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THE MAHLER MEASURE AND ITS AREAL ANALOG
FOR TOTALLY POSITIVE ALGEBRAIC INTEGERS.

V. FLAMMANG

Abstract
Using the method of explicit auxiliary functions, we first improve the known lower bounds of the absolute Mahler measure of totally positive algebraic integers. In 2008, I. Pritsker defined a natural areal analog of the Mahler measure that we call the Pritsker measure. We study the spectrum of the absolute Pritsker measure for totally positive algebraic integers and find the four smallest points. Finally, we give inequalities involving the Mahler measure and the Pritsker measure of totally positive algebraic integers. The polynomials involved in the auxiliary functions are found by our recursive algorithm.

1 Introduction

The Mahler measure of a polynomial $P(z) = a_0 z^n + \ldots + a_n = a_0 \prod_{j=1}^n (z - \alpha_j) \in \mathbb{C}[X]$, $a_0 \neq 0$, as defined by D. H. Lehmer [L] in 1933, is

$$M(P) = |a_0| \prod_{j=1}^n \max(1, |\alpha_j|).$$

In 1962, K. Mahler [M] gave the following definition

$$M(P) = \exp \left( \int_0^1 \log |P(e^{2\pi it})| dt \right),$$

which is equivalent to Lehmer’s definition by Jensen’s formula [J]

$$\int_0^1 \log |e^{2\pi it} - \alpha| dt = \log \max(1, |\alpha|).$$

If $\alpha$ is an algebraic integer, then the Mahler measure of $\alpha$, denoted by $M(\alpha)$, is the Mahler measure of its minimal polynomial $P$ in $\mathbb{Z}[z]$. The absolute Mahler measure of $\alpha$ is defined by

$$\Omega(\alpha) = M(\alpha)^{1/\deg(\alpha)}.$$ 

If $\alpha$ is an algebraic integer and $M(\alpha) = 1$, then a classical theorem of Kronecker [K] tells us that $\alpha$ is a root of unity. It suggests the question: $\inf_{\text{a root of unity}} M(\alpha) > 1$? It is known as the Lehmer’s problem and it is still open. Another formulation can be given as follows. Does there exist an absolute constant $c > 0$ such that: if $M(\alpha) > 1$ then $M(\alpha) > 1 + c$? The smallest known value is due to Lehmer himself and is $M(P) = 1.176280 \ldots$ where $P(z) = z^{10} + z^9 - z^7 - z^6 - z^5 - z^4 - z^3 + z + 1$.

In this paper, we are interested in totally positive algebraic integers $\alpha$, i.e., algebraic integers all of whose conjugates are positive real numbers. Let $\mathcal{L}$ be the set of the quantities $\Omega(\alpha)$ where $\alpha$ is a totally positive algebraic integer. In 1973, A. Schinzel [Sc] showed that all totally positive algebraic integers $\alpha$, different from 0 and 1, satisfy $\Omega(\alpha) \geq \frac{1 + \sqrt{5}}{2}$ and the equality holds if $\alpha$ is a root of the polynomial $x^2 - 3x + 1$. It means that $\frac{1 + \sqrt{5}}{2}$ is the smallest element of
the spectrum of the absolute Mahler measure of totally positive algebraic integers. In 1981, C. Smyth [Sm1] showed that, if \( \alpha \) is a totally positive algebraic integer, then, with a finite set of exceptions, \( \Omega(\alpha) \geq 1.717177 \ldots \). His result uses the method of explicit auxiliary functions with heuristic search of polynomials and allows him to find the three following points of \( \mathcal{L} \). He also showed that \( \mathcal{L} \) is dense in \([1.727305, \ldots, \infty)\). In 1994, with the same method and thanks to numerical improvements, we obtained [F1] that, if \( \alpha \) is a positive algebraic integer, then, with a finite set of exceptions, \( \Omega(\alpha) \geq 1.720678 \ldots \). This lower bound gives the two following points of the spectrum. Our recursive algorithm, developed in [F2] from Wu’s algorithm [Wu] substitutes the heuristic search with a systematic search by induction of suitable polynomials used in the auxiliary functions. We prove the following

**Theorem 1.** If \( \alpha \) is a nonzero totally positive algebraic integer whose minimal polynomial is different from \( x-1, x^2-3x+1, x^3-7x^2+13x^2-7x+1, x^6-11x^5+41x^4-63x^3+41x^2-11x+1, x^8-15x^7+83x^6-220x^5+303x^4-220x^3+83x^2-15x+1, x^8-15x^7+84x^6-225x^5+311x^4-225x^3+84x^2-15x+1 \) and \( x^{16}-31x^{15}+413x^{14}-3141x^{13}+15261x^{12}-50187x^{11}+115410x^{10}-189036x^9+222621x^8-189036x^7+115410x^6-50187x^5+15261x^4-3141x^3+413x^2-31x+1 \), then we have

\[
\Omega(\alpha) \geq 1.722069.
\]

This constant improves those of Q. Mu and Q. Wu [MW] (2013) and is the best constant to our knowledge.

