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Ciprian G. Gal. Nonlocal Cahn-Hilliard equations with fractional dynamic boundary conditions. 2016.
hal-01230112v2

HAL Id: hal-01230112

<https://hal.science/hal-01230112v2>

Preprint submitted on 1 Nov 2016

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NONLOCAL CAHN-HILLIARD EQUATIONS WITH FRACTIONAL DYNAMIC BOUNDARY CONDITIONS

CIPRIAN G. GAL

ABSTRACT. We consider a nonlocal version of the Cahn-Hilliard equation characterized by the presence of a fractional diffusion operator, and which is subject to fractional dynamic boundary conditions. Our system generalizes the classical system in which the dynamic boundary condition was used to describe any relaxation dynamics of the order-parameter at the walls. The proposed fractional dynamic boundary condition appears to be more general in the sense that it incorporates nonlocal effects which were completely ignored in the classical approach. We aim to deduce well-posedness and regularity results as well as to establish the existence of finite dimensional attractors for this system.

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1. INTRODUCTION

The classical form of the Cahn-Hilliard equation [6] developed in the later 1950's has been successfully applied in a variety of phenomena in science (see also [29]). In particular, it is frequently used as a model for phase separation phenomena in two-component materials which occupy a given domain $\Omega \subset \mathbb{R}^d$, $d \leq 3$. The standard form of the classical Cahn-Hilliard equation is given by

$$(1.1) \quad \partial_t \varphi = \Delta \mu, \quad \mu = -\Delta \varphi + F'(\varphi), \quad \text{in } \Omega \times (0, \infty),$$

after we have scaled up all the relevant physical constants to one, and is usually subjected to no-flux boundary conditions

$$(1.2) \quad \partial_\nu \mu = 0 \quad \text{on } \partial\Omega \times (0, \infty),$$

$$(1.3) \quad \partial_\nu \varphi = 0 \quad \text{on } \partial\Omega \times (0, \infty).$$

In this context, one has ± 1 corresponding to the pure phases of the material while $\varphi \in (-1, 1)$ corresponds to the transition in the interface between the two material phases. A physically relevant choice for F is a singular logarithmic potential which is often approximated by a regular polynomial potential, typically, the double-well potential $F(r) = (r^2 - 1)^2$. We point out that (1.2) implies the conservation of mass for (1.1) while (1.3) implies that the diffused interface, which separates the two phases of the material, intersects the solid wall at a perfect contact angle of $\pi/2$. This, unfortunately, is a strong assumption for many materials and binary systems. We refer the reader for further discussion to [7, 17] and the references therein. In order to remedy this situation one needs

1991 *Mathematics Subject Classification.* 35R09, 37L30, 82C24.

Key words and phrases. nonlocal Cahn-Hilliard equation, fractional dynamic boundary conditions, regional fractional Laplacian, fractional Wentzell Laplacian, well-posedness, regularity, global attractor.

to allow a dynamic contact angle to deviate from the stationary one of $\pi/2$, and so the contact line can move also relative to the solid boundary $\partial\Omega$ of the domain. From this point of view, it was proposed in the 1990's (see [5, 13]) that the dynamics near the solid wall is better described by dynamic boundary conditions, of the form

$$(1.4) \quad \partial_t \psi - \Delta_{\partial\Omega} \psi + \partial_\nu \varphi + \beta \psi + G'(\psi) = 0, \text{ on } \partial\Omega \times (0, \infty),$$

where $\psi = \varphi|_{\partial\Omega}$, in the trace sense whenever the latter exists. Generally, $\beta > 0$ is a constant, $\Delta_{\partial\Omega}$ denotes the Laplace-Beltrami operator on $\partial\Omega$, and the nonlinear function G characterizes the possible preferential attraction (or repulsion) of one of the components by the walls $\partial\Omega$. Thus, in more general binary systems (1.4) must replace the condition (1.3) to account for relaxation dynamics on the boundary $\partial\Omega$. We recall that various other modifications of (1.4) and (1.2) have also been proposed recently to account for the interaction of the two-phase material with the solid wall $\partial\Omega$. See, for instance, [7, Remark 1.4] and the corresponding references therein. We note that an important feature of (1.1)-(1.2) and (1.4) is that any sufficiently smooth solution (φ, ψ) satisfies the energy identity

$$\frac{d}{dt} E_{\text{tot}}(\varphi(t), \psi(t)) + \int_{\Omega} |\nabla \mu(x, t)|^2 dx + \int_{\partial\Omega} |\partial_t \psi(x, t)|^2 dS = 0, \text{ for all } t \geq 0,$$

where the free (bulk plus surface) energy is given by

$$(1.5) \quad E_{\text{tot}}(\varphi, \psi) = \int_{\Omega} (|\nabla \varphi|^2 + F(\varphi)) dx + \int_{\partial\Omega} (|\nabla_{\partial\Omega} \psi|^2 + \beta \psi^2 + G(\psi)) dS.$$

As far as theoretical results are concerned for the classical Cahn-Hilliard systems (1.1)-(1.3), and (1.1)-(1.2) together with (1.4), we now have a rather complete picture on well-posedness, regularity and longtime behaviour in terms of global attractors and the global asymptotic stability as time goes to infinity of trajectories associated with these systems. We refer the reader to the survey paper [7] for an extensive list of references.

Although the derivation of the Cahn-Hilliard system comprising (1.1), (1.2) and (1.4) may appear physically sound, it was only introduced as an approximation where mass currents/fluxes in Ω and on $\partial\Omega$ were driven by local chemical potential functionals containing derivatives of the density up to second order; this is in agreement with the presence of gradient terms in (1.5). This derivation was performed in [5, 13] starting out from a nonlocal microscopic description of the kinetics of phase changes for a lattice gas of interacting particles, by ignoring any nonlocal contribution to the total free energy in the continuum limit, and by restricting oneself only to the nearest-neighbour interactions. Thus, the interaction between any two nearest neighbours was effectively assumed to be short-ranged. Interestingly this was already pointed out by the authors of [5, 13]. Note that such approximations of first order in the free energy functional, which eventually lead to the classical system (1.1)-(1.3), were already assumed by Cahn and Hilliard in their influential paper [6] but there the interactions with the boundary walls were also ignored. Besides the interaction was once again assumed to be short ranged. Hence, a rigorous derivation was required in order to develop a phase-field model which can account for both short ranged interactions as well as any long range interactions between any two distant particles, and without making any kind of approximations. Such rigorous analysis was performed only recently by Giacomini and Lebowitz [21] starting out from a microscopic model for a lattice gas with long-range potentials of the form $J(x) = J(-x)$ with $J \in C^2(\mathbb{R}^d)$. It turns out that in place of the approximation (1.1) one obtains instead the following *nonlocal* version of the Cahn-Hilliard equation:

$$(1.6) \quad \partial_t \varphi = \Delta \mu, \quad \mu = \int_{\Omega} J(x-y) (\varphi(x) - \varphi(y)) dy + F'(\varphi), \text{ in } \Omega \times (0, \infty).$$

More precisely, in this formulation the radially symmetric kernel J encodes both the local and nonlocal information in a manner in which any (microscopic) particle interaction takes place throughout the domain Ω . It is interesting to point out that for (1.6) the

corresponding free energy takes on the form

$$(1.7) \quad E_{nonloc}(\varphi) = \int_{\Omega} \int_{\Omega} J(x-y) |\varphi(x) - \varphi(y)|^2 dx dy + \int_{\Omega} F(\varphi) dx.$$

Rigorous mathematical results for the nonlocal Cahn-Hilliard equation (1.6) together with (1.2), have been established only recently. We refer the reader to [1, 3, 4, 8, 14] for well-posedness results, to [1, 11, 12, 16] for regularity and longtime behaviour in terms of global attractors, and to [16, 26, 27] for issues concerning the global asymptotic stability of trajectories as time goes to infinity.

However, in the general formulation of Giacomini and Lebowitz [21], the interaction between particles inside the bulk domain Ω and particles on $\partial\Omega$ was ignored, and so any (stationary) boundary condition for φ was somewhat encoded in the properties of the interaction kernel J . This is in particular true for a strongly singular kernel $J \notin L^1_{loc}(\mathbb{R}^d)$ (see [1, 18, 22, 23]) and it covers the case when a (regional) fractional Laplace operator $(-\Delta)^s_{\Omega}$, $s \in (0, 1)$, appears in the formulation of the chemical potential in (1.6) (of course, in that case the integral in the second equation of (1.6) must be understood in a principal valued sense). We recall that the use of fractional operators is quite natural for systems that describe long ranged interactions, when particles are allowed to take "arbitrarily" large steps (up to the system size) with a small finite probability for each such step, in addition to any short-ranged interactions between close neighbours. In this sense, normal (classical) diffusion in the Cahn-Hilliard equation (1.1) describes phase-transitions in systems close to equilibrium, whereas anomalous diffusion in the presence of fractional Laplace operators in (1.6) seems to be inherent in phase-transitions phenomena that is far from equilibrium. Returning to our first point of this paragraph, our goal is to consider the case of phase-separation phenomena that also accounts for the presence of particle interactions with the wall $\partial\Omega$. We follow an idea consistent with the point of view of [13], except that now our (dynamic) boundary condition incorporates any of the aforementioned nonlocal effects (see Appendix), and may simply reduce to the classical description (1.4) under certain conditions. To give a precise formulation of the nonlocal value problem we wish to investigate, we first recall the definition for the *regional fractional Laplace operator*, as given by, for $u \in \mathcal{L}^1(\Omega)$,

$$(1.8) \quad \begin{aligned} (-\Delta)^s_{\Omega} u(x) &= C_{d,s} \text{P.V.} \int_{\Omega} \frac{u(x) - u(y)}{|x - y|^{d+2s}} dy \\ &= C_{d,s} \lim_{\varepsilon \downarrow 0} \int_{\{y \in \Omega, |y-x| > \varepsilon\}} \frac{u(x) - u(y)}{|x - y|^{d+2s}} dy, \quad x \in \Omega, \end{aligned}$$

provided that the limit exists. Here,

$$\mathcal{L}^1(\Omega) := \{u : \Omega \rightarrow \mathbb{R} \text{ measurable, } \int_{\Omega} \frac{|u(x)|}{(1 + |x|)^{d+2s}} dx < \infty\}$$

and

$$(1.9) \quad C_{d,s} = \frac{s 2^{2s} \Gamma\left(\frac{d+2s}{2}\right)}{\pi^{\frac{d}{2}} \Gamma(1-s)},$$

where Γ denotes the usual Gamma function. We refer to [22, 23, 33] (cf. also Section 2.2 below) for a class of functions for which the limit in (1.8) exists.

Our interest lies with the (nonlocal) boundary value problem:

$$(1.10) \quad \partial_t \varphi = \Delta \mu, \quad \mu = (-\Delta)^s_{\Omega} \varphi + F'(\varphi), \quad \text{in } \Omega \times (0, \infty),$$

subject to the boundary conditions:

$$(1.11) \quad \partial_{\nu} \mu = 0, \quad \text{on } \partial\Omega \times (0, \infty),$$

$$(1.12) \quad \partial_t \psi + (-\Delta)^l_{\partial\Omega} \psi + C_s N^{2-2s} \varphi + \beta \psi + G'(\psi) = 0, \quad \text{on } \partial\Omega \times (0, \infty),$$

such that $\psi = \varphi|_{\partial\Omega}$, and

$$(1.13) \quad \varphi|_{t=0} = \varphi_0 \text{ in } \Omega, \quad \psi|_{t=0} = \psi_0 \text{ on } \partial\Omega.$$

In (1.12), $C_s N^{2-2s}$, $s \in (1/2, 1)$, denotes the so-called fractional normal derivative and $(-\Delta)_{\partial\Omega}^l$, $l \in (0, 1)$ corresponds to the fractional surface diffusion operator on $\partial\Omega$ (see Section 2). We observe that at least formally when $s = l = 1$ the boundary condition (1.12) translates to the classical boundary condition (1.4), and (1.10) is formally the classical (local) Cahn-Hilliard equation (1.1). A further physical justification for (1.10)-(1.13) using the simplest model of an interacting particle systems is given in the Appendix (see Section 5).

It is our goal to give a comprehensive treatment of (1.10)-(1.13) in terms of existence, regularity and stability (with respect to the initial data) of properly defined solutions. We will also discuss and derive sufficient conditions for the nonlinearities F , G , so that (1.10)-(1.13) possesses finite dimensional global and exponential attractors. Our main goal is to relax the usual framework (see [7]) associated with the classical Cahn-Hilliard equations (1.1)-(1.2) subject to classical dynamic boundary conditions (1.4), in the sense that we wish to consider the more general case when Ω is a bounded domain with Lipschitz continuous boundary. This is in particular important when the domains of the operators involved in the weak and strong formulations associated with the problem no longer enjoy explicit characterizations in terms of standard Sobolev spaces. Indeed, this is not at all expected and in fact impossible in the nonlocal framework that includes fractional diffusion operators like the *regional fractional Laplace* $(-\Delta)_\Omega^s$ and the fractional Laplace-Beltrami boundary operator $(-\Delta)_{\partial\Omega}^l$. These features make the usual constructive arguments based on the Galerkin approximation scheme for the classical (local) Cahn-Hilliard equation very difficult to apply, as well as place additional regularity constraints on the parameters involved in the nonlocal counter-part. To resolve these issues in the nonlocal framework, we develop a time-implicit discretization scheme for an abstract initial value problem in a generic setting, that includes all the problems considered in this article and much more. In fact, such abstract schemes have already proved their usefulness in the treatment of other nonlocal problems, see [15]. We then take advantage of several recent results of Gal and Warma [19, 20] suited for nonlocal operators associated with the fractional dynamic boundary condition (1.12) to develop a suitable solution theory for problem (1.10)-(1.13). Such results will also become crucial in the last part of this contribution where we will show that our problem admits an exponential attractor, as well as a finite dimensional global attractor.

The article is organized as follows. In Section 2 we define the functional setting needed for our approach: we first consider in subsection 2.1 an abstract boundary value for two (generic) self-adjoint operators A, B and solve this problem by appealing to a backward Euler scheme and by employing fixed point arguments. Then in subsection 2.2 we give a proper definition for the fractional dynamic boundary condition in (1.12) as well as define all the corresponding operators involved in the subsequent analysis. In Section 3 we define what we mean by a weak and strong solution of (1.10)-(1.13) and then provide and prove appropriate well-posedness results. Section 4 is dedicated to the regularity of weak solutions as well as the existence of finite dimensional attractors for (1.10)-(1.13). Section 6 contains some additional remarks and some open questions.

2. FUNCTIONAL FRAMEWORK AND WELL-POSEDNESS

2.1. A backward Euler scheme. Let us first introduce an abstract initial-boundary value problem:

$$(2.1) \quad \partial_t u(t) + A\mu(t) = 0, \text{ in } H_A \times (0, T)$$

$$(2.2) \quad \partial_t U(t) + BU(t) + f(U(t)) = \mu(t), \text{ in } H_B \times (0, T),$$

subject to the condition

$$(2.3) \quad U|_{t=0} = U_0 \text{ in } H_B.$$

Let X be a locally compact metric space and m a Radon measure on X such that $\text{supp}(m) = X$. We let $H_A = L^2(X, m)$ be a given Hilbert space with inner product $(\cdot, \cdot)_{H_A}$ and assume that $A \geq 0$ is a self-adjoint operator on H_A such that 0 is an eigenvalue of A . Similarly, let $H = L^2(Y, m_Y)$ for some Radon measure m_Y that is supported on a locally

compact metric space Y . Then consider the (product) Hilbert space $H_B = H_A \times H$, endowed with the natural product norm $\|U\|_{H_B}^2 = \|u\|_{H_A}^2 + \|\bar{u}\|_H^2$, for all vector-valued functions $U = (u, \bar{u}) \in H_B$. In (2.2) we further identify $\mu \in H_A$ with its pair $(\mu, 0) \in H_B$. On H_B we define a densely defined operator B that is positive and self-adjoint. The matrix-valued function $f(U) = (f_1(u), f_2(\bar{u}))$ is assumed to belong to $C_b^1(\mathbb{R}^2, \mathbb{R}^2)$. Furthermore, we let $V_A = D((A + I)^{1/2})$ and $V_B = D(B^{1/2})$ and endow them with the natural inner products of V_A and V_B , respectively. Finally, we also assume that $(I + A)^{-1}$ and B^{-1} are compact operators on H_A and H_B , respectively. Without loss of generality, we also assume that the constant (function) $1 \in D(B) = \{U \in V_B : BU \in H_B\}$ since this is our case of interest. The application we have mind for the abstract problem (2.1)-(2.3) in Sections 3, 4 is as follows: $X = \Omega$ is a bounded domain in $\mathbb{R}^d$, $d \leq 3$ with Lipschitz continuous boundary $Y = \partial\Omega$, where m is the usual Lebesgue measure on X and m_Y is the standard surface measure on $\partial\Omega$. Furthermore, A and B are the (weak) Neumann Laplacian and fractional "Wentzell" Laplacian, respectively (cf. Section 2.2 below).

