess their portfolios' mortality risk has considerably increased. The new regulation and norms, Solvency II, shed light on the need of life tables that best reflect the experience of insured portfolios in order to quantify reliably the underlying mortality risk. In this context and following the work of Bühlmann and Gisler (2005) and Hardy and Panjer (1998), we propose a credibility approach which consists on reviewing, as new observations arrive, the assumption on the mortality curve. Unlike the methodology considered in Hardy and Panjer (1998) that consists on updating the aggregate deaths we have chosen to add an age structure on these deaths. Formally, we use a Makeham graduation model. Such an adjustment allows to add a structure in the mortality pattern which is useful when portfolios are of limited size so as to ensure a good representation over the entire age bands considered. We investigate the divergences in the mortality forecasts generated by the classical credibility approaches of mortality including Hardy and Panjer (1998) and the Poisson-Gamma model on portfolios originating from various French insurance companies.

Keywords: Credibility, Makeham law, Mortality, Life insurance, Graduation, Extrapolation

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1 Introduction

Recently, interest from life insurers to assess their experienced mortality risk has considerably increased. The new regulation and norms, Solvency II, shed light on the need of life tables that best reflect the experience of insured portfolios so as to reliably quantify the underlying mortality risk. Insurers, in France for example, are used to rely on regulatory life tables for pricing purposes, which are sometimes too conservative. In general, ill-suited mortality assumptions and life tables, especially being too conservative, lead to two effects:

(i) Increase of *Best Estimate* technical provisions (and thus decrease *Basic Own-Funds*);

(ii) Increase of the base figure used for calculating the capital charge for mortality risk (15 % increase scenario of the conditional death rates under the Solvency II framework).

Therefore, the question of which mortality table can be considered for pricing and reserving purposes is of substantial importance. A first attempt, to handle this issue, is to use the available data at the portfolio level and build a specific mortality table. However, practitioners may face technical difficulties related to the size of the portfolio and the heterogeneity of the guarantees (for the same underlying risk). For instance, an insurer may detain a fairly big portfolio but with insureds holding different policies: pure endowment contracts, unit-linked contracts with minimum death guarantees, loan insurance and so on. In such a case, it is difficult to build mortality tables only based on the sole experience of each policy. Especially since it may induce significant impacts on the technical reserves if the table has to be updated more frequently over time. In this paper, we consider an insurer with exposures to different policies and aiming at establishing an experience-based mortality table for each policy.

In the academic literature, various methodologies have been proposed to built and graduate mortality rates at the insured portfolio level. They are usually divided into non-parametric and parametric techniques. The latter are very useful in practice especially when there is sufficient data, see Forfar *et al.* (1988) for a comprehensive introduction to the use of parametric models for graduation. These approaches fit the parametric structure to the mortality of interest over a given period. The graduated mortality is then used to project future liabilities related to the underlying population. By doing so, the evolution of the flow of data related to latest available information is not taken into account. This should be, for example, used to update the graduated mortality. However, if one decides to re-calibrate the parametric model each year, the forecasts are likely to be unstable. This is mainly due to the instability of parameters estimation due to the lack of sufficient data.

In this context and following the work of Bühlmann and Gisler (2005) and Hardy and Panjer (1998), we propose a credibility approach which consists on reviewing, as new observations arrive, the parameters of a Makeham fit. The framework considered in Hardy and Panjer (1998) focuses on the update of the aggregate deaths recorded over the whole portfolio. However, such an approach may be not effective in situations where the insurer liability is highly dependent on the age structure of the underlying portfolio. Thus, using an adjustment makes possible to add a structure in the mortality pattern which is useful when portfolios are of limited size so as to ensure a good representation over the entire age-band considered. Note that, adding an age structure is also beneficial given the heterogeneity observed in the cost of the guarantees according to the age. To recap, as we can see in Section 5, the proposed adjustment approach is intended to enhance the predictive ability of the credibility-based revisions at the age-level and not on the aggregate portfolio level.

The remainder of the paper is organized as follows. Section 2 has still an introductory purpose. It specifies the notation, assumptions and the Makeham settings used in the following. Section 3 introduces the Makeham credibility approach and assess the estimation of the credibility model. Section 4 describes the classical credibility approaches of mortality including Hardy and Panjer (1998) and the Poisson-Gamma model. Section 5 presents an application with experience data originating from French insurance companies. Finally, some remarks in Section 6 conclude the paper.

2 A Credibility Model for Makeham's Law

2.1 Data Structure and Notation. We suppose that we have at our disposal age-specific mortality statistics originating from n portfolios. For each portfolio $i \in \{1, \cdots, n\}$, we observe the deaths of exposures over a period T_i. Denote the number of individuals at attained age x during calendar year $t = 1, \cdots, T_i$ by $L^i_{x,t}$ and $D^i_{x,t}$ represents the number of deaths recorded. We also introduce the following notation,

$$D^i_{x,\bullet} = \sum_{t=1}^{T_i} D^i_{x,t}, \quad L^i_{x,\bullet} = \sum_{t=1}^{T_i} L^i_{x,t}, \quad \text{and} \quad D^i_{\bullet,t} = \sum_{x=\underline{x}}^{\overline{x}} D^i_{x,t}, \quad L^i_{\bullet,t} = \sum_{x=\underline{x}}^{\overline{x}} L^i_{x,t},$$

which refer respectively to the aggregate deaths and individuals over the age-band $\{\underline{x}, \underline{x}+1, \ldots, \overline{x}\}$ and calendar years 1 to T_i for each portfolio i. Henceforth, the "$\bullet$" indexation refers to the summation over the index of interest. For example, $D^\bullet_{x,\bullet}$ refers to the aggregate deaths over the period $[1, T_i]$ and over the n portfolios, i.e. $D^\bullet_{x,\bullet} = \sum_{i=1}^n \sum_{t=1}^{T_i} D^i_{x,t}$.

2.2 Mortality Law. We consider the (first) Makeham law of mortality, which generalizes the Gompertz law. Omitting the time dependency, Makeham (1867) assumes that the force of mortality φ^i_x at attained age x during calendar year t has the following form:

$$\varphi_x = A + B \times C^x, \tag{2.1}$$

with A, B and C are some constants. These parameters capture the essential properties of the progression of mortality. For instance, the dominant effect, i.e. the aging effect, over the age is captured by the multiplicative component factor $B \times C^x$. The non-age dependent parameter A can be interpreted as the non-senescent mortality, for instance, due to accidents. Both of these capture the exponential increase in the forces of mortality observed for adult mortality, see Bongaarts (2005) for more details.

Various modification of the above law have been proposed, especially, to encounter for the time dependency of the mortality, see for e.g. Keyfitz (1981) among others. Indeed, as soon as age-specific mortality patterns over time are concerned, the time series records of the latter show a discernible downward trend with minor fluctuations around. In order to correct this deficiency in the model (2.1), we suppose that the time trend is incorporated in the parameter B^i denoted henceforth B^i_t. Therefore, the force of mortality $\varphi^i_{x,t}$ for portfolio i writes now as the following expression

$$\varphi^i_{x,t} = A^i + B^i_t (C^i)^x. \tag{2.2}$$

This model should capture the behavior of the probability of death over years through the time-dependent parameter B_t^i. This also make possible the prediction of future mortality, i.e. for $t = T+1, T+2, \ldots$, through the study of the time series B_t^i for $t = 1, \ldots, T$. When it comes to small portfolios, the model in (2.2) is not easy to implement. Indeed, as discussed later in this paper, the temporal behavior the factor B^i cannot be accurately extracted. Nevertheless, one can use the estimated values of B_t^i even over the few periods to predict the future behavior of B^i.

Note that in order to estimate the parameters of the model (2.2), given the growth of the forces of mortality with the age, we must have a constant C greater than 1 and a positive B. Then,

$$\begin{aligned} q_{x,t}^i &= 1 - \exp\left(-\int_x^{x+1} \varphi_{y,t}\, dy\right) = 1 - \exp\left(-\int_x^{x+1} A^i + B_t^i \times (C^i)^y\, dy\right) \\ &= 1 - \exp(-A^i) \exp\left(-\frac{B_t^i}{\ln C^i}(C^i)^x (C^i - 1)\right), \end{aligned} \tag{2.3}$$

where $q_{x,t}^i$ denotes the one-year probability of death at attained age x during calendar year t for portfolio i. Consistent estimates $\widehat{A^i}, \widehat{B_t^i}$ and $\widehat{C^i}$ of the parameters are obtained by minimizing the following weighted distance:

$$\sum_{x=\underline{x}}^{\overline{x}} \frac{L_{x,t}^i}{q_{x,t}^i(1 - q_{x,t}^i)} (q_{x,t}^i - \widehat{q}_{x,t}^i)^2,$$

with $\widehat{q}_{x,t}^i = D_{x,t}^i / L_{x,t}^i$ is the crude mortality rates.

2.3 Differential Mortality Law. It is common in modeling specific portfolio's mortality to consider an adjustment with regard to a baseline mortality. Generally, this implicitly assumes that both populations share common features up to a random effect. Relational models stipulate a deterministic relationship in the form $q_x^i = f(q_x^{\mathrm{b}})$ links the two mortalities, where q_x^{b} refers to the *baseline mortality*. The function $f : [0,1] \to [0,1]$ is a known and deterministic function, see Delwarde *et al.* (2004) for more details. A simple example would suggest that the death rate is common for all companies. Specifically, $q_x^i = q_x^{\mathrm{b}}$ for any $i \in \{1, \cdots, n\}$. However, such an assumption does not appreciate the specific characteristic of each portfolio's portfolio. In other words, portfolios having lives in poorer or better conditions than the baseline mortality do not behave in a similar fashion than the baseline mortality. This implies that one should encounter for differential mortality that arises due to portfolio specific features, e.g. particular socioeconomic groups involved, average income level, etc. However, when it comes to the study of the mortality at a single portfolio level, some specific issues arise:

(i) Size of populations: Insured population are generally of small size, so none or very few deaths are observable at some ages. This may not only bias the estimation of the force of mortality but also lead to a mis-estimation of the parameters in (2.3). This may cause high fluctuations for $q_{x,t}^i$ and consequently for A^i, B_t^i and C^i.

(ii) Length of historical data: Available age-specific mortality statistics lacks of deepness. This makes difficult to isolate a possible time trend as it may be captured by B_t^i. The latter may be fluctuating due to the small size of the dataset as noted before.

(iii) Scale of available data: Insured portfolios show a typical behavior compared to a national mortality. The mortality of insured population is significantly lower than the national population from which it is drawn. This could make the use of a baseline mortality based on national demographic statistics as a substitute useless as it may not have the same characteristics of the initial population.

All these characteristics make forecasting of future mortality evolution problematic. In order to overcome these issues when implementing and fitting the model (2.3) for each portfolio $i \in \{1, \ldots, n\}$ we will make the following assumptions:

(i) The baseline mortality $q^{\mathrm{b}}_{x,t}$ is described by the Makeham model in (2.3).

(ii) The age effect is similar on the n portfolios and companies specific model is assumed to share the same parameters A^i and C^i. Those are set equal to the baseline ones, i.e. $A^i = A^{\mathrm{b}}$ and $C^i = C^{\mathrm{b}}$ for any $i \in \{1, \cdots, n\}$.

(iii) The time-dependent parameter B^i_t is fitted at each period. This is given by the following formula:

$$B^i_t = \frac{D^i_{\bullet,t} - A^{\mathrm{b}} L^i_{\bullet,t}}{\sum_{x=\underline{x}}^{\overline{x}} (C^{\mathrm{b}})^x L^i_{x,t}}.$$

The assumptions (i) and (ii) allows to overcome potential estimation bias of the parameters A^i and C^i. Indeed, basing the estimation on a large population allows to avoid erroneous inferences of the parameters. Also, if the portfolio i is a subset of the baseline population composed of the aggregated portfolios, we may think that both the non-senescent factor A^i and the slope C^i are equivalent and thus normalized with the baseline mortality. Empirical evidence of a normalized slope can be found in Thatcher (1999). It is shown that relative rate of increase is the same at all ages and is a shared feature with over subset populations, see also the empirical study of Zhu and Li (2013). As for the non-senescent parameter, the assumption is relevant to the extent that this effect is generally of small impact and sometimes ignored (especially for industrialized countries), see Gavrilova and Gavrilov (2011). The unique parameter that captures the specific mortality at the portfolio level is B^i_t, which would *a priori* not be the same over companies due to the heterogeneity of the underlying populations as explained above. This can be regarded as an unobservable random factor and similar to the so-called *frailty factor*. Such a methodology is widely understood in the literature as well as in life insurance practice. Assumption (iii) gives an estimate of the time-dependent parameter. By time-dependent we only track the fluctuation of B^i_t over time that might be caused by the small size and length of data. Thus our main aim is to *sequentially* adjust the estimation of B^i_t over time in view of the flow of information at our disposal.