The polynomials involved in this result are read from Table 1.

We conjecture that the following point of the spectrum has the minimal polynomial \( x^6-12x^5+44x^4-67x^3+44x^2-12x+1 \) and the Mahler measure 1.722325.

Our works on the Mahler measure lead us naturally to be interested in a new height of polynomials introduced by I. Pritsker [P] in 2008. He replaced the normalized arc length measure on the unit circumference by the normalized areal measure on the unit disk \( D \) and so, for \( P \in \mathbb{C}[z] \), defined:

\[
||P||_0 = \exp \left( \frac{1}{\pi} \int \int_D \log |P(z)| \ dA \right).
\]

This height is the natural areal analog of the Mahler measure and we call it the Pritsker measure.

Let \( P(z) = a_n \prod_{i=1}^{d}(z - \alpha_i) = \sum_{k=1}^{d} a_k z^k \in \mathbb{C} \). He showed that

\[
||P||_0 = M(P) \exp \left( \frac{1}{2} \sum_{|\alpha_i| < 1} (|\alpha_i|^2 - 1) \right).
\]

From this identity, he obtained immediately the following inequalities:

\[
\forall P \in \mathbb{C}[z], \ e^{-d/2}M(P) \leq ||P||_0 \leq M(P).
\]

Moreover, he proved that an irreducible polynomial \( P(z) \in \mathbb{Z}[z] \) with \( P(0) \neq 0 \) is cyclotomic if and only if \( ||P||_0 = 1 \). This result is a direct analog of Kronecker’s theorem for the Mahler measure and suggests the question: does there exist an absolute constant \( c > 0 \) such that, if \( ||P||_0 > 1 \), then \( ||P||_0 > 1+c! \)? The answer to this question is negative. Consider the polynomials \( P_n(z) = n z^n - 1 \). Pritsker showed that \( ||P_n||_0 \to 1 \) as \( n \to \infty \).
Pritsker’s works inspired firstly S. Choi and C. Samuels [CS]. In 2012, they improved two inequalities in the case of polynomials with “small” Pritsker measure. Secondly, in 2013, H. Huang [H] defined, among others, a Fock-space analog of the Mahler measure for which he gives an equivalent version of Lehmer’s conjecture.

If \( \alpha \) is an algebraic integer, then the Pritsker measure of \( \alpha \), denoted by \( ||\alpha||_0 \), is the Pritsker measure of its minimal polynomial. The absolute Pritsker measure of \( \alpha \) is defined by \( N_0(\alpha) = ||\alpha||_0^{1/\deg(\alpha)} \).

In this paper, we study the set of the quantities \( N_0(\alpha) \) where \( \alpha \) is a totally positive algebraic integer. We prove the following

**Theorem 2.** If \( \alpha \) is a totally algebraic integer whose minimal polynomial is different from \( x, x-1, x^2-3x+1, x^3-6x^2+5x-1, x^4-7x^3+13x^2-7x+1 \) and \( x^5-11x^4+29x^3-26x^2+9x-1 \), then we have:

\[
N_0(\alpha) \geq 1.380047.
\]

From this results derive the four smallest points of the spectrum:

1. \( 1.306937 \ldots = N_0(x^2-3x+1) \)
2. \( 1.337938 \ldots = N_0(x^3-6x^2+5x-1) \)
3. \( 1.370929 \ldots = N_0(x^4-7x^3+13x^2-7x+1) \)
4. \( 1.375486 \ldots = N_0(x^5-11x^4+29x^3-26x^2+9x-1) \)

We conjecture that the following point has minimal polynomial \( x^5-13x^4+32x^3-27x^2+9x-1 \) and absolute Pritsker measure 1.3816081 \ldots.

