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V Flammang

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THE TOTAL DISTANCE FOR TOTALLY POSITIVE ALGEBRAIC INTEGERS

V. FLAMMANG

Abstract

Let $P(x)$ be a degree d polynomial with zeros $\alpha_1, \dots, \alpha_d$. In 2013, K. Stulov and R. Yang defined the *total distance of P* as $\text{td}(P) = \sum_{i=1}^d ||\alpha_i| - 1|$. In this paper, using the method of explicit auxiliary functions, we study the spectrum of the absolute total distance for totally positive algebraic integers and find its five smallest points. Moreover, we establish inequalities involving the total distance and two standard measures of totally positive algebraic integers and also the trace of totally positive algebraic integers. We give numerical examples to show that our bounds are quite good. The polynomials involved in the auxiliary functions are found by our recursive algorithm.

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1 Introduction

1.1 The total distance

We recall the definitions of two standard measures of polynomials:

let $P = a_0x^d + a_1x^{d-1} + \dots + a_d = a_0(x - \alpha_1)\dots(x - \alpha_d)$ be a polynomial with complex coefficients. Then

- the *length of P* is $L(P) = \sum_{i=0}^d |a_i|$,
- the *Mahler measure of P* is $M(P) = |a_0| \prod_{i=1}^d \max(1, |\alpha_i|)$.

In 2013, K. Stulov and R. Yang [SY] defined a less usual measure, called the *total distance of P* , defined as follows

$$\text{td}(P) = \sum_{i=1}^d ||\alpha_i| - 1|.$$

If α is an algebraic integer then the measure of α is the measure of its minimal polynomial P in $\mathbb{Z}[x]$. If α is a nonzero algebraic integer and $M(\alpha) = 1$ then a classical theorem of Kronecker [K] tells us that α is a root of unity. This suggests the question: $\inf_{\alpha \text{ not a root of unity}} M(\alpha) > 1$?

This is known as *Lehmer's problem*. It is still open. Another formulation of the problem is: does there exist an absolute constant $c > 0$ such that: if $M(\alpha) > 1$ then $M(\alpha) > 1 + c$?

The smallest known value is due to Lehmer himself and is $M(P) = 1.176280\dots$ where $P(x) = x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1$.

In their paper, K. Stulov and R. Yang established the following inequalities for a complex polynomial P with $|a_d| = |a_0| = 1$:

$$m(P) \leq \text{td}(P) \leq 2(M(P) - 1),$$

where $m(P) = \log(M(P))$. Thus, a third version of Lehmer's question can be formulated as follows: does there exist an absolute constant $c' > 0$ such that: if $\text{td}(\alpha) > 0$, with $|a_d| = |a_0| = 1$ then $\text{td}(\alpha) > c'$?

In this paper, we are interested in totally positive algebraic integers α , i.e., algebraic integers all of whose conjugates are positive real numbers. We define the *absolute total distance* of α as $\text{Td}(\alpha) = \frac{\text{td}(\alpha)}{\deg(\alpha)}$. As part of a study of the set $\mathcal{Td}$ of such quantities, we prove the following:

Theorem 1. *If α is a nonzero totally positive algebraic integer whose minimal polynomial is different from $x - 1$, $x^2 - 3x + 1$, $x^3 - 5x^2 + 6x - 1$, $x^5 - 9x^4 + 27x^3 - 32x^2 + 13x - 1$, $x^5 - 9x^4 + 28x^3 - 35x^2 + 15x - 1$ and $x^6 - 11x^5 + 45x^4 - 84x^3 + 70x^2 - 21x + 1$ then we have*

$$\text{Td}(\alpha) \geq 1.318853.$$

The polynomials involved in this result can be read from Table 1.

We deduce from this result the first five points of the spectrum of Td which are:

$$\begin{aligned} 1.118034\dots &= \text{Td}(x^2 - 3x + 1), \\ 1.201291\dots &= \text{Td}(x^3 - 5x^2 + 6x - 1), \\ 1.291483\dots &= \text{Td}(x^5 - 9x^4 + 28x^3 - 35x^2 + 15x - 1), \\ 1.312968\dots &= \text{Td}(x^6 - 11x^5 + 45x^4 - 84x^3 + 70x^2 - 21x + 1), \\ 1.317559\dots &= \text{Td}(x^5 - 9x^4 + 27x^3 - 32x^2 + 13x - 1). \end{aligned}$$

We conjecture that the next smallest point of the spectrum has minimal polynomial $x^9 - 17x^8 + 120x^7 - 456x^6 + 1011x^5 - 1324x^4 + 986x^3 - 376x^2 + 57x - 1$ and absolute total distance $1.329113\dots$

Now, it is natural to compare the total distance with the two standard measures. We establish the following inequalities:

Theorem 2. *If α is a nonzero totally positive algebraic integer of degree d whose minimal polynomial is different from $x - 1$, $x - 2$, $x^2 - 3x + 1$ and $x^2 - 5x + 5$ then we have:*

$$0.227804 \, m(\alpha) + 1.077638d \leq \text{td}(\alpha) \leq 86.546972 \, m(\alpha) - 43.686533d.$$

Theorem 3. *If α is a nonzero totally positive algebraic integer of degree d whose minimal polynomial is different from $x - 1$, $x - 2$, $x - 3$ and $x^2 - 3x + 1$ then we have:*

$$0.026162 \, \text{td}(\alpha) + 0.689186d \leq \log L(\alpha) \leq 1.084996 \, \text{td}(\alpha) - 0.149578d.$$