Definition 2.1. *Let $T > 0$ be given, but otherwise arbitrary. Our notion of strong solution for (2.1)-(2.3) is as follows: the pair (U, μ) satisfies*

$$\begin{cases} U \in L^2(0, T; D(B)) \cap L^\infty(0, T; V_B), \\ \partial_t U \in L^2(0, T; H_B), \\ f(U) \in L^2(0, T; H_B), \\ \mu \in L^2(0, T; D(A)), \end{cases}$$

such that $\partial_t u(t) + A\mu(t) = 0$ in H_A -sense, $\partial_t U(t) + BU(t) + f(U(t)) = \mu(t)$, in the H_B -sense, for a.e. $t \in (0, T)$. The initial condition $U|_{t=0} = U_0$ is also satisfied in a strong H_B -sense.

The proof of the following result is based on a time-discretization scheme and compactness arguments.

Theorem 2.2. *Let f_1, f_2 be globally Lipschitz functions over $\mathbb{R}$ and assume there exist $c_f, C_f > 0$ such that $f(U) \cdot U \geq -c_f |U|^2 - C_f$, with $2c_f \leq C_B$ where $\|U\|_{H_B}^2 \leq C_B \|U\|_{V_B}^2$. Then for every $U_0 \in V_B$, the abstract nonlinear problem (2.1)-(2.3) possesses at least one strong solution in the sense of Definition 2.1.*

Remark 2.1. Note that (2.2) is a system of equations for the vector-valued function $U = (u, \bar{u})$. Theorem 2.2 can be also modified for the case of a single equation when one merely identifies U with u and lets $H_A = H_B$.

Remark 2.2. We also observe that, as a consequence of Definition 2.1 we have $U \in C_w([0, T]; V_B)$, where the latter space stands for the space of weakly continuous functions from $[0, T]$ with values in V_B . We point out that in fact more can be said about the solution of Definition 2.1. Indeed, $U \in C([0, T]; V_B)$ owing to the fact that by (2.2), we have

$$(2.4) \quad \frac{d}{dt} \|U(t)\|_{V_B}^2 = -2 \|BU\|_{H_B}^2 - 2(f(U), BU)_{H_B} + 2(\mu, BU)_{H_B},$$

which follows by testing (2.2) with $BU \in H_B$. Since all the terms on the right-hand side of (2.4) belong to $L^1(0, T)$, we find the absolute continuity of $\|U(t)\|_{V_B}^2$ on $[0, T]$. This, together with weak continuity, gives the desired strong continuity.

Theorem 2.2 will become crucial in the subsequent Section 3 in the existence of solutions for (1.10)-(1.13) when we shall specialize the operators A, B . Our goal in this subsection is to provide a complete proof of Theorem 2.2. We shall employ a technique based on semi-discretization in time, which can be used to construct explicit approximate solutions that will converge to a solution of the original problem under natural conditions on the physical parameters.

We will divide this programme into several key steps. We first construct our time-difference scheme. To this end, let n be an integer and P be a partition of the time interval $[0, T]$ such that $P = \{t_0, t_1, \dots, t_n\}$ with $0 = t_0 < t_1 < \dots < t_m < \dots < t_n = T$, where $t_m = mh$ for $0 \leq m \leq n$ and $h = T/n$ is the step size. In applications one may also

choose variable step sizes for every subinterval (t_{m-1}, t_m) but in our case a fixed step size suffices for the sake of simplicity. Then for $1 \leq m \leq n$, we consider the time difference equations associated with (2.1)-(2.3):

$$(2.5) \quad \delta_t u^m = -A\mu^m \text{ in } H_A,$$

$$(2.6) \quad \delta_t U^m + BU^m + f(U^m) = \mu^m \text{ in } H_B,$$

such that

$$(2.7) \quad U^0 = U_0 \in V_B.$$

Here we have set

$$\delta_t u^m = \frac{u^m - u^{m-1}}{h}, \quad \delta_t U^m = (\delta_t u^m, \delta_t \bar{u}^m)$$

($\delta_t \bar{u}^m$ is defined in a similar way) and we think of U^m and μ^m as approximations of the functions $U(x, t_m)$ and $\mu(x, t_m)$, respectively.

By analogy with Definition 2.1 we say that (2.5)-(2.7) has a strong solution in the following sense.

Definition 2.3. *We say (U^m, μ^m) is a strong solution if $U^m \in D(B)$, $\mu^m \in D(A)$, for every $1 \leq m \leq n$, the initial condition $U^0 = U_0$ is satisfied in the strong H_B -sense, and the relations in (2.5)-(2.6) are satisfied in the strong sense of H_A and H_B , respectively.*

Step 1. Our first goal is to prove the existence of at least one solution (U^m, μ^m) to the system (2.5)-(2.7), for a fixed m and assuming that U^{m-1} is already known. To show this claim, we apply the Leray-Schauder degree theory to prove the existence of a solution to the following nonlinear system:

$$(2.8) \quad u + hA\mu = g \text{ in } H_A,$$

$$(2.9) \quad hBU + U + hf(U) = h\mu + l \text{ in } H_B,$$

where we have set $(U, \mu) = (U^m, \mu^m)$ and $g = u^{m-1}$, $l = U^{m-1}$. We wish to formulate problem (2.8)-(2.9) in terms of a fixed point for a nonlinear map $S(1, U, \mu) = (U, \mu)$, where $S(\lambda, \cdot, \cdot)$ is a compact homotopy depending on the parameter $\lambda \in [0, 1]$ defined as follows:

$$S(\lambda, \cdot, \cdot) : [0, 1] \times V_B \times V_A \rightarrow V_B \times V_A$$

such that $S(\lambda, V, w) = (U, \mu)$ is a solution of

$$(2.10) \quad v + hA\mu = \lambda g, \text{ in } H_A,$$

$$(2.11) \quad hBU + V = hw + \lambda l - h\lambda f(V), \text{ in } H_B.$$

In this step we may also assume without loss of generality that $A = A^*$ is positive. Indeed if $A \geq 0$, we could replace equation (2.8) by $u + h(A + I)\mu = g + h\mu$, and then (2.10) by $v + h(A + I)\mu = \lambda g + hw$. Next, we observe that if $g \in H_A$ and $v \in H_A$ then $\mu \in D(A)$ is the unique solution of (2.10). Moreover, if $l \in H_B$ and $H_A \ni w = (w, 0) \in H_B$ and $V \in H_B$, which yields that $f(V) \in H_B$ owing to $f \in C_b^1$, then (2.11) has a unique solution $U \in D(B)$. Here we have endowed both $D(A)$ and $D(B)$ with their corresponding graph norms. Since the injections $D(A) \subset V_A$ and $D(B) \subset V_B$ are continuous, the mapping $S(\lambda, \cdot, \cdot)$ is well-defined.

Lemma 2.4. *S is continuous as a mapping from $[0, 1] \times V_B \times V_A \rightarrow V_B \times V_A$.*

Proof. Let $\lambda_n \rightarrow \lambda \in [0, 1]$ and $(V_n, w_n) \rightarrow (V, w)$ strongly in $V_B \times V_A$. Write $(U_n^{\lambda_n}, \mu_n^{\lambda_n}) = S(\lambda_n, V_n, w_n)$, $(U_n^\lambda, \mu_n^\lambda) = S(\lambda, V_n, w_n)$ and $(U^\lambda, \mu^\lambda) = S(\lambda, V, w)$, respectively. From (2.10)-(2.11) it follows that $(U_n^{\lambda_n} - U_n^\lambda, \mu_n^{\lambda_n} - \mu_n^\lambda)$ satisfies the system

$$(2.12) \quad hA(\mu_n^{\lambda_n} - \mu_n^\lambda) = (\lambda_n - \lambda)g, \text{ in } H_A,$$

$$(2.13) \quad hB(U_n^{\lambda_n} - U_n^\lambda) = (\lambda_n - \lambda)l - h(\lambda_n - \lambda)f(V_n), \text{ in } H_B.$$

Taking the corresponding inner products of (2.12)-(2.13) in H_A , H_B , with $\mu_n^{\lambda_n} - \mu_n^\lambda$ and $U_n^{\lambda_n} - U_n^\lambda$, respectively, we easily deduce the estimates:

$$(2.14) \quad \left\| \mu_n^{\lambda_n} - \mu_n^\lambda \right\|_{V_A} \leq C |\lambda_n - \lambda| \|g\|_{H_A},$$

$$(2.15) \quad \left\| U_n^{\lambda_n} - U_n^\lambda \right\|_{V_B} \leq C |\lambda_n - \lambda| \left(\|l\|_{H_B} + \|V_n\|_{H_B} \right),$$

for some constant $C > 0$, independent of n, λ_n, λ . Note that since $g \in H_A$ and $l \in H_B$ and $\{V_n\} \subset H_B$ is bounded, we have as $n \rightarrow \infty$ that $\mu_n^{\lambda_n} - \mu_n^\lambda \rightarrow 0$ strongly in V_A , as well as $U_n^{\lambda_n} - U_n^\lambda \rightarrow 0$ strongly in V_B . On the other hand, from (2.10)-(2.11) we see that $(U_n^\lambda - U^\lambda, \mu_n^\lambda - \mu^\lambda)$ satisfies

$$(2.16) \quad hA(\mu_n^\lambda - \mu^\lambda) = -(v_n - v), \text{ in } H_A,$$

$$(2.17) \quad hB(U_n^\lambda - U^\lambda) + (V_n - V) = h(w_n - w) - h\lambda(f(V_n) - f(V)), \text{ in } H_B.$$

As before, taking the inner products of (2.16)-(2.17) in H_A, H_B , with $\mu_n^\lambda - \mu^\lambda$ and $U_n^\lambda - U^\lambda$, respectively, we immediately get

$$(2.18) \quad \left\| \mu_n^\lambda - \mu^\lambda \right\|_{V_A} \leq C \|v_n - v\|_{H_A} \leq C \|V_n - V\|_{H_B}$$

and

$$(2.19) \quad \left\| U_n^\lambda - U^\lambda \right\|_{V_B} \leq C \left(\|V_n - V\|_{H_B} + \|w_n - w\|_{H_A} \right),$$

owing to the fact that $f \in C_b^1$, for some constant $C > 0$ independent of n . Thus, if the right-hand sides of (2.18)-(2.19) go to zero as $n \rightarrow \infty$, then $\mu_n^\lambda - \mu^\lambda \rightarrow 0$, strongly in V_A -sense, and $U_n^\lambda - U^\lambda \rightarrow 0$ strongly in V_B . This completes the proof in view of (2.14)-(2.15) and (2.18)-(2.19). ■

Lemma 2.5. $S(\lambda, \cdot, \cdot)$ is also compact.

Proof. Let $\{(\lambda_n, V_n, w_n)\}$ be any bounded sequence in $[0, 1] \times V_B \times V_A$ and consider $(U_n, \mu_n) = S(\lambda_n, V_n, w_n)$ as a solution of (2.10)-(2.11). Exploiting the same arguments as in (2.14)-(2.15) and (2.18)-(2.19), we easily see that $\{(U_n, \mu_n)\}$ is bounded in $V_B \times V_A$ uniformly with respect to $n \in \mathbb{N}$. Then taking the inner products of the equations for (U_n, μ_n) with $A\mu_n$ and BU_n in H_A and H_B , respectively, and using the aforementioned fact, we also arrive at the fact that $\{(U_n, \mu_n)\}$ is bounded in $D(B) \times D(A)$ uniformly in n , owing to $f \in C_b^1$. Since $(I + A)^{-1}$ and B^{-1} are compact (self-adjoint) operators, the injections $D(B) \subset V_B$, $D(A) \subset V_A$ are dense and compact. This yields in particular the compactness of $S(\lambda, \cdot, \cdot)$ in $V_B \times V_A$. ■

Lemma 2.6. The Leray-Schauder degree for the mapping $S(1, \cdot, \cdot)$ is

$$D(Id - S(1, \cdot, \cdot), B(0, M), 0) = 1,$$

for some (open) ball

$$B(0, M) = \left\{ (U, \mu) \in V_B \times V_A : \|(U, \mu)\|_{V_B \times V_A} < M \right\},$$

for some $M > 0$. In particular, problem (2.8)-(2.9) has at least one (strong) solution.

Proof. For a definition of the Leray-Schauder degree, please refer to [9, Chapter 2, Section 8.3]. In order to prove the claim, it suffices to show that any possible fixed point of $S(\lambda, \cdot, \cdot)$ can be estimated independently of $\lambda \in [0, 1]$. More precisely, we claim that if $(U, \mu) \in V_B \times V_A$ is such that $S(\lambda, U, \mu) = (U, \mu)$, for some $\lambda \in [0, 1]$, then there exists a constant $M > 0$ such that

$$(2.20) \quad \|(U, \mu)\|_{V_B \times V_A} < M.$$

First, let us consider the linear system of equations given by $u_* + hA\mu_* = 0$ in H_A , and $hBU_* + U_* = h\mu_*$, in H_B . Observing that $S(0, U_*, \mu_*) = (U_*, \mu_*)$ is the unique solution of such a system, our first requirement is that $M > 0$ is large enough such that the ball $B(0, M)$ contains this unique solution. On the other hand, we recall that any fixed point (U, μ) of (2.10)-(2.11) solves the system

$$(2.21) \quad u + hA\mu = \lambda g, \text{ in } H_A,$$

$$(2.22) \quad hBU + U = h\mu + \lambda l - h\lambda f(U), \text{ in } H_B.$$

Let us now take the inner product in H_A of (2.21) with $h\mu$, the inner product in H_B of (2.22) with U and then add the resulting relations. We deduce

$$(2.23) \quad \begin{aligned} & h \|U\|_{V_B}^2 + \|U\|_{H_B}^2 + h^2 \left\| A^{1/2} \mu \right\|_{H_A}^2 \\ & = \lambda h (g, \mu)_{H_A} + \lambda (l, U)_{H_B} - \lambda h (f(U), U)_{H_B}. \end{aligned}$$

Since by assumption $f(U) \cdot U \geq -c_f |U|^2 - C_f$, we obtain from (2.23) for every $\lambda \in [0, 1]$ that

$$\begin{aligned} & h \left(\frac{C_B}{2} - c_f \right) \|U\|_{H_B}^2 + \frac{h}{2} \|U\|_{V_B}^2 + \|U\|_{H_B}^2 + h^2 \left\| A^{1/2} \mu \right\|_{H_A}^2 \\ & \leq |h| \left(\|g\|_{H_A} \left\| \mu - (\mu, 1)_{H_A} \right\|_{H_A} + \left| (g, 1)_{H_A} (\mu, 1)_{H_A} \right| \right) \\ & + \left(\varepsilon \|U\|_{H_B}^2 + C_\varepsilon \|l\|_{H_B}^2 \right) \\ & \leq \left(C_{\varepsilon, h} \|g\|_{H_A}^2 + \varepsilon h^2 \left\| A^{1/2} \mu \right\|_{H_A}^2 \right) + C \|g\|_{H_A} \left| (\mu, 1)_{H_A} \right| \\ & + \left(\varepsilon \|U\|_{H_B}^2 + C_\varepsilon \|l\|_{H_B}^2 \right), \end{aligned}$$

for any $\varepsilon > 0$, for some constants $C, C_\varepsilon > 0$ independent of $\lambda \in [0, 1]$. Therefore, choosing a sufficiently small $\varepsilon \leq 1/2$, we derive the estimate:

$$(2.24) \quad \|U\|_{V_B}^2 + h^2 \left\| A^{1/2} \mu \right\|_{H_A}^2 \leq C_h \left(\|g\|_{H_A}^2 + \|l\|_{H_B}^2 + \|g\|_{H_A} \left| (\mu, 1)_{H_A} \right| \right).$$

Next, taking the inner product in H_B of (2.22) with $V \equiv 1$ we find

$$(\mu, 1)_{H_A} = h \left(B^{1/2} U, B^{1/2} 1 \right)_{H_B} + (U, 1)_{H_B} - \lambda (l(x), 1)_{H_B} + h \lambda (f(U), 1)_{H_B}.$$

By virtue of (2.24) and the fact that f is globally Lipschitzian on $\mathbb{R}^2$, we have for every $\lambda \in [0, 1]$,

$$(2.25) \quad \left| (\mu, 1)_{H_A} \right| \leq C \left(1 + \|U\|_{V_B} + \|l\|_{H_B} \right),$$

for some $C > 0$ independent of λ . Estimate (2.25) together with (2.24) then yields by token of elementary Young inequalities,

$$(2.26) \quad \|U\|_{V_B}^2 + \|\mu\|_{V_A}^2 \leq C \left(1 + \|g\|_{H_A}^2 + \|l\|_{H_B}^2 \right),$$

for $C = C(h, A, B, c_f, C_f) > 0$ independent of λ . Thus, if we take $M > 0$ large enough such that $M > C(1 + \|g\|_{H_A}^2 + \|l\|_{H_B}^2)$, (2.20) ensures that $S(\lambda, U, \mu) \neq (U, \mu)$ for any $(U, \mu) \in \partial B(0, M)$, for all $\lambda \in [0, 1]$. Finally, the homotopy invariance of the degree $D(Id - S(\lambda, \cdot, \cdot), B(0, M), 0)$, for $\lambda \in [0, 1]$, implies that

$$D(Id - S(1, \cdot, \cdot), B(0, M), 0) = D(Id - S(0, \cdot, \cdot), B(0, M), 0) = 1,$$

which is enough to conclude that problem (2.8)-(2.9) has at least one solution $(U, \mu) \in V_B \times V_A$. The latter also immediately yields the regularity $(U, \mu) \in D(B) \times D(A)$ owing once again to the facts that $g \in H_A$, $l \in H_B$ and f is globally Lipschitz. ■

Remark 2.3. It turns out that each solution (U, μ) of (2.1)-(2.3) is also unique, a fact which can be easily checked. We will not need this property here and so we leave the details to be verified by the interested reader.