After studying these two measures separately, it is natural to compare them. We prove the following inequalities:

**Theorem 3.** If \( \alpha \) is a totally positive algebraic integer of degree \( d \) whose minimal polynomial is different from \( x, x-1, x-2, 2x-1, x^2-3x+1, x^2-5x+5, x^3-8x^2+6x-1, x^3-6x^2+5x-1 \) and \( x^3-5x^2+6x-1 \), then we have:

\[
1.292012^d M(P)^0.104208 \leq ||P||_0 \leq 0.637361^d M(P)^1.55575.
\]

In Section 2, we explain the method of explicit auxiliary functions. We link them with a generalization of the classical integer transfinite diameter. Then we detail how our recursive algorithm enables us to get the constant of Theorem 1 for the Mahler measure. Section 3 deals with the Pritsker measure. In Section 4, we only give the auxiliary functions involved in the above inequalities. For the rest of the proof, we proceed as for those of Theorem 1. Finally, we give some numerical examples where our inequalities are better than Pritsker’s ones. All the computations are done on a MacBookPro with the languages Pari and Pascal.

## 2 The Mahler measure of totally positive algebraic integers

### 2.1 The explicit auxiliary function

Let \( \alpha \) be a totally positive algebraic integer, \( \alpha = \alpha_1, ..., \alpha_d \) be its conjugates and \( P \) its minimal polynomial.
The auxiliary function involved in Theorem 1 has the following type:

for $x > 0$, $f(x) = \log \max(1, x) - c_0 \log x - \sum_{1 \leq j \leq J} c_j \log |Q_j(x)| \geq m$

where the $c_j$ are positive real numbers and the polynomials $Q_j$ are non-zero polynomials in $\mathbb{Z}[x]$. Then we have

$$\sum_{i=1}^{d} f(\alpha_i) \geq md$$

i.e.,

$$\log M(\alpha) \geq md + \sum_{1 \leq j \leq J} c_j \log |\prod_{i=1}^{d} Q_j(\alpha_i)|.$$  

We assume that $P$ does not divide any $Q_j$, then $\prod_{i=1}^{d} Q_j(\alpha_i)$ is a nonzero integer because it is the resultant of $P$ and $Q_j$.

Therefore, if $\alpha$ is not a root of $Q_j$, we have

$$\Omega(\alpha) \geq e^m.$$  

It is possible to reduce the domain of study of the function $f$. If we consider the function $g(x) = 1/2[f(x) + f(1/x)]$, we get a minimum greater or equal to that given by $f$. But $g$ is invariant under the application $x \to 1/x$ so it is sufficient to study $g$ on $(0,1)$. Thus, without loss of generality, we can limit our study to auxiliary functions invariant under this transformation. This implies that we can take for $Q_j$ reciprocal polynomials, i.e., $Q_j(x) = x^{\deg Q_j}Q_j(1/x)$. The condition $f(x) = f(1/x)$ gives $2c_0 + \sum_{1 \leq j \leq J} c_j \deg(Q_j) = 1$.

We denote $\deg(Q_j) = 2d_j$ for $1 \leq j \leq J$.

On $(0,1)$, the auxiliary function $f$ can be written

$$f(x) = -c_0 \log x - \sum_{1 \leq j \leq J} c_j \log \left| \frac{Q_j(x)}{x^{d_j}} \right| - \sum_{1 \leq j \leq J} c_j \log x^{d_j} \geq m.$$  

Thus, if we put $y = x + \frac{1}{x} - 2$ and $x = (2 + y - \sqrt{y^2 + 4y})/2$, $f(x)$ becomes

for $y > 0$, $g(y) = \frac{1}{2} \log x - \sum_{1 \leq j \leq J} c_j \log |U_j(y)| \geq m \quad (1)$

where $\deg(U_j) = d_j$. Moreover, we impose the condition $\sum_{1 \leq j \leq J} c_j \deg(U_j) < \frac{1}{2}$ in order that the supremum of the function $g$ to be finite.

The main problem is to find a good list of polynomials $Q_j$ which gives a value of $m$ as large as possible. Thus, we link the auxiliary function with a generalization of the integer transfinite diameter in order to find the polynomials with our recursive algorithm.
2.2 Auxiliary functions and integer transfinite diameter

In this section, we shall need the following definition: Let \( K \) be a compact subset of \( \mathbb{C} \).

If \( \varphi \) is a positive function defined on \( K \), the \( \varphi \)-generalized integer transfinite diameter of \( K \) is defined as

\[
t_{Z,\varphi}(K) = \lim_{n \to \infty} \inf \, \sup_{P \in \mathbb{Z}[X], \ y \in K} \left( \frac{1}{n} \varphi(x) \right).
\]

This weighted version of the integer transfinite diameter was introduced by F. Amoroso [A] and is an important tool in the study of rational approximations of logarithms of rational numbers.