Let $P = x^d + a_1x^{d-1} + \dots + a_d = (x - \alpha_1)\dots(x - \alpha_d)$ be a monic polynomial with complex coefficients. The *trace* of P is defined by $\text{trace}(P) = \sum_{i=1}^d \alpha_i = -a_1$. Although the trace is not a measure in general, it may be considered as a measure for minimal polynomials of totally positive algebraic integers. So, we can use the trace in our inequalities and we prove:

Theorem 4. *If α is a nonzero totally positive algebraic integer of degree d whose minimal polynomial is different from $x - 1$, $x - 2$, $x^2 - 3x + 1$, $x^2 - 4x + 2$, $x^3 - 6x^2 + 8x - 2$ and $x^3 - 6x^2 + 8x - 1$ then we have:*

$$0.236643 \, \text{td}(\alpha) + 1.307819d \leq \text{trace}(\alpha) \leq 2.211970 \, \text{td}(\alpha) - 0.635402d.$$

In Table 2, Table 3 and Table 4, we give numerical examples where we can see that our bounds are quite good. In particular, Table 2 shows that our bounds are better than those of K. Stulov and R. Yang for the set of polynomials that we consider. For reciprocal polynomials P i.e., $P(x) = x^{\deg(P)}P(1/x)$, K. Stulov and R. Yang proved the following stronger lower bound: $\text{td}(P) \geq 2m(P)$. We give in Table 5 a set of reciprocal polynomials for which our lower bound is still better than their bound.

1.2 Explicit auxiliary function

The *absolute trace* of a totally positive algebraic integer α of degree $d \geq 2$ whose conjugates are $\alpha_1 = \alpha, \dots, \alpha_d$ is $\mathcal{T}r\alpha = \text{trace}(\alpha)/d = \sum_{i=1}^d \alpha_i/d$. The *Schur-Siegel-Smyth problem* is the following: fix $\rho < 2$, then show that all but finitely many totally positive algebraic integer α have $\mathcal{T}r\alpha > \rho$. The problem was solved by I. Schur in 1918 for $\rho < \sqrt{e}$ [Sc] then by C.L. Siegel in 1943 for $\rho < 1.7337$ [Si]. In 1984, C.J. Smyth developed the method of explicit auxiliary functions and solved the problem for $\rho < 1.7719$ [Sm]. All subsequent results were obtained thanks to this method. For a complete survey of this problem, see [ABP]. Until 2003, the polynomials involved in the auxiliary functions were found by heuristic methods. In 2003, when studying the linear independence measure of logarithms of rational numbers, Q. Wu [Wu] developed an algorithm by which the suitable polynomials are found by a systematic search. The author [F2] improved this algorithm in 2006 and now the suitable polynomials are found by induction. The best result for the trace problem to this day is obtained by the author for $\rho < 1.792519$ [F4]. The method of explicit auxiliary functions is also used to get the greatest lower bound of the absolute measure of an algebraic integer α , all of whose conjugates lie in a sector $S_\theta = \{z \in \mathbb{C} \text{ such that } |\arg z| \leq \theta\}$ with $0 < \theta < \pi$. M. Langevin [La] proved that there exists a function $c(\theta)$ on $[0, \pi)$, always > 1 , such that if $\alpha \neq 0$ is not a root of unity, whose conjugates all lie in S_θ , then $\Omega(\alpha) = M(\alpha)^{1/d} \geq c(\theta)$. Note that this result is not numerical. G. Rhin and C. Smyth [RS] succeeded in finding the exact value of $c(\theta)$ for θ in nine subintervals of $[0, 2\pi/3]$ and conjectured that $c(\theta)$ is a "staircase" function of θ , which is constant except for finitely many left discontinuities in any closed subinterval of $[0, \pi)$. The polynomials involved in their auxiliary functions were found by heuristic methods. In 2004, thanks to Wu's algorithm [Wu], G. Rhin and Q. Wu [RW] gave the exact value of $c(\theta)$ for four new subintervals of $[0, \pi)$ and extended four existing subintervals. In 2013, the author and G. Rhin [FR] found for the first time a complete subinterval and a fourteenth subinterval. These improvements are due to our recursive algorithm. The author [F1] computed the greatest lower bound $c(\theta)$ of the absolute trace for θ belonging to seven subintervals of $[0, 90^\circ)$. Q. Wu and L. Wang [WW] used the method of auxiliary functions to improve the irrationality measure of the number $\log 3$.

In Section 2, we explain the method of explicit auxiliary functions. We link it with a generalization of the classical integer transfinite diameter. Then we detail how our recursive algorithm enables us to get the constant of Theorem 1 for the total distance. In Section 3, we only give the auxiliary functions involved in the above inequalities. For the rest of the proof, we proceed as in Theorem 1. All the computations are done on a MacBookPro with the languages Pari and Pascal.

2 The total distance of totally positive algebraic integers

2.1 The explicit auxiliary function

Let α be a totally positive algebraic integer, $\alpha = \alpha_1, \dots, \alpha_d$ be its conjugates and P be its minimal polynomial. The auxiliary function involved in Theorem 1 has the following type:

$$\text{for } x > 0, f(x) = |x - 1| - \sum_{0 \leq j \leq J} c_j \log |Q_j(x)| \quad (1)$$

where the c_j are positive real numbers and the polynomials Q_j are nonzero polynomials in $\mathbb{Z}[x]$.