Step 2. In this step we are interested in deducing a priori estimates for solutions of (2.5)-(2.7), which are uniform with respect to the step size h . In what follows, $C > 0$ will denote a constant that is independent of h but which only depends on f , and the operators A, B . Since $A = A^* \geq 0$ and $1 \in \text{Null}(A)$, we may assume that $(\delta_t u^m, 1)_{H_A} = 0$; this implies that $(u^m, 1)_{H_A} = (u^{m-1}, 1)_{H_A} = (u_0, 1)_{H_A}$ for every $1 \leq m \leq n$. Next, we take the inner

product in H_A of (2.5) with $\mu^m - (\mu^m, 1)_{H_A}$ and the inner product in H_B of (2.6) with $\delta_t U^m$. Adding the relations that we obtain, we find

$$(2.27) \quad \begin{aligned} & \|\delta_t U^m\|_{H_B}^2 + \|A^{1/2} \mu^m\|_{H_A}^2 + \frac{1}{h} (BU^m, U^m - U^{m-1})_{H_B} \\ & = - (f(U^m), \delta_t U^m)_{H_B}. \end{aligned}$$

The basic identity $2x(x-y) = x^2 - y^2 + (x-y)^2$ gives

$$\begin{aligned} & \frac{1}{h} (BU^m, U^m - U^{m-1})_{H_B} \\ & = \frac{1}{2h} \left(\|B^{1/2} U^m\|_{H_B}^2 - \|B^{1/2} U^{m-1}\|_{H_B}^2 + \|B^{1/2} (U^m - U^{m-1})\|_{H_B}^2 \right), \end{aligned}$$

while for $f \in C_b^1$, we get

$$(f(U^m), \delta_t U^m)_{H_B} \leq \varepsilon \|\delta_t U^m\|_{H_B}^2 + C_\varepsilon (\|U^m\|_{H_B}^2 + 1).$$

Therefore, for a sufficiently small $\varepsilon > 0$, from (2.27) we obtain after we add the relations for $m = 1, 2, \dots, r$,

$$(2.28) \quad \begin{aligned} & 2h \sum_{m=1}^r \left(\|\delta_t U^m\|_{H_B}^2 + \|A^{1/2} \mu^m\|_{H_A}^2 \right) + \|U^r\|_{V_B}^2 + \sum_{m=1}^r \|U^m - U^{m-1}\|_{V_B}^2 \\ & \leq C \left(h \sum_{m=1}^r \|U^m\|_{H_B}^2 + 1 + \|U_0\|_{V_B}^2 \right), \end{aligned}$$

for $1 \leq r \leq n$ and $|h| \leq 1$. We can estimate $(\mu^m, 1)_{H_A}$ by taking the inner product in H_B of (2.6) with $V \equiv 1$. Indeed, we have

$$(\mu^m, 1)_{H_A} = (f(U^m), 1)_{H_B} + (B^{1/2} U^m, B^{1/2} 1)_{H_B} + (\delta_t U^m, 1)_{H_B}.$$

A repeated application of Young's inequality gives

$$(2.29) \quad h \sum_{m=1}^r \left| (\mu^m, 1)_{H_A} \right|^2 \leq C \left(h \sum_{m=1}^r \|U^m\|_{V_B}^2 + 1 + \|U_0\|_{V_B}^2 \right),$$

recalling once again that (2.28) is satisfied. We can combine the two estimates of (2.28) and (2.29). We find

$$(2.30) \quad \begin{aligned} & h \sum_{m=1}^r \left(\|\delta_t U^m\|_{H_B}^2 + \|\mu^m\|_{V_A}^2 \right) + \|U^r\|_{V_B}^2 + \sum_{m=1}^r \|U^m - U^{m-1}\|_{V_B}^2 \\ & \leq C \left(1 + \|U_0\|_{V_B}^2 + h \sum_{m=1}^r \|U^m\|_{V_B}^2 \right). \end{aligned}$$

Now let us recall Gronwall's lemma in discrete form.

Proposition 2.7. *Let $\{y_r\}$ and $\{a_r\}$ be two nonnegative sequences and c_0 a positive constant. If $y_r \leq c_0 + \sum_{m=1}^r a_m y_m$ for $r \geq 1$, then*

$$(2.31) \quad y_r \leq c_0 \exp \left(\sum_{m=1}^r a_m \right), \text{ for } r \geq 1.$$

Apply now (2.31) to (2.30) to conclude that

$$\|U^r\|_{V_B}^2 \leq e^{CT} \left(1 + \|U_0\|_{V_B}^2 \right), \text{ for every } 1 \leq r \leq n.$$

Also going back to (2.30), we also obtain the following estimates that are uniform with respect to h :

$$(2.32) \quad h \sum_{m=1}^r \left(\|\delta_t U^m\|_{H_B}^2 + \|\mu^m\|_{V_A}^2 \right) \leq C_0, \quad \sum_{m=1}^r \|U^m - U^{m-1}\|_{V_B}^2 \leq C_0,$$

$$(2.33) \quad \sup_{1 \leq r \leq n} \|U^r\|_{V_B}^2 \leq C_0, \quad \sup_{1 \leq r \leq n} \|f(U^r)\|_{V_B}^2 \leq C_0$$

for some constant $C_0 > 0$ which depends only on $C, T > 0$ and the initial datum $U_0 \in V_B$. The last inequality in (2.33) is immediate on account of the first one and the fact that f is globally Lipschitz.

Next, we need higher-order estimates for (U^m, μ^m) . We take the inner product in H_A of (2.5) with $A\mu^m$ and use Young and Cauchy-Schwarz inequalities to observe that

$$\|A\mu^m\|_{H_A}^2 \leq C \|\delta_t \mu^m\|_{H_A}^2 \leq C \|\delta_t U^m\|_{H_B}^2.$$

This estimate together with (2.32) yields

$$(2.34) \quad h \sum_{m=1}^r \|\mu^m\|_{D(A)}^2 \leq C_0, \text{ for all } 1 \leq r \leq n.$$

Analogously, we take the inner product in H_B of equation (2.6) with BU^m , use the fact that $f \in C_b^1$ and exploit once more (2.32)-(2.33) to easily arrive at the estimate

$$(2.35) \quad h \sum_{m=1}^r \|U^m\|_{D(B)}^2 \leq C_0, \text{ for all } 1 \leq r \leq n.$$

Step 3. We can now complete the proof of Theorem 2.2. To this end, we need to introduce the corresponding piecewise constant interpolating functions (U_h, μ_h) and also the corresponding linear interpolate functions $\tilde{U}_h$ as follows. Let $P = \{t_0, t_1, \dots, t_n\}$ be a partition with $0 = t_0 < t_1 < \dots < t_m < \dots < t_n = T$, where $t_m = mh$ for $0 \leq m \leq n$ and $h = T/n$ is the step size. For the vectors $\{\mu^m\}_{m=0}^n \subset H_A$, $\{U^m\}_{m=0}^n \subset H_B = H_A \times H$, consider the following interpolating functions

$$(2.36) \quad \mu_h(\cdot, t) = \mu^m, \quad U_h(\cdot, t) = U^m, \quad \text{for } t \in [t_{m-1}, t_m]$$

and

$$(2.37) \quad \tilde{U}_h(\cdot, t) = U^m + \frac{(t - t_{m-1})}{h} (U^m - U^{m-1}), \quad t \in [t_{m-1}, t_m].$$

With these definitions we may now rewrite the entire problem (2.5)-(2.7) in terms of U_h, μ_h and $\tilde{U}_h$ as follows:

$$(2.38) \quad \partial_t \tilde{u}_h + A\mu_h = 0 \text{ in } H_A \times (0, T),$$

$$(2.39) \quad \partial_t \tilde{U}_h + BU_h + f(U_h) = \mu_h \text{ in } H_B \times (0, T),$$

such that

$$(2.40) \quad \tilde{U}_h(\cdot, 0) = U_0 \text{ in } H_B.$$

By rewriting the estimates obtained in *Step 2* in terms of the interpolations functions U_h, μ_h and $\tilde{U}_h$, we deduce the following result.

Lemma 2.8. *We have the estimates:*

$$(2.41) \quad \|U_h\|_{L^\infty(0, T; V_B)} \leq C_0, \quad \|\mu_h\|_{L^2(0, T; D(A))} \leq C_0,$$

$$(2.42) \quad \|U_h\|_{L^2(0, T; D(B))} \leq C_0, \quad \|f(U_h)\|_{L^\infty(0, T; H_B)} \leq C_0,$$

$$(2.43) \quad \|\tilde{U}_h\|_{L^\infty(0, T; V_B)} \leq C_0, \quad \|\tilde{U}_h\|_{L^\infty(0, T; D(B))} \leq C_0$$

and

$$(2.44) \quad \|\partial_t \tilde{U}_h\|_{L^2(0, T; H_B)} \leq C_0.$$

Proof. The first of (2.41) and the last of (2.42) are immediate consequences of (2.33). To prove the second of (2.41), we observe

$$\begin{aligned} \|\mu_h\|_{L^2(0, T; D(A))}^2 &= \sum_{m=1}^n \int_{(m-1)h}^{mh} \left(\|(A + I)\mu^m\|_{H_A}^2 \right) dt \\ &\leq h \sum_{m=1}^n \|\mu^m\|_{D(A)}^2 \leq C_0, \end{aligned}$$

by virtue of (2.34). In a similar fashion one can deduce the first estimate of (2.42), owing to (2.35). Also by token of the first of (2.32), we have

$$\|\partial_t \tilde{U}_h\|_{L^2(0,T;H_B)}^2 = \sum_{m=1}^n \int_{(m-1)h}^{mh} \|\delta_t U^m\|_{H_B}^2 dt \leq h \sum_{m=1}^n \|\delta_t U^m\|_{H_B}^2 \leq C_0.$$

By a similar argument, using (2.33) and (2.35) we obtain the other estimates in (2.43). ■

We now pass to the limit in (2.38)-(2.39) as the step size $h \rightarrow 0^+$. According to the uniform estimates (2.41)-(2.44), there exist subsequences (still labelled by $U_h, \mu_h, \tilde{U}_h$) such that as $h \rightarrow 0^+$, we have the following convergence properties:

$$(2.45) \quad U_h \rightarrow U \text{ weakly } * \text{ in } L^\infty(0, T; V_B),$$

$$(2.46) \quad U_h \rightarrow U \text{ weakly in } L^2(0, T; D(B)),$$

$$(2.47) \quad \mu_h \rightarrow \mu \text{ weakly in } L^2(0, T; D(A)),$$

$$(2.48) \quad \partial_t \tilde{U}_h \rightarrow \partial_t U \text{ weakly in } L^2(0, T; H_B),$$

$$(2.49) \quad \tilde{U}_h \rightarrow \tilde{U} \text{ weakly } * \text{ in } L^\infty(0, T; V_B),$$

$$(2.50) \quad \tilde{U}_h \rightarrow \tilde{U} \text{ weakly in } L^2(0, T; D(B)).$$

These convergence relations and the compact embedding

$$L^2(0, T; D(B)) \cap H^1(0, T; H_B) \subset L^2(0, T; V_B)$$

also allow us to conclude

$$(2.51) \quad U_h \rightarrow U \text{ strongly in } L^2(0, T; V_B) \subset L^2(0, T; H_B).$$

By definition, we further observe

$$\begin{aligned} \|\tilde{U}_h - U_h\|_{L^2(0,T;H_B)}^2 &= \sum_{m=1}^n \int_{(m-1)h}^{mh} (t - t_m)^2 \|\delta_t U^m\|_{H_B}^2 dt \\ &= \frac{h^2}{3} \left(h \sum_{m=1}^n \|\delta_t U^m\|_{H_B}^2 \right) \\ &= \frac{h^2}{3} \|\partial_t \tilde{U}_h\|_{L^2(0,T;H_B)}^2. \end{aligned}$$

Therefore, by (2.44) we may infer that $\|\tilde{U}_h - U_h\|_{L^2(0,T;H_B)} \leq C_0 h$. Thus, passing to the limit as $h \rightarrow 0$, and recalling (2.46), (2.50) we find that $\tilde{U} \equiv U$ in H_B , a.e. on $(0, T)$. Finally, also with help from (2.51) we can then pass to the limit in the nonlinear term of (2.39) to find that $f(U_h) \rightarrow f(U)$ strongly in $L^2(0, T; H_B)$. Subsequently, one may employ once again (2.45)-(2.48) to pass to the limit in the remaining linear terms of (2.38)-(2.39) in a standard fashion. Thus, we have proved that the limit function (U, μ) yields a strong solution in the sense of Definition 2.1. The proof of Theorem 2.2 is now complete.

We can prove further regularity properties for the strong solution of Theorem 2.2.

Theorem 2.9. *Let the assumptions of Theorem 2.2 be satisfied and assume $U_0 \in D(B)$. Then the strong solution (U, μ) of (2.1)-(2.3) has the following regularity:*

$$(2.52) \quad U \in L^\infty(0, T; D(B)) \cap W^{1,\infty}(0, T; H_B), \quad \mu \in L^\infty(0, T; D(A)).$$

Proof. In order to deduce (2.52), we need to (formally) differentiate the equations of (2.1)-(2.3) with respect to $t > 0$ and then set $V := \partial_t U$, $w = \partial_t \mu$. In this case, (V, w) solves the following system

$$(2.53) \quad \begin{cases} \partial_t v + Aw = 0, & \text{in } H_A \times (0, T), \\ \partial_t V + BV + f'(U)V = w, & \text{in } H_B \times (0, T), \end{cases}$$

subject to the initial condition

$$(2.54) \quad V|_{t=0} := V_0 = \begin{pmatrix} \mu_0 \\ 0 \end{pmatrix} - BU_0 - f(U_0).$$

Here in (2.53)-(2.54) we have once again identified $H_A \ni w$ with $(w, 0) \in H_B$, and let μ_0 be the solution of the following abstract boundary value problem

$$(2.55) \quad (A + I)\mu_0 = (BU_0)_1 + f_1(u_0) \text{ in } H_A,$$

where $U_0 = (u_0, \bar{u}_0) \in D(B)$, f_1 is globally Lipschitz in $\mathbb{R}$, and there is the operator representation

$$BU = \begin{pmatrix} (BU)_1 \\ (BU)_2 \end{pmatrix}, \text{ for } U \in D(B).$$

Note that for $U_0 \in D(B)$, we have $\mu_0 \in D(A) \subset H_A$ by (2.55), and so in (2.54) $V_0 \in H_B$. Moreover, the matrix-valued function $f'(U) = (f'_1(u), f'_2(\bar{u}))$ belongs to $C_b(\mathbb{R}^2, \mathbb{R}^2)$ by assumption.