Inside the auxiliary function (1), we replace the numbers \( c_j \) by rational numbers \( a_j/q \) where \( q \) is a common denominator of the \( c_j \) for \( 1 \leq j \leq J \). Then we can write:

\[
\text{for } y > 0, \ g(y) = -\frac{1}{2} \log x - \frac{r}{t} \log |Q(y)| \geq m \quad (2)
\]

where \( Q = \prod_{j=1}^{J} U_j^{a_j} \in \mathbb{Z}[X] \) is of degree \( r = \sum_{i=1}^{J} a_i \deg U_j \) and \( t = \sum_{i=1}^{J} c_j \deg U_j \) is a positive rational number < 1/2. We want to get a function whose minimum \( m \) is as large as possible. Thus we search a polynomial \( Q \in \mathbb{Z}[X] \) such that

\[
\sup_{y>0} |Q(y)|^{t/r} x^{1/2} \leq e^{-m}.
\]

If we suppose that \( t < 1/2 \) is fixed, it is clear that we need an effective upper bound for the quantity

\[
t_{Z,\varphi}((0, \infty)) = \lim_{n \to \infty} \inf \, \sup_{P \in \mathbb{Z}[X], \ y > 0} \left( \frac{1}{n} \varphi(x) \right)
\]

where we use the weight \( \varphi(x) = x^{1/2} \).

Even if we replace the compact subset \( K \) by the infinite interval \((0, \infty)\), the weight \( \varphi \) ensures that the quantity \( t_{Z,\varphi}((0, \infty)) \) is finite.

2.3 Construction of the auxiliary function

The improvement compared with Wu’s algorithm is that our polynomials are obtained by induction. Suppose that we have \( Q_1, Q_2, ..., Q_J \). Then we use the semi-infinite linear programming (introduced into number theory by C. Smyth [Sm2]) to optimize \( f \) for this set of polynomials (i.e., to get the greatest possible \( m \)). We obtain the numbers \( c_1, c_2, ..., c_J \) and \( f \) in the form (2)

with \( t = \sum_{i=1}^{J} c_j \deg(Q_j) \).

For several value of \( k \), we search a polynomial \( R(y) = \sum_{l=0}^{k} a_l y^l \in \mathbb{Z}[y] \) such that

\[
\sup_{y>0} |Q(y)R(y)|^{t/r} x^{1/2} \leq e^{-m},
\]

i.e., such that

\[
\sup_{y>0} |Q(y)R(y)|x^{(r+k)/2t}
\]

is as small as possible.
We apply LLL to the linear forms
\[ Q(y_i)V(y_i)x_i^{(r+k)/2t} \]
where \( x_i \) are control points uniformly distributed in the interval (0,1) and \( y_i = x_i + 1/x_i - 2 \), including the points where \( f \) has its least local minima. We get a polynomial \( R \) whose factors \( R_j \) are good candidates to enlarge the set of polynomials \( (Q_1, Q_2, ... , Q_J) \). We only keep the polynomials \( R_j \) which have a nonzero coefficient \( c_j \) in the new optimized auxiliary function \( f \). After optimization, some previous polynomials \( Q_j \) may have a zero coefficient \( c_j \) and so are removed.

In order to get the constant of Theorem 1, we take \( k \) from 8 to 20 successively.

3 The Pritsker measure of totally positive algebraic integers

The auxiliary function involved in Theorem 2 is of the following type:

\[
\text{for } x > 0, \quad f(x) = \log \max(1, x) + \frac{1}{2} \left( \min(1, x^2) - 1 \right) - c_0 \log x - \sum_{1 \leq j \leq J} c_j \log |Q_j(x)| \geq m
\]

where the \( c_j \) are positive real numbers and the polynomials \( Q_j \) are non zero polynomials in \( \mathbb{Z}[x] \).

Thus, for several values of \( k \), we seek a polynomial \( R(x) = \sum_{l=0}^{k} a_l x^l \in \mathbb{Z}[x] \) such that

\[
\sup_{x > 0} |Q(x)R(x)| \left| \frac{1}{t \pi} \right| \left| \max(1, x) \exp \left( \frac{1}{2} \left( \min(1, x^2) - 1 \right) \right) \right|^{-1} \leq e^{-m}
\]

i.e.,

\[
\sup_{x > 0} |Q(x)R(x)| \left| \max(1, x) \exp \left( \frac{1}{2} \left( \min(1, x^2) - 1 \right) \right) \right|^{(r+k)/t}
\]

is as small as possible.

We apply LLL to the linear forms

\[
|Q(x_i)R(x_i)| \left| \max(1, x_i) \exp \left( \frac{1}{2} \left( \min(1, x_i^2) - 1 \right) \right) \right|^{-(r+k)/t}
\]

where the \( x_i \) are control points uniformly distributed in the interval (0,70).

Then, we proceed as described in Section 1.