3 Credibility of the Makeham Mortality

3.1 Next Period Prediction. In the following, we are interested in the behavior, over time, of the random variable

$$X^i_t = \frac{B^i_t}{B^{\mathrm{b}}_t}, \tag{3.1}$$

and specifically on its next period prediction X^i_{T+1} merging information from other portfolios $j = 1, \cdots, n$ with $j \neq i$. Specifically, suppose that we are at the end of the year T, i.e. at time $T+1$, and we want predict the next period deaths $D^i_{x,T+1}$ in the portfolio (equivalently the probability of death $q^i_{x,T+1}$). Naturally, we can assume that this ratio is constant over time and thus invoke a widespread practice that applies a single factor of reduction/increase to the baseline mortality. On the other hand, one could propose a dynamic model on the same line as

Plat (2009). The latter proposes a modeling framework of the relative ratio of an experienced mortality (death rates) to a baseline and consider that this can be diffused using either an autoregressive model or a decomposition similar to the one introduced by Lee and Carter (1992). Other methodologies have been also proposed, see Ngai and Sherris (2011) and Hyndman *et al.* (2013) among others. However, random effects that constitute the decomposition of the experienced mortality have to be projected using their temporal and statistical features. In our case, we are not only interested in handling populations of small size but also with potentially limited historic period of observation. Therefore, such a methodology would typically not be useful in our setting as it requires a long experience.

Note that the behavior of X_t^i is broadly related to the so-called basis risk. This refers to the fact that the evolution of the policyholders mortality is usually different from that of the national population (baseline), due to some selection effects. This selection effect has different impacts on different insurance companies portfolios, as mortality improvements and accelerations are very heterogeneous in the insurance industry, see Barrieu *et al.* (2012).

3.2 Heterogeneity and Makeham's Law Adjustment. As noted above, we are interested in the accurate adjustment of the portfolio-dependent parameter in (2.3), i.e. B_t^i. Given the specific parameterization of the problem, one may think of the n portfolios as a subset of the reference population and thus each population is characterized by a risk profile Θ_i. In addition, it is beneficial to borrowing information across the different portfolios to enhance the knowledge and estimation of the mortality at the single portfolio level. Furthermore, these subpopulations may, for example, share a common mortality feature, while showing some specificity in their mortality profile. This can be seen as a random variable effect or heterogeneity characterizing the specific profile of each portfolio, for $i = 1, \cdots, n$. Therefore, we implicitly assume that each portfolio is endowed by a risk profile θ_i which is a realization of a random variable Θ.

In view of the various stylized facts presented above and in order to predict $D_{x,T+1}^i$, for each age x, we focus on the projection of X_{T+1}^i. Therefore, we suppose that this relative trend level of portfolio i with respect to the baseline mortality (trend) is characterized by the risk profile θ_i which is a realization of Θ_i. In other words, X_t^i is viewed as a function of a random element Θ_i representing the unobserved characteristics of the portfolio mortality trend (with respect to the baseline). By doing so, we implicitly take into account the heterogeneity of the portfolio i's portfolio mortality profile. It thus remains to predict X_{T+1}^i taking into account this random heterogeneity. By doing so, we naturally invoke the use of a credibility approach to estimate X_{T+1}^i.

3.3 Credibility Based Adjustment. As noted above, the objective is to estimate the next period projection of the relative ratio for each portfolio i. More precisely, in view the available data up to time T_i, i.e. X_t^i, for $t = 1, \cdots, T_i$, one aims to find the best estimate of $\mathbb{E}[X_{T_i+1}^i | \Theta_i] = \mu(\Theta_i)$, which is unknown. Let $\widehat{\mu}(\Theta_i)$ be this estimation. For this purpose and using the usual credibility setting, we shall make the following hypotheses:

(H1) Conditionally on Θ_i, the random variables X_t^i, for $t \in \{1, \ldots, T_i\}$, are independent with mean and variance given as follows

$$\mathbb{E}\left[X_t^i \,|\, \Theta_i\right] = \mu(\Theta_i) \quad \text{and} \quad \mathbb{V}\mathrm{ar}\left[X_t^i \,|\, \Theta_i\right] = \frac{\sigma^2(\Theta_i)}{\omega_t^i} \;,$$

for some functions $\mu\big(\Theta_i\big)$ and $\sigma^2\big(\Theta_i\big)$ and where

$$\omega_t^i = \frac{\sum_{x=\underline{x}}^{\overline{x}} (C^{\mathrm{b}})^x \, L_{x,t}^i}{\sum_{i=1}^n \sum_{x=\underline{x}}^{\overline{x}} (C^{\mathrm{b}})^x \, L_{x,t}^i},$$

measures the *weight* given to the period t experience from the portfolio i.

(H2) The pairs $\big(\Theta_i, X_t^i\big)$, $\big(\Theta_k, X_l^k\big)$, $k \neq i$ are independent and identically distributed.

The first assumption (H1) implies that for each risk profile i (portfolio), the *true* relative ratio $\mu\big(\Theta_i\big)$ (conditionally on the knowledge of the risk profile Θ_i) does not change over time and its variance given Θ_i, $\mathbb{V}\mathrm{ar}\big[X_t^i \,|\, \Theta_i\big]$ changes in proportion to the relative size of the portfolio ω_t^i. The latter expresses different concerns outlined earlier. Specifically, it links the variability of the estimation of the parameter B_t^i to the size of the underlying population: very small portfolios are subject to larger variability on the estimation of B_t^i and vice versa.

The second assumption (H2) means that the risk profiles are independent. The successive realizations of the relative ratio X_t^i for any portfolio are independent of each other except through the risk parameter Θ_i. Moreover, using the random variable X_t^i instead of B_t^i permits to avoid data adjustment for Intuitively, assumption (H2) implicitly suggests that portfolios are comparable as they random sub-groups of a reference (national) population, but not entirely similar which induces the conditional independence.

In view of these assumptions, the following results are straightforward:

(i) The expected prediction of X_{T+1}^i unconditionally on the risk profile Θ_i is given by $\mathbb{E}\big[X_{T+1}^i\big] = \mathbb{E}\big[\widehat{X}_{T+1}(\Theta_i)\big] = 1$. In other words, in the absence of any information on the heterogeneity level on the parameter B_t^i, the best next-period prediction of the latter is the reference one, i.e. $\mathbb{E}\big[B_t^i\big] = B_t^{\mathrm{b}}$.

(ii) Using the law of total variance, the dependence structure of portfolio i associated risk factor over time, is, for $l, t \in \{1, \ldots, T\}$,

$$\begin{aligned}
\mathbb{C}\mathrm{ov}\big(X_l^i, X_t^i\big) &= \mathbb{C}\mathrm{ov}\big(\mathbb{E}\big[X_l^i|\Theta_i\big], \mathbb{E}\big[X_t^i|\Theta_i\big]\big) + \mathbb{E}\big[\mathbb{C}\mathrm{ov}\big(X_l^i, X_t^i|\Theta_i\big)\big] \\
&= \mathbb{V}\mathrm{ar}\big[\mu\big(\Theta_i\big)\big] + \mathbb{E}\big[\mathbb{C}\mathrm{ov}\big(X_l^i, X_t^i|\Theta_i\big)\big] \\
&= \begin{cases} \tau^2 & \text{if } l \neq t \\ \tau^2 + \dfrac{\sigma^2}{\omega_t^i} & \text{if } l = t, \end{cases}
\end{aligned} \tag{3.2}$$

where $\mathbb{V}\mathrm{ar}\big[\mu\big(\Theta_i\big)\big] = \mathbb{V}\mathrm{ar}\big[\Theta_i\big] := \tau^2$, while $\mathbb{E}\big[\sigma^2\big(\Theta_i\big)\big] = \mathbb{E}\big[\Theta_i\big] := \sigma^2$.

3.4 Credibility Estimator. Following the Bühlmann-Straub credibility approach, the aim is to find the *best estimate* of the actual to expected mortality ratio $\mathbb{E}\big[X_{T_i+1}^i \,|\, \Theta_i\big] = \mu\big(\Theta_i\big)$ which is linear in the observations. For each portfolio, due to the assumption (H2), $\widehat{\mu}(\Theta_i)$ depends only on the observations and the linear credibility estimator is of the form

$$\widehat{\mu}\big(\Theta_i\big) = \widehat{a}_0^i + \sum_{t=1}^{T_i} \widehat{a}_t^i \, X_t^i \ , \tag{3.3}$$

where the coefficients $\widehat{a}_t^i$, for $t = 0, \cdots, T_i$, are those minimizing the mean squared errors criterion

$$(\widehat{a}_t^i)_{t=0,\cdots,T_i} = \underset{(a_t^i)_{(t=0,\cdots,T_i)}}{\text{argmin}} \left\{ \mathbb{E}\Big[\big(\widehat{\mu}(\Theta_i) - a_0^i - \sum_{t=1}^{T_i} a_t^i X_t^i \big)^2 \Big] \right\}.$$

In view of Equation 3.2, taking the derivatives of the above criterion with respect to the $a_{i,t}$'s and equating to zero gives,

$$\widehat{a}_0^i = 1 - \frac{\tau^2\,\omega_\bullet^i}{\sigma^2 + \tau^2\,\omega_\bullet^i} \quad \text{and} \quad \widehat{a}_t^i = \frac{\tau^2\,\omega_t^i}{\sigma^2 + \tau^2\,\omega_\bullet^i}, \quad \text{with} \quad \omega_\bullet^i = \sum_{t=1}^{T_i} \omega_t^i\ . \tag{3.4}$$

Then, substituting Equation 3.4 into (3.3), leads to the following the Bühlmann-Straub credibility estimator of $X_{T_i+1}^i$

$$\widehat{X}_{T_i+1}^i(\Theta_i) = \alpha^i\,X_\bullet^i + (1-\alpha^i), \quad \text{with } \alpha^i = \omega_\bullet^i\tau^2/(\omega_\bullet^i\tau^2 + \sigma^2), \tag{3.5}$$

where $X_\bullet^i = (\sum_{t=1}^{T_i} \omega_t^i X_t^i)/\omega_\bullet^i$. Note that the ratio σ^2/τ^2 represents the credibility coefficient. The parameter α_i is called the *credibility factor* or *credibility weight* for portfolio i and takes values in $[0,1]$. For each portfolio i, note that the larger the volume of historical data, the larger α_i will be, see Equation 3.5.