Then we have

$$\sum_{i=1}^d f(\alpha_i) \geq md$$

where m denotes the minimum of the function f . Thus,

$$\text{td}(\alpha) \geq md + \sum_{1 \leq j \leq J} c_j \log \left| \prod_{i=1}^d Q_j(\alpha_i) \right|.$$

We assume that P does not divide any Q_j , then $\prod_{i=1}^d Q_j(\alpha_i)$ is a nonzero integer because it is the resultant of P and Q_j . Therefore, if α is not a root of Q_j , we have

$$\text{Td}(\alpha) \geq m.$$

The main problem is to find a good list of polynomials Q_j which gives a value of m as large as possible. To do this, we link the auxiliary function with a generalization of the integer transfinite diameter in order to find the polynomials with our recursive algorithm.

2.2 Auxiliary functions and integer transfinite diameter

In this section, we need the following definition: let K be a compact subset of $\mathbb{C}$. If φ is a positive function defined on K , the φ -integer transfinite diameter of K is defined as

$$t_{\mathbb{Z},\varphi}(K) = \liminf_{\substack{n \geq 1 \\ n \rightarrow \infty}} \inf_{\substack{P \in \mathbb{Z}[X] \\ \deg(P) = n}} \sup_{x \in K} |P(x)|^{\frac{1}{n}} \varphi(x).$$

This weighted version of the integer transfinite diameter was introduced by F. Amoroso [A] and is an important tool in the study of rational approximations of logarithms of rational numbers. Inside the auxiliary function (1), we replace the numbers c_j by rational numbers a_j/q where q is a common denominator of the c_j for $1 \leq j \leq J$. Then we may write:

$$\text{for } x > 0, \quad f(x) = |x - 1| - \frac{t}{r} \log |Q(x)| \quad (2)$$

where $Q = \prod_{j=1}^J Q_j^{a_j} \in \mathbb{Z}[X]$ is of degree $r = \sum_{j=1}^J a_j \deg Q_j$ and $t = \sum_{j=1}^J c_j \deg Q_j$. We search for a polynomial $Q \in \mathbb{Z}[X]$ such that

$$\sup_{x>0} |Q(x)|^{t/r} \exp(-|x - 1|) \leq e^{-m},$$

where m is as large as possible. If we suppose that t is fixed, it is clear that we need an effective upper bound for the quantity

$$t_{\mathbb{Z},\varphi}((0, \infty)) = \liminf_{\substack{r \geq 1 \\ r \rightarrow +\infty}} \inf_{\substack{P \in \mathbb{Z}[X] \\ \deg(P) = r}} \sup_{x > 0} |P(x)|^{\frac{t}{r}} \varphi(x)$$

where we use the weight $\varphi(x) = \exp(-|x - 1|)$.

Even if we replace the compact subset K by the half line $(0, \infty)$, the weight φ ensures that the quantity $t_{\mathbb{Z},\varphi}((0, \infty))$ is finite.

2.3 Construction of the auxiliary function

The polynomials involved in the auxiliary function are found by our recursive algorithm developed in [F1] from Wu's algorithm [Wu]. It replaces the heuristic search by a systematic search by induction for suitable polynomials. Suppose that we have $Q_1, Q_2, \dots, Q_J$. Then we use semi-infinite linear programming (introduced into number theory by C. Smyth [Sm]) to optimize f for this set of polynomials (i.e., to get the greatest possible m). We obtain the positive real numbers $c_1, c_2, \dots, c_J$ and then f in the form (2) as above.

For several values of k , we search for a polynomial $R(x) = \sum_{l=0}^k a_l x^l \in \mathbb{Z}[x]$ such that

$$\sup_{x>0} |Q(x)R(x)|^{\frac{t}{r+k}} \exp(-|x-1|) \leq e^{-m},$$

i.e., such that

$$\sup_{x>0} |Q(x)R(x)| \exp(-|x-1|(r+k)/t)$$

is as small as possible.

Next, we apply LLL to the linear forms

$$Q(x_i)R(x_i) \exp(-|x_i-1|(r+k)/t)$$

where the x_i are control points in the half line $(0, \infty)$. After several trials, we find that a set of 70 points uniformly distributed in the interval $(0, 70)$ is convenient for the first step of LLL. After semi-infinite linear programming, this set will be augmented by the points where f has its least local minima. We get a polynomial R whose factors R_j are good candidates to enlarge the set of polynomials $(Q_1, Q_2, \dots, Q_J)$. We only keep the polynomials Q_j which have a nonzero coefficient in the newly optimized auxiliary function f . After optimization, some previous polynomials Q_j may have a zero coefficient and so are removed.

In order to get the constant of Theorem 1, we take k from 4 to 20 successively.

3 Comparison of measures

The method of explicit auxiliary functions has already been used to establish inequalities involving the Mahler measure, the length and the trace of totally positive polynomials, see [F3].

The auxiliary function used for the lower bound (resp. for the upper bound) of Theorem 2 is of the following type:

$$\text{for } x > 0, f(x) = \pm |x-1| \mp c_0 \log \max(1, x) - \sum_{1 \leq j \leq J} c_j \log |Q_j(x)|,$$

where the c_j are positive real numbers and the polynomials Q_j are nonzero polynomials in $\mathbb{Z}[x]$.