A bound for $V \in L^\infty(0, T; H_B)$ is equivalent to a bound for $U \in W^{1,\infty}(0, T; H_B)$ in (2.52). In order to derive this bound rigorously, we shall instead work with a time-discretization of (2.53) by connecting the problem to the one employed earlier in (2.5)-(2.7). To this end, for the same approximations (U^m, μ^m) of problem (2.5)-(2.6), we further define for $1 \leq m \leq n$, $V^m = (U^m - U^{m-1})/h$ and $w^m = (\mu^m - \mu^{m-1})/h$ and then set

$$\delta_t V^m := \frac{V^m - V^{m-1}}{h}, \text{ i.e., } \delta_t V^m = (\delta_t v^m, \delta_t \bar{v}^m).$$

For $1 \leq m \leq n$, consider the system of time-difference equations

$$(2.56) \quad \delta_t v^m = -Aw^m \text{ in } H_A,$$

$$(2.57) \quad \delta_t V^m + BV^m + f'(U^m)V^m = w^m \text{ in } H_B,$$

such that

$$(2.58) \quad V^0 = V_0 \in H_B.$$

Recall first that $(u^m, 1)_{H_A} = (u^{m-1}, 1)_{H_A} = (u_0, 1)_{H_A}$ for every $1 \leq m \leq n$; this yields $(v^m, 1)_{H_A} = 0$. We are interested in obtaining the required uniform bound for $V^m \in L^\infty(0, T; H_B)$ with respect to the step size h . Thus, we take the inner product in H_A of (2.56) with $A^{-1}(v^m)$ and the inner product in H_B of (2.57) with V^m . Adding the relations that we obtain, we find

$$(2.59) \quad \begin{aligned} & \frac{1}{2h} \left(\|V^m\|_{H_B}^2 - \|V^{m-1}\|_{H_B}^2 + \|V^m - V^{m-1}\|_{H_B}^2 \right) \\ & + \frac{1}{2h} \left(\|v^m\|_{V_A^*}^2 - \|v^{m-1}\|_{V_A^*}^2 + \|v^m - v^{m-1}\|_{V_A^*}^2 \right) \\ & + (BV^m, V^m)_{H_B} \\ & = - \left(f'(U^m)V^m, V^m \right)_{H_B}, \end{aligned}$$

owing to the identity $2x(x - y) = x^2 - y^2 + (x - y)^2$. Since $f \in C_b^1$, we derive further from (2.59) that

$$(2.60) \quad \|V^m\|_{H_B}^2 - \|V^{m-1}\|_{H_B}^2 + \|V^m - V^{m-1}\|_{H_B}^2 + h\|V^m\|_{V_B}^2 \leq C_f h \|V^m\|_{H_B}^2.$$

Now sum (2.60) over all $m = 1, 2, \dots, r$; it follows that

$$(2.61) \quad \begin{aligned} & \|V^r\|_{H_B}^2 + \sum_{m=1}^r \left(\|V^m - V^{m-1}\|_{H_B}^2 + h\|V^m\|_{V_B}^2 \right) \\ & \leq C \left(h \sum_{m=1}^r \|V^m\|_{H_B}^2 + \|V^0\|_{H_B}^2 \right). \end{aligned}$$

In view of the Gronwall's inequality in discrete form (see Proposition 2.7), we infer that

$$(2.62) \quad \max_{1 \leq r \leq n} \|V^r\|_{H_B}^2 \leq C_0$$

for some constant $C_0 > 0$ which depends only on $C, T > 0$ and the initial datum $V^0 \in H_B$. Now returning to (2.61) by virtue of (2.62), we also derive

$$(2.63) \quad \sum_{m=1}^r \left(\|V^m - V^{m-1}\|_{H_B}^2 + h \|V^m\|_{V_B}^2 \right) \leq C_0$$

uniformly with respect to h . As usual, the a posteriori analysis of the time-discretization (2.56)-(2.57) relies on an appropriate system of equations. For a partition $P = \{t_0, \dots, t_n\}$ and interpolating functions of (2.36)-(2.37), it suffices then to define new functions $\tilde{V}_h(\cdot, t)$, $w_h(\cdot, t)$ which are continuous, affine on each interval $[t_{m-1}, t_m]$, and equal to $V^m = (U^m - U^{m-1})/h$ and $w^m = (\mu^m - \mu^{m-1})/h$, respectively, at $t = t_m$ (also, $\tilde{V}_h(\cdot, 0) = V_0$ and $\mu_h(\cdot, 0) = \mu_0$). In particular, we can define new functions $V_h(\cdot, t) = \partial_t \tilde{U}_h(\cdot, t)$, $w_h(\cdot, t) = \partial_t \tilde{\mu}_h(\cdot, t)$, where

$$\tilde{\mu}_h(\cdot, t) = \mu^m + \frac{(t - t_m)}{h} (\mu^m - \mu^{m-1}), \quad t \in [t_{m-1}, t_m],$$

as well as

$$\tilde{V}_h(\cdot, t) = V^m + \frac{(t - t_m)}{h} (V^m - V^{m-1}), \quad t \in [t_{m-1}, t_m].$$

With these definitions we can rewrite (2.56)-(2.58) in terms of V_h, w_h and $\tilde{V}_h$ as follows:

$$(2.64) \quad \partial_t \tilde{v}_h + A w_h = 0 \text{ in } H_A \times (0, T),$$

$$(2.65) \quad \partial_t \tilde{V}_h + B V_h + f'(U_h) V_h = w_h \text{ in } H_B \times (0, T),$$

such that

$$(2.66) \quad \tilde{V}_h(\cdot, 0) = V_0 \text{ in } H_B.$$

Exactly as in Lemma 2.8, by rewriting the estimates obtained in (2.62)-(2.63) in terms of the function V_h , we deduce the following estimate

$$\|V_h\|_{L^\infty(0, T; H_B) \cap L^2(0, T; V_B)} \leq C_0.$$

Passing to a suitable subsequence as $h \rightarrow 0^+$, we can then infer

$$(2.67) \quad \partial_t \tilde{U}_h \rightarrow \partial_t U \text{ weakly-}^* \text{ in } L^\infty(0, T; H_B),$$

$$(2.68) \quad \partial_t \tilde{U}_h \rightarrow \partial_t U \text{ weakly in } L^2(0, T; V_B).$$

Recalling now that $f(U_h) \in L^\infty(0, T; H_B)$ (see (2.42)), we can use (2.39) to write

$$\left(\partial_t \tilde{U}_h, 1 \right)_{H_B} + \left(B^{1/2} U_h, B^{1/2} 1 \right)_{H_B} + (f(U_h), 1)_{H_B} = (\mu_h, 1)_{H_A}$$

and observe that $(\mu_h, 1)_{H_A} \in L^\infty(0, T)$, on account of (2.67) and (2.45). Moreover, by (2.38)-(2.39) note that μ_h solves the following abstract boundary value problem

$$(2.69) \quad \mu_h + A \mu_h = (B U_h)_1 + f_1(u_h) \text{ in } H_A \times (0, T).$$

Also by virtue of (2.38), if we set $\hat{\mu}_h := \mu_h - (\mu_h, 1)_{H_A} / \|1\|_{H_A}^2$ we have

$$(\hat{\mu}_h, \hat{\mu}_h)_{H_A} = (A^{-1} \partial_t \tilde{u}_h, \hat{\mu}_h)_{H_A}$$

owing to the fact that $(\partial_t \tilde{u}_h, 1)_{H_A} = 0$. Therefore, $\hat{\mu}_h \in L^\infty(0, T; H_A)$ provided that $\partial_t \tilde{u}_h \in L^\infty(0, T; H_A) \subset L^\infty(0, T; V_A^*)$, which actually is satisfied by (2.67). It follows that the bound $\mu_h \in L^\infty(0, T; H_A)$ is also uniform with respect to the step size h . We can exploit once again (2.57) and (2.67) to deduce by comparison in (2.39) that $B U_h \in L^\infty(0, T; H_B)$ uniformly in $h > 0$. Owing to the global Lipschitz continuity of f_1 , from (2.69) we also get the uniform bound $\mu_h \in L^\infty(0, T; D(A))$. These uniform estimates entail that as we pass along a subsequence as $h \rightarrow 0$, we additionally have the convergence properties:

$$\begin{cases} U_h \rightarrow U \text{ weakly-}^* \text{ in } L^\infty(0, T; D(B)), \\ \mu_h \rightarrow \mu \text{ weakly-}^* \text{ in } L^\infty(0, T; D(A)). \end{cases}$$

Thus, the final claim in the statement of the theorem easily follows. ■

Remark 2.4. As in Remark 2.2, one can show by similar arguments that any strong solution satisfying the assumptions of Theorem 2.9 also satisfies $U \in C([0, T]; D(B))$ and $\partial_t U \in C([0, T]; H_B)$.

2.2. The fractional dynamic boundary condition. We need to introduce some notation and preliminary results related to some auxiliary problems. We denote by $\|\cdot\|_{L^p}$ and $\|\cdot\|_{L^p(\partial\Omega)}$ the norms on $L^p(\Omega)$ and $L^p(\partial\Omega)$, respectively. In the case $p = 2$, $(\cdot, \cdot)$ and $(\cdot, \cdot)_{\partial\Omega}$ stand for the usual scalar products in $L^2(\Omega)$ and $L^2(\partial\Omega)$, respectively. The norms on $H^s(\Omega)$ and $H^s(\partial\Omega)$ are indicated by $\|\cdot\|_{H^s}$ and $\|\cdot\|_{H^s(\partial\Omega)}$, respectively, for any $s \in \mathbb{R}$. Let now $\Omega \subset \mathbb{R}^d$, $d \leq 3$, be a bounded domain with Lipschitz continuous boundary $\partial\Omega$ (i.e., Ω is of class $C^{0,1}$). For $s \in (0, 1)$, we denote by

$$H^s(\Omega) := \{\varphi \in L^2(\Omega) : \int_{\Omega} \int_{\Omega} \frac{|\varphi(x) - \varphi(y)|^2}{|x - y|^{d+2s}} dx dy < \infty\}$$

the fractional order Sobolev space endowed with the norm

$$\|u\|_{H^s} = \left(\frac{C_{d,s}}{2} \int_{\Omega} \int_{\Omega} \frac{|\varphi(x) - \varphi(y)|^2}{|x - y|^{d+2s}} dx dy + \int_{\partial\Omega} \frac{\beta}{2} |\varphi|^2 dS \right)^{1/2}$$

having defined $C_{d,s}$ as in (1.9), for some $\beta > 0$. It is well-known that for $0 < s \leq 1/2$, the spaces $H^s(\Omega)$ and $H_0^s(\Omega) = \overline{\mathcal{D}(\Omega)}^{\|\cdot\|_{H^s}}$ coincide with equivalent norms, see [2, Corollary 2.8 and Remark 2.3]. We also let $\tilde{H}^s(\Omega) = \overline{H^s(\Omega) \cap C_c(\bar{\Omega})}^{\|\cdot\|_{H^s}}$ and note that $\tilde{H}^s(\Omega)$ contains $H_0^s(\Omega)$ as a closed subspace. By definition, $H_0^s(\Omega)$ is also the smallest closed subspace of $H^s(\Omega)$ containing $\mathcal{D}(\Omega)$. In order to talk about "traces" for $H_0^s(\Omega)$ in the case when $s \in (0, 1/2]$, we need to exploit a different (potential-theoretic) description of $H_0^s(\Omega)$ using the notion of relative capacity (see, e.g., [33, Definition 3.2]; cf. also [2]). Without going into too much detail, let us mention that the most common property of the relative capacity is that it measures small sets more precisely than the usual Lebesgue measure on Ω . Thus, the potential description gives $H_0^s(\Omega) = \{u \in \tilde{H}^s(\Omega) \mid \tilde{u} = 0 \text{ r.q.e. on } \partial\Omega\}$, where $\tilde{u}$ denotes the relative quasi-continuous version of u (see, e.g., [33, Theorem 4.5]). Here, the property $\tilde{u} = 0$ r.q.e. on $\partial\Omega$ means that there exists a polar set P (i.e., whose relative capacity is zero) such that $\tilde{u} = 0$ almost everywhere on $\partial\Omega \setminus P$, see [33, Definition 3.3]. Also, every $u \in \tilde{H}^s(\Omega)$ admits a unique (up to a relatively polar set P) relatively quasi-continuous version $\tilde{u} : \bar{\Omega} \rightarrow \mathbb{R}$ such that $\tilde{u} = u$ a.e. in Ω (see, for instance, [33, Theorem 3.7]). Therefore, every $\varphi \in H_0^s(\Omega)$ is zero S -a.e. on $\partial\Omega$ (i.e., more precisely, its relative quasi-continuous version $\tilde{\varphi} = 0$ on $\partial\Omega$, up to a relatively polar set) where S denotes the usual Lebesgue surface measure on $\partial\Omega$. Thus we can, more than formally, say that the "trace" for $\varphi \in H_0^s(\Omega)$ is actually null when $0 < s \leq 1/2$, so that there are no nontrivial boundary conditions for ψ in that case. Therefore, from this point on we shall simply assume that $1/2 < s < 1$ for φ to possess a nontrivial trace $\psi = \varphi|_{\partial\Omega}$, but which is well-defined in the usual sense. Indeed, when $1/2 < s < 1$, the trace $\psi = \varphi|_{\partial\Omega}$ is well defined as a bounded operator from $H^s(\Omega) \rightarrow L^q(\partial\Omega)$, for every $\varphi \in H^s(\Omega) \cap C(\bar{\Omega})$, where $q = 2(d-1)/(d-2s)$ if $d \geq 2$ and $s \in (1/2, 1)$, and $q = \infty$ if $d = 1$ and $s \in (1/2, 1)$. Furthermore, by standard Sobolev function theory the injection $H^s(\Omega)$ into $L^{2d/(d-2s)}(\Omega)$ is continuous if $d \geq 2$ and $s \in (1/2, 1)$, and it is continuous in $C(\bar{\Omega})$ when $d = 1$ and $s \in (1/2, 1)$. Clearly, the continuous injection $H^s(\Omega) \subset L^2(\Omega)$ is also compact (see [30]). Next, for $l \in (0, 1)$, we let

$$H^l(\partial\Omega) := \{\psi \in L^2(\partial\Omega) : \int_{\partial\Omega} \int_{\partial\Omega} \frac{|\psi(x) - \psi(y)|^2}{|x - y|^{d-1+2l}} dS_x dS_y < \infty\}$$

be the fractional order Sobolev space endowed with the norm

$$\|\psi\|_{H^l(\partial\Omega)} := \left(\frac{\beta}{2} \int_{\partial\Omega} |\psi|^2 dS + \frac{C_{d-1,l}}{2} \int_{\partial\Omega} \int_{\partial\Omega} \frac{|\psi(x) - \psi(y)|^2}{|x - y|^{d-1+2l}} dS_x dS_y \right)^{1/2}.$$

Similarly, we have $H^l(\partial\Omega) \subset L^{p_*}(\partial\Omega)$ is continuous with $p_* := \frac{2(d-1)}{d-1-2l} > 2$ when $d > 1+2l$, $H^l(\partial\Omega) \subset L^\infty(\partial\Omega)$ is continuous provided that $d < 1+2l$ and $H^l(\partial\Omega) \subset L^r(\partial\Omega)$, for any

$r \in [2, \infty)$ when $d = 1 + 2l$. Clearly, the continuous injection $H^l(\partial\Omega) \subset L^2(\partial\Omega)$ is also compact. To this end, for $1/2 < s < 1$ and $l \in (0, 1)$, we define another scale of Hilbert spaces

$$V^{s,l} := \{U = (\varphi, \psi) : \varphi \in H^s(\Omega), \psi = \varphi|_{\partial\Omega} \in H^l(\partial\Omega)\}$$

endowed with norm

$$\|U\|_{V^{s,l}}^2 = \|\varphi\|_{H^s}^2 + \|\psi\|_{H^l(\partial\Omega)}^2.$$

We shall now recall an integration by parts formula that was derived in [22, Theorem 3.3] for the regional fractional Laplacian (1.8) in smooth domains. For this, consider the following constant

$$C_s := \frac{C_{1,s}}{2s(2s-1)} \int_0^\infty \frac{|t-1|^{1-2s} - (t \vee 1)^{1-2s}}{t^{2-2s}} dt,$$

where $t \vee 1 = \max\{t, 1\}$.

Theorem 2.10. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain of class $\mathcal{C}^{1,1}$ and recall that $1/2 < s < 1$. Define the set*

$$C_{2s}^2(\overline{\Omega}) := \{u : u(x) = f(x)\rho(x)^{2s-1} + g(x), \forall x \in \Omega, \text{ for some } f, g \in C^2(\overline{\Omega})\},$$

where $\rho(x) := \text{dist}(x, \partial\Omega)$, $x \in \Omega$. For $u \in C_{2s}^2(\overline{\Omega})$ and $z \in \partial\Omega$, we define the fractional normal derivative $N^{2-2s}u$ of the function u by

$$(2.70) \quad N^{2-2s}u(z) = -\lim_{t \downarrow 0} \frac{du(z - \nu(z)t)}{dt} t^{2-2s}, \text{ for } z \in \partial\Omega.$$

Then, for every $u \in C_{2s}^2(\overline{\Omega})$ and $v \in H^s(\Omega)$, one has $(-\Delta)_\Omega^s u \in L^2(\Omega)$, $N^{2-2s}u \in L^2(\partial\Omega)$ and

$$(2.71) \quad \begin{aligned} \int_\Omega v(-\Delta)_\Omega^s u dx &= \frac{C_{d,s}}{2} \int_\Omega \int_\Omega \frac{(v(x) - v(y))(u(x) - u(y))}{|x - y|^{d+2s}} dx dy \\ &\quad - C_s \int_{\partial\Omega} v N^{2-2s} u dS. \end{aligned}$$

One can also introduce a weak version for the fractional normal derivative on non-smooth domains Ω that are only of class $\mathcal{C}^{0,1}$.

Definition 2.11. *Recall that $1/2 < s < 1$.*

(i) *Let $u \in W^{s,2}(\Omega)$. We say that $(-\Delta)_\Omega^s u \in L^2(\Omega)$ if there exists $w \in L^2(\Omega)$ such that*

$$\frac{C_{d,s}}{2} \int_\Omega \int_\Omega \frac{(v(x) - v(y))(u(x) - u(y))}{|x - y|^{N+2s}} dx dy = \int_\Omega w v dx$$

for all $v \in C_c^\infty(\Omega)$ (and, so for all $v \in H_0^s(\Omega)$ by density). In that case, one writes $(-\Delta)_\Omega^s u = w$.