To obtain the constant of Theorem 2, we take \( k \) from 4 to 35.

4 Comparison of the two measures

The auxiliary function used for the lower bound of Theorem 3 is of the following type:

\[
\text{for } x > 0, \quad f(x) = \log \max(1, x) + \frac{1}{2} \left( \min(1, x^2) - 1 \right) - c_0 \log \max(1, x) - \sum_{1 \leq j \leq J} c_j \log |Q_j(x)| \geq m
\]

where the \( c_j \) are positive real numbers and the polynomials \( Q_j \) are non zero polynomials in \( \mathbb{Z}[x] \).
The auxiliary function used for the upper bound of Theorem 3 is of the following type:

\[ f(x) = -\log \max(1, x) - \frac{1}{2} (\min(1, x)^2 - 1) + c_0 \log \max(1, x) - \sum_{1 \leq j \leq J} c_j \log |Q_j(x)| \geq m \]

where the \(c_j\) are positive real numbers and the polynomials \(Q_j\) are non zero polynomials in \(\mathbb{Z}[x]\).

In our recursive algorithm, we take \(k\) from 4 to 10 only in order to have a small number of exceptions in the inequalities.

We consider the totally positive algebraic integers that appeared in the proof of Theorem 1. For each of them, we compute the bounds obtained by I. Pritsker and our bounds. The results are recorded in Table 3. We see that our bounds are better than Pritsker’s ones for this set of totally positive algebraic integers.
Table 1: Polynomials involved in Theorem 1 with their coefficients.

<table>
<thead>
<tr>
<th>j</th>
<th>$c_j$</th>
<th>$U_j$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.15896787</td>
<td>z</td>
</tr>
<tr>
<td>2</td>
<td>0.04241521</td>
<td>z – 1</td>
</tr>
<tr>
<td>3</td>
<td>0.00020790</td>
<td>z – 2</td>
</tr>
<tr>
<td>4</td>
<td>0.01491572</td>
<td>z^2 – 3z + 1</td>
</tr>
<tr>
<td>5</td>
<td>0.00092761</td>
<td>z^2 – 4z + 1</td>
</tr>
<tr>
<td>6</td>
<td>0.00251285</td>
<td>z^3 – 5z^2 + 6z – 1</td>
</tr>
<tr>
<td>7</td>
<td>0.0125715</td>
<td>z^3 – 6z^2 + 5z – 1</td>
</tr>
<tr>
<td>8</td>
<td>0.00520917</td>
<td>z^4 – 7z^3 + 13z^2 – 7z + 1</td>
</tr>
<tr>
<td>9</td>
<td>0.00012028</td>
<td>z^4 – 8z^3 + 15z^2 – 8z + 1</td>
</tr>
<tr>
<td>10</td>
<td>0.0133227</td>
<td>z^4 – 7z^3 + 14z^2 – 8z + 1</td>
</tr>
<tr>
<td>11</td>
<td>0.00039481</td>
<td>z^4 – 8z^3 + 14z^2 – 7z + 1</td>
</tr>
<tr>
<td>12</td>
<td>0.00018247</td>
<td>z^5 – 11z^4 + 42z^3 – 67z^2 + 45z^2 – 12z + 1</td>
</tr>
<tr>
<td>13</td>
<td>0.00035109</td>
<td>z^7 – 13z^6 + 61z^5 – 13z^4 + 136z^3 – 66z^2 + 14z – 1</td>
</tr>
<tr>
<td>14</td>
<td>0.00020876</td>
<td>z^7 – 13z^6 + 61z^5 – 13z^4 + 136z^3 – 66z^2 + 14z – 1</td>
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<tr>
<td>15</td>
<td>0.00006016</td>
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</tr>
<tr>
<td>16</td>
<td>0.00016459</td>
<td>z^7 – 13z^6 + 62z^5 – 130z^4 + 40z^3 – 67z^2 + 14z – 1</td>
</tr>
<tr>
<td>17</td>
<td>0.00016285</td>
<td>z^7 – 12z^6 + 55z^5 – 120z^4 + 129z^3 – 65z^2 + 14z – 1</td>
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<tr>
<td>18</td>
<td>0.00134672</td>
<td>z^8 – 15z^7 + 85z^6 – 220z^5 + 303z^4 – 220z^3 + 83z^2 – 15z + 1</td>
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<tr>
<td>19</td>
<td>0.00007718</td>
<td>z^8 – 15z^7 + 85z^6 – 242z^5 + 328z^4 – 242z^3 + 91z^2 – 16z + 1</td>
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<tr>
<td>20</td>
<td>0.00014420</td>
<td>2z^8 – 267z^7 + 128z^6 – 308z^5 + 391z^4 – 265z^3 + 94z^2 – 16z + 1</td>
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<td>21</td>
<td>0.00012982</td>
<td>z^11 – 21z^10 + 182z^9 – 853z^8 + 2386z^7 – 4151z^6 + 4545z^5 – 5309z^4 + 1298z^3 – 315z^2 + 40z – 2</td>
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<tr>
<td>22</td>
<td>0.00003860</td>
<td>z^13 – 21z^12 + 181z^11 – 837z^10 + 2287z^9 – 3841z^8 + 4002z^7 – 2557z^6 + 973z^5 – 209z^4 + 23z^3 – 1</td>
</tr>
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<td>23</td>
<td>0.00001784</td>
<td>2z^{12} – 42z^{11} + 369z^{10} – 1778z^9 + 5192z^8 – 9614z^7 + 11504z^6 – 8919z^5 + 4435z^4 – 1378z^3 + 254z^2 – 25z + 1</td>
</tr>
<tr>
<td>24</td>
<td>0.00008883</td>
<td>z^{13} – 25z^{12} + 266z^{11} – 1582z^{10} + 5817z^9 – 13835z^8 + 21682z^7 – 22449z^6 + 15233z^5 – 46667z^4 + 18333z^3 – 302z^2 + 27z – 1</td>
</tr>
<tr>
<td>25</td>
<td>0.00026384</td>
<td>z^{10} – 31z^9 + 413z^8 – 3411z^7 + 15261z^6 – 50187z^5 + 115410z^4 – 189036z^3 + 222621z^2 – 189036z^2 + 115410z^3 – 50187z^4 + 15261z^5 – 3141z^6 – 31z^7 + 1</td>
</tr>
<tr>
<td>26</td>
<td>0.00008705</td>
<td>z^{17} – 33z^{16} + 482z^{15} – 4118z^{14} + 22929z^{13} – 87820z^{12} + 238254z^{11} – 465043z^{10} + 657575z^9 – 674220z^8 + 499573z^7 – 265657z^6 + 100250z^5 – 26365z^4 + 4688z^3 – 53z^2 + 35z – 1</td>
</tr>
<tr>
<td>27</td>
<td>0.00007497</td>
<td>z^{10} – 36z^9 + 583z^8 – 5620z^7 + 36130z^6 – 163492z^5 + 53804z^4 – 131216z^3 + 239543z^2 + 328762z^1 + 309277z^0 – 2624445z^8 + 1512731z^7 – 643804z^6 + 199609z^5 – 44154z^4 + 6741z^3 – 671z^2 + 69z – 1</td>
</tr>
</tbody>
</table>
Table 2: Polynomials involved in Theorem 2 with their coefficient.
\[ x^{11} - 34x^{12} + 435x^{11} - 2711x^{10} + 9425x^9 - 19827x^8 + 26588x^7 - 23525x^6 + 13991x^5 - 5606x^4 + 1488x^3 - 250x^2 + 24x - 1 \]