The auxiliary function used for the lower bound (resp. for the upper bound) of Theorem 3 is of the following type:

$$\text{for } x > 0, f(x) = \pm \log(x+1) \mp c_0 |x-1| - \sum_{1 \leq j \leq J} c_j \log |Q_j(x)|,$$

where the c_j are positive real numbers and the polynomials Q_j are nonzero polynomials in $\mathbb{Z}[x]$.

The auxiliary function used for the lower bound (resp. for the upper bound) of Theorem 4 is of the following type:

$$\text{for } x > 0, f(x) = \pm x \mp c_0 |x-1| - \sum_{1 \leq j \leq J} c_j \log |Q_j(x)|,$$

where the c_j are positive real numbers and the polynomials Q_j are nonzero polynomials in $\mathbb{Z}[x]$.

In our recursive algorithm, we take k from 4 to 10 only in order to have a small number of exceptions to the validity of the inequalities.

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Table 1: Polynomials Q_j involved in Theorem 1 with their coefficients c_j

	c_j	Polynomial Q_j
1	0.10655700	x
2	0.71262251	$x - 1$
3	0.20232450	$x - 2$
4	0.11567245	$x^2 - 3x + 1$
5	0.00866176	$x^2 - 4x + 2$
6	0.00500940	$x^2 - 5x + 5$
7	0.00253941	$x^3 - 6x^2 + 9x - 1$
8	0.07317471	$x^3 - 5x^2 + 6x - 1$
9	0.00434234	$x^3 - 6x^2 + 9x - 3$
10	0.00406945	$x^4 - 7x^3 + 14x^2 - 8x + 1$
11	0.00732231	$x^5 - 9x^4 + 27x^3 - 32x^2 + 13x - 1$
12	0.01494744	$x^5 - 9x^4 + 28x^3 - 35x^2 + 15x - 1$
13	0.01489508	$x^5 - 9x^4 + 27x^3 - 31x^2 + 12x - 1$
14	0.00052501	$x^5 - 9x^4 + 26x^3 - 29x^2 + 11x - 1$
15	0.00035586	$x^6 - 11x^5 + 46x^4 - 90x^3 + 81x^2 - 27x + 1$
16	0.00183029	$x^6 - 11x^5 + 44x^4 - 79x^3 + 63x^2 - 18x + 1$
17	0.00722208	$x^6 - 11x^5 + 45x^4 - 84x^3 + 70x^2 - 21x + 1$
18	0.00230977	$x^8 - 14x^7 + 78x^6 - 221x^5 + 337x^4 - 267x^3 + 94x^2 - 8x - 1$
19	0.00070739	$x^8 - 15x^7 + 89x^6 - 268x^5 + 438x^4 - 384x^3 + 166x^2 - 29x + 1$
20	0.00370284	$x^8 - 14x^7 + 78x^6 - 222x^5 + 345x^4 - 289x^3 + 120x^2 - 21x + 1$
21	0.00155673	$2x^8 - 26x^7 + 136x^6 - 367x^5 + 544x^4 - 435x^3 + 171x^2 - 27x + 1$
22	0.00217993	$x^9 - 17x^8 + 119x^7 - 444x^6 + 954x^5 - 1188x^4 + 817x^3 - 276x^2 + 37x - 1$
23	0.00160414	$x^9 - 17x^8 + 119x^7 - 444x^6 + 954x^5 - 1187x^4 + 812x^3 - 268x^2 + 33x - 1$
24	0.00067700	$x^9 - 17x^8 + 119x^7 - 444x^6 + 953x^5 - 1181x^4 + 800x^3 - 259x^2 + 31x - 1$
25	0.00337174	$x^9 - 17x^8 + 120x^7 - 456x^6 + 1011x^5 - 1324x^4 + 986x^3 - 376x^2 + 57x - 1$
26	0.00031596	$x^{10} - 18x^9 + 135x^8 - 550x^7 + 1330x^6 - 1959x^5 + 1737x^4 - 886x^3 + 238x^2 - 28x + 1$
27	0.00753897	$x^{10} - 18x^9 + 136x^8 - 562x^7 + 1388x^6 - 2104x^5 + 1937x^4 - 1036x^3 + 294x^2 - 36x + 1$
28	0.00415571	$x^{10} - 19x^9 + 152x^8 - 668x^7 + 1764x^6 - 2873x^5 + 2841x^4 - 1602x^3 + 446x^2 - 44x + 1$
29	0.00152277	$x^{10} - 18x^9 + 136x^8 - 562x^7 + 1387x^6 - 2096x^5 + 1913x^4 - 1002x^3 + 271x^2 - 30x + 1$
30	0.00038875	$x^{10} - 18x^9 + 135x^8 - 549x^7 + 1320x^6 - 1921x^5 + 1667x^4 - 820x^3 + 208x^2 - 23x + 1$
31	0.00021183	$x^{10} - 17x^9 + 121x^8 - 470x^7 + 1090x^6 - 1556x^5 + 1369x^4 - 734x^3 + 237x^2 - 43x + 3$
