Comparison of measures of totally positive polynomials
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COMPARISON OF MEASURES OF TOTALLY POSITIVE POLYNOMIALS

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Abstract

In this note, explicit auxiliary functions are used to get upper and lower bounds for the Mahler measure of monic irreducible totally positive polynomials with integer coefficients. These bounds involve the length and the trace of the polynomial.

1 Introduction

Let \( P = a_0 x^d + a_1 x^{d-1} + \ldots + a_d = a_0(x-\alpha_1)\ldots(x-\alpha_d) \) be a polynomial with complex coefficients.

We define:

- the trace of \( P \) as \( \text{trace}(P) = \sum_{i=1}^{d} \alpha_i \),
- the length of \( P \) as \( L(P) = \sum_{i=0}^{d} |a_i| \),
- the Mahler measure of \( P \) as \( M(P) = |a_0| \prod_{i=1}^{d} \max(1, |\alpha_i|) \).

We have the well-known inequality: \( 2^{-d} L(P) \leq M(P) \leq L(P) \) (for more details, see for example [Mi]).

Now, we consider a monic polynomial \( P \) and totally positive (that is, its roots are all positive real numbers). In this case, \( L(P) = |P(-1)| = \prod_{i=1}^{d} (1 + \alpha_i) \). Then, we have the basic inequality \( \log L(P) \leq \text{trace}(P) \).

In this paper, we prove the following results

**Theorem 1.** If \( P \) is a totally positive monic irreducible polynomial of degree \( d \) with integer coefficients, different from \( x, x-1, x^2-3x+1, x^3-7x^2+13x-7+1, x^2-4x+1, x^6-12x^5+44x^4-67x^3+44x^2-12x+1 \) and \( x^8-15x^7+83x^6-220x^5+303x^4-220x^3+83x^2-15x+1 \) then

\[
\text{(1.1)} \quad \max \left( 2^{-d} L(P), 1.058358^d L(P)^{0.562454} \right) \leq M(P) \leq \min \left( L(P), 0.379128^d L(P)^{1.803995} \right).
\]

**Theorem 2.** If \( P \) is a totally positive monic irreducible polynomial of degree \( d \) with integer coefficients, different from \( x-1, x-2, x-3, x^2-3x+1, x^3-5x^2+6x-1 \) and \( x^3-5x^2+5x+1 \) then

\[
\text{(1.2)} \quad \log L(P)^{\frac{1}{d}} \leq \min \left( \frac{1}{d} \text{trace}(P), 0.051012 + 0.472699 \frac{1}{d} \text{trace}(P) \right).
\]
Theorem 3. If $P$ is a totally positive monic irreducible polynomial of degree $d$ with integer coefficients and with all roots in $(0, 1000)$, different from $x$ and $x - 1$ then

\begin{equation}
\log L(P) \geq 0.801729 + 0.0019901 \frac{1}{d} \text{trace}(P).
\end{equation}

The proofs of these theorems use the principle of explicit auxiliary function that was introduced into Number Theory by C. J. Smyth [Sm1]. The method is based on the fact that the resultant of two polynomials in $\mathbb{Z}[X]$ with no common roots is a nonzero integer.

For example, to get the lower bound in the inequality (1.1), we use the auxiliary function:

\begin{equation}
\text{for } x > 0, \quad f(x) = \log \max(1, x) - c_0 \log(x + 1) - \sum_{1 \leq j \leq J} c_j \log |Q_j(x)| \geq m
\end{equation}

where the $c_j$ are positive real numbers and the polynomials $Q_j$ are nonzero elements of $\mathbb{Z}[X]$. Then

$$
\sum_{i=1}^{d} f(\alpha_i) \geq md,
$$

that is,

$$
\log M(P) \geq md + c_0 \log L(P) + \sum_{1 \leq j \leq J} c_j \log \left| \prod_{i=1}^{d} Q_j(\alpha_i) \right|.
$$

We assume that $P$ does not divide any $Q_j$, then $\prod_{i=1}^{d} Q_j(\alpha_i)$ is a nonzero integer because it is the resultant of $P$ and $Q_j$.