\[ x^{11} - 29x^{12} + 328x^{11} - 1895x^{10} + 6311x^9 - 13029x^8 + 17502x^7 - 15800x^6 + 9749x^5 - 4114x^4 + 1165x^3 - 211x^2 + 22x - 1 \]

\[ x^{11} - 36x^{12} + 462x^{11} - 2852x^{10} + 9799x^9 - 20384x^8 + 27079x^7 - 23786x^6 + 14073x^5 - 5629x^4 + 1489x^3 - 250x^2 + 24x - 1 \]

\[ x^{11} - 32x^{12} + 390x^{11} - 2419x^{10} + 8586x^9 - 18561x^8 + 25487x^7 - 22953x^6 + 13816x^5 - 5577x^4 + 1486x^3 - 250x^2 + 24x - 1 \]

\[ x^{11} - 31x^{12} + 385x^{11} - 2533x^{10} + 9924x^9 - 24695x^8 + 40656x^7 - 45466x^6 + 35083x^5 - 18777x^4 + 6923x^3 - 1718x^2 + 273x^2 - 25x + 1 \]

\[ x^{11} - 34x^{12} + 463x^{11} - 381x^{12} + 1494x^{11} - 42607x^{10} + 81569x^9 - 107671x^8 + 9766x^7 - 65514x^6 + 30501x^5 - 959x^4 + 2221x^3 - 321x^2 + 27x - 1 \]

\[ x^{11} - 38x^{12} + 545x^{11} - 4031x^{10} + 17628x^9 - 49170x^8 + 91713x^9 - 117996x^8 + 106826x^7 - 6876x^6 + 31492x^5 - 10150x^4 + 2242x^3 - 322x^2 + 27x - 1 \]