32	0.00009780	$x^{11} - 20x^{10} + 171x^9 - 819x^8 + 2418x^7 - 4560x^6 + 5502x^5 - 4135x^4 + 1822x^3 - 420x^2 + 40x - 1$
33	0.00012607	$x^{11} - 21x^{10} + 189x^9 - 954x^8 + 2967x^7 - 5879x^6 + 7412x^5 - 5760x^4 + 2571x^3 - 573x^2 + 46x - 1$
34	0.00072532	$x^{12} - 23x^{11} + 231x^{10} - 1332x^9 + 4878x^8 - 11854x^7 + 19398x^6 - 21250x^5 + 15190x^4 - 6738x^3 + 1695x^2 - 202x + 7$
35	0.00030507	$x^{12} - 22x^{11} + 210x^{10} - 1142x^9 + 3906x^8 - 8753x^7 + 12982x^6 - 12579x^5 + 7676x^4 - 2759x^3 + 523x^2 - 43x + 1$
36	0.00016745	$x^{12} - 23x^{11} + 230x^{10} - 1314x^9 + 4741x^8 - 11278x^7 + 17930x^6 - 18906x^5 + 12849x^4 - 5324x^3 + 1218x^2 - 127x + 4$
37	0.00145980	$x^{13} - 24x^{12} + 252x^{11} - 1523x^{10} + 5865x^9 - 15051x^8 + 26160x^7 - 30702x^6 + 23858x^5 - 11830x^4 + 3529x^3 - 575x^2 + 43x - 1$
38	0.00083711	$x^{14} - 26x^{13} + 300x^{12} - 2027x^{11} + 8911x^{10} - 26781x^9 + 56259x^8 - 82985x^7 + 85077x^6 - 59065x^5 + 26496x^4 - 7089x^3 + 981x^2 - 54x + 1$
39	0.00040496	$x^{14} - 26x^{13} + 299x^{12} - 2006x^{11} + 8720x^{10} - 25793x^9 + 53052x^8 - 76192x^7 + 75625x^6 - 50595x^5 + 21850x^4 - 5679x^3 + 790x^2 - 48x + 1$
40	0.00031272	$x^{18} - 35x^{17} + 560x^{16} - 5430x^{15} + 35664x^{14} - 167990x^{13} + 586104x^{12} - 1542459x^{11} + 3089299x^{10} - 4717066x^9 + 5466151x^8 - 4755434x^7 + 3050835x^6 - 1405114x^5 + 446641x^4 - 92341x^3 + 11292x^2 - 691x + 15$
41	0.00011511	$x^{19} - 35x^{18} + 560x^{17} - 5431x^{16} + 35692x^{15} - 168343x^{14} + 588756x^{13} - 1555701x^{12} + 3135698x^{11} - 4834464x^{10} + 5683528x^9 - 5050713x^8 + 3343089x^7 - 1612543x^6 + 549476x^5 - 126561x^4 + 18484x^3 - 1556x^2 + 65x - 1$
42	0.00042098	$x^{19} - 35x^{18} + 560x^{17} - 5431x^{16} + 35692x^{15} - 168343x^{14} + 588756x^{13} - 1555701x^{12} + 3135698x^{11} - 4834465x^{10} + 5683542x^9 - 5050795x^8 + 3343351x^7 - 1613042x^6 + 550057x^5 - 126968x^4 + 18645x^3 - 1587x^2 + 67x - 1$
43	0.00034312	$x^{19} - 35x^{18} + 561x^{17} - 5461x^{16} + 36098x^{15} - 171626x^{14} + 606463x^{13} - 1623033x^{12} + 3321706x^{11} - 5213487x^{10} + 6255933x^9 - 5688941x^8 + 3862060x^7 - 1913635x^6 + 669910x^5 - 158075x^4 + 23448x^3 - 1962x^2 + 77x - 1$
44	0.00085592	$x^{19} - 35x^{18} + 560x^{17} - 5431x^{16} + 35692x^{15} - 168344x^{14} + 588778x^{13} - 1555913x^{12} + 3136877x^{11} - 4838661x^{10} + 5693555x^9 - 5067109x^8 + 3361466x^7 - 1626464x^6 + 556386x^5 - 128692x^4 + 18858x^3 - 1588x^2 + 66x - 1$
45	0.00124561	$x^{19} - 35x^{18} + 561x^{17} - 5461x^{16} + 36099x^{15} - 171649x^{14} + 606698x^{13} - 1624441x^{12} + 3327206x^{11} - 5228221x^{10} + 6283668x^9 - 5725945x^8 + 3896983x^7 - 1936732x^6 + 680478x^5 - 161381x^4 + 24145x^3 - 2055x^2 + 83x - 1$
46	0.00031896	$x^{19} - 35x^{18} + 561x^{17} - 5460x^{16} + 36071x^{15} - 171298x^{14} + 604092x^{13} - 1611668x^{12} + 3283605x^{11} - 5121678x^{10} + 6095068x^9 - 5483972x^8 + 3674067x^7 - 1792070x^6 + 616291x^5 - 142721x^4 + 20802x^3 - 1722x^2 + 69x - 1$