(ii) *Let $u \in H^s(\Omega)$ such that $(-\Delta)_\Omega^s u \in L^2(\Omega)$. We say that u has a (weak) fractional normal derivative in $L^2(\partial\Omega)$ if there exists $g \in L^2(\partial\Omega)$ such that*

$$(2.72) \quad \int_\Omega v(-\Delta)_\Omega^s u dx = \frac{C_{d,s}}{2} \int_\Omega \int_\Omega \frac{(v(x) - v(y))(u(x) - u(y))}{|x - y|^{d+2s}} dx dy - \int_{\partial\Omega} g v dS$$

for all $v \in H^s(\Omega)$. In this case, g is uniquely determined by (2.72); we may simply write $g = C_s N^{2-2s}u$ and refer to g as the fractional normal derivative of u .

(iii) *Consequently, the following Green identity holds:*

$$(2.73) \quad \begin{aligned} \int_\Omega v(-\Delta)_\Omega^s u dx &= \frac{C_{d,s}}{2} \int_\Omega \int_\Omega \frac{(v(x) - v(y))(u(x) - u(y))}{|x - y|^{d+2s}} dx dy \\ &\quad - C_s \int_{\partial\Omega} v N^{2-2s} u dS, \end{aligned}$$

for all $v \in H^s(\Omega)$ whenever $u \in W^{s,2}(\Omega)$, $(-\Delta)_\Omega^s u \in L^2(\Omega)$ and $N^{2-2s}u$ exists as an element of $L^2(\partial\Omega)$.

Remark 2.5. If Ω is a bounded domain of class $\mathcal{C}^{1,1}$ and if $u \in C_{2s}^2(\overline{\Omega})$, then $N^{2-2s}u$ coincides exactly with function given by definition (2.70). In that case, $N^{2-2s}u \in L^2(\partial\Omega)$ and $(-\Delta)_\Omega^s u \in L^2(\Omega)$; see, for instance, [22].

To give a proper definition of a fractional power for the Laplace-Beltrami operator acting on $\partial\Omega$, we let $H^{-l}(\partial\Omega) = (H^l(\partial\Omega))^*$ denote the dual of the reflexive Hilbert space $H^l(\partial\Omega)$, and consider the operator $D_l : H^l(\partial\Omega) \rightarrow H^{-l}(\partial\Omega)$ by the following definition:

$$(2.74) \quad \langle D_l u, v \rangle := \frac{C_{d-1,l}}{2} \int_{\partial\Omega} \int_{\partial\Omega} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{d-1+2l}} dS_x dS_y,$$

where $\langle \cdot, \cdot \rangle$ denotes the duality map between $H^{-l}(\partial\Omega)$ and $H^l(\partial\Omega)$. It turns out that D_l is bounded and the quadratic form associated with the right-hand side of (2.74) defines a bilinear (symmetric) continuous form for $u, v \in H^l(\partial\Omega)$. Next, we can define a fractional power of the Laplace-Beltrami operator as a principal-valued integral

$$\begin{aligned} (-\Delta)_{\partial\Omega}^l u &= C_{d-1,l} \text{P.V.} \int_{\partial\Omega} \frac{u(x) - u(y)}{|x - y|^{d-1+2l}} dS_y \\ &= C_{d-1,l} \lim_{\varepsilon \rightarrow 0^+} \int_{\partial\Omega: |x-y| > \varepsilon} \frac{u(x) - u(y)}{|x - y|^{d-1+2l}} dS_y, \end{aligned}$$

whenever the latter exists, and observe that $\langle D_l u, v \rangle = \langle (-\Delta)_{\partial\Omega}^l u, v \rangle$, for all $u, v \in H^l(\partial\Omega)$. Viewing $L^2(\partial\Omega)$ as a subspace of $H^{-l}(\partial\Omega)$, and denoting the part of the operator D_l in $L^2(\partial\Omega)$ by D_l again, we then have the following characterization:

$$(2.75) \quad D(D_l) = \{u \in H^l(\partial\Omega), (-\Delta)_{\partial\Omega}^l u \in L^2(\partial\Omega)\}, \quad D_l u = (-\Delta)_{\partial\Omega}^l u.$$

We also observe that

$$(2.76) \quad \frac{C_{d-1,l}}{2} \int_{\partial\Omega} \int_{\partial\Omega} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{d-1+2l}} dS_x dS_y = \int_{\partial\Omega} D_l u(x) v(x) dS_x,$$

for $u \in D(D_l)$ and $v \in H^l(\partial\Omega)$. We refer the reader for more details about this operator to [19, Section 2.4] (the proofs of Propositions 2.9-2.11). Finally, we can now give the definition for the linear *fractional Wentzell (dynamic)* boundary condition associated with condition (1.12).

Theorem 2.12. [19, Theorem 2.12] *Assume that Ω is of class $C^{0,1}$ and let $s \in (1/2, 1)$ and $l \in (0, 1)$. We define a bilinear symmetric form*

$$a_W : V^{s,l} \times V^{s,l} \rightarrow \mathbb{R}_+,$$

as follows:

$$(2.77) \quad \begin{aligned} a_W(U, V) &= \frac{C_{d,s}}{2} \int_{\Omega} \int_{\Omega} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{d+2s}} dx dy + \int_{\partial\Omega} \beta u v dS \\ &\quad + \frac{C_{d-1,l}}{2} \int_{\partial\Omega} \int_{\partial\Omega} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{d-1+2l}} dS_x dS_y, \end{aligned}$$

for all $U = (u, u|_{\partial\Omega})$, $V = (v, v|_{\partial\Omega}) \in V^{s,l}$. Let $A_W^{s,l}$ be the closed linear self-adjoint operator on $L^2(\Omega) \times L^2(\partial\Omega)$ associated with this form a_W . Then, the following are satisfied:

(i) The domain $D(A_W^{s,l})$ of $A_W^{s,l}$ consists of functions $U = (u, u|_{\partial\Omega}) \in V^{s,l}$ such that $(-\Delta)_{\Omega}^s u \in L^2(\Omega)$, $D_l u|_{\partial\Omega} \in L^2(\partial\Omega)$, $N^{2-2s}u$ exists as an element of $L^2(\partial\Omega)$, and

$$C_s N^{2-2s}u + D_l u|_{\partial\Omega} + \beta u|_{\partial\Omega} \in L^2(\partial\Omega).$$

In this case, $A_W^{s,l}$ has the following operator representation:

$$A_W^{s,l} \begin{pmatrix} u \\ u|_{\partial\Omega} \end{pmatrix} = \begin{pmatrix} (-\Delta)_{\Omega}^s u \\ C_s N^{2-2s}u + D_l u|_{\partial\Omega} + \beta u|_{\partial\Omega} \end{pmatrix}.$$

(ii) Also, $A_W^{s,l}$ has compact resolvent and the first eigenvalue $\lambda_{W,1}^{s,l}$ is positive. Finally, the injection $D(A_W^{s,l}) \subset L^\infty(\Omega) \times L^\infty(\partial\Omega)$ is continuous provided that $d = 1$, and if $d = 2, 3$:

$$\begin{cases} s > \frac{d}{4} \text{ if } l \geq \frac{s(d-1)}{d}, \text{ or } l > \frac{d-1}{4} \text{ if } s - \frac{1}{2} \leq l < \frac{s(d-1)}{d}, \\ \text{or } s > \frac{d+1}{4} \text{ if } l \leq s - \frac{1}{2}. \end{cases}$$

We call $A_W^{s,l}$, the realization of the *regional fractional Laplace* operator $(-\Delta)_\Omega^s$ with the *fractional Wentzell* type boundary condition. We conclude this subsection with a regularity result for a nonlinear elliptic boundary value problem associated with the fractional Wentzell operator $A_W^{s,l}$.

Theorem 2.13. [20, Section 3] *Assume $s \in (1/2, 1) \cap (d/4, 1)$ and $l \in ((d-1)/4, 1)$ if $d \leq 3$. Consider the following nonlocal boundary value problem:*

$$(2.78) \quad \begin{cases} (-\Delta)_\Omega^s u + \theta(u) = h_1 & \text{in } \Omega, \\ D_l u + C_s N^{2-2s} u + \beta u + \varpi(u) = h_2 & \text{on } \partial\Omega, \end{cases}$$

where $(h_1, h_2) \in L^2(\Omega) \times L^2(\partial\Omega)$.

(i) Assume that $\theta, \varpi \in C(\mathbb{R})$ satisfy the following conditions:

$$(2.79) \quad \theta(x)x \geq -\alpha_0|x| - \alpha_1, \quad \varpi(x)x \geq -\beta_0|x| - \beta_1,$$

for all $x \in \mathbb{R}$ with $|x| \geq x_0$, for some $x_0 > 0$, and some positive constants on the right-hand side of (2.79). Then every bounded solution $U = (u, u|_{\partial\Omega}) \in V^{s,l} \cap (L^\infty(\Omega) \times L^\infty(\partial\Omega))$ of (2.78) satisfies the (explicit) estimate:

$$\|u\|_{L^\infty} + \|u\|_{L^\infty(\partial\Omega)} \leq C \left(1 + \|h_1\|_{L^2} + \|h_2\|_{L^2(\partial\Omega)} \right),$$

for some constant $C > 0$ independent of U , h_1 and h_2 .

(ii) Consequently, any bounded solution of (2.78) is also a strong solution (i.e., $(-\Delta)_\Omega^s u \in L^2(\Omega)$, $N^{2-2s}u \in L^2(\partial\Omega)$ and $D_l u \in L^2(\partial\Omega)$), and the following estimate holds:

$$\int_\Omega |(-\Delta)_\Omega^s u|^2 dx + \int_{\partial\Omega} |D_l u|^2 + |N^{2-2s}u|^2 dS \leq Q(1 + \|h_1\|_{L^2} + \|h_2\|_{L^2(\partial\Omega)}),$$

for some positive function Q independent of U , h_1 and h_2 .

Remark 2.6. For further clarification, we say that U is a bounded solution of (2.78) if $U \in V^{s,l} \cap (L^\infty(\Omega) \times L^\infty(\partial\Omega))$ and

$$a_W(U, V) + \int_\Omega \theta(u) v dx + \int_{\partial\Omega} \varpi(u) v dS = \int_\Omega h_1 v dx + \int_{\partial\Omega} h_2 v dS,$$

for all $V \in V^{s,l}$.

Finally, for a domain Ω with Lipschitz continuous boundary $\partial\Omega$, we define the (weak) Neumann Laplacian $A_N \mu = -\Delta \mu$ with domain given by

$$(2.80) \quad D(A_N) = \{\mu \in H^1(\Omega) : \Delta \mu \in L^2(\Omega), \partial_\nu \mu = 0 \text{ on } \partial\Omega\}.$$

The boundary condition in (2.80) is understood in the following (variational) sense:

$$(2.81) \quad \int_\Omega \Delta \mu \omega dx = - \int_\Omega \nabla \mu \cdot \nabla \omega, \text{ for all } \omega \in H^1(\Omega),$$

whenever $\Delta \mu \in L^2(\Omega)$ and $\mu \in H^1(\Omega)$. By results in spectral theory it is well known that $A = A^* \geq 0$ on $H_A = L^2(\Omega)$ and $(I + A)^{-1}$ is compact in $L^2(\Omega)$.

3. WEAK AND STRONG SOLUTIONS

Now let us give the assumptions on the potential functions F, G .

(H1) $F, G \in C^2(\mathbb{R})$ and there exists $c_F, c_G > 0$ such that

$$F''(x) \geq -c_F, \quad G''(x) \geq -c_G, \quad \forall x \in \mathbb{R}.$$

(H2) There exist $c_f, c_g > 0$ and $p, q \in (1, 2]$ such that

$$|F'(x)|^p \leq c_f (|F(x)| + 1), \quad |G'(x)|^q \leq c_g (|G(x)| + 1), \quad \forall x \in \mathbb{R}.$$

(H3) There exist $C_i > 0$, $i = 1, \dots, 4$, such that

$$F(x) \geq C_1 |x|^{p/(p-1)} - C_2, \quad G(x) \geq C_3 |x|^{q/(q-1)} - C_4, \quad \forall x \in \mathbb{R},$$

for some $p, q \in (1, 2]$.

Remark 3.1. We observe that (H1) implies that both potentials are quadratic perturbations of some strictly convex functions. Assumption (H2) is satisfied by potentials of arbitrary polynomial growth of orders $\bar{p} = p/(p-1)$ and $\bar{q} = q/(q-1)$, respectively. In particular, for the double-well potential $F(x) = (x^2 - 1)^2$ one can take $p = 4/3$.

Let us define the (weak) energy space

$$Y_m = \left\{ U = (\varphi, \psi) \in V^{s,l} : F(\varphi) \in L^1(\Omega), G(\psi) \in L^1(\partial\Omega), |\langle \varphi \rangle| \leq m \right\},$$

for some given $m > 0$, where we have also set

$$\langle \varphi \rangle = |\Omega|^{-1} \int_{\Omega} \varphi(x) dx.$$

We equip Y_m with the following metric

$$\begin{aligned} d(U_1, U_2) = & \|U_1 - U_2\|_{V^{s,l}} + \left| \int_{\Omega} F(\varphi_1) - F(\varphi_2) dx \right|^{1/2} \\ & + \left| \int_{\partial\Omega} G(\psi_1) - G(\psi_2) dS \right|^{1/2}. \end{aligned}$$

Our definition of a weak solution for the boundary value problem (1.10)-(1.13) is as follows.

Definition 3.1. Let $(\varphi_0, \psi_0) \in Y_m$ and $0 < T < +\infty$ be given. We say $U = (\varphi, \psi)$ is a weak solution of (1.10)-(1.13) if

- $\psi(t) = \varphi|_{\partial\Omega}(t)$, for all $0 < t < T$, and the functions (φ, ψ) and μ satisfy
- (3.1) $(\varphi, \psi) \in L^\infty(0, T; Y_m)$, $\partial_t \varphi \in L^2(0, T; (H^1(\Omega))^*)$
- (3.2) $\mu \in L^2(0, T; H^1(\Omega))$, $\partial_t \psi \in L^2(0, T; L^2(\partial\Omega))$,
- (3.3) $F(\varphi) \in L^\infty(0, T; L^1(\Omega))$, $G(\psi) \in L^\infty(0, T; L^1(\partial\Omega))$,
- (3.4) $F'(\varphi) \in L^\infty(0, T; L^p(\Omega))$, $G'(\psi) \in L^\infty(0, T; L^q(\partial\Omega))$.
- For every $V = (v, \tilde{v}) \in V^{s,l} \cap (L^{\bar{p}}(\Omega) \times L^{\bar{q}}(\partial\Omega))$, $\omega \in H^1(\Omega)$, a.e. $t \in (0, T)$ we have
- (3.5) $\langle \partial_t \varphi(t), \omega \rangle + (\nabla \mu(t), \nabla \omega) = 0$,
- (3.6) $a_W(U(t), V) + (\partial_t \psi(t), \tilde{v})_{\partial\Omega} + \langle F'(\varphi(t)), v \rangle + \langle G'(\psi(t)), \tilde{v} \rangle = (\mu, v).$
- We have $\varphi(0) = \varphi_0$ in Ω and $\psi(0) = \psi_0$ on $\partial\Omega$.

Remark 3.2. Setting $\omega \equiv 1$ in (3.5) we deduce the conservation of mass $(\varphi(t), 1) = (\varphi_0, 1)$ for all $t \geq 0$. The initial conditions $\varphi(0) = \varphi_0$, $\psi(0) = \psi_0$ also make sense in a strong L^2 -sense since by (3.1)-(3.2) we have $(\varphi, \psi) \in C([0, T]; L^2(\Omega) \times L^2(\partial\Omega))$. In (3.6), each $\langle \cdot, \cdot \rangle$ is understood as the duality map between $L^p(\Omega)$ and $L^{\bar{p}}(\Omega)$, and between $L^q(\partial\Omega)$ and $L^{\bar{q}}(\partial\Omega)$, respectively. Here we recall once again that $1/\bar{p} + 1/p = 1$ and $1/\bar{q} + 1/q = 1$.

We also define what we mean by a strong solution. To this end, we define a strong energy space

$$Z_m = \left\{ (\varphi_0, \psi_0) \in D(A_W^{s,l}) : \mu_0 \in H^1(\Omega), |\langle \varphi \rangle| \leq m \right\}$$

and endow it with the norm

$$(3.7) \quad \|(\varphi_0, \psi_0)\|_{Z_m}^2 = \|(\varphi_0, \psi_0)\|_{D(A_W^{s,l})}^2 + \|\mu_0\|_{H^1}^2,$$

where μ_0 is computed via the equation

$$\mu_0 = (-\Delta)_\Omega^s \varphi_0 + F'(\varphi_0) \text{ in } \Omega.$$

In (3.7), the norm in $Z_m = D(A_W^{s,l}) \times H^1(\Omega)$ is meant for the triplet $(\varphi_0, \psi_0, \mu_0)$, but we kept the notion $\|(\varphi_0, \psi_0)\|_{Z_m}$ instead of $\|(\varphi_0, \psi_0, \mu_0)\|_{Z_m}$ in order to simplify the exposition.