3.5 Estimators of the Structure Parameters. As the risk parameters, Θ_i, for $i \in \{1,\ldots,n\}$, are assumed to be identically distributed, their moments are identical. Therefore τ^2 and σ^2 are the same for all portfolios and measure the residual heterogeneity of the risk profiles and the pure randomness respectively. These parameters are the key determinants of the credibility estimator, i.e. Equation 3.5. In the following, special attention is addressed to the estimation of these quantities. Recall the definition of the structure parameters,

$$\sigma^2 = \mathbb{E}\big[\sigma^2(\Theta_i)\big] = \omega_t^i\,\mathbb{E}\big[\mathbb{V}\text{ar}\big[X_t^i|\Theta_i\big]\big], \quad \text{and} \quad \tau^2 = \mathbb{V}\text{ar}\big[\mathbb{E}[X_t^i|\Theta_i]\big].$$

Then, it is reasonable to propose the estimators $\widehat{\sigma}^2$ and $\widehat{\widehat{\tau}}^2$ in the same vein as Bühlmann and Gisler (2005) based on the observations X_t^i:

$$\widehat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^n s_i^2, \quad \text{with} \quad s_i^2 = \frac{1}{T_i - 1}\sum_{t=1}^{T_i} \omega_t^i \big(X_t^i - X_\bullet^i\big)^2, \tag{3.6}$$

$$\text{and} \quad \widehat{\widehat{\tau}}^2 = \frac{\omega_\bullet^\bullet}{(\omega_\bullet^\bullet)^2 - \sum_{i=1}^n (\omega_\bullet^i)^2} \left\{ \sum_{i=1}^n \omega_\bullet^i \big(X_\bullet^i - X_\bullet^\bullet\big)^2 - (n-1)\widehat{\sigma}^2 \right\},$$

$$\text{with} \quad X_\bullet^\bullet = \frac{1}{\omega_\bullet^\bullet}\sum_{i=1}^n \omega_\bullet^i X_\bullet^i \quad \text{and} \quad \omega_\bullet^\bullet = \sum_{i=1}^n \omega_\bullet^i.$$

These estimators are unbiased and consistent, see Bühlmann and Gisler (2005) for more details. Note that $\widehat{\widehat{\tau}}^2$ can be negative. This would mean that there would be no difference between the risks. In this case, $\widehat{\tau}^2$ is set to 0. Hence we use as estimator $\widehat{\tau}^2 = \max\big(\widehat{\widehat{\tau}}^2, 0\big)$.

3.6 Empirical Credibility Estimator. The empirical credibility estimator is obtained from the credibility formula (3.3) by replacing the structural parameters σ^2 and τ^2 by their estimators derived in Subsection 3.5. Hence, we have

$$\begin{cases} \widehat{\widehat{X}}^i_{T_i+1} & = \widehat{\alpha}^i X^i_{\bullet} + (1-\widehat{\alpha}^i), \\ \widehat{\alpha}^i & = \dfrac{\widehat{\tau}^2 \omega^i_{\bullet}}{\widehat{\sigma}^2 + \widehat{\tau}^2 \omega^i_{\bullet}}. \end{cases} \tag{3.7}$$

It follows from Equation 3.5, that the mortality time varying coefficient is successively updated as follows

$$\widehat{B}^i_{T_i+1} = \widehat{B}^{\mathrm{b}}_{T+1}\big(1 + \widehat{\alpha}^i\,(X^i_{\bullet} - 1)\big), \tag{3.8}$$

and similarly, the forces of mortality and the probabilities of death are given respectively by

$$\widehat{\varphi}^i_{x,T_i+1} = \Big\{\widehat{\alpha}^i(1 - X^i_{\bullet})\Big\}\widehat{A}^{\mathrm{b}} + \Big\{\widehat{\alpha}^i(X^i_{\bullet} - 1) + 1\Big\}\widehat{\varphi}^{\mathrm{b}}_{x,T+1},$$

$$\text{and} \quad \widehat{q}^i_{x,T_i+1} = \widehat{q}^{\mathrm{b}}_{x,T+1}\left(\frac{1 - \widehat{q}^{\mathrm{b}}_{x,T+1}}{\exp(-\widehat{A}^{\mathrm{b}})}\right)^{\dfrac{1}{\widehat{\alpha}^i(X^i_{\bullet} - 1)}}. \tag{3.9}$$

4 Classical Credibility Approaches to Mortality

Next, we wish to compare our model to the Hardy and Panjer (1998) and Poisson-Gamma credibility analysis to mortality. The actual to expected mortality ratio is the key observation that is the focus of the two following approaches. Specifically, the *a priori* expected number of deaths for portfolio i in calendar year t in the age-band $\big[\,\underline{x}\,,\overline{x}\,\big]$ is denoted by

$$\omega^i_t = \mathbb{E}\big[D^i_{\bullet,t}\big] = \sum_{x=\underline{x}}^{\overline{x}} q^{\mathrm{b}}_{x,t}\, L^i_{x,t}\,.$$

The actual to expected mortality ratios denoted by X^i_t are computed for each calendar year t in aggregate for each portfolio i,

$$X^i_t = \frac{D^i_{\bullet,t}}{\mathbb{E}\big[D^i_{\bullet,t}\big]} = \frac{D^i_{\bullet,t}}{\omega^i_t}.$$

Both the Hardy and Panjer (1998) and the Poisson-Gamma credibility approaches are using the Bühlmann-Straub set-up, see Subsection 3.3. Again, the key determinants of the credibility estimator (3.5) are the structure parameters, i.e. the variance part of the credibility premium $\mathbb{E}\big[\sigma^2(\Theta_i)\big]$ denoted by σ^2 and the fluctuation part $\mathbb{V}\big[\mu(\Theta_i)\big]$ denoted by τ^2.

4.1 The Hardy-Panjer Approach. As in general we have no knowledge or, at least, no exact knowledge of the parametric distributions for the number of deaths or of the structure distribution, we need estimators for the two components of the credibility estimator (3.5), i.e. estimators for τ^2 and σ^2. The Hardy and Panjer (1998) credibility approach to mortality estimates the structure parameters from the aggregated data using the estimators derived in

Centeno (1989). They estimate $\mathbb{E}\big[\sigma^2(\Theta_i)\big]$ using the following estimator, denoted by $\widehat{\sigma}_0^2$:

$$\widehat{\sigma}_0^2 = \frac{1}{\sum_{i=1}^n C_i} \sum_{i=1}^n C_i\, s_i^2, \quad \text{where} \quad C_i = \frac{1}{1 + \frac{2}{T_i - 1}\,\phi} \quad \text{with} \quad \phi = \frac{\mathbb{E}\big[\sigma^4(\Theta_i)\big]}{\mathbb{V}\big[\sigma^2(\Theta_i)\big]}.$$

Again, as the risk parameters, $\{\Theta_i\}_{i=1}^n$, are assumed to be identically distributed, the factors $\mathbb{E}\big[\sigma^4(\Theta_i)\big]$ and $\mathbb{V}\big[\sigma^2(\Theta_i)\big]$ are independent of the portfolio. Hence, the only portfolio dependent variable in C_i is T_i, the number of years data available for the portfolio.

Both methods give the same result. In addition, the latter approach, derived from Centeno (1989), allows to obtain a credibility estimator for the variance part of the credibility premium which has the form: $\widetilde{\sigma}_i{}^2 = C_i\, s_i^2 + (1 - C_i)\, \widehat{\sigma}_0^2$.

The estimate of $\mathbb{V}\big[\mu\big(\Theta_i\big)\big]$ denoted by $\widehat{\tau}^2$ is

$$\widehat{\tau}^2 = \frac{\omega_{\bullet}^{\bullet}\, W - \widehat{\sigma}^2}{\omega_{\bullet}^{\bullet}\, \Omega}, \quad \text{where } \Omega = \frac{1}{\left(\sum_{i=1}^n T_i\right) - 1} \sum_{i=1}^n \frac{\omega_{\bullet}^i}{\omega_{\bullet}^{\bullet}} \left(1 - \frac{\omega_{\bullet}^i}{\omega_{\bullet}^{\bullet}}\right),$$

$$\text{and} \quad W = \frac{1}{\left(\sum_{i=1}^n T_i\right) - 1} \sum_{i=1}^n \sum_{t=1}^{T_i} \frac{\omega_t^i}{\omega_{\bullet}^{\bullet}} \big(X_t^i - X_{\bullet}^{\bullet}\big)^2, \text{with} \quad \omega_{\bullet}^{\bullet} = \sum_{i=1}^n \omega_{\bullet}^i, \quad \text{and} \quad X_{\bullet}^{\bullet} = \frac{1}{\omega_{\bullet}^{\bullet}} \sum_{i=1}^n \omega_{\bullet}^i X_{\bullet}^i.$$

Then, the estimate of $\mathbb{E}\big[\sigma^4(\Theta_i)\big]$ denoted by $\widehat{\sigma}^4$ is

$$\widehat{\sigma}^4 = \frac{1}{\sum_{i=1}^n (T_i + 1)} \sum_{i=1}^n (T_i - 1)\big(s_i^2\big)^2,$$

and the estimate of $\mathbb{V}\big[\sigma^2(\Theta_i)\big]$ denoted by $\widehat{\upsilon}_{\sigma^2}$ is

$$\widehat{\upsilon}_{\sigma^2} = \frac{1}{R} \left(\sum_{i=1}^n (T_i - 1)\big(s_i^2 - \beta^2\big)^2 - 2\,\widehat{\sigma}^4 (n-1) \right),$$

$$\text{where} \quad R = \sum_{i=1}^n (T_i - 1) - \frac{\sum_{i=1}^n (T_i - 1)^2}{\sum_{i=1}^n (T_i - 1)}, \quad \text{and} \quad \beta^2 = \frac{1}{n} \sum_{i=1}^n s_i^2.$$

4.2 The Poisson-Gamma Approach. A priori, we could assume that $\mathbb{E}\big[\Theta_i\big] = 1$ so that the baseline mortality produces the a priori expected number of deaths,

$$\mathbb{E}\big[D_{\bullet,t}^i\big] = \mathbb{E}\big[\omega_t^i\, \Theta_i\big] = \omega_t^i.$$

We suppose here that the parametric distribution for the number of deaths $D_{\bullet,t}^i$ is Poisson conditional to the relative risk level Θ_i, so that

$$\mathbb{E}\big[D_{\bullet,t}^i\,|\Theta_i\big] = \mathbb{V}\big[D_{\bullet,t}^i\,|\Theta_i\big] = \omega_t^i\, \Theta_i.$$

Then, under assumption H1, Subsection 3.3, the conditional mean and variance of the actual to expected mortality ratios become:

$$\mathbb{E}\big[X_t^i\,|\,\Theta_i\big] = \mu(\Theta_i) = \Theta_i \quad \text{and} \quad \mathbb{V}\big[X_t^i\,|\,\Theta_i\big] = \frac{\sigma^2(\Theta_i)}{\omega_t^i} = \frac{\Theta_i}{\omega_t^i},$$

and the p.d.e with respect to $a_{i,0}$ and $a_{i,t}$, Equation 3.4 are:

$$a_{i,0} = 1 - \frac{\tau^2\,\omega^i_\bullet}{1+\tau^2\,\omega^i_\bullet} \quad \text{and} \quad a_{i,t} = \frac{\tau^2\,\omega^i_t}{1+\tau^2\,\omega^i_\bullet}\,, \quad \text{since} \quad \sigma^2 = \mathbb{E}\big[\Theta_i\big] = 1.$$

Then the linear credibility estimator is given by

$$\widehat{\mu}\big(\Theta_i\big) = \widehat{X}^i_{T_i+1} = \frac{1}{1+\tau^2\,\omega^i_\bullet} + \frac{\tau^2\,\omega^i_\bullet}{1+\tau^2\,\omega^i_\bullet}\,\frac{1}{\omega^i_\bullet}\sum_{t=1}^{T_i}\omega_{i,t}\,X_{i,t}\,. \tag{4.1}$$

And, the expected number of deaths for portfolio i for next year T_i+1 is

$$\omega^i_{T_i+1}\,\widehat{X}^i_{T_i+1} = \omega^i_{T_i+1}\frac{1+\tau^2\,D^i_{\bullet,\bullet}}{1+\tau^2\,\omega^i_\bullet}.$$

Then, we need to obtain the structure parameter $\tau^2 = \mathbb{V}\big[\Theta_i\big]$. As the distribution of the total number of deaths in portfolio i is $D^i_{\bullet,\bullet} \sim \mathcal{MP}\big(\omega^i_\bullet\Theta_i\big)$ and using the variance decomposition principle,

$$\begin{aligned}\mathbb{V}\big[D^i_{\bullet,\bullet}\big] &= \mathbb{V}\big[\mathbb{E}\big[D^i_{\bullet,\bullet}\,|\,\Theta_i\big]\big] + \mathbb{E}\big[\mathbb{V}\big[D^i_{\bullet,\bullet}\,|\,\Theta_i\big]\big] = \mathbb{V}\big[\omega^i_\bullet\Theta_i\big] + \mathbb{E}\big[\omega^i_\bullet\Theta_i\big]\\ &= \tau^2\,(\omega^i_\bullet)^2 + \omega^i_\bullet\,.\end{aligned}$$

And, $\sum_{i=1}^n \mathbb{V}\big[D^i_{\bullet,\bullet}\big] = \omega^2\,\sum_{i=1}^n(\omega^i_\bullet)^2 + \sum_{i=1}^n\omega^i_\bullet$, leads to $\tau^2 = \sum_{i=1}^n\big(\mathbb{V}\big[D^i_\bullet\big] - \omega^i_\bullet\big)/\sum_{i=1}^n(\delta^i_\bullet)^2$. Thus, the estimator of τ^2 writes

$$\widehat{\tau}^2 = \frac{\sum_{i=1}^n\big(\big(D^i_{\bullet,\bullet} - \omega^i_\bullet\big)^2 - D^i_{\bullet,\bullet}\big)}{\sum_{i=1}^n(\omega^i_\bullet)^2}.$$

5 Numerical Analysis

5.1 Data Quantitative Analysis. The data come from studies conducted by *Institut des Actuaires*. These studies include in total 14 portfolio covering the period 2007-2011 with each companies contributing data for at least 4 of a possible 5 years. Table 1 presents the observed characteristics of the male population of the portfolios. For this dataset, we are considering respectively $T_i = 3$ and $T_i = 4$ for all companies. The remaining years serve to test the predictive feature of the model through an in-sample analysis. The age band for all companies ranges from 30 to 95 years old. Figure 1 shows the age distribution of two portfolios. It graphically depicts the heterogeneity observed between the portfolios with insureds holding different policies.