Therefore, if $P$ does not divide any $Q_j$, we have

$$
M(P) \geq e^{md} L(P)^{c_0}.
$$

To get the upper bound for $M(P)$, we use the auxiliary function:

\text{for } x > 0, \quad f(x) = -\log \max(1, x) + c_0 \log(x + 1) - \sum_{1 \leq j \leq J} c_j \log |Q_j(x)| \geq m.

The upper bound in the inequality (1.2) is obtained with the auxiliary function:

\text{for } x > 0, \quad f(x) = -\log (x + 1) + c_0 x - \sum_{1 \leq j \leq J} c_j \log |Q_j(x)| \geq m.

In general, it is not possible to get a lower bound for $L(P)$ involving $\text{trace}(P)$ with an auxiliary function. But, if we assume that the polynomial $P$ has all its roots in an interval not too large for instance $(0,1000)$ then we can get a result of the type (1.3) with the auxiliary function:

\text{for } x \in (0, 1000), \quad f(x) = \log (x + 1) - c_0 x - \sum_{1 \leq j \leq J} c_j \log |Q_j(x)| \geq m.

Usually, the main problem is then to find a good list of polynomials $Q_j$ which gives a value of $m$ as large as possible. This is feasible by an inductive version of Wu’s algorithm (for details, see [F2]) and it can happen that a great number of polynomials are useful (for example, 35 to get a lower bound for the trace, see [F2]). But here, we want to have only a few exceptional polynomials so that we stop the algorithm fairly quickly and thus we accept that our $m$ is not the best possible, nevertheless sufficiently large to give good inequalities.
In Section 2 we explain how to construct the auxiliary function (1.4). The same method works for the other auxiliary functions. We also give a table of all polynomials involved in the different auxiliary functions and their coefficients. In Section 3, we give numerical examples for a particular family of polynomials. All the computations are done on a MacBook Pro Macintosh with the languages Pascal and Pari [Pari].

2 Construction of the explicit auxiliary function

2.1 Rewriting the auxiliary function

Inside the auxiliary function (1.4) we replace the numbers $c_j$ by rational numbers.

So we may write:

\[(2.1) \quad \text{for } x > 0, \quad f(x) = \log \max(1, x) - c_0 \log(x + 1) - \frac{t}{r} \log |Q(x)| \geq m.\]

where $Q \in \mathbb{Z}[X]$ is of degree $r$ and $t$ is a positive real number. We want to get a function $f$ whose minimum $m$ on $(0, \infty)$ is sufficiently large. Thus we search a polynomial $Q \in \mathbb{Z}[X]$ such that

\[\sup_{x > 0} \frac{|Q(x)|^{t/r} \max(1, x)}{(x + 1)^{c_0}} \leq e^{-m}.\]

If we suppose that $t$ is fixed, we need to get an effective upper bound for the quantity

\[t_{\mathbb{Z}, \varphi}([0, \infty)) = \lim \inf \inf_{r \to +\infty} \sup_{\deg(P) = n > 0} |P(x)|^{\frac{1}{r}} \varphi(x),\]

in which we use the weight $\varphi(x) = \frac{\max(1, x)}{(x + 1)^{c_0}}$.

It is clear that this quantity is closely related to the usual integer transfinite diameter of an interval $I = [a, b]$ which is defined as

\[t_{\mathbb{Z}}(I) = \lim \inf_{n \to +\infty} \inf_{\deg(P) = n} |P|^{\frac{1}{\infty, I}} \]

where $|P|_{\infty, I} = \sup_{t \in I} |P(t)|$ for all $P \in \mathbb{Z}[x]$.