\[ x^{11} - 38x^{12} + 604x^{11} - 539x^{10} + 29359x^9 - 106776x^8 + 267161x^7 - 472631x^6 + 602676x^5 - 561207x^4 + 384361x^3 - 193778x^2 + 71447x^1 - 18966x^4 + 3510x^3 - 429x^2 + 31x - 1 \]

\[ x^{11} - 44x^{12} + 81x^{11} - 836x^{10} + 54453x^9 - 23925x^8 + 739805x^7 - 1655854x^6 + 2735704x^5 - 3382218x^4 + 3157621x^3 - 2236949x^2 + 1203101x^1 - 489025x^0 + 148906x^5 - 33106x^4 + 5230x^3 - 553x^2 + 35x - 1 \]

\[ x^{20} - 47x^{19} + 931x^{18} - 10364x^{17} + 73053x^{16} - 348423x^{15} + 1173249x^{14} - 2872845x^{13} + 9226603x^{12} - 7178644x^{11} + 7300013x^{10} - 6078050x^9 + 3788444x^8 - 18227289x^7 + 672900x^6 - 983607x^5 + 39165x^4 - 5814x^3 + 90x^2 - 36x + 1 \]

\[ x^{20} - 48x^{19} + 972x^{18} - 11057x^{17} + 79497x^{16} - 385761x^{15} + 1317937x^{14} - 3251303x^{13} + 993613x^{12} - 8296063x^{11} + 8175572x^{10} - 7035412x^9 + 4371546x^8 - 2089175x^7 + 763517x^6 - 210855x^5 + 43105x^4 - 6304x^3 + 622x^2 - 37x + 1 \]

\[ x^{20} - 47x^{19} + 93x^{18} - 10468x^{17} + 74154x^{16} - 35968x^{15} + 1198175x^{14} - 2938041x^{13} + 534852x^{12} - 734596x^{11} + 700694x^{10} - 6208329x^9 + 3863186x^8 - 1854174x^7 + 682581x^6 - 190467x^5 + 39468x^4 - 5870x^3 + 591x^2 - 36x + 1 \]

\[ x^{20} - 48x^{19} + 966x^{18} - 10871x^{17} + 77142x^{16} - 369133x^{15} + 1243595x^{14} - 303999x^{13} + 551254x^{12} - 753822x^{11} + 7865425x^{10} - 6312278x^9 + 3911092x^8 - 1870040x^7 + 686243x^6 - 191024x^5 + 9518x^4 - 5872x^3 + 591x^2 - 36x + 1 \]

\[ x^{20} - 47x^{19} + 93x^{18} - 10437x^{17} + 74205x^{16} - 355205x^{15} + 1202014x^{14} - 2953319x^{13} + 5382278x^{12} + 7935811x^{11} + 770793x^{10} - 6243945x^9 + 3881218x^8 - 1800622x^7 + 844168x^6 - 190722x^5 + 39492x^4 - 581x^3 + 591x^2 - 36x + 1 \]
Table 3: Comparison of Pritsker’s bounds and the author’s bounds. $P$ denote the minimal polynomial of a totally positive algebraic integer.