5.2 The Baselines Mortality. We consider two prospective tables as baselines for our credibility models. One is the national demographic projections for the French population over the period 2007-2060, provided by the French National Office for Statistics, INSEE, see Blanpain and Chardon (2010). These projections are based on assumptions concerning fertility, mortality and migrations. We choose the baseline scenario among a total of 27 scenarios. The baseline scenario is based on the assumption that until 2060, the total fertility rate is remaining at a very high level (1.95). The decrease in sex and age-specific mortality rates is greater for men over 85 years old. The baseline assumption on migration consists in projecting a constant annual net-migration balance of 100,000 inhabitants. The second external reference table, denoted

Table 1: *Observed characteristics of portfolios population.*

	Period of observation		Mean age		Average exposure	Mean age at death
	Beginning	End	In	Out		
1	1/1/07	12/31/11	36.96	39.74	2.77	68.78
2	1/1/07	12/31/11	69.3	73.35	4.05	80.34
3	1/1/07	12/31/10	40.16	43.1	2.94	71.77
4	1/1/07	12/31/11	37.5	41.13	3.63	54.08
5	1/1/07	12/31/11	36.9	39.1	2.2	59.31
6	1/1/07	12/31/10	48.5	52.11	3.62	82.34
7	1/1/07	12/31/11	66.65	71.29	4.64	73.68
8	1/1/07	4/13/11	67.51	71.38	3.86	80.72
9	1/1/07	6/30/11	45.97	49.6	3.62	73.17
10	1/1/07	12/31/11	62.97	67.64	4.67	79.77
11	1/1/07	12/31/11	38.89	42	3.11	56.44
12	1/1/07	12/31/11	37.05	39.2	2.15	57.41
13	1/1/07	12/31/11	43.01	46.89	3.88	71.03
14	1/1/07	12/31/11	50.12	54.16	4.04	72.37

IA2013, is a market table constructed for the French insurance market provided by *Institute des Actuaires*, see Tomas and Planchet (2013). It is worth to mention that this table is derived on mortality trends originating from the INSEE table and covers the period 2007-2060.

Following, assumption (i) in Subsection 2.3, the baseline mortality $q^{\mathrm{b}}_{x,t}$ is described by the Makeham model in (2.3). Table 2 presents the estimated parameters for each of the baselines considered.

Table 2: *Estimated parameters of the Makeham model* (2.3) *for the baselines of mortality considered, male population.*

	INSEE		IA2013	
	2007–2009	**2007–2010**	**2007–2009**	**2007–2010**
$\widehat{A}^{\mathrm{b}}_T$	$4.2835e-03$	$4.2787e-03$	$2.1577e-04$	$2.4355e-04$
$\widehat{B}^{\mathrm{b}}_T$	$7.9564e-07$	$7.7199e-07$	$4.0863e-06$	$3.9935e-06$
$\widehat{C}^{\mathrm{b}}_T$	1.1484	1.1487	1.1211	1.1213

5.3 Adjustment of the Makeham model. Following assumptions (ii) in Subsection 2.3, we fit the Makeham model (2.3) for the baselines of mortality considered so as to estimate B^{b}_t for each calendar year while the parameters $\widehat{A}^{\mathrm{b}}_t = \widehat{A}^{\mathrm{b}}_T$ and $\widehat{C}^{\mathrm{b}}_t = \widehat{C}^{\mathrm{b}}_T$ remain fixed. Table 3 presents the estimated parameters for each year and baselines considered.

5.4 Proximity Between the Observations and the Model. We assess the overall deviation with the observed mortality by comparing criteria measuring the distance between the observations and the models with the χ^2 applied by Forfar *et al.* (1988), the mean average

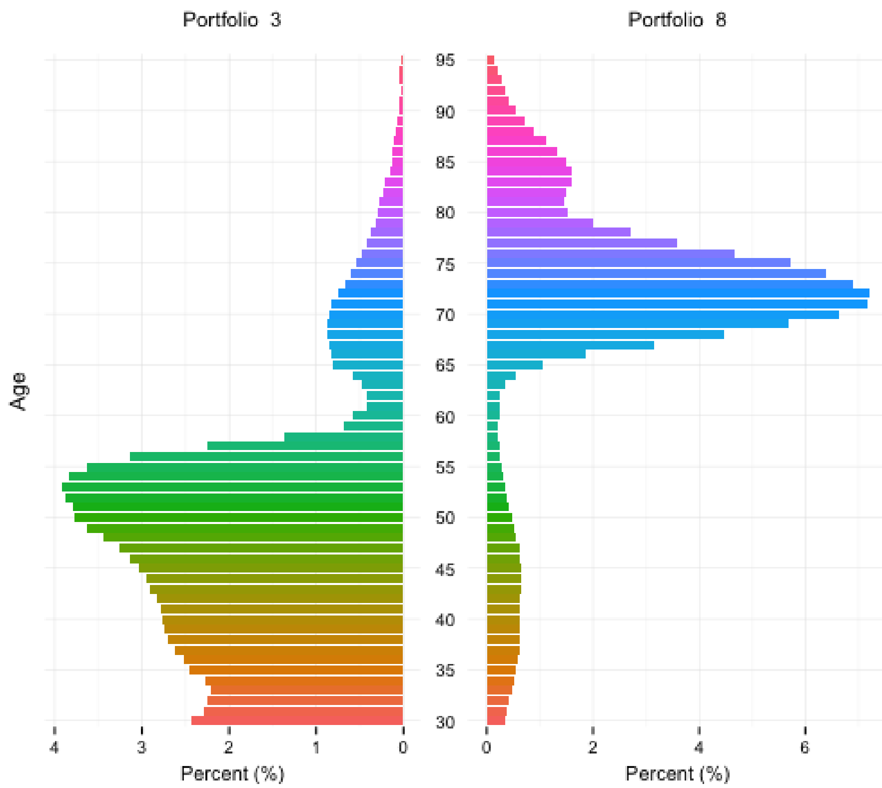


Figure 1: *Distribution of age groups in portfolios 3 (left panel) and 8 (right panel), male population.*

Table 3: *Estimated parameters of the Makeham model* (2.3) *for each year and baselines of mortality considered, male population.*

	INSEE		**IA2013**	
	2007–2009	**2007–2010**	**2007–2009**	**2007–2010**
$\widehat{A}^{\mathrm{b}}_{T}$	$4.2835e-03$	$4.2787e-03$	$2.1577e-04$	$2.4355e-04$
$\widehat{B}^{\mathrm{b}}_{2007}$	$8.0826e-07$	$7.9035e-07$	$4.1740e-06$	$4.1204e-04$
$\widehat{B}^{\mathrm{b}}_{2008}$	$7.9554e-07$	$7.7790e-07$	$4.0843e-06$	$4.0319e-06$
$\widehat{B}^{\mathrm{b}}_{2009}$	$7.8318e-07$	$7.658e-07$	$4.0009e-06$	$3.9496e-06$
$\widehat{B}^{\mathrm{b}}_{2010}$	–	$7.5406e-07$	–	$3.8729e-06$
$\widehat{C}^{\mathrm{b}}_{T}$	1.1484	1.1487	1.1211	1.1213

percentage error (MAPE) applied by Felipe *et al.* (2002) as well as the standardized mortality ratio (SMR) and the number of standardized residuals larger then 2 and 3, see Tomas and Planchet (2014). In addition, we find useful to use the SMR test proposed by Liddell (1984) and the likelihood ratio test. The tests and quantities summarizing the proximity between the observations and the model are described in the following. The χ^2 allows to measure the quality of the fit of the model. It writes,

$$\chi^2 = \sum_{(x,t)} \frac{\big(D_{x,t} - L_{x,t}\, \widehat{q}_x(t)\big)^2}{L_{x,t}\, \widehat{q}_x(t)\big(1 - \widehat{q}_x(t)\big)}.$$

The MAPE is the average of the absolute values of the deviations from the observations,

$$\text{MAPE} = \frac{\sum_{(x,t)} \big|\big(D_{x,t}/L_{x,t} - \widehat{q}_x(t)\big)/\big(D_{x,t}/L_{x,t}\big)\big|}{\sum_{(x,t)} D_{x,t}} \times 100.$$

We can also determine if the fit corresponds to the underlying mortality law (null hypothesis $\mathcal{H}_0$) with the likelihood ratio test. The statistic, ξ^{LR}, writes

$$\xi^{\text{LR}} = \sum_{(x,t)} \left(D_{x,t} \ln \left(\frac{D_{x,t}}{L_{x,t}\, \widehat{q}_x(t)} \right) + \big(L_{x,t} - D_{x,t}\big) \ln \left(\frac{L_{x,t} - D_{x,t}}{L_{x,t} - L_{x,t}\, \widehat{q}_x(t)} \right) . \right) .$$

If $\mathcal{H}_0$ is true, this statistic follows a χ^2 law with a number of degrees of freedom equal to the number of observations n: $\xi^{\text{LR}} \sim \chi^2(n)$. Hence, the null hypothesis $\mathcal{H}_0$ is rejected if $\xi^{\text{LR}} > \chi^2_{1-\alpha}(n)$, where $\chi^2_{1-\alpha}(n)$ is the $(1-\alpha)$ quantile of the χ^2 distribution with n degrees of freedom. The p-value is the lowest value of the type I error (α) for which we reject the test. We will privilege the model having the p-value $= \mathbb{P}\big[\chi^2_{1-\alpha}(n) > \xi^{\text{LR}}\big] = 1 - \text{F}_{\chi^2(n)}(\xi^{\text{LR}})$ closest to 1. The SMR is computed as the ratio between the observed and fitted number of deaths:

$$\text{SMR} = \frac{\sum_{(x,t)} D_{x,t}}{\sum_{(x,t)} L_{x,t}\, \widehat{q}_x(t)}.$$

Hence, if SMR > 1, the fitted deaths are under-estimated and vice-versa if SMR < 1. Note that we can consider the SMR as a global criterion which does not take the age structure into account, compared to the chi2 and MAPE for instance. We can also apply a test to determine if the SMR is significatively different from 1. Liddell (1984) proposes to compute the statistic,

$$\xi^{\text{SMR}} = \begin{cases} 3 \times D^{\frac{1}{2}} \big(1 - (9D)^{-1} - (D/E)^{\frac{1}{3}}\big) & \text{If SMR } > 1, \\ 3 \times D^{*\,\frac{1}{2}} \big((D^*/E)^{\frac{1}{3}} + (9D^*)^{-1} - 1\big) & \text{If SMR } < 1, \end{cases}$$

where $D = \sum_{(x,t)} D_{x,t}$, $D^* = \sum_{(x,t)} D_{x,t} + 1$ and $E = \sum_{(x,t)} L_{x,t}\, \widehat{q}_x(t)$. If the SMR is not significatively different from 1 (null hypothesis $\mathcal{H}_0$), this statistic follows a standard Normal law, $\xi^{\text{SMR}} \sim \text{N}(0,1)$. Thus, the null hypothesis $\mathcal{H}_0$ is rejected if $\xi^{\text{SMR}} > \text{N}_{1-\alpha}(0,1)$, where $\text{N}_{1-\alpha}(0,1)$ is the $(1-\alpha)$ quantile of the standard Normal distribution. The p-value is given by p-value $= 1 - \text{F}_{\text{N}(0,1)}(\xi^{\text{SMR}})$.