Table 2: Comparison of the bounds of Stulov and Yang and the author's bounds. Here P denotes the minimal polynomial of a totally positive algebraic integer of degree d .

P	The lower bound of Stulov and Yang	The author's lower bound	The author's upper bound	The upper bound of Stulov and Yang
P	$m(P)$	$0.227804 \ m(P) + 1.077638 \ d$	$86.546972 \ m(P) - 43.686533 \ d$	$2 \ M(P) - 1$
P_1	2.1678737	4.8044023	5.3307043	15.479363
P_2	2.0879857	4.7862035	5.4562194	14.137292
P_3	2.8102287	6.0283713	6.5877984	31.227436
P_4	2.8837927	6.0451295	6.4574148	33.763933
P_5	2.8096787	6.0282461	6.7109713	31.209166
P_6	2.7481251	6.0142239	6.7611431	29.226662
P_7	3.4600638	7.2540444	7.9852747	61.638016
P_8	3.5325158	7.2705492	7.8778103	66.419852
P_9	3.3692819	7.2333639	8.4370661	56.115309
P_{10}	4.0647148	8.4694243	10.133815	114.49658
P_{11}	4.1344755	8.4853161	9.9760434	122.91365
P_{12}	4.6480352	9.6799450	10.973399	206.75939
P_{13}	4.4586497	9.6368022	10.172583	170.74161
P_{14}	5.4858197	10.948434	12.480484	480.49324
P_{15}	5.4175259	10.932876	11.962019	448.64192
P_{16}	5.4904408	10.949486	12.464721	482.72805
P_{17}	5.5064050	10.953123	12.519260	490.52841
P_{18}	6.0204376	12.147860	13.500532	821.51749
P_{19}	5.6605471	12.065875	13.366620	572.61157
P_{20}	5.5713426	12.045554	13.433823	523.57335
P_{21}	5.5661460	12.044370	13.488010	520.84922
P_{22}	6.8257165	14.486582	16.412179	1840.4725
P_{23}	6.8465121	14.491319	16.464912	1879.1890
P_{24}	7.6494198	15.751862	17.801539	4196.8542
P_{25}	8.2171482	16.958831	18.935244	7405.8489
P_{26}	8.9419509	19.279220	21.778250	15290.198
P_{27}	11.016533	22.984732	25.502603	121742.50

where

$$\begin{aligned}
P_1 &= x^4 - 7x^3 + 14x^2 - 8x + 1 \\
P_2 &= x^4 - 7x^3 + 13x^2 - 7x + 1 \\
P_3 &= x^5 - 9x^4 + 27x^3 - 32x^2 + 13x - 1 \\
P_4 &= x^5 - 9x^4 + 28x^3 - 35x^2 + 15x - 1 \\
P_5 &= x^5 - 9x^4 + 27x^3 - 31x^2 + 12x - 1 \\
P_6 &= x^5 - 9x^4 + 26x^3 - 29x^2 + 11x - 1 \\
P_7 &= x^6 - 11x^5 + 44x^4 - 79x^3 + 63x^2 - 18x + 1 \\
P_8 &= x^6 - 11x^5 + 45x^4 - 84x^3 + 70x^2 - 21x + 1 \\
P_9 &= x^6 - 11x^5 + 43x^4 - 72x^3 + 51x^2 - 14x + 1 \\
P_{10} &= x^7 - 13x^6 + 63x^5 - 143x^4 + 157x^3 - 78x^2 + 16x - 1 \\
P_{11} &= x^7 - 13x^6 + 64x^5 - 150x^4 + 172x^3 - 89x^2 + 18x - 1 \\
P_{12} &= x^8 - 15x^7 + 89x^6 - 268x^5 + 438x^4 - 384x^3 + 166x^2 - 29x + 1 \\
P_{13} &= x^8 - 14x^7 + 78x^6 - 221x^5 + 337x^4 - 267x^3 + 94x^2 - 8x - 1 \\
P_{14} &= x^9 - 17x^8 + 119x^7 - 444x^6 + 954x^5 - 1188x^4 + 817x^3 - 276x^2 + 37x - 1 \\
P_{15} &= x^9 - 17x^8 + 120x^7 - 456x^6 + 1011x^5 - 1324x^4 + 986x^3 - 376x^2 + 57x - 1 \\
P_{16} &= x^9 - 17x^8 + 119x^7 - 444x^6 + 954x^5 - 1187x^4 + 812x^3 - 268x^2 + 33x - 1 \\
P_{17} &= x^9 - 17x^8 + 119x^7 - 444x^6 + 953x^5 - 1181x^4 + 800x^3 - 259x^2 + 31x - 1 \\
P_{18} &= x^{10} - 19x^9 + 152x^8 - 668x^7 + 1764x^6 - 2873x^5 + 2841x^4 - 1602x^3 + 446x^2 - 44x + 1 \\
P_{19} &= x^{10} - 18x^9 + 135x^8 - 549x^7 + 1320x^6 - 1920x^5 + 1662x^4 - 813x^3 + 206x^2 - 24x + 1 \\
P_{20} &= x^{10} - 18x^9 + 134x^8 - 538x^7 + 1273x^6 - 1822x^5 + 1560x^4 - 766x^3 + 200x^2 - 24x + 1 \\
P_{21} &= x^{10} - 18x^9 + 134x^8 - 537x^7 + 1265x^6 - 1798x^5 + 1526x^4 - 743x^3 + 194x^2 - 24x + 1 \\
P_{22} &= x^{12} - 22x^{11} + 208x^{10} - 1108x^9 + 3667x^8 - 7851x^7 + 10995x^6 - 9977x^5 + 5702x^4 \\
&\quad - 1952x^3 + 368x^2 - 33x + 1 \\
P_{23} &= x^{12} - 22x^{11} + 208x^{10} - 1108x^9 + 3666x^8 - 7840x^7 + 10948x^6 - 9877x^5 + 5589x^4 \\
&\quad - 1885x^3 + 349x^2 - 31x + 1 \\
P_{24} &= x^{13} - 24x^{12} + 252x^{11} - 1523x^{10} + 5865x^9 - 15051x^8 + 26160x^7 - 30702x^6 + 23858x^5 \\
&\quad - 11830x^4 + 3529x^3 - 575x^2 + 43x - 1 \\
P_{25} &= x^{14} - 26x^{13} + 299x^{12} - 2006x^{11} + 8720x^{10} - 25793x^9 + 53052x^8 - 76192x^7 + 75625x^6 \\
&\quad - 50595x^5 + 21850x^4 - 5679x^3 + 790x^2 - 48x + 1 \\
P_{26} &= x^{16} - 29x^{15} + 375x^{14} - 2859x^{13} + 14320x^{12} - 49711x^{11} + 123001x^{10} - 219813x^9 \\
&\quad + 284591x^8 - 265614x^7 + 176517x^6 - 81849x^5 + 25677x^4 - 5199x^3 + 630x^2 - 40x + 1 \\
P_{27} &= x^{19} - 35x^{18} + 560x^{17} - 5431x^{16} + 35692x^{15} - 168343x^{14} + 588756x^{13} - 1555701x^{12} \\
&\quad + 3135698x^{11} - 4834465x^{10} + 5683542x^9 - 5050795x^8 + 3343351x^7 - 1613042x^6 \\
&\quad + 550057x^5 - 126968x^4 + 18645x^3 - 1587x^2 + 67x - 1
\end{aligned}$$