2.2 How to find the polynomials $Q_j$

Consider the auxiliary function

\[f(x) = \log \max(1, x) - c_0 \log(x + 1) - \sum_{1 \leq j \leq J} c_j \log |Q_j(x)| \geq m.\]

The main idea is to find the polynomials $Q_j$ by induction. We first optimize the auxiliary function $f_1 = \log \max(1, x) - c_0 \log(1 + x) - c_1 \log x$. Then we take $t = c_0 \deg(x + 1) + c_1 \deg(x)$. Suppose that we have $Q_1, ..., Q_J$ and an optimal function $f$ for this set of polynomials in the form (2.1) with $t = \sum_{j=0}^{J} c_j \deg(Q_j)$. We seek a polynomial $R(x) = \sum_{i=0}^{k} a_i x^i \in \mathbb{Z}[x]$ of degree $k$ ($k = 10$ for example) such that

\[\sup_{x \in I} |Q(x) R(x)|^{\frac{1}{r+1}} \frac{(1 + x)^{c_0}}{\max(1, x)} \leq e^{-m},\]
that is, such that
\[ \sup_{x \in I} |Q(x)R(x)| \left( \frac{(1 + x)^{c_0}}{\max(1, x)} \right)^{r + k} \]
is as small as possible. We apply LLL to the linear forms in the unknown coefficients \( a_i \)
\[ Q(x_i)R(x_i) \left( \frac{(1 + x_i)^{c_0}}{\max(1, x_i)} \right)^{r + k}. \]
The numbers \( x_i \) are suitable points in \( I = [0, 50] \) here, including the points where \( f \) has its least local minima. We get a polynomial \( R \) whose irreducible factors \( R_j \) are good candidates to enlarge the set of polynomials \( (Q_1, ..., Q_J) \). We only keep the polynomials \( R_j \) which have a nonzero coefficient \( c_j \) in the new optimized auxiliary function \( f \). After optimization, some previous polynomials \( Q_j \) may have a zero coefficient and are removed.

### 2.3 Optimization of the \( c_j \)

For the optimization of the auxiliary function we use the semi-infinite linear programming method due to C. J. Smyth [Sm1]. We recall it briefly. We define by induction a sequence of finite sets \( X_n, n \geq 0, \) with \( X_n \subset [0, \infty) \). We start with an arbitrary set of points \( X_0 \) of cardinal greater than \( J \). At each step \( n \geq 0 \), we compute the best values for \( c_j \) by linear programming on the set \( X_n \). We get a function \( f_n \) whose minimum \( m_n = \min_{x \in X_n} f_n(x) \) is greater than \( m'_n = \min_{x > 0} f_n(x) \). We add to \( X_n \) the points of \( [0, \infty) \) where \( f_n \) has a local minimum smaller than \( m_n + \epsilon_n \), where \((\epsilon_n)_{n \geq 0}\) is a decreasing sequence of positive numbers tending to 0 when \( n \) is increasing and chosen such that the set \( X_n \) does not increase too quickly. We stop for instance when \( m_n - m'_n < 10^{-6} \). If \( k \) steps are necessary, we take \( m = m'_k \).

### 2.4 Numerical results

The polynomials used in the different auxiliary functions are:

\[
\begin{align*}
Q_1 &= x \\
Q_2 &= x - 1 \\
Q_3 &= x - 2 \\
Q_4 &= x - 3 \\
Q_5 &= x^2 - 3x + 1 \\
Q_6 &= x^2 - 4x + 1 \\
Q_7 &= x^3 - 5x^2 + 5x + 1 \\
Q_8 &= x^3 - 5x^2 + 6x - 1 \\
Q_9 &= x^4 - 7x^3 + 13x^2 - 7x + 1 \\
Q_{10} &= x^6 - 12x^5 + 44x^4 - 67x^3 + 44x^2 - 12x + 1 \\
Q_{11} &= x^8 - 15x^7 + 83x^6 - 220x^5 + 303x^4 - 220x^3 + 83x^2 - 15x + 1.
\end{align*}
\]

Table 1 gives for each inequality of the different theorems the polynomials \( Q_j \) and their coefficients \( c_j, 1 \leq j \leq J \), the coefficient \( c_0 \) and the value of \( m \).
Table 1: Polynomials $Q_j$ and their coefficients $c_j$, $1 \leq j \leq J$, the coefficient $c_0$ and the value of $m$ for each inequality of Theorems 1, 2 and 3.