5.5 In-Sample Numerical Analysis. We fitted the approaches over a history covering 3 and 4 years (2007-2009 and 2007-2010 respectively) and compared the overall deviation between the observations and the models (for the year 2010 and 2011 respectively). Table 4 displays the estimates of the structure parameters for the three approaches.

Table 5 presents the tests and quantities summarizing the overall deviation between the observations and the credibility analysis for the male population of portfolio 1 obtained by the

Table 4: *Estimates of the structure parameters, male population.*

		Hardy-Panjer		Poisson-Gamma		Makeham-Credibility	
		INSEE	IA2013	INSEE	IA2013	INSEE	IA2103
2007-09	$\widehat{\mu}_0$	3.5521	16.3290	1	1	1	1
	$\widehat{\sigma}^2$	44.4032	92.1668	1	1	4.0552e-04	2.3198e-03
	$\widehat{\tau}^2$	6.8368	44.0092	10.7485	367.5029	0.1935	3.5960e-02
2007-10	$\widehat{\mu}_0$	3.6495	15.7865	1	1	1	1
	$\widehat{\sigma}^2$	65.9649	116.0159	1	1	5.1034e-04	2.6285e-03
	$\widehat{\tau}^2$	7.0772	43.4966	10.9684	338.4440	0.2217	5.0281e-02

Hardy-Panjer, Poisson-Gamma and the Makeham credibility approaches with the two baselines mortality considered for the year 2010. Tables 6, 7 and 8, 9 in Appendix A and B display the results for all the portfolios and for the years 2010 and 2011 respectively.

Table 5: *Tests and quantities summarizing the deviation between the observations and the models for portfolio 1, calendar year* 2010*, male population.*

		INSEE			IA2103		
		Hardy-Panjer	Poisson-Gamma	Makeham-Credibility	Hardy-Panjer	Poisson-Gamma	Makeham-Credibility
Standardized	> 2	60	60	35	46	46	15
residuals	> 3	48	48	28	32	32	5
χ^2		5481.86	5542.82	3569.97	1705.25	1747.25	208.81
MAPE	(%)	233.22	230.94	373.89	117.01	115.42	42.35
Likelihood	ξ^{LR}	946.98	947.72	443.16	463.48	468.46	88.03
ratio test	p-value	0	0	0	0	0	0.0364
	SMR	1.1792	1.1919	0.5265	1.7629	1.7957	1.0532
SMR test	ξ^{SMR}	4.0379	4.2939	12.1893	13.0352	13.4202	1.2845
	p-value	0	0	0	0	0	0.0995

The Hardy-Panjer and Poisson-Gamma approaches produce relatively similar graduations. However, we notice some differences with the Makeham credibility model which displays more favorable results whatever the baseline mortality considered for the two periods fitted.

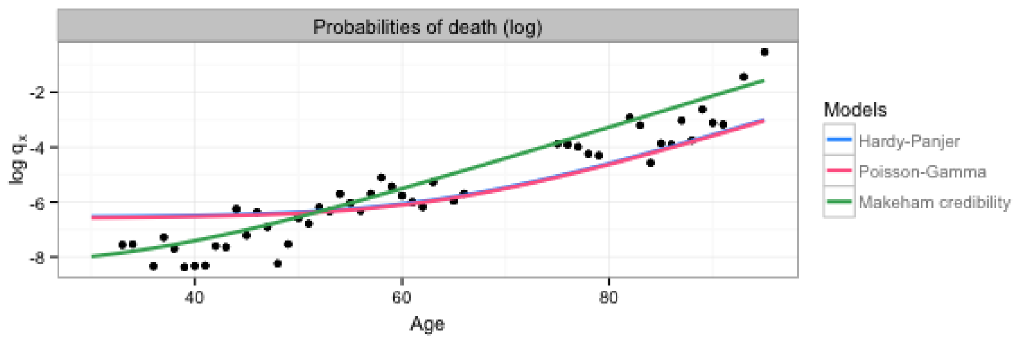
It is also apparent that using the market baseline mortality IA2013 produces better results than the national demographic projections originating from INSEE, see Subsection 5.2. It illustrates the importance of using an adequate baseline mortality when adjusting the models.

When looking at criteria and quantities which take the age structure of the error into account, the Makeham credibility approach is a benefit. The quality of the fit increases, sometimes drastically, compared to the Hardy-Panjer and Poisson-Gamma model in terms of having the minimum χ^2 and MAPE values. The Makeham credibility model leads to the lowest number of standardized residuals lower than 2 and 3. It exhibits as well the highest p-value for the likelihood ratio test.

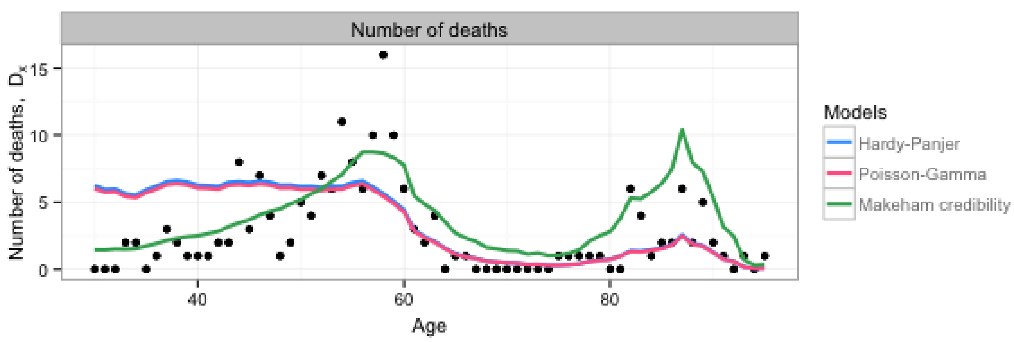
Even when we considering a global indicator of the quality of the fit such as the SMR which does not take the age structure into account, the Makeham credibility model seems to perform better than the Hardy-Panjer and Poisson-Gamma approaches. The statistic ξ^{SMR} of the SMR test is the smaller 8 times over 14 for the year 2010, see Tables 6 and 7 in Appendix A, and 6 times over 12 for the year 2011, see Tables 8 and 9 in Appendix B.

We also notice that the the Makeham credibility model has tendency to over-estimate the total number of deaths, having a SMR lower than 1 for 9 portfolios over 14 in 2010 and for 8 portfolios over 12 in 2011.

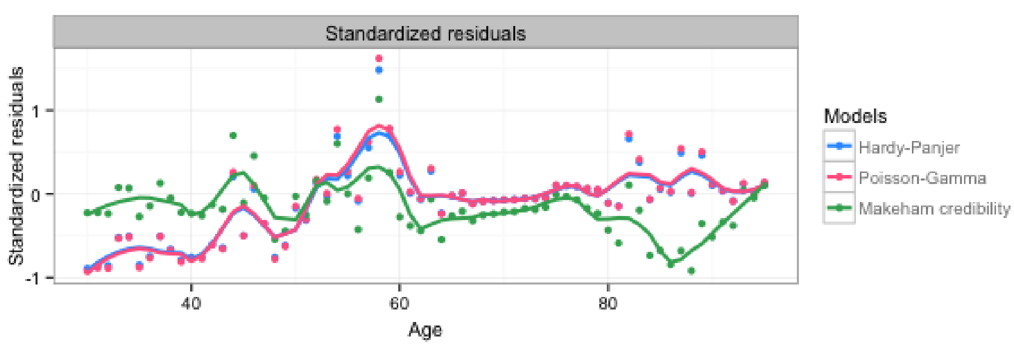
In the following, these quantitative diagnostics are supplemented by a range of visual comparisons. Besides the tests and quantities, the comparison involves graphical analysis. It consists of representing graphically the fitted values against the observations for the years 2010 and 2011. For clarity, the graphical comparisons only consider the market baseline mortality IA2013 as it leads to better results than using the national demographic projections.



(a) *Fitted probabilities of death in the log scale.*



(b) *Fitted number of deaths.*



(c) *Standardized residuals.*

Figure 2: *Fitted values against the observations for portfolio 1 for the year 2010, male population.*

Figure 2a displays the the fitted probabilities of death in the log scale for portfolio 1 for the year 2010. Figure 3 and 4 in Appendix C and D display the comparisons for all the portfolios and for the years 2010 and 2011 respectively. It gives us the opportunity to visualize the similarities and differences between the fits obtained by the approaches. It is again apparent that the Hardy-Panjer and Poisson-Gamma models lead to similar results. In addition, we observe that these approaches have a tendency to strongly overestimate the probabilities of death for the age band $[30, 60]$ and reciprocally underestimate them for the age band $[60, 95]$. This is explained by the fact that the age structure is not taken in account by the Hardy-Panjer and Poisson-Gamma approaches, conversely to the Makeham credibility model. We can visualize this lack of fit in the plots of the fitted number of deaths, Figure 2b for portfolio 1 and Figure 5 and 6 for all portfolios in Appendix E and F.

In conjunction with looking to the plots of the fits, we should study the residuals plots. Such residual plots provide a powerful diagnostic that nicely complements the analysis. The diagnostic plots can show lack of fit locally and we have the opportunity to judge the lack of fit based on our knowledge on the data and of the performance of the models. We superimposed a smooth curve on the standardized residuals. This smooth helps search for clusters of residuals that may indicate a lack of fit. The plots of the standardized residuals, for the male population, are display in Figure 2c for portfolio 1 and Figure 7 and 8 in Appendix G and H for all the portfolios and for the years 2010 and 2011 respectively.
The standardized residuals, obtained by the Hardy-Panjer and Poisson-Gamma models, present a high curvature for most of the portfolios in Figure 7 and 8. It indicates a clear lack of fit. These models overestimate the number of deaths for the age band $[30, 60]$ et underestimate them for the age band $[60, 95]$, as observed in the plots of the fits previously. Conversely, no strong patterns appear in the standardized residuals retrieved for the Makeham credibility model. The smooth curves over the standardized residuals is meanly flat, meaning that no systematic reproducible lack of fit has been detected and that the Makeham credibility model captures adequately the variability of the data.

6 Concluding Remarks

We considered the periodic adjustment of a mortality graduated curve using a Makeham parametric model. This relies on the revision of a single parameter the two remaining been fixed. The framework considered here is closely related to the one introduced in Hardy and Panjer (1998). The main difference is the age-structure included through the parametric Makeham model. By doing so, we showed that adding an age structure enhances the predictive ability of the death forecast especially when we consider age-sensitive proxies. If one is only interested in predicting deaths at the aggregate portfolio level our methodology yields to the same forecast as in the Hardy and Panjer (1998) framework. Moreover, we should note that in our methodology especially using the ratio of the considered Makeham parameters allows to overcome the de-trending step recommended in Hardy and Panjer (1998).

In order to assess the predictive power of our methodology, various other measures of risk and goodness-of-fit should be taken into account. Especially, we should consider the age-structure's impact on the prices and reserves and potential benefit of our model compared to the current market practice. There are also several piratical we do not address here which we openly acknowledge and leave for future research.

Appendix

A Tests and quantities summarizing the deviation between the observations and the models for the year 2010

Tables 6 and 7 present the tests and quantities summarizing the overall deviation between the observations and the credibility analysis for the male population obtained by the Hardy-Panjer, Poisson-Gamma and the Makeham credibility approaches with the two baselines mortality considered for the year 2010.

B Tests and quantities summarizing the deviation between the observations and the models for the year 2011

Tables 8 and 9 present the tests and quantities summarizing the overall deviation between the observations and the credibility analysis for the male population obtained by the Hardy-Panjer, Poisson-Gamma and the Makeham credibility approaches with the two baselines mortality considered for the year 2011.

C fitted probabilities of death in the log scale for the year 2010

Figure 3 displays the fitted probabilities of death in the log scale for the male population for the year 2010.