Table 3: Comparison of the trace and the total distance of some polynomials P which are minimal polynomials of totally positive algebraic integers of degree d .

P	0.236643	$\text{td}(P) + 1.077638 d$	$\text{trace}(P)$	2.211970	$\text{td}(P) - 0.635402d$
P_1		6.4927498	7		9.7538396
P_2		6.5224521	7		10.113062
P_3		8.0980514	9		11.975985
P_4		8.0671970	9		11.602829
P_5		8.1271994	9		12.328504
P_6		8.1390722	9		12.472095
P_7		9.7365734	11		14.599902
P_8		9.7111427	11		14.292340
P_9		9.8434866	11		15.892924
P_{10}		11.552829	13		19.373356
P_{11}		11.515494	13		18.921816
P_{12}		13.059330	15		20.400591
P_{13}		12.869823	14		18.108666
P_{14}		14.723790	17		23.338209
P_{15}		14.601099	17		21.854367
P_{16}		14.720060	17		23.293094
P_{17}		14.732966	17		23.449186
P_{18}		16.272996	19		24.881930
P_{19}		16.241307	18		24.498675
P_{20}		16.257210	18		24.691011
P_{21}		16.270033	18		24.846094
P_{22}		19.577655	22		30.463744
P_{23}		19.590134	22		30.614665
P_{24}		21.214257	24		33.064432
P_{25}		22.790359	26		34.933438
P_{26}		26.078774	29		40.318803
P_{27}		30.883573	35		46.850926

Table 4: Comparison of the length and the total distance of some polynomials P which are minimal polynomials of totally positive algebraic integers of degree d .

P	0.236643	$\text{td}(P) + 1.077638 d$	$\text{trace}(P)$	2.211970	$\text{td}(P) - 0.635402d$
P_1		2.8962059	3.4339872		5.1854808
P_2		2.8994896	3.3672958		5.3216642
P_3		3.6182800	4.4188406		6.3998449
P_4		3.6148689	4.4886364		6.2583793
P_5		3.6215024	4.3944492		6.5334870
P_6		3.6228150	4.3438054		6.5879232
P_7		4.3440268	5.3798974		7.7665231
P_8		4.3412153	5.4510385		7.6499246
P_9		4.3558465	5.2626902		8.2567150
P_{10}		5.0894229	6.1569790		9.9481027
P_{11}		5.0852952	6.2304814		9.7769212
P_{12}		5.8005741	7.2377782		10.709470
P_{13}		5.7796231	6.9265770		9.8405877
P_{14}		6.5291884	8.2568668		12.195073
P_{15}		6.5156243	8.3777012		11.632541
P_{16}		6.5287760	8.2521854		12.177970
P_{17}		6.5302029	8.2443340		12.237145
P_{18}		7.2450609	9.2506182		13.152243
P_{19}		7.2415575	8.8022217		13.006949
P_{20}		7.2433157	8.7541607		13.079865
P_{21}		7.2447333	8.7388957		13.138657
P_{22}		8.6996074	10.642683		16.012213
P_{23}		8.7009870	10.634051		16.069428
P_{24}		9.4251419	11.690352		17.370085
P_{25}		10.143988	12.678190		18.450572
P_{26}		11.596739	14.038836		21.236066
P_{27}		13.761733	17.100651		24.828240

Table 5: Comparison of the bounds of Stulov and Yang and the author's for some reciprocal polynomials P which are minimal polynomials of totally positive reciprocal algebraic integers of degree d .

P	The improved lower bound of Stulov and Yang	The author's lower bound	The author's upper bound	The upper bound of Stulov and Yang
P	$2 \text{ m}(P)$	$0.227804 \text{ m}(P) + 1.077638 \text{ d}$	$\text{td}(P)$	$86.546972 \text{ m}(P) - 43.686533 \text{ d}$
P_1	4.1759713	5.2134895	5.4562194	5.9627053
P_2	4.4523454	5.4285522	6.4412053	17.922373
P_3	6.4766953	7.9857783	8.7017102	18.149987
P_4	6.5241080	8.0226728	9.6390906	20.201699
P_5	8.5857452	10.608914	11.898492	22.042862
P_6	8.7932213	10.770363	12.885760	31.021073
P_7	8.6714949	10.675641	11.919817	25.753549
P_8	8.7121732	10.707295	12.864449	27.513840
P_9	13.076541	16.067387	18.359886	41.629114
P_{10}	15.285923	18.768598	21.581474	49.863733
P_{11}	15.332635	18.804946	21.602688	51.885084
P_{12}	15.254839	18.744409	21.582923	48.518583
P_{13}	15.265027	18.752337	21.585202	48.959471
P_{14}	17.355512	21.361027	24.785407	52.048999
P_{15}	17.451738	21.435906	24.801640	56.213016
P_{16}	24.134495	29.582033	34.490281	83.280024
P_{17}	28.588126	35.011585	40.931773	101.25801