Definition 3.2. Let $0 < T < +\infty$ be given. We say $U = (\varphi, \psi)$ is a strong solution of (1.10)-(1.13) if it is a weak solution in the sense of Definition 3.1, and (φ, ψ) and μ satisfy

$$(3.8) \quad (\varphi, \psi) \in L^\infty(0, T; L^\infty(\Omega) \times L^\infty(\partial\Omega)) \cap L^\infty(0, T; D(A_W^{s,l}),$$

$$(3.9) \quad (\partial_t \varphi, \partial_t \psi) \in L^2(0, T; V^{s,l}), \quad \mu \in L^\infty(0, T; H^1(\Omega)) \cap L^2(0, T; D(A_N)).$$

In particular, for the strong solution we have $\partial_t \varphi = \Delta \mu$, a.e. in $\Omega \times (0, T)$, while the boundary condition $\partial_\nu \mu = 0$ is understood in the sense of (2.81), a.e. on $(0, T)$. Finally, we have

$$(3.10) \quad \mu = (-\Delta)_\Omega^s \varphi + F'(\varphi), \quad \text{a.e. in } \Omega \times (0, T)$$

and

$$(3.11) \quad \partial_t \psi + D_l \psi + C_s N^{2-2s} \varphi + \beta \psi + G'(\psi) = 0, \quad \text{a.e. on } \partial\Omega \times (0, T).$$

The first goal of this subsection is to prove the existence of at least one strong energy solution to problem (1.10)-(1.13) by passing to the limit as $\epsilon, \alpha \rightarrow 0$ in a regularized version of the system, admitting a sufficiently smooth approximate solution $(\varphi_{\epsilon, \alpha}, \psi_{\epsilon, \alpha})$. The existence of such smooth solutions will be ensured by the abstract result of Theorem 2.2 which was proven by means of a backward Euler scheme. Then, we will derive additional uniform estimates for the solutions $(\varphi_{\epsilon, \alpha}, \psi_{\epsilon, \alpha})$ as $\epsilon, \alpha \rightarrow 0$.

Theorem 3.3. Let $(\varphi_0, \psi_0) \in Z_m$ for $s \in (1/2, 1) \cap (d/4, 1)$ and $l \in ((d-1)/4, 1)$ if $d \leq 3$. Assume that F, G satisfy the assumption (H1). Then there exists at least one strong solution $U = (\varphi, \psi) \in L^\infty(0, T; Z_m)$ to problem (1.10)-(1.13) in the sense of Definition 3.2.

Remark 3.3. In the range provided for

$$s \in (1/2, 1) \cap (d/4, 1) \quad \text{and} \quad l \in ((d-1)/4, 1),$$

we have by part (ii) of Theorem 2.12 that $D(A_W^{s,l}) \subset L^\infty(\Omega) \times L^\infty(\partial\Omega)$, and therefore for any $(\varphi, \psi) \in L^\infty(0, T; Z_m)$ the nonlinearities $F'(\varphi) \in L^\infty((0, T) \times \Omega)$ and $G'(\psi) \in L^\infty((0, T) \times \partial\Omega)$ are well-defined.

Proof. Step 1. An approximating problem $\mathbf{P}_{\epsilon, \alpha}$ associated with (1.10)-(1.13) can be formulated abstractly in (3.15)-(3.17). For the sake of notation, we let $f = F'$ and $g = G'$.

- We take, for $\epsilon \in (0, 1)$, smooth functions $f_\epsilon \rightarrow f, g_\epsilon \rightarrow g$ uniformly on compact intervals of $\mathbb{R}$, with the following properties:

$$(3.12) \quad |f_\epsilon^{(j)}(x)| \leq c_{f, \epsilon}, \quad |g_\epsilon^{(j)}(x)| \leq c_{g, \epsilon}, \quad \text{for all } x \in \mathbb{R},$$

for $j = 0, 1$ and some positive constants $c_{f, \epsilon}, c_{g, \epsilon}$ which may behave like ϵ^{-m} as $\epsilon \rightarrow 0$, for some $m = m(l) \in \mathbb{N}$. Since $F_\epsilon(s) = \int_0^s f_\epsilon(\zeta) d\zeta$ and $G_\epsilon(s) = \int_0^s g_\epsilon(\zeta) d\zeta$, we may also assume

$$(3.13) \quad \limsup_{\epsilon \rightarrow 0} F_\epsilon(x) = F(x), \quad \limsup_{\epsilon \rightarrow 0} G_\epsilon(x) = G(x), \quad \text{for all } x \in \mathbb{R},$$

such that

$$(3.14) \quad F_\epsilon''(x) \geq -c_F \quad \text{and} \quad G_\epsilon''(x) \geq -c_G, \quad \text{for all } x \in \mathbb{R}$$

owing to assumption (H1). Such constructions can be easily realized in a standard way as follows: one lets $F_\epsilon(x) = F(x)$ for $|x| \leq \epsilon^{-1}$,

$$F_\epsilon(x) = \sum_{i=1}^2 F^{(i)}(\epsilon^{-1}) (x - \epsilon^{-1})^i, \quad \text{for } x > \epsilon^{-1},$$

and

$$F_\epsilon(x) = \sum_{i=1}^2 F^{(i)}(-\epsilon^{-1}) (x + \epsilon^{-1})^i, \quad \text{for } x < -\epsilon^{-1}.$$

A similar construction applies to G_ϵ (see, for instance, [7] and the references therein). In particular, we observe that (3.12) and (3.14) imply that both f_ϵ, g_ϵ are

globally Lipschitz functions on $\mathbb{R}$, and that $(f_\epsilon(x), g_\epsilon(\bar{x})) \cdot (x, \bar{x}) \geq -C_1(x^2 + \bar{x}^2) - C_2$, for all $(x, \bar{x}) \in \mathbb{R}^2$, for some sufficiently large constants $0 < C_1 \leq (2\lambda_{W,1}^{s,l})^{-1}$, $C_2 > 0$. We also preliminarily observe that, due to (H1), F and G (as well as F_ϵ, G_ϵ) are merely quadratic perturbations of the convex functions $\tilde{F}(x) = F(x) + c_F x^2$, $\tilde{G}(x) = G(x) + c_G x^2$ (respectively, $\tilde{F}_\epsilon = F_\epsilon + c_F x^2$, $\tilde{G}_\epsilon = G_\epsilon + c_G x^2$).

- We insert an *artificial* viscosity $\alpha \in (0, 1)$ into the problem by adding the viscous term $\alpha \partial_t \varphi$ in the definition of the chemical potential in (1.10), i.e., we let

$$\mu = (-\Delta)_\Omega^s \varphi + \alpha \partial_t \varphi + F'(\varphi).$$

More precisely, our approximated problem $\mathbf{P}_{\epsilon,\alpha}$, $\epsilon, \alpha \in (0, 1)$, consists of finding a function $(\varphi_{\epsilon,\alpha}, \psi_{\epsilon,\alpha})$ which has the regularity stated in (3.8)-(3.9), such that

$$\partial_t \varphi_{\epsilon,\alpha} \in L^2(0, T; L^2(\Omega)),$$

and

$$(3.15) \quad \partial_t \varphi_{\epsilon,\sigma} + A_N \mu_{\epsilon,\alpha} = 0,$$

$$(3.16) \quad \begin{pmatrix} \alpha \partial_t \varphi_{\epsilon,\sigma} \\ \partial_t \psi_{\epsilon,\alpha} \end{pmatrix} + A_W^{s,l} \begin{pmatrix} \varphi_{\epsilon,\alpha} \\ \psi_{\epsilon,\alpha} \end{pmatrix} + \begin{pmatrix} f_\epsilon(\varphi_{\epsilon,\alpha}) \\ g_\epsilon(\psi_{\epsilon,\alpha}) \end{pmatrix} = \begin{pmatrix} \mu_{\epsilon,\alpha} \\ 0 \end{pmatrix},$$

subject to the initial conditions

$$(3.17) \quad \varphi_{\epsilon,\alpha}(0) = \varphi_0, \quad \psi_{\epsilon,\alpha}(0) = \psi_0.$$

In (3.15)-(3.16), we have let $B = A_W^{s,l}$ with $D(B) = D(A_W^{s,l})$ and $A = A_N$ with domain $D(A) = D(A_N)$. We recall that $A = A^* \geq 0$ on $H_A = L^2(\Omega)$ and $(I + A)^{-1}$ is compact in $L^2(\Omega)$ provided that Ω is of class $\mathcal{C}^{0,1}$, while $B = B^* > 0$ on $H_B = L^2(\Omega) \times L^2(\partial\Omega)$ and B^{-1} is compact by Theorem 2.12. We also prepare a fixed initial datum $\mu_{\epsilon,\alpha}(0) \in L^2(\Omega)$ that solves

$$(3.18) \quad \mu_{\epsilon,\alpha}(0) + \alpha A_N \mu_{\epsilon,\alpha}(0) = \mu_0 := (-\Delta)_\Omega^s \varphi_0 + F'(\varphi_0) \text{ in } L^2(\Omega),$$

whenever it is assumed that $(\varphi_0, \psi_0) \in Z_m \subset D(A_W^{s,l})$. Observe that the latter also yields that $\mu_{\epsilon,\alpha}(0) \in H^1(\Omega)$ uniformly with respect to $\epsilon, \alpha > 0$. Indeed, from (3.18) we immediately see that

$$\|\nabla \mu_{\epsilon,\alpha}(0)\|_{L^2}^2 + \alpha \|A_N \mu_{\epsilon,\alpha}(0)\|_{L^2}^2 = (\nabla \mu_0, \nabla \mu_{\epsilon,\alpha}(0))$$

so that $\nabla \mu_{\epsilon,\alpha}(0) \in L^2(\Omega)^d$ and $\sqrt{\alpha} A_N \mu_{\epsilon,\alpha}(0) \in L^2(\Omega)$ uniformly in (ϵ, α) provided that $\mu_0 \in H^1(\Omega)$ (the latter is clearly satisfied by assumption, cf. the definition of Z_m). In particular,

$$(3.19) \quad \|\mu_{\epsilon,\alpha}(0)\|_{H^1}^2 + \alpha \|A_N \mu_{\epsilon,\alpha}(0)\|_{L^2}^2 \leq C \|(\varphi_0, \psi_0)\|_{Z_m}^2,$$

for some $C > 0$ independent of α, ϵ . By taking a sufficiently small $\epsilon \leq 1/M_0$, if necessary where $\|(\varphi_0, \psi_0)\|_{L^\infty \times L^\infty(\partial\Omega)} \leq M_0$ (since in that case $g_\epsilon(\psi_0) \equiv g(\psi_0)$), we also specify a fixed set of data $v_{\epsilon,\alpha}(0) \in L^2(\partial\Omega)$ such that

$$(3.20) \quad v_{\epsilon,\alpha}(0) = -D_l \psi_0 + C_s N^{2-2s} \varphi_0 + \beta \psi_0 + g(\psi_0).$$

Then, it is clear that

$$(3.21) \quad \|v_{\epsilon,\alpha}(0)\|_{L^2(\partial\Omega)}^2 \leq Q \left(\|(\varphi_0, \psi_0)\|_{Z_m}^2 \right),$$

for some function $Q > 0$ independent of (ϵ, α) .

Therefore, the abstract results of Theorem 2.2 and Theorem 2.9 yield the existence of at least one smooth solution $U_{\epsilon,\alpha} = (\varphi_{\epsilon,\alpha}, \psi_{\epsilon,\alpha})$ to problem $\mathbf{P}_{\epsilon,\alpha}$ in the sense of Definition 2.1 by setting $V_B = V^{s,l}$ and $V_A = H^1(\Omega)$ (we also note that the proof of Theorem 2.2 was given in the case $\alpha = 1$, but clearly it can be easily adapted to accommodate the case $\alpha > 0$). In particular, we deduce the existence of a smooth function

$$\begin{cases} U_{\epsilon,\alpha} \in L^\infty(0, T; D(A_W^{s,l})) \cap W^{1,\infty}(0, T; L^2(\Omega) \times L^2(\partial\Omega)), \\ \mu_{\epsilon,\alpha} \in L^\infty(0, T; D(A_N)), \end{cases}$$

satisfying the equations

$$(3.22) \quad \partial_t \varphi_{\epsilon, \alpha}(t) + A_N \mu_{\epsilon, \alpha}(t) = 0,$$

$$(3.23) \quad (-\Delta)_\Omega^s \varphi_{\epsilon, \alpha} + f_\epsilon(\varphi_{\epsilon, \alpha}) + \alpha \partial_t \varphi_{\epsilon, \alpha} = \mu_{\epsilon, \alpha},$$

$$(3.24) \quad \partial_t \psi_{\epsilon, \alpha} + D_l \psi_{\epsilon, \alpha} + C_s N^{2-2s} \varphi_{\epsilon, \alpha} + \beta \psi_{\epsilon, \alpha} + g_\epsilon(\psi_{\epsilon, \alpha}) = 0$$

in the sense of $L^2(0, T; L^2(\Omega))$ and $L^2(0, T; L^2(\partial\Omega))$, respectively. Note that in the range of parameters provided for (s, l) in the statement of the theorem, we also have by part (iii) of Theorem 2.12 that

$$(3.25) \quad U_{\epsilon, \alpha} \in L^\infty(0, T; L^\infty(\Omega) \times L^\infty(\partial\Omega)).$$

Step 2. Now, we derive uniform estimates for $U_{\epsilon, \alpha}$ and $\mu_{\epsilon, \alpha}$ with respect to $0 < \epsilon \leq 1/M_0$, $\alpha \in (0, 1)$. We point out that all test functions used in this step are admissible in working with (3.22)-(3.24) on account of the regularity of $U_{\epsilon, \alpha}$ and $\mu_{\epsilon, \alpha}$. For practical purposes, in this proof $C, Q > 0$ denote a positive constant and function that are independent of $\alpha, \epsilon, \varphi, \psi$. Such parameters may vary even from line to line. Further dependencies of the constants and functions on other parameters will be pointed out as needed.

We begin by testing (3.22) in $L^2(\Omega)$ with $\mu_{\epsilon, \alpha}$. Then we test (3.23) and (3.24) by $\partial_t \varphi_{\epsilon, \alpha}$ and $\partial_t \psi_{\epsilon, \alpha}$ in $L^2(\Omega)$ and in $L^2(\partial\Omega)$, respectively. By virtue of the integration by parts formula (see Definition 2.11, (iii)) and taking the sum of all equations that we obtain, we have the following identity:

$$(3.26) \quad \frac{d}{dt} E_{\epsilon, \alpha}(t) + \alpha \int_\Omega |\partial_t \varphi_{\epsilon, \alpha}(t)|^2 dx + \int_\Omega |\nabla \mu_{\epsilon, \alpha}(t)|^2 dx + \int_{\partial\Omega} |\partial_t \psi_{\epsilon, \alpha}(t)|^2 dS = 0,$$

for all $t \geq 0$, where we have set

$$(3.27) \quad E_{\epsilon, \alpha}(t) = a_W((\varphi_{\epsilon, \alpha}(t), \psi_{\epsilon, \alpha}(t))) + \int_\Omega F_\epsilon(\varphi_{\epsilon, \alpha}(t)) dx + \int_{\partial\Omega} G_\epsilon(\psi_{\epsilon, \alpha}(t)) dS.$$

According to (3.17), since by density $D(A_W^{s,l}) \subset V^{s,l}$ and recalling Remark 3.3, we have $(\varphi_{\epsilon, \alpha}(0), \psi_{\epsilon, \alpha}(0)) \in V^{s,l} \cap (L^\infty(\Omega) \times L^\infty(\partial\Omega))$ uniformly in (ϵ, α) , then $E_{\epsilon, \alpha}(0) \leq C$ uniformly in (ϵ, α) , also owing to (3.13); indeed, (3.13) yields

$$\int_\Omega F_\epsilon(\varphi_0) dx \leq C, \int_{\partial\Omega} G_\epsilon(\psi_0) dS \leq C.$$

Thus integrating (3.26) over the time interval $(0, T)$, we deduce

$$(3.28) \quad (\varphi_{\epsilon, \alpha}, \psi_{\epsilon, \alpha}) \text{ is uniformly bounded in } L^\infty(0, T; V^{s,l}),$$

$$(3.29) \quad F_\epsilon(\varphi_{\epsilon, \alpha}) \text{ is uniformly bounded in } L^\infty(0, T; L^1(\Omega)),$$

$$(3.30) \quad G_\epsilon(\psi_{\epsilon, \alpha}) \text{ is uniformly bounded in } L^\infty(0, T; L^1(\partial\Omega)),$$

$$(3.31) \quad \sqrt{\alpha} \partial_t \varphi_{\epsilon, \alpha} \text{ is uniformly bounded in } L^2(0, T; L^2(\Omega)),$$

$$(3.32) \quad \partial_t \psi_{\epsilon, \alpha} \text{ is uniformly bounded in } L^2(0, T; L^2(\partial\Omega)),$$

$$(3.33) \quad \nabla \mu_{\epsilon, \alpha} \text{ is uniformly bounded in } L^2(0, T; L^2(\Omega)^d).$$