| $P$ | $\exp(-d/2) M(P)$ | $1.292012 M(P)^{0.104208}$ | $||P||_0$ | $0.6373614 M(P)^{1.553575}$ | $M(P)$ |
|-----|-------------------|---------------------------|---------|----------------------------|---------|
| $P_1$ | 1.2560190 | 1.8971700 | 2.5485223 | 2.73702205 | 3.4142136 |
| $P_2$ | 1.6030410 | 2.6487513 | 3.2037594 | 5.5412214 | 7.1842101 |
| $P_3$ | 1.0997725 | 3.4638739 | 3.5323256 | 4.2290481 | 8.0064659 |
| $P_4$ | 1.1872782 | 3.3928090 | 3.6776663 | 4.7887919 | 8.7396813 |
| $P_5$ | 1.2816200 | 4.7940717 | 4.9253736 | 7.5181727 | 15.613331 |
| $P_6$ | 1.3637370 | 4.8251981 | 5.0341295 | 8.2796922 | 16.613718 |
| $P_7$ | 1.3628782 | 4.8249215 | 5.2018670 | 8.2720203 | 16.604583 |
| $P_8$ | 1.4594204 | 9.0027274 | 10.204362 | 17.670141 | 48.329366 |
| $P_9$ | 1.4594204 | 9.0027274 | 10.204362 | 17.670141 | 48.329366 |
| $P_{10}$ | 1.8860326 | 9.2465478 | 11.649260 | 26.318821 | 62.458623 |
| $P_{11}$ | 1.4012628 | 8.9646575 | 10.125036 | 16.588582 | 46.403452 |
| $P_{12}$ | 1.9127553 | 9.2601144 | 14.584773 | 26.900422 | 63.341755 |
| $P_{13}$ | 1.7598460 | 9.1795726 | 11.320272 | 23.615559 | 58.248292 |
| $P_{14}$ | 1.7888136 | 9.1596936 | 11.281170 | 24.241466 | 59.237570 |
| $P_{15}$ | 1.7428906 | 12.482526 | 14.412289 | 32.266459 | 95.158601 |
| $P_{16}$ | 1.6630924 | 12.421712 | 13.695129 | 30.000620 | 90.801766 |
| $P_{17}$ | 2.0587843 | 12.701090 | 15.740401 | 41.796323 | 112.40582 |
| $P_{18}$ | 1.7442265 | 12.483523 | 14.327744 | 32.304892 | 95.231542 |
| $P_{19}$ | 1.8286485 | 12.545188 | 14.901925 | 34.767411 | 99.842791 |
| $P_{20}$ | 1.7614652 | 23.151136 | 25.727951 | 63.006289 | 261.42461 |
| $P_{21}$ | 1.8901100 | 45.110638 | 49.272933 | 130.61455 | 746.38764 |
| $P_{22}$ | 2.0192595 | 43.505462 | 52.161849 | 149.63011 | 814.62741 |

where

$P_1 = x^2 - 4x + 2$,
$P_2 = x^3 - 8x^2 + 6x - 1$,
$P_3 = x^4 - 7x^3 + 13x^2 - 7x + 1$,
$P_4 = x^4 - 8x^3 + 14x^2 - 7x + 1$,
$P_5 = x^5 - 11x^4 + 29x^3 - 26x^2 + 9x - 1$,
$P_6 = x^6 - 13x^5 + 32x^4 - 27x^3 + 9x - 1$,
$P_7 = x^6 - 12x^5 + 31x^4 - 27x^3 + 9x - 1$,
$P_8 = x^7 - 16x^6 + 75x^5 - 148x^4 + 137x^3 - 62x^2 + 13x - 1$,
$P_9 = x^7 - 16x^6 + 75x^5 - 148x^4 + 137x^3 - 62x^2 + 13x - 1$,
$P_{10} = x^7 - 18x^6 + 89x^5 - 172x^4 + 150x^3 - 64x^2 + 13x - 1$,
$P_{11} = x^7 - 14x^6 + 66x^5 - 136x^4 + 131x^3 - 61x^2 + 13x - 1$,
$P_{12} = x^7 - 17x^6 + 90x^5 - 201x^4 + 214x^3 - 115x^2 + 30x - 3$,
$P_{13} = x^7 - 16x^6 + 78x^5 - 157x^4 + 143x^3 - 63x^2 + 13x - 1$,
$P_{14} = x^7 - 16x^6 + 80x^5 - 160x^4 + 144x^3 - 63x^2 + 13x - 1$,
$P_{15} = x^8 - 19x^7 + 121x^6 - 312x^5 + 386x^4 - 251x^3 + 87x^2 - 15x + 1$,
$P_{16} = x^8 - 21x^7 + 124x^6 - 309x^5 + 378x^4 - 246x^3 + 86x^2 - 15x + 1$,
$P_{17} = x^8 - 21x^7 + 130x^6 - 334x^5 + 407x^4 - 259x^3 + 88x^2 - 15x + 1$,
$P_{18} = x^8 - 21x^7 + 128x^6 - 321x^5 + 391x^4 - 252x^3 + 87x^2 - 15x + 1$,
$P_{19} = x^8 - 23x^7 + 139x^6 - 342x^5 + 409x^4 - 259x^3 + 88x^2 - 15x + 1$,
$P_{20} = x^{10} - 24x^9 + 194x^8 - 743x^7 + 1526x^6 - 1798x^5 + 1265x^4 - 537x^3 + 134x^2 - 18x + 1$,
$P_{21} = x^{12} - 26x^{11} + 265x^{10} - 1388x^9 + 4177x^8 - 7677x^7 + 8944x^6 - 6752x^5 + 3322x^4 - 1050x^3$
$+ 204x^2 - 22x + 1$,
$P_{22} = x^{14} - 29x^{13} + 309x^{12} - 1629x^{11} + 4833x^{10} - 8678x^9 + 9852x^8 - 7250x^7 + 3483x^6 - 1078x^5$
$+ 206x^4 - 22x^3 + 1$. 


References


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