where

$$\begin{aligned}
P_1 &= x^4 - 7x^3 + 13x^2 - 7x + 1 \\
P_2 &= x^4 - 8x^3 + 15x^2 - 8x + 1 \\
P_3 &= x^6 - 11x^5 + 41x^4 - 63x^3 + 41x^2 - 11x + 1 \\
P_4 &= x^6 - 12x^5 + 44x^4 - 67x^3 + 44x^2 - 12x + 1 \\
P_5 &= x^8 - 15x^7 + 83x^6 - 220x^5 + 303x^4 - 220x^3 + 83x^2 - 15x + 1 \\
P_6 &= x^8 - 16x^7 + 91x^6 - 244x^5 + 337x^4 - 244x^3 + 91x^2 - 16x + 1 \\
P_7 &= x^8 - 15x^7 + 84x^6 - 225x^5 + 311x^4 - 225x^3 + 84x^2 - 15x + 1 \\
P_8 &= x^8 - 16x^7 + 90x^6 - 239x^5 + 329x^4 - 239x^3 + 90x^2 - 16x + 1 \\
P_9 &= x^{12} - 23x^{11} + 218x^{10} - 1118x^9 + 3438x^8 - 6651x^7 + 8271x^6 - 6651x^5 + 3438x^4 - 1118x^3 + 218x^2 - 23x + 1 \\
P_{10} &= x^{14} - 27x^{13} + 309x^{12} - 1977x^{11} + 7871x^{10} - 20564x^9 + 36261x^8 - 43749x^7 + 36261x^6 - 20564x^5 + 7871x^4 \\
&\quad - 1977x^3 + 309x^2 - 27x + 1 \\
P_{11} &= x^{14} - 27x^{13} + 309x^{12} - 1979x^{11} + 7894x^{10} - 20668x^9 + 36503x^8 - 44067x^7 + 36503x^6 - 20668x^5 + 7894x^4 \\
&\quad - 1979x^3 + 309x^2 - 27x + 1 \\
P_{12} &= x^{14} - 27x^{13} + 308x^{12} - 1963x^{11} + 7790x^{10} - 20307x^9 + 35763x^8 - 43131x^7 + 35763x^6 - 20307x^5 + 7790x^4 \\
&\quad - 1963x^3 + 308x^2 - 27x + 1 \\
P_{13} &= x^{14} - 27x^{13} + 308x^{12} - 1964x^{11} + 7800x^{10} - 20348x^9 + 35853x^8 - 43247x^7 + 35853x^6 - 20348x^5 + 7800x^4 \\
&\quad - 1964x^3 + 308x^2 - 27x + 1 \\
P_{14} &= x^{16} - 31x^{15} + 413x^{14} - 3141x^{13} + 15261x^{12} - 50187x^{11} + 115410x^{10} - 189036x^9 + 222621x^8 - 189036x^7 \\
&\quad + 115410x^6 - 50187x^5 + 15261x^4 - 3141x^3 + 413x^2 - 31x + 1 \\
P_{15} &= x^{16} - 31x^{15} + 415x^{14} - 3177x^{13} + 15538x^{12} - 51389x^{11} + 118680x^{10} - 194903x^9 + 229733x^8 - 194903x^7 \\
&\quad + 118680x^6 - 51389x^5 + 15538x^4 - 3177x^3 + 415x^2 - 31x + 1 \\
P_{16} &= x^{22} - 43x^{21} + 832x^{20} - 9625x^{19} + 74627x^{18} - 412074x^{17} + 1680988x^{16} - 5187203x^{15} + 12299616x^{14} - 22643382x^{13} \\
&\quad + 32578283x^{12} - 36764041x^{11} + 32578283x^{10} - 22643382x^9 + 12299616x^8 - 5187203x^7 + 1680988x^6 - 412074x^5 \\
&\quad + 74627x^4 - 9625x^3 + 832x^2 - 43x + 1 \\
P_{17} &= x^{26} - 51x^{25} + 1191x^{24} - 16934x^{23} + 164453x^{22} - 1160191x^{21} + 6175143x^{20} - 25425203x^{19} + 82394564x^{18} \\
&\quad - 212746984x^{17} + 441495979x^{16} - 740812263x^{15} + 1009086208x^{14} - 1118311827x^{13} + 1009086208x^{12} - 740812263x^{11} \\
&\quad + 441495979x^{10} - 212746984x^9 + 82394564x^8 - 25425203x^7 + 6175143x^6 - 1160191x^5 + 164453x^4 - 16934x^3 \\
&\quad + 1191x^2 - 51x + 1
\end{aligned}$$

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UMR CNRS 7502. IECL, Université de Lorraine, site de Metz, Département de Mathématiques,
UFR MIM, Ile du Saulcy, CS 50128. 57045 METZ cedex 01. FRANCE
E-mail address :valerie.flammang@univ-lorraine.fr