D fitted probabilities of death in the log scale for the year 2011

Figure 4 displays the fitted probabilities of death in the log scale for the male population for the year 2011.

E Fitted number of deaths for the year 2010

Figure 5 displays the fitted number of deaths for the male population for the year 2010.

F Fitted number of deaths for the year 2010

Figure 6 displays the fitted number of deaths for the male population for the year 2010.

G Standardized residuals for the year 2010

Figure 7 displays the standardized residuals for the male population for the year 2010.

H Standardized residuals for the year 2011

Figure 8 displays the standardized residuals for the male population for the year 2011.

Table 6: *Tests and quantities summarizing the deviation between the observations and the model, calendar year* 2010*, male population.*

			INSEE			IA2103		
			Hardy-Panjer	Poisson-Gamma	Makeham-Credibility	Hardy-Panjer	Poisson-Gamma	Makeham-Credibility
Portfolio 1	Standardized	> 2	60	60	35	46	46	15
	residuals	> 3	48	48	28	32	32	5
	χ^2		5481.86	5542.82	3569.97	1705.25	1747.25	208.81
	MAPE (%)		233.22	230.94	373.89	117.01	115.42	42.35
	Likelihood	ξ^{LR}	946.98	947.72	443.16	463.48	468.46	88.03
	ratio test	p-value	0	0	0	0	0	0.0364
		SMR	1.1792	1.1919	0.5265	1.7629	1.7957	1.0532
	SMR test	ξ^{SMR}	4.0379	4.2939	12.1893	13.0352	13.4202	1.2845
		p-value	0	0	0	0	0	0.0995
Portfolio 2	Standardized	> 2	9	11	0	1	1	0
	residuals	> 3	2	1	0	0	0	0
	χ^2		102.84	101.50	29.62	41.54	40.41	30.75
	MAPE (%)		108.16	116.37	48.80	47.18	48.09	54.70
	Likelihood	ξ^{LR}	90.3	94.99	33.8	36.43	36.77	33.35
	ratio test	p-value	3e-04	1e-04	0.9517	0.908	0.901	0.9573
		SMR	0.6421	0.6014	0.8764	1.0149	0.9868	0.8567
	SMR test	ξ^{SMR}	3.6844	4.2907	0.9805	0.074	0.0207	1.1681
		p-value	1e-04	0	0.1634	0.4705	0.4918	0.1214
Portfolio 3	Standardized	> 2	34	32	11	7	7	4
	residuals	> 3	9	9	5	0	0	0
	χ^2		416.19	420.04	161.66	110.28	110.89	64.16
	MAPE (%)		156.33	154.14	76.78	64.99	64.67	45.48
	Likelihood	ξ^{LR}	239.44	236.76	115.84	91.13	90.85	38.51
	ratio test	p-value	0	0	1e-04	0.0219	0.023	0.9973
		SMR	0.5361	0.5451	0.8955	0.8989	0.9052	1.1212
	SMR test	ξ^{SMR}	7.0465	6.8379	1.0892	1.049	0.9746	1.1174
		p-value	0	0	0.138	0.1471	0.1649	0.1319
Portfolio 4	Standardized	> 2	20	19	15	8	5	2
	residuals	> 3	2	1	3	0	0	0
	χ^2		183.96	181.13	199.86	83.98	83.32	41.51
	MAPE (%)		201.49	196.70	189.51	92.75	90.01	44.33
	Likelihood	ξ^{LR}	212.87	208	174.37	101.22	98.75	36.28
	ratio test	p-value	0	0	0	0	1e-04	0.9406
		SMR	0.3590	0.3665	0.4332	0.6161	0.6326	1.0677
	SMR test	ξ^{SMR}	11.537	11.2597	8.408	4.9251	4.6315	0.5798
		p-value	0	0	0	0	0	0.2810
Portfolio 5	Standardized	> 2	8	9	8	8	10	13
	residuals	> 3	8	8	7	8	8	6
	χ^2		368.00	470.94	205.33	259.26	366.90	209.05
	MAPE (%)		72.14	78.00	67.45	79.85	85.30	82.04
	Likelihood	ξ^{LR}	63.85	63.4	59.53	52.94	55.85	43.15
	ratio test	p-value	0.1069	0.1141	0.1930	0.3992	0.2977	0.7746
		SMR	1.4167	1.7941	1.2442	2.1557	2.9553	3.1797
	SMR test	ξ^{SMR}	1.6446	2.6956	1.0308	3.4572	4.6617	4.9234
		p-value	0.05	0.0035	0.1513	3e-04	0	0
Portfolio 6	Standardized	> 2	62	62	56	50	50	24
	residuals	> 3	61	61	50	44	44	7
	χ^2		7615.50	7615.40	1538.42	1364.75	1364.60	256.21
	MAPE (%)		2558.24	2558.59	652.41	631.01	631.13	145.45
	Likelihood	ξ^{LR}	7417.14	7417.88	1707.66	1575.04	1575.17	272.40
	ratio test	p-value	0	0	0	0	0	0
		SMR	0.5444	0.5443	0.9802	0.9337	0.9335	0.9829
	SMR test	ξ^{SMR}	42.5496	42.56	1.2532	4.3697	4.3813	1.0796
		p-value	0	0	0.1051	0	0	0.1402
Portfolio 7	Standardized	> 2	51	51	5	16	16	4
	residuals	> 3	44	44	0	1	1	1
	χ^2		1501.03	1504.85	114.94	163.58	164.90	77.81
	MAPE (%)		515.81	516.42	72.78	96.70	96.99	29.18
	Likelihood	ξ^{LR}	1417.16	1420.27	145.97	201.82	202.79	60.31
	ratio test	p-value	0	0	0	0	0	0.6743
		SMR	0.5941	0.5934	0.909	0.8941	0.8923	0.9264
	SMR test	ξ^{SMR}	33.38	33.4583	5.6836	6.688	6.8078	4.5385
		p-value	0	0	0	0	0	0

Table 7: *Tests and quantities summarizing the deviation between the observations and the model, calendar year* 2010*, male population.*

			INSEE			IA2103		
			Hardy-Panjer	Poisson-Gamma	Makeham-Credibility	Hardy-Panjer	Poisson-Gamma	Makeham-Credibility
Portfolio 8	Standardized	> 2	62	62	5	26	25	4
	residuals	> 3	59	59	1	6	6	0
	χ^2		2962.00	2967.84	115.90	274.12	275.25	85.69
	MAPE (%)		837.99	839.75	67.40	116.81	117.16	23.93
	Likelihood	ξ^{LR}	2455.66	2462.18	130.73	247.92	248.89	55.14
	ratio test	p-value	0	0	0	0	0	0.8273
		SMR	0.5638	0.5627	0.9673	0.8972	0.8953	0.9811
	SMR test	ξ^{SMR}	37.8094	37.9526	1.9899	6.6063	6.7345	1.1323
		p-value	0	0	0.0233	0	0	0.1287
Portfolio 9	Standardized	> 2	63	63	40	45	45	12
	residuals	> 3	59	59	30	38	38	3
	χ^2		5759.31	5759.28	591.79	741.90	742.08	147.99
	MAPE (%)		754.36	754.31	192.26	198.83	198.97	22.01
	Likelihood	ξ^{LR}	3443.71	3443.52	427.46	502.06	502.38	77.09
	ratio test	p-value	0	0	0	0	0	0.1653
		SMR	0.5262	0.5262	0.9084	0.8627	0.8622	0.9078
	SMR test	ξ^{SMR}	41.6671	41.6629	5.6716	8.7994	8.8355	5.7137
		p-value	0	0	0	0	0	0
Portfolio 10	Standardized	> 2	48	48	1	6	7	3
	residuals	> 3	33	33	0	1	1	1
	χ^2		669.50	672.46	80.63	121.74	122.90	86.65
	MAPE (%)		504.88	509.44	75.75	110.69	112.55	55.72
	Likelihood	ξ^{LR}	631.46	636.74	82.38	114.75	116.28	48.65
	ratio test	p-value	0	0	0.0839	2e-04	1e-04	0.9461
		SMR	0.5336	0.5292	0.8434	0.8352	0.8263	0.91
	SMR test	ξ^{SMR}	16.4396	16.6765	4.1025	4.344	4.6133	2.2303
		p-value	0	0	0	0	0	0.0129
Portfolio 11	Standardized	> 2	43	43	23	33	33	2
	residuals	> 3	37	37	20	17	17	1
	χ^2		1387.49	1391.13	695.98	380.60	383.26	74.55
	MAPE (%)		257.02	255.43	464.55	125.94	124.91	46.19
	Likelihood	ξ^{LR}	429.18	426.67	338.9	161.53	161.07	39.42
	ratio test	p-value	0	0	0	0	0	0.9949
		SMR	0.5373	0.5407	0.4887	0.9009	0.9094	1.092
	SMR test	ξ^{SMR}	14.8749	14.7085	14.254	2.2628	2.0519	1.8578
		p-value	0	0	0	0.0118	0.0201	0.0316
Portfolio 12	Standardized	> 2	33	33	25	17	18	4
	residuals	> 3	17	17	20	3	3	0
	χ^2		588.18	592.25	449.62	161.43	164.16	91.65
	MAPE (%)		241.27	236.71	514.74	111.99	108.62	89.85
	Likelihood	ξ^{LR}	274.67	270.12	329.89	122.89	120.3	96.49
	ratio test	p-value	0	0	0	0	1e-04	0.0085
		SMR	0.4877	0.4971	0.3291	0.7957	0.8243	0.7125
	SMR test	ξ^{SMR}	10.8436	10.5217	15.3598	3.1391	2.6305	4.7701
		p-value	0	0	0	8e-04	0.0043	0
Portfolio 13	Standardized	> 2	55	55	41	27	27	19
	residuals	> 3	44	44	30	16	16	11
	χ^2		2162.97	2162.79	761.52	331.68	331.72	252.49
	MAPE (%)		478.97	478.32	200.96	136.82	136.60	46.85
	Likelihood	ξ^{LR}	1360.75	1359.18	469.14	241.26	241.02	136.5
	ratio test	p-value	0	0	0	0	0	0
		SMR	0.5378	0.5385	0.9215	0.8966	0.8979	0.8966
	SMR test	ξ^{SMR}	24.9715	24.9134	2.9868	4.0137	3.9601	4.0113
		p-value	0	0	0.0014	0	0	0
Portfolio 14	Standardized	> 2	50	50	23	23	23	12
	residuals	> 3	38	38	7	5	5	1
	χ^2		970.86	970.89	268.98	239.70	239.35	119.91
	MAPE (%)		492.88	492.65	153.55	170.57	171.60	57.14
	Likelihood	ξ^{LR}	742.64	742.36	200.35	207.33	208.11	69.04
	ratio test	p-value	0	0	0	0	0	0.3750
		SMR	0.5329	0.5331	0.9491	0.8529	0.848	1.0419
	SMR test	ξ^{SMR}	15.6678	15.6546	1.1518	3.6326	3.7699	0.8978
		p-value	0	0	0.1247	1e-04	1e-04	0.1847

Table 8: *Tests and quantities summarizing the deviation between the observations and the model, calendar year* 2011, *male population.*