In order to deduce higher-order estimates for $(\phi_{\epsilon, \alpha}, \psi_{\epsilon, \alpha})$ we need to differentiate the equations (3.22)-(3.24) with respect to t , and set $u_{\epsilon, \alpha} = \partial_t \varphi_{\epsilon, \alpha}$, $v_{\epsilon, \alpha} = \partial_t \psi_{\epsilon, \alpha}$, $\omega_{\epsilon, \alpha} = \partial_t \mu_{\epsilon, \alpha}$. Then we (formally) have

$$(3.34) \quad \partial_t u_{\epsilon, \alpha}(t) + A_N \omega_{\epsilon, \alpha}(t) = 0,$$

$$(3.35) \quad (-\Delta)_\Omega^s u_{\epsilon, \alpha} + f'_\epsilon(\varphi_{\epsilon, \alpha}) u_{\epsilon, \alpha} + \alpha \partial_t u_{\epsilon, \alpha} = \omega_{\epsilon, \alpha},$$

$$(3.36) \quad \partial_t v_{\epsilon, \alpha} + D_l v_{\epsilon, \alpha} + C_s N^{2-2s} u_{\epsilon, \alpha} + \beta v_{\epsilon, \alpha} + g'_\epsilon(\psi_{\epsilon, \alpha}) v_{\epsilon, \alpha} = 0,$$

subject to the initial conditions

$$(3.37) \quad u_{\epsilon, \alpha}|_{t=0} = -A_N \mu_{\epsilon, \alpha}(0), \quad v_{\epsilon, \alpha}|_{t=0} = v_{\epsilon, \alpha}(0),$$

given by (3.18), (3.20). Here, we note the equations in (3.34)-(3.36) are understood only formally in $L^2((0, T) \times \Omega)$ and $L^2((0, T) \times \partial\Omega)$ since the functions involved may not admit sufficient regularity. However, for our prepared (regular) initial data $(u_{\epsilon, \alpha}(0), v_{\epsilon, \alpha}(0)) \in$

$L^2(\Omega) \times L^2(\partial\Omega)$ we can reformulate (3.34)-(3.36) in the strong L^2 -sense of problem (2.56)-(2.58) using the time discretization scheme introduced in the proof of Theorem 2.9. More precisely, for that scheme all the estimates found in (3.40)-(3.42) below can be performed rigorously using the Rothe functions introduced for $u_{\epsilon,\alpha}$, $v_{\epsilon,\alpha}$ in the proof of Theorem 2.9. For this reason, we can refrain from explicitly showing the same arguments to avoid further redundancy. Thus, we can proceed formally to test equation (3.34) in $L^2(\Omega)$ by $A_N^{-1}u_{\epsilon,\alpha}$, and equations (3.35)-(3.36) by $u_{\epsilon,\alpha}$, $v_{\epsilon,\alpha}$ in $L^2(\Omega)$ and $L^2(\partial\Omega)$, respectively. Then taking the sum of the equations we obtain, we deduce the identity

$$(3.38) \quad \begin{aligned} & \frac{d}{dt} \left(\alpha \|u_{\epsilon,\alpha}(t)\|_{L^2}^2 + \|v_{\epsilon,\alpha}(t)\|_{L^2(\partial\Omega)}^2 + \|u_{\epsilon,\alpha}(t)\|_{H^{-1}}^2 \right) \\ & + 2 \|(u_{\epsilon,\alpha}(t), v_{\epsilon,\alpha}(t))\|_{V^{s,l}}^2 \\ & = -2(f'_\epsilon(\varphi_{\epsilon,\alpha}(t)) u_{\epsilon,\alpha}(t), u_{\epsilon,\alpha}(t)) - 2(g'_\epsilon(\psi_{\epsilon,\alpha}(t)) v_{\epsilon,\alpha}(t), v_{\epsilon,\alpha}(t))_{\partial\Omega}, \end{aligned}$$

by virtue of the first Green identity (2.73). We note again that such an identity is not formal at the level of the discretization introduced in the proof of Theorem 2.9. By assumption (H1), and so by (3.14), $f'_\epsilon(\varphi_{\epsilon,\alpha}) \geq -c_F$ and $g'_\epsilon(\psi_{\epsilon,\alpha}) \geq -c_G$. Consequently, applying Gronwall's inequality to (3.38) and employing the interpolation inequality

$$(3.39) \quad \|u\|_{L^2}^2 \leq C(s) \|u\|_{H^s}^{2/(1+s)} \|u\|_{H^{-1}}^{2s/(1+s)}, \text{ for } s \in (1/2, 1),$$

we have

$$(3.40) \quad \begin{aligned} & \alpha \|u_{\epsilon,\alpha}(t)\|_{L^2}^2 + \|v_{\epsilon,\alpha}(t)\|_{L^2(\partial\Omega)}^2 + \|u_{\epsilon,\alpha}(t)\|_{H^{-1}}^2 \\ & + \int_0^t \|(u_{\epsilon,\alpha}(s), v_{\epsilon,\alpha}(s))\|_{V^{s,l}}^2 ds \\ & \leq C e^{Ct} \left(\alpha \|u_{\epsilon,\alpha}(0)\|_{L^2}^2 + \|v_{\epsilon,\alpha}(0)\|_{L^2(\partial\Omega)}^2 + \|u_{\epsilon,\alpha}(0)\|_{H^{-1}}^2 \right), \end{aligned}$$

for all $t \geq 0$. By virtue of (3.22)-(3.24), we note that (3.40) also automatically gives

$$(3.41) \quad \alpha \|A_N \mu_{\epsilon,\alpha}(t)\|_{L^2}^2 + \|\nabla \mu_{\epsilon,\alpha}(t)\|_{L^2}^2 \leq Q \left(\|(\varphi_0, \psi_0)\|_{Z_m}^2 \right) e^{Ct}$$

as well as

$$(3.42) \quad \alpha \|u_{\epsilon,\alpha}(t)\|_{L^2}^2 + \|v_{\epsilon,\alpha}(t)\|_{L^2(\partial\Omega)}^2 + \|u_{\epsilon,\alpha}(t)\|_{H^{-1}}^2 \leq Q \left(\|(\varphi_0, \psi_0)\|_{Z_m}^2 \right) e^{Ct},$$

where in the inequality of (3.40) we have actually inserted (3.19), (3.21).

In order to derive a uniform bound for $\mu_{\epsilon,\alpha}$ in $L^\infty(0, T; H^1(\Omega))$ we test both equations of (3.23)-(3.24) by $\varphi_{\epsilon,\alpha} - \langle \varphi_0 \rangle$ and $\psi_{\epsilon,\alpha} - \langle \varphi_0 \rangle$, respectively, and recall that $\langle \varphi_{\epsilon,\alpha} \rangle = \langle \varphi_0 \rangle$. If we further set $\bar{U}_{\epsilon,\alpha} = (\varphi_{\epsilon,\alpha} - \langle \varphi_0 \rangle, \psi_{\epsilon,\alpha} - \langle \varphi_0 \rangle)$, for $t \geq 0$ we deduce that

$$(3.43) \quad \begin{aligned} & a_W(\bar{U}_{\epsilon,\alpha}(t), \bar{U}_{\epsilon,\alpha}(t)) + (f_\epsilon(\varphi_{\epsilon,\alpha}(t)), \varphi_{\epsilon,\alpha}(t) - \langle \varphi_0 \rangle) \\ & + (g_\epsilon(\psi_{\epsilon,\alpha}(t)), \psi_{\epsilon,\alpha}(t) - \langle \varphi_0 \rangle)_{\partial\Omega} \\ & = (\mu_{\epsilon,\alpha}(t), \varphi_{\epsilon,\alpha}(t) - \langle \varphi_0 \rangle) - \alpha (u_{\epsilon,\alpha}(t), \varphi_{\epsilon,\alpha}(t) - \langle \varphi_0 \rangle) \\ & - (v_{\epsilon,\alpha}(t), \psi_{\epsilon,\alpha}(t) - \langle \varphi_0 \rangle)_{\partial\Omega}. \end{aligned}$$

Since the first term on the right-hand side of (3.43) can be estimated in terms of

$$C_\varepsilon \|\nabla \mu_{\epsilon,\alpha}\|_{L^2}^2 + \varepsilon \|\varphi_{\epsilon,\alpha}\|_{H^s}^2 \leq C_\varepsilon \|u_{\epsilon,\alpha}\|_{H^{-1}}^2 + \varepsilon \|\varphi_{\epsilon,\alpha} - \langle \varphi_0 \rangle\|_{H^s}^2,$$

for any $\varepsilon > 0$ (owing to (3.22), $H^1(\Omega) \subset H^s(\Omega)$, $s \in (1/2, 1)$, and the fact that $\langle u_{\epsilon,\alpha}(t) \rangle = 0$), from (3.43) and (3.40)-(3.42), together with the uniform estimate (3.19) on $\mu_{\epsilon,\alpha}(0)$, we have

$$(3.44) \quad \begin{aligned} & a_W(\bar{U}_{\epsilon,\alpha}(t), \bar{U}_{\epsilon,\alpha}(t)) + (f_\epsilon(\varphi_{\epsilon,\alpha}(t)), \varphi_{\epsilon,\alpha}(t) - \langle \varphi_0 \rangle) \\ & + (g_\epsilon(\psi_{\epsilon,\alpha}(t)), \psi_{\epsilon,\alpha}(t) - \langle \varphi_0 \rangle)_{\partial\Omega} \\ & \leq Q \left(\|(\varphi_0, \psi_0)\|_{Z_m}^2 \right) e^{Ct}, \end{aligned}$$

for all $t \geq 0$. We now note that the assumption (H1), together with (3.14), allow us to further observe that

$$(3.45) \quad \begin{cases} f_\epsilon(x)(x - \langle \varphi_0 \rangle) \geq \frac{1}{2} |f_\epsilon(x)| (1 + |x|) - C_m (1 + x^2), \\ g_\epsilon(x)(x - \langle \varphi_0 \rangle) \geq \frac{1}{2} |g_\epsilon(x)| (1 + |x|) - C_m (1 + x^2), \end{cases}$$

for some $C_m = C_m(f, g, \langle \varphi_0 \rangle) > 0$ independent of (ϵ, α) . Thus, in view (3.45) we get from (3.44) that

$$(3.46) \quad \|f_\epsilon(\varphi_{\epsilon, \alpha}(t))\|_{L^1} + \|g_\epsilon(\psi_{\epsilon, \alpha}(t))\|_{L^1(\partial\Omega)} \leq Q \left(\|(\varphi_0, \psi_0)\|_{Z_m}^2 \right) e^{Ct},$$

on account of the regularity estimate (3.28). Next, integrating equations (3.23) and (3.24) over Ω and $\partial\Omega$, respectively, we further see that

$$(3.47) \quad \begin{aligned} \langle \mu_{\epsilon, \alpha}(t) \rangle &= \langle f_\epsilon(\varphi_{\epsilon, \alpha}(t)) \rangle \\ &\quad + |\Omega|^{-1} \left((g_\epsilon(\psi_{\epsilon, \alpha}(t)) + \beta \psi_{\epsilon, \alpha}(t) - \partial_t \psi_{\epsilon, \alpha}(t), 1)_{\partial\Omega} \right) \end{aligned}$$

owing to the fact that $\langle \partial_t \phi_{\epsilon, \alpha}(t) \rangle = 0$. Thus, since all the terms on the right-hand side of (3.47) are already estimated in (3.46), (3.40)-(3.42), we find that

$$(3.48) \quad |\langle \mu_{\epsilon, \alpha}(t) \rangle| \leq Q \left(\|(\varphi_0, \psi_0)\|_{Z_m}^2 \right) e^{Ct}.$$

Consequently, from equation (3.22) we deduce

$$(3.49) \quad \partial_t \varphi_{\epsilon, \alpha} \text{ is uniformly bounded in } L^\infty(0, T; (H^1(\Omega))^*).$$

Collecting now (3.48), (3.40) together with (3.42), we have deduced that

$$(3.50) \quad \mu_{\epsilon, \alpha} \text{ is uniformly bounded in } L^\infty(0, T; H^1(\Omega)),$$

$$(3.51) \quad \sqrt{\alpha} \mu_{\epsilon, \alpha} \text{ is uniformly bounded in } L^\infty(0, T; D(A_N)),$$

$$(3.52) \quad \partial_t \psi_{\epsilon, \alpha} \text{ is uniformly bounded in } L^\infty(0, T; L^2(\partial\Omega)),$$

$$(3.53) \quad \alpha^{1/2} \partial_t \varphi_{\epsilon, \alpha} \text{ is uniformly bounded in } L^\infty(0, T; L^2(\Omega)).$$

In order to deduce the uniform $L^\infty(\Omega) \times L^\infty(\partial\Omega)$ -estimate for $(\varphi_{\epsilon, \alpha}, \psi_{\epsilon, \alpha})$, we exploit an "elliptic" regularity result for

$$(3.54) \quad \begin{cases} (-\Delta)_\Omega^s \varphi_{\epsilon, \alpha} + f_\epsilon(\varphi_{\epsilon, \alpha}) = h_1 := \mu_{\epsilon, \alpha} - \alpha \partial_t \varphi_{\epsilon, \alpha}, \\ D_l \psi_{\epsilon, \alpha} + C_s N^{2-2s} \varphi_{\epsilon, \alpha} + \beta \psi_{\epsilon, \alpha} + g_\epsilon(\psi_{\epsilon, \alpha}) = h_2 := -\partial_t \psi_{\epsilon, \alpha}. \end{cases}$$

Indeed, according to (3.42), (3.41) (see also (3.50)-(3.53)), we have for every $\alpha \in (0, 1)$, $\epsilon > 0$,

$$(3.55) \quad \|h_1(t)\|_{L^2}^2 + \|h_2(t)\|_{L^2(\partial\Omega)}^2 \leq Q \left(\|(\varphi_0, \psi_0)\|_{Z_m}^2 \right) e^{Ct}.$$

On the other hand, observe that owing to (H1) and (3.14), $f_\epsilon(\varphi_{\epsilon, \alpha}) \varphi_{\epsilon, \alpha} \geq -C |\varphi_{\epsilon, \alpha}| - C$, $g_\epsilon(\psi_{\epsilon, \alpha}) \psi_{\epsilon, \alpha} \geq -C |\psi_{\epsilon, \alpha}| - C$, for a sufficiently large $C > 0$ independent of (ϵ, α) . Therefore, recalling (3.25) we can apply the elliptic regularity result of Theorem 2.13, (i), to infer

$$(3.56) \quad \|\varphi_{\epsilon, \alpha}(t)\|_{L^\infty}^2 + \|\psi_{\epsilon, \alpha}(t)\|_{L^\infty(\partial\Omega)}^2 \leq Q \left(\|(\varphi_0, \psi_0)\|_{Z_m}^2 \right) e^{Ct}.$$

Finally, having obtained (3.56) we can interpret the nonlinearities in (3.54) as bounded external forces (note that we may assume that $|f_\epsilon(s)| \leq |f(s)|$, for sufficiently large $|s| \geq s_0$, as both f_ϵ and f are linear perturbations of some strictly monotone functions; the same applies to g_ϵ and g), and deduce according to the application of part (ii) of Theorem 2.13,

$$(3.57) \quad \|(\varphi_{\epsilon, \alpha}(t), \psi_{\epsilon, \alpha}(t))\|_{D(A_W^{s,l})}^2 \leq Q \left(\|(\varphi_0, \psi_0)\|_{Z_m}^2 \right) e^{Ct},$$

for some positive function Q independent of (ϵ, α) . Consequently, collecting (3.57), (3.41) and (3.48) we have derived

$$(3.58) \quad \|(\varphi_{\epsilon, \alpha}(t), \psi_{\epsilon, \alpha}(t))\|_{Z_m}^2 \leq Q \left(\|(\varphi_0, \psi_0)\|_{Z_m}^2 \right) e^{Ct}.$$

Step 3. In this step, we can finally pass to the limit as $(\epsilon, \alpha) \rightarrow (0, 0)$ in the sequence of solutions $U_{\epsilon, \alpha}$ satisfying the approximated problem $\mathbf{P}_{\epsilon, \alpha}$. Recalling (3.50)-(3.53), (3.49), (3.56), (3.58) and (3.28)-(3.33), we deduce, up to subsequences, the following convergences:

$$(3.59) \quad \partial_t \psi_{\epsilon, \alpha} \rightarrow \partial_t \psi \quad \text{weakly-* in } L^\infty(0, T; L^2(\partial\Omega)),$$

$$(3.60) \quad (\partial_t \varphi_{\epsilon, \alpha}, \partial_t \psi_{\epsilon, \alpha}) \rightarrow (\partial_t \varphi, \partial_t \psi) \quad \text{weakly in } L^2(0, T; V^{s, l}),$$

$$(3.61) \quad (\varphi_{\epsilon, \alpha}, \psi_{\epsilon, \alpha}) \rightarrow (\varphi, \psi) \quad \text{weakly-* in } L^\infty(0, T; D(A_W^{s, l})),$$

$$(3.62) \quad \mu_{\epsilon, \alpha} \rightarrow \mu \quad \text{weakly-* in } L^\infty(0, T; H^1(\Omega)),$$

$$(3.63) \quad \sqrt{\alpha} \partial_t \varphi_{\epsilon, \alpha} \rightarrow 0 \quad \text{strongly in } L^2(0, T; L^2(\Omega)),$$

$$(3.64) \quad \partial_t \varphi_{\epsilon, \alpha} \rightarrow \partial_t \varphi \quad \text{weakly in } L^2(0, T; (H^1(\Omega))^*).$$

On the other hand, due to (3.59)-(3.61) and classical compactness theorems, we also have

$$(3.65) \quad \varphi_{\epsilon, \alpha} \rightarrow \varphi \quad \text{strongly in } C(0, T; L^2(\Omega)),$$

$$(3.66) \quad \psi_{\epsilon, \alpha} \rightarrow \psi \quad \text{strongly in } C(0, T; L^2(\partial\Omega)).$$

Furthermore, on account of (3.65)-(3.66) and (3.61), the fact that f_ϵ and g_ϵ are continuously differentiable and that $f_\epsilon \rightarrow f$, $g_\epsilon \rightarrow g$ uniformly on compact subsets of $\mathbb{R}$, it is not difficult to show

$$(3.67) \quad f_\epsilon(\varphi_{\epsilon, \alpha}) \rightarrow f(\varphi) \quad \text{strongly in } C(0, T; L^2(\Omega)),$$

$$(3.68) \quad g_\epsilon(\psi_{\epsilon, \alpha}) \rightarrow g(\psi) \quad \text{strongly in } C(0, T; L^2(\partial\Omega)).$$

We now have all the ingredients to deduce that (φ, ψ) is a strong solution to (1.10)-(1.13) on $(0, T)$, for any $T > 0$. Indeed, we can pass to the limit in the approximating problem for $(\varphi_{\epsilon, \alpha}, \psi_{\epsilon, \alpha})$, and find that (φ, ψ) is a strong solution such that $(\varphi, \psi) \in L^\infty(0, T; Z_m)$. Since $(\varphi, \psi) \in L^2(0, T; V^{s, l})$ by (3.60), it also follows that $\mu \in L^2(0, T; D(A_N))$. Finally, we recover the continuity properties $\varphi(0) = \varphi_0$ in Ω and $\psi(0) = \psi_0$ on $\partial\Omega$ by means of standard arguments. ■

We now verify the stability of strong solutions of problem (1.10)-(1.13) with respect to the initial data in Z_m .