<table>
<thead>
<tr>
<th>Theorem</th>
<th>$Q_j$</th>
<th>$c_j$</th>
<th>$c_0$</th>
<th>$m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Theorem 1 left hand-side</td>
<td>$Q_1$</td>
<td>$0.026208$</td>
<td>$0.271988$</td>
<td>$0.023069$</td>
</tr>
<tr>
<td></td>
<td>$Q_9$</td>
<td>$0.000883$</td>
<td>$0.000895$</td>
<td>$0.003261$</td>
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<td></td>
<td>$Q_{10}$</td>
<td>$0.371232$</td>
<td>$0.030766$</td>
<td>$1.058358$</td>
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<tr>
<td></td>
<td>$Q_{11}$</td>
<td>$0.023069$</td>
<td>$0.030766$</td>
<td>$0.003261$</td>
</tr>
<tr>
<td>Theorem 1 right hand-side</td>
<td>$Q_1$</td>
<td>$0.026208$</td>
<td>$0.271988$</td>
<td>$0.023069$</td>
</tr>
<tr>
<td></td>
<td>$Q_6$</td>
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<td>$0.062529$</td>
<td>$0.004051$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$0.046880$</td>
<td>$0.001968$</td>
<td>$0.018628$</td>
</tr>
<tr>
<td>Theorem 2</td>
<td>$Q_2$</td>
<td>$0.371232$</td>
<td>$0.030766$</td>
<td>$1.058358$</td>
</tr>
<tr>
<td></td>
<td>$Q_3$</td>
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<td>$0.004051$</td>
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<tr>
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<td>$Q_4$</td>
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<td>$0.001968$</td>
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<tr>
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<td>$Q_5$</td>
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<td>$0.196606$</td>
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</tr>
<tr>
<td></td>
<td>$Q_7$</td>
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</tr>
<tr>
<td></td>
<td>$Q_8$</td>
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<td>$0.001968$</td>
<td>$0.018628$</td>
</tr>
</tbody>
</table>

3 Numerical example

The Gorshkov-Wirsing polynomials are defined as follows:

$$P_0(X) = X - 1 \quad \text{and} \quad P_n(X) = X^{\deg(P_{n-1})} P_{n-1} \left( X + \frac{1}{X} - 2 \right) \quad \text{for} \ n \geq 1.$$ 

C.J. Smyth [Sm2] showed that the sequence $\left( M(P_n) \left( \frac{1}{\deg(P_n)} \right) \right)_{n \geq 0}$ has a limit point $l = 1.727305...$.

The author proved [F3] that the sequence $\left( L(P_n) \left( \frac{1}{\deg(P_n)} \right) \right)_{n \geq 0}$ has a limit point $l' = 2.376841...$.

It is easy to see that the sequence $\left( \frac{1}{\deg(P_n)} \right)_{n \geq 0}$ has a limit point $l'' = 2$.

Previously, the author [F1] obtained, without explicit auxiliary functions, the following theorem:

**Theorem 4.** Let $P$ be a totally positive monic irreducible polynomial of degree $d$ with integer coefficients and not divisible by $x$ and $x - 1$. We have:

$$\left( \frac{1 + \sqrt{5}}{2} \right)^d \left( \frac{L(P)}{\sqrt{5}} \right)^{d - \sqrt{5}} \leq M(P) \leq \left( \frac{1 + \sqrt{5}}{2} \right)^d \left( \frac{L(P)}{\sqrt{5}} \right)^{\sqrt{5}}.$$ 

For the family of polynomials $(P_n)_{n \geq 0}$, it gives:

$$1.687734 \leq \lim_{n \to \infty} M(P_n) \left( \frac{1}{\deg(P_n)} \right) = 1.727305... \leq 1.854643.$$ 

Whereas, by Theorem 1, we obtain the better inequalities:

$$1.722326 \leq \lim_{n \to \infty} M(P_n) \left( \frac{1}{\deg(P_n)} \right) = 1.727305... \leq 1.807488.$$ 

By Theorem 2 and 3, we get:

$$0.851529 \leq \lim_{n \to \infty} \log L(P_n) \left( \frac{1}{\deg(P_n)} \right) = 0.865755... \leq 0.99641.$$ 

Thus, we see that, for this particular family of polynomials, the inequalities are quite good.

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References


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