			INSEE			IA2103		
			Hardy-Panjer	Poisson-Gamma	Makeham-Credibility	Hardy-Panjer	Poisson-Gamma	Makeham-Credibility
Portfolio 1	Standardized	> 2	58	57	56	48	48	13
Portfolio 1	residuals	> 3	52	52	48	35	35	5
Portfolio 1	χ^2		5574.22	5621.44	3126.56	1901.24	1928.68	259.40
Portfolio 1	MAPE	(%)	201.74	200.25	178.37	102.66	102.00	32.87
Portfolio 1	Likelihood	ξ^{LR}	1027.03	1027.4	806.13	524.68	528.16	106.43
Portfolio 1	ratio test	p-value	0	0	0	0	0	0.0012
Portfolio 1		SMR	1.1124	1.1216	1.2011	1.7371	1.7557	1.1256
Portfolio 1	SMR test	ξ^{SMR}	2.935	3.1588	4.9944	14.192	14.441	3.2557
Portfolio 1		p-value	0.0017	8e-04	0	0	0	6e-04
Portfolio 2	Standardized	> 2	4	4	0	3	2	1
Portfolio 2	residuals	> 3	2	1	0	0	0	0
Portfolio 2	χ^2		77.89	78.43	29.07	34.89	33.64	30.94
Portfolio 2	MAPE	(%)	114.12	124.28	52.48	48.03	49.12	53.99
Portfolio 2	Likelihood	ξ^{LR}	66.67	71.07	29.92	28.72	28.99	28.16
Portfolio 2	ratio test	p-value	0.0385	0.0169	0.9811	0.9877	0.9864	0.9901
Portfolio 2		SMR	0.6545	0.6097	0.8668	1.0371	1.0016	0.905
Portfolio 2	SMR test	ξ^{SMR}	3.7113	4.3984	1.1388	0.2609	0.0268	0.7649
Portfolio 2		p-value	1e-04	0	0.1274	0.3971	0.5107	0.2222
Portfolio 4	Standardized	> 2	20	20	15	13	12	6
Portfolio 4	residuals	> 3	4	4	3	4	4	1
Portfolio 4	χ^2		250.57	250.57	1026.72	130.12	132.89	79.00
Portfolio 4	MAPE	(%)	202.16	196.53	226.84	95.39	92.49	44.88
Portfolio 4	Likelihood	ξ^{LR}	173.25	168.79	172.37	90.66	89.04	51.08
Portfolio 4	ratio test	p-value	0	0	0	7e-04	0.0011	0.51
Portfolio 4		SMR	0.4852	0.498	0.5443	0.826	0.8534	1.4047
Portfolio 4	SMR test	ξ^{SMR}	8.9106	8.5491	6.3742	2.1049	1.7255	3.4889
Portfolio 4		p-value	0	0	0	0.0177	0.0422	2e-04
Portfolio 5	Standardized	> 2	8	8	8	10	12	17
Portfolio 5	residuals	> 3	8	8	6	8	8	12
Portfolio 5	χ^2		706.87	851.26	262.78	473.68	573.94	348.18
Portfolio 5	MAPE	(%)	77.15	80.93	77.56	85.66	88.04	90.42
Portfolio 5	Likelihood	ξ^{LR}	64.56	65.02	52.53	56.7	58.91	50.61
Portfolio 5	ratio test	p-value	0.1133	0.1061	0.4534	0.3041	0.2374	0.5288
Portfolio 5		SMR	1.714	2.0544	1.8163	2.857	3.4243	5.0206
Portfolio 5	SMR test	ξ^{SMR}	2.4494	3.1986	2.6942	4.4512	5.0828	6.2982
Portfolio 5		p-value	0.0072	7e-04	0.0035	0	0	0
Portfolio 7	Standardized	> 2	55	55	15	21	21	11
Portfolio 7	residuals	> 3	45	45	8	3	3	9
Portfolio 7	χ^2		1593.26	1597.79	236.83	221.64	223.56	195.00
Portfolio 7	MAPE	(%)	620.28	621.10	95.95	135.39	135.71	37.25
Portfolio 7	Likelihood	ξ^{LR}	1448.73	1452.33	201.91	227.8	229.08	118.01
Portfolio 7	ratio test	p-value	0	0	0	0	0	1e-04
Portfolio 7		SMR	0.5775	0.5768	0.811	0.8455	0.844	0.8229
Portfolio 7	SMR test	ξ^{SMR}	35.3923	35.4792	2.74	10.1297	10.2409	11.8209
Portfolio 7		p-value	0	0	0	0	0	0
Portfolio 8	Standardized	> 2	65	65	37	50	50	29
Portfolio 8	residuals	> 3	63	63	29	29	29	29
Portfolio 8	χ^2		4987.77	5002.11	2485.39	2575.63	2583.90	2414.25
Portfolio 8	MAPE	(%)	788.87	790.77	292.03	323.78	324.61	263.21
Portfolio 8	Likelihood	ξ^{LR}	4970.14	4984.29	1891.04	2059.63	2066.49	1765.46
Portfolio 8	ratio test	p-value	0	0	0	0	0	0
Portfolio 8		SMR	0.1483	0.148	0.2404	0.2315	0.2311	0.2431
Portfolio 8	SMR test	ξ^{SMR}	82.2167	82.3412	56.2334	58.1115	58.2101	55.6816
Portfolio 8		p-value	0	0	0	0	0	0
Portfolio 9	Standardized	> 2	59	59	64	59	59	44
Portfolio 9	residuals	> 3	54	54	62	55	55	36
Portfolio 9	χ^2		4718.93	4718.46	1511.79	1572.53	1573.97	1502.87
Portfolio 9	MAPE	(%)	1124.24	1124.13	349.82	368.08	368.29	125.64
Portfolio 9	Likelihood	ξ^{LR}	4311.65	4311.18	1207.84	1283.35	1284.47	985.20
Portfolio 9	ratio test	p-value	0	0	0	0	0	0
Portfolio 9		SMR	0.2613	0.2613	0.4243	0.4232	0.423	0.4185
Portfolio 9	SMR test	ξ^{SMR}	70.5073	70.5015	41.3364	41.481	41.5056	42.0967
Portfolio 9		p-value	0	0	0	0	0	0

Table 9: *Tests and quantities summarizing the deviation between the observations and the model, calendar year* 2011*, male population.*

			INSEE			IA2103		
			Hardy-Panjer	Poisson-Gamma	Makeham-Credibility	Hardy-Panjer	Poisson-Gamma	Makeham-Credibility
Portfolio 10	Standardized	>2	50	50	3	5	5	5
	residuals	>3	34	33	1	1	1	1
	χ^2		635.65	638.50	74.42	115.82	116.47	97.88
	MAPE (%)		408.56	412.51	51.54	89.68	91.03	46.14
	Likelihood	ξ^{LR}	613.89	619.28	73.18	112.54	113.73	47.99
	ratio test	p-value	0	0	0.2542	3e-04	2e-04	0.9535
		SMR	0.5666	0.5617	0.9229	0.8708	0.8623	0.9596
	SMR test	ξ^{SMR}	15.1709	15.4268	1.9491	3.4132	3.6619	0.9826
		p-value	0	0	0.0256	3e-04	1e-04	0.1629
Portfolio 11	Standardized	>2	43	43	24	35	35	4
	residuals	>3	41	41	22	17	19	0
	χ^2		1379.61	1382.88	926.73	415.32	417.53	76.48
	MAPE (%)		299.83	297.80	555.88	152.87 151.69	46.97	
	Likelihood	ξ^{LR}	511.89	508.51	429.79	214.06	213.07	52.92
	ratio test	p-value	0	0	0	0	0	0.8779
		SMR	0.4927	0.4961	0.4443	0.8291	0.8369	1.0183
	SMR test	ξ^{SMR}	16.7405	16.5554	15.6387	4.0301	3.8212	0.3648
		p-value	0	0	0	0	1e-04	0.3576
Portfolio 12	Standardized	>2	35	35	21	10	11	4
	residuals	>3	16	15	16	1	1	0
	χ^2		470.25	471.58	263.73	130.05	129.23	90.74
	MAPE (%)		231.00	226.18	470.92	110.54	107.22	95.27
	Likelihood	ξ^{LR}	317.04	310.88	337.99	144.41	140.05	114.53
	ratio test	p-value	0	0	0	0	0	2e-04
		SMR	0.3668	0.3745	0.2039	0.5981	0.6188	0.5426
	SMR test	ξ^{SMR}	12.9497	12.6324	17.505	6.0626	5.624	7.3459
		p-value	0	1e-04	0	0	0	0
Portfolio 13	Standardized	>2	56	56	39	28	28	23
	residuals	>3	49	50	29	19	19	10
	χ^2		2058.43	2057.75	678.98	351.56	351.36	263.55
	MAPE (%)		589.24	588.36	245.40	180.91	180.61	54.62
	Likelihood	ξ^{LR}	1316.24	1314.45	414.88	237.69	237.35	141.71
	ratio test	p-value	0	0	0	0	0	0
		SMR	0.5092	0.5099	0.8679	0.8392	0.8404	0.8316
	SMR test	ξ^{SMR}	27.2355	27.1712	5.2064	6.4792	6.4261	6.8303
		p-value	0	0	0	0	0	0
Portfolio 14	Standardized	>2	48	48	21	24	24	7
	residuals	>3	36	36	5	5	6	0
	χ^2		862.31	862.27	248.72	227.86	227.95	85.92
	MAPE (%)		445.95	445.66	135.88	159.74	160.60	53.53
	Likelihood	ξ^{LR}	709.24	708.85	186.3	204.38	205.17	57.2
	ratio test	p-value	0	0	0	0	0	0.7717
		SMR	.5019	0.5022	0.9239	0.7916	0.7879	0.9385
	SMR test	ξ^{SMR}	16.5241	16.5063	1.6821	5.1598	5.2678	1.3381
		p-value	0	0	0.0463	0	0	0.0904

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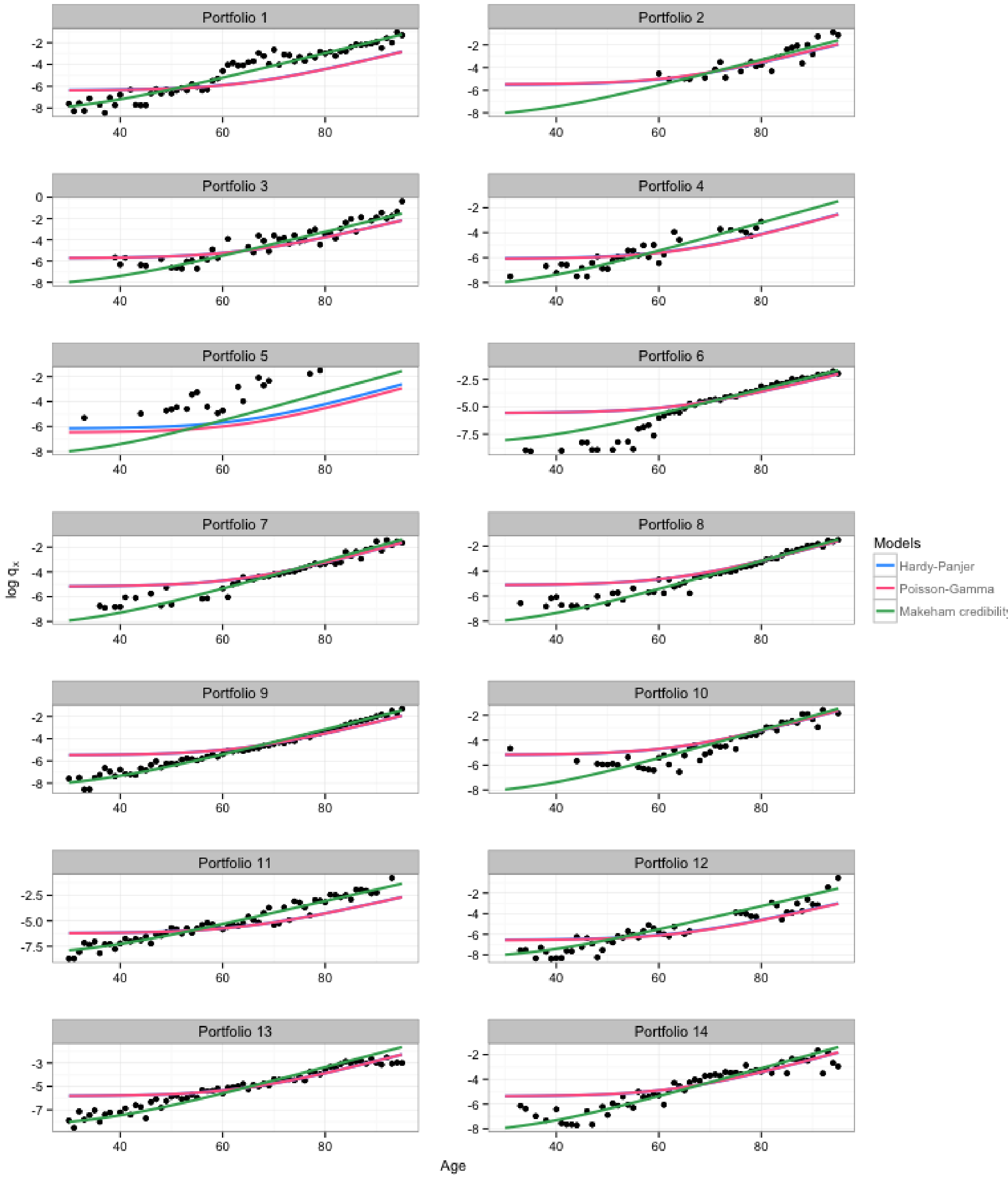

Figure 3: *Fitted probability of death, log scale, for the year 2010, male population*

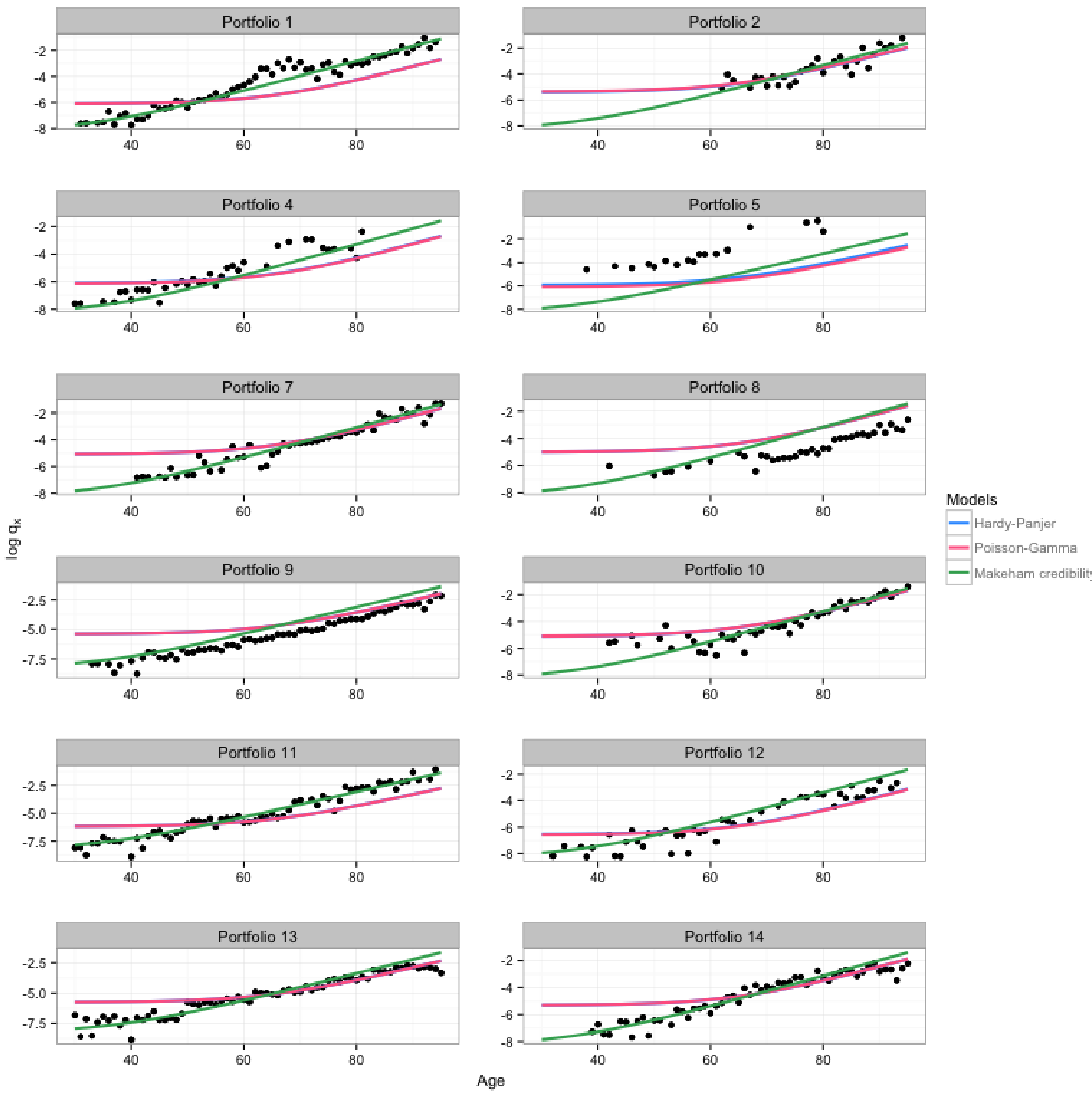

Figure 4: *Fitted probability of death, log scale, for the year 2011, male population*

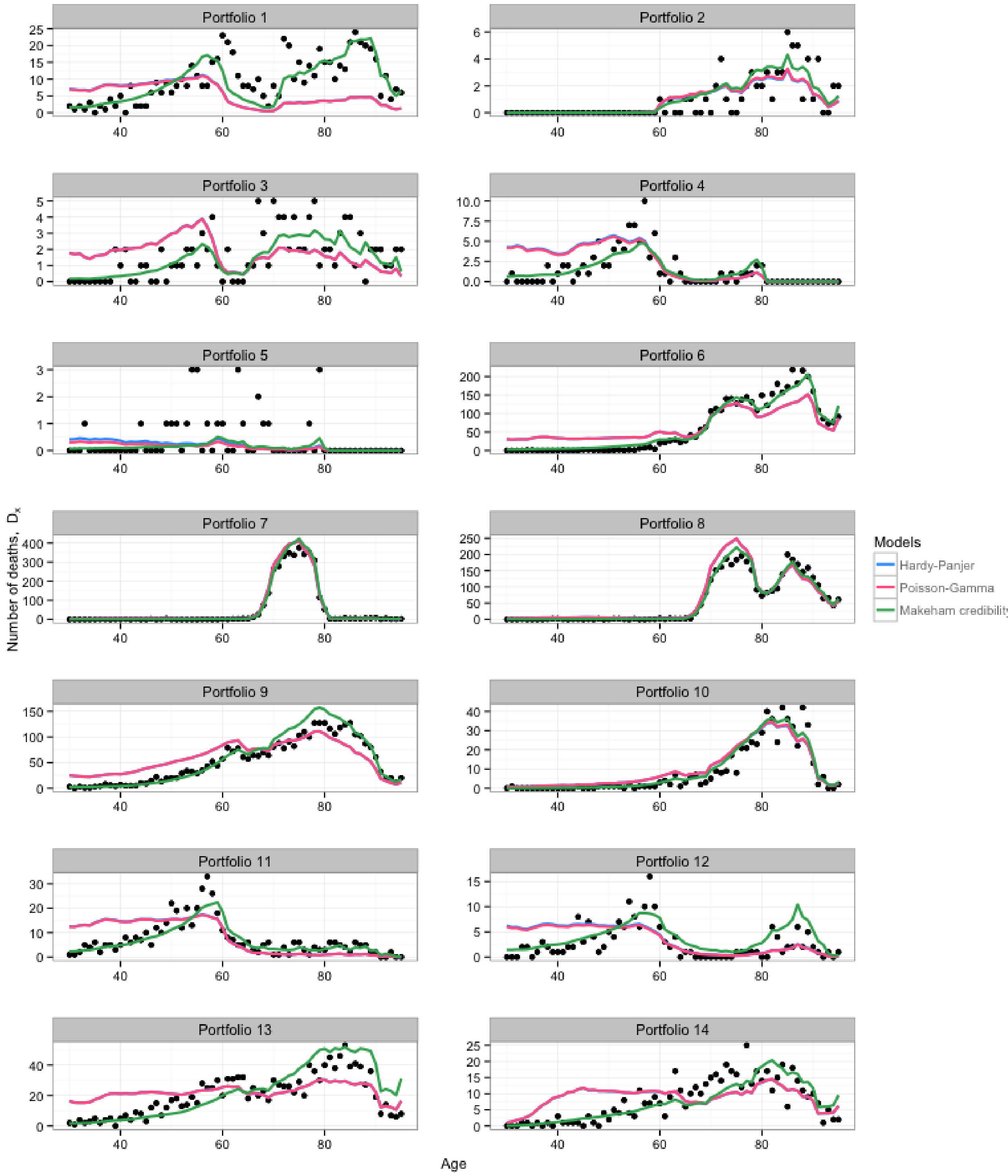

Figure 5: *Fitted number of deaths for the year 2010, male population*

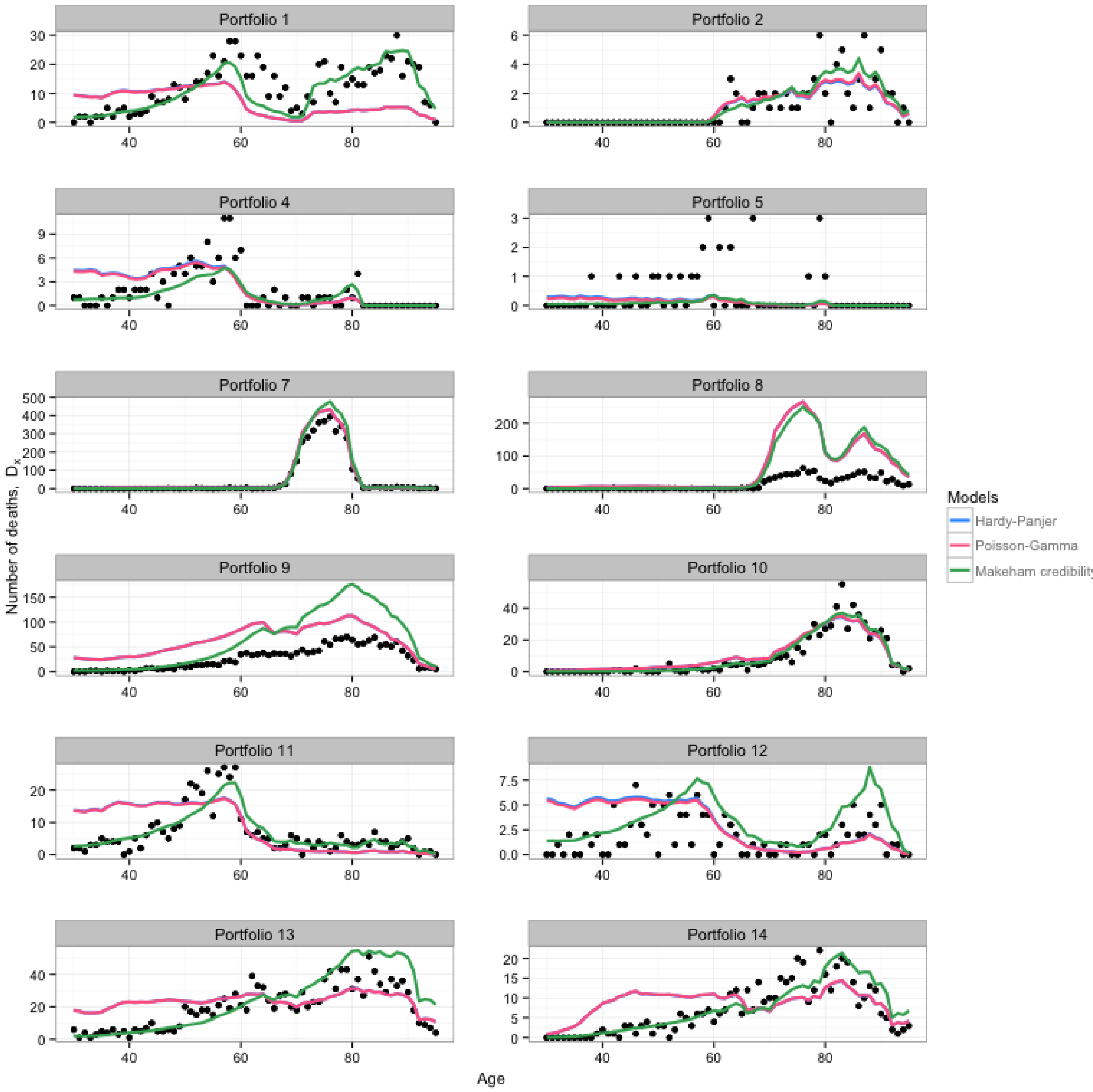

Figure 6: *Fitted number of deaths for the year 2011, male population*

Figure 7: *Standardized residuals, calendar year 2010, male population*

Figure 8: *Standardized residuals, calendar year 2011, male population*