Theorem 3.4. *Let the assumptions of Theorem 3.3 be satisfied. Let $U_i = (\varphi_i, \psi_i)$ be any two strong solutions corresponding to the initial data*

$$U_i(0) = (\varphi_i(0), \psi_i(0)), \text{ with } \langle \varphi_1(0) \rangle = \langle \varphi_2(0) \rangle.$$

Then the following estimate holds:

$$(3.69) \quad \|\varphi_1(t) - \varphi_2(t)\|_{H^{-1}}^2 + \|\psi_1(t) - \psi_2(t)\|_{L^2(\partial\Omega)}^2 + \int_0^t \|(U_1(s) - U_2(s))\|_{V^{s, l}}^2 ds \\ \leq C e^{Ct} \left(\|\varphi_1(0) - \varphi_2(0)\|_{H^{-1}}^2 + \|\psi_1(0) - \psi_2(0)\|_{L^2(\partial\Omega)}^2 \right),$$

where $C > 0$ is independent of (φ_i, ψ_i) , time and the initial data.

Proof. Set $(\bar{\mu}, \bar{\varphi}, \bar{\psi})$ such that $\bar{\mu} := \mu_1 - \mu_2$, $\bar{\varphi} := \varphi_1 - \varphi_2$ and $\bar{\psi} := \psi_1 - \psi_2$. Then this function satisfies the system

$$(3.70) \quad \partial_t \bar{\varphi} + A_N \bar{\mu} = 0,$$

$$(3.71) \quad \begin{pmatrix} 0 \\ \partial_t \bar{\psi} \end{pmatrix} + A_W^{s, l} \begin{pmatrix} \bar{\varphi} \\ \bar{\psi} \end{pmatrix} + \begin{pmatrix} f(\varphi_1) - f(\varphi_2) \\ g(\psi_1) - g(\psi_2) \end{pmatrix} = \begin{pmatrix} \bar{\mu} \\ 0 \end{pmatrix},$$

subject to the initial conditions

$$(3.72) \quad \bar{\varphi}(0) = \varphi_{10} - \varphi_{20}, \quad \bar{\psi}(0) = \psi_{10} - \psi_{20}.$$

By assumption, we also have $\langle \bar{\varphi}(0) \rangle = 0$ and so $\langle \bar{\varphi}(t) \rangle \equiv 0$, as well. Testing equation (3.70) by $A_N^{-1}(\bar{\varphi})$ and equation (3.71) by $\bar{U} = (\bar{\varphi}, \bar{\psi})$, we deduce

$$(3.73) \quad \frac{1}{2} \frac{d}{dt} \left(\|\bar{\varphi}(t)\|_{H^{-1}}^2 + \|\bar{\psi}(t)\|_{L^2(\partial\Omega)}^2 \right) + a_W(U(t), U(t))$$

$$= - (f(\varphi_1(t)) - f(\varphi_2(t)), \bar{\varphi}(t)) - (g(\psi_1(t)) - g(\psi_2(t)), \bar{\psi}(t))_{\partial\Omega}.$$

Recalling (H1), we have $f'(x) \geq -C_F$ and $g'(x) \geq -C_G$, and therefore, we get in (3.73) that

$$(3.74) \quad \begin{aligned} & \frac{d}{dt} \left(\|\bar{\varphi}(t)\|_{H^{-1}}^2 + \|\bar{\psi}(t)\|_{L^2(\partial\Omega)}^2 \right) + 2a_W(U(t), U(t)) \\ & \leq C \left(\|\bar{\varphi}(t)\|_{L^2}^2 + \|\bar{\psi}(t)\|_{L^2(\partial\Omega)}^2 \right), \end{aligned}$$

for some constant $C > 0$ depending only on C_F, C_G . Interpolating exactly as in (3.39) to estimate the L^2 -norm of $\bar{\varphi}$ and applying Gronwall's inequality to the resulting inequality, we immediately deduce (3.69), which finishes the proof. ■

We conclude this section with some results on the existence of weak solutions for problem (1.10)-(1.13).

Theorem 3.5. *Let F, G satisfy the assumptions (H1)-(H3) and assume that $s \in (1/2, 1) \cap (d/4, 1)$ and $l \in ((d-1)/4, 1)$ for $d \leq 3$. For every initial datum $U_0 = (\varphi_0, \psi_0) \in Y_m$, there exists at least one weak solution $U = (\varphi, \psi)$ in the sense of Definition 3.1, and in addition,*

$$\varphi \in L^\infty(0, T; L^{\bar{p}}(\Omega)), \quad \psi \in L^\infty(0, T; L^{\bar{q}}(\partial\Omega)).$$

As a consequence, the weak solution is also unique.

Proof. Choose a smooth sequence of data $U_{0n} \in Z_m$ such that $U_{0n} \rightarrow U_0 \in Y_m$, i.e., $U_{0n} \rightarrow U_0$ in $V^{s,l}$ and $F(\varphi_{0n}) \rightarrow F(\varphi_0)$ in $L^1(\Omega)$, $G(\psi_{0n}) \rightarrow G(\psi_0)$ in $L^1(\partial\Omega)$ (here we can use implicitly that $D(A_W^{s,l})$ is dense in $V^{s,l}$). The corresponding strong solution $U_n = (\varphi_n, \psi_n)$ exists by Theorem 3.3 and is unique by Theorem 3.4 for every $n \in \mathbb{N}$. Arguing as in the proof of (3.24)-(3.26), we find

$$(3.75) \quad \frac{d}{dt} E_n(t) + \int_{\Omega} |\nabla \mu_n(t)|^2 dx + \int_{\partial\Omega} |\partial_t \psi_n(t)|^2 dS = 0,$$

for all $t \geq 0$, where we have set

$$(3.76) \quad E_n(t) = a_W((\varphi_n(t), \psi_n(t))) + \int_{\Omega} F(\varphi_n(t)) dx + \int_{\partial\Omega} G(\psi_n(t)) dS.$$

As before, owing to the fact that F, G are quadratic perturbations of strictly convex functions (see (H1)), we observe

$$\limsup_{n \rightarrow \infty} \int_{\Omega} F(\varphi_{0n}) dx \leq \int_{\Omega} F(\varphi_0) dx, \quad \limsup_{n \rightarrow \infty} \int_{\partial\Omega} G(\psi_{0n}) dS \leq \int_{\partial\Omega} G(\psi_0) dS.$$

Integrating now (3.75) over the time interval $(0, T)$, for any $T > 0$ we deduce

$$(3.77) \quad (\varphi_n, \psi_n) \text{ is uniformly bounded in } L^\infty(0, T; V^{s,l}),$$

$$(3.78) \quad F(\varphi_n) \text{ is uniformly bounded in } L^\infty(0, T; L^1(\Omega)),$$

$$(3.79) \quad G(\psi_n) \text{ is uniformly bounded in } L^\infty(0, T; L^1(\partial\Omega)),$$

$$(3.80) \quad \partial_t \psi_n \text{ is uniformly bounded in } L^2(0, T; L^2(\Omega)),$$

$$(3.81) \quad \nabla \mu_n \text{ is uniformly bounded in } L^2(0, T; L^2(\Omega)^d).$$

Furthermore, we can exploit these uniform estimates and assumption (H2), to infer from (3.78), (3.79) that

$$(3.82) \quad F'(\varphi_n) \text{ is uniformly bounded in } L^\infty(0, T; L^p(\Omega)),$$

$$(3.83) \quad G'(\psi_n) \text{ is uniformly bounded in } L^\infty(0, T; L^q(\partial\Omega)).$$

In fact, recall that $F(x) = \tilde{F}(x) - c_F x^2$, $G(x) = \tilde{G}(x) - c_G x^2$, such that $\tilde{F}'' \geq c_F$ and $\tilde{G}'' \geq c_G$. We also note that, by assumption (H2), both $\tilde{F}$ and $\tilde{G}$ are at most of polynomial growth of orders $\bar{p}$ and $\bar{q}$, respectively. In particular, for the (strictly) convex potentials $\tilde{F}, \tilde{G}$, one can find constants $C_f, C_g > 0$ such that $\tilde{F}(x) \geq C_f |x|^{\bar{p}} - c_f$ and

$\tilde{G}(x) \geq C_g |x|^{\bar{q}} - c_g$ (see assumption (H3)). Therefore, (3.78), (3.79) together with (3.77) also yield that

$$(3.84) \quad (\varphi_n, \psi_n) \text{ is uniformly bounded in } L^\infty(0, T; L^{\bar{p}}(\Omega) \times L^{\bar{q}}(\partial\Omega)).$$

Next, let us recall the identity (3.47) which actually holds by any smooth solution (φ_n, ψ_n) ,

$$(3.85) \quad \langle \mu_n(t) \rangle = \langle f(\varphi_n(t)) \rangle + |\Omega|^{-1} ((g(\psi_n(t)) + \beta\psi_n(t) - \partial_t \psi_n(t), 1)_{\partial\Omega}).$$

By virtue of estimates (3.80), (3.77) and (3.82)-(3.83) we then get $\langle \mu_n \rangle \in L^2(0, T)$ uniformly in n . Together with (3.81), we can confirm that

$$(3.86) \quad \mu_n \text{ is uniformly bounded in } L^2(0, T; H^1(\Omega))$$

and, thus, by a comparison argument in (1.10), also

$$(3.87) \quad \partial_t \varphi_n \text{ is uniformly bounded in } L^2(0, T; (H^1(\Omega))^*).$$

The foregoing estimates (3.77)-(3.87) and on account of the compact embedding

$$L^\infty(0, T; V^{s,l}) \cap H^1(0, T; (H^1(\Omega))^* \times L^2(\partial\Omega)) \subset C([0, T]; L^2(\Omega) \times L^2(\partial\Omega)),$$

we infer the existence of

$$(3.88) \quad \begin{cases} U = (\varphi, \psi) \in L^\infty(0, T; V^{s,l}) \cap H^1(0, T; (H^1(\Omega))^* \times L^2(\partial\Omega)), \\ \mu \in L^2(0, T; H^1(\Omega)), \quad U \in L^\infty(0, T; L^{\bar{p}}(\Omega) \times L^{\bar{q}}(\partial\Omega)), \end{cases}$$

such that, as $n \rightarrow \infty$,

$$\begin{cases} \partial_t \varphi_n \rightarrow \partial_t \varphi \text{ weakly in } L^2(0, T; (H^1(\Omega))^*), \\ \partial_t \psi_n \rightarrow \partial_t \psi \text{ weakly in } L^2(0, T; L^2(\partial\Omega)), \\ (\varphi_n, \psi_n) \rightarrow (\varphi, \psi) \text{ weakly-}^* \text{ in } L^\infty(0, T; V^{s,l}), \\ \mu_n \rightarrow \mu \text{ weakly in } L^2(0, T; H^1(\Omega)), \\ (\varphi_n, \psi_n) \rightarrow (\varphi, \psi) \text{ strongly in } C([0, T]; L^2(\Omega) \times L^2(\partial\Omega)). \end{cases}$$

In particular, the last convergence together with the continuity of F', G' and (3.82)-(3.83), by the dominated convergence theorem we obtain

$$(3.89) \quad F'(\varphi_n) \rightarrow F'(\varphi) \text{ weakly in } L^p(0, T; L^p(\Omega)),$$

as well as

$$(3.90) \quad G'(\psi_n) \rightarrow G'(\psi) \text{ weakly in } L^q(0, T; L^q(\Omega)).$$

Thus, on account of (3.89) and (3.90) we can pass to the limit as n goes to infinity in the problem satisfied by the strong solution (φ_n, ψ_n) . We easily obtain that the limit function in (3.88) is the desired weak solution in the sense of Definition 3.1. We leave the simple details to the interested reader. For the proof of uniqueness we go to the proof of Theorem 3.4 where we observe that the test functions $A_N^{-1}(\bar{\varphi})$, $\bar{U} = (\bar{\varphi}, \bar{\psi})$ used there are in fact admissible in the weak formulation (3.5)-(3.6) on account of the regularity (3.88). This completes the proof. ■

The existence of a unique weak solution can also be established under slightly different conditions. Indeed, we can replace both assumptions (H2) and (H3) by the following one:

(H4) There exist $x_0 > 0$ and some $c_f, c_g > 0$ (which may depend on x_0) such that

$$(3.91) \quad |F'(x)| \leq c_f |x|^m, \quad |G'(x)| \leq c_g |x|^n, \quad \forall |x| \geq x_0,$$

with

$$\begin{cases} m = \frac{d+2s}{d-2s} \text{ whenever } d \geq 2, \\ n = q_* - 1 = \frac{q_*}{q_*} \text{ whenever } d \geq 2, \end{cases}$$

where $q_* = \max \left\{ \frac{2}{1-2l}, \frac{1}{1-s} \right\}$ when $l \in (1/4, 1/2)$ and $d = 2$, and

$$q_* = \max \left\{ \frac{2}{1-l}, \frac{4}{3-2s} \right\}$$

whenever $d = 3$. Here we also recall that we are in the range $s \in (1/2, 1) \cap (d/4, 1)$ and $l \in ((d-1)/4, 1)$ if $d \leq 3$. No growth assumptions are required for $f = F'$, $g = G'$ in dimension $d = 1$, and for $g = G'$ in dimension $d = 2$ whenever $l \in (1/2, 1)$

(also when $d = 2$ and $l = 1/2$, g may satisfy (3.91) with an arbitrary but fixed polynomial growth).

Remark 3.4. Let (H4) hold. We observe that for $(\varphi, \psi) \in L^\infty(0, T; V^{s,l})$, by the embedding theorems stated in subsection 2.2, we have

$$(\varphi, \psi) \in L^\infty(0, T; L^\infty(\Omega) \times L^\infty(\partial\Omega))$$

in dimension $d = 1$, and $\psi \in L^\infty(0, T; L^\infty(\partial\Omega))$ in dimension $d = 2$ whenever $l \in (1/2, 1)$. Thus in this case the nonlinearities $F'(\varphi) \in L^\infty(0, T; L^\infty(\Omega))$, $G'(\psi) \in L^\infty(0, T; L^\infty(\partial\Omega))$ are well-defined. In the remaining cases when $d \geq 2$, we note that $V^{s,l} \subset L^{2d/(d-2s)}(\Omega) \times L^{q*}(\partial\Omega)$. Once again this yields for

$$(\varphi, \psi) \in L^\infty(0, T; V^{s,l}) \subset L^\infty(0, T; L^{2d/(d-2s)}(\Omega) \times L^{q*}(\partial\Omega)),$$

by virtue of the growth conditions (3.91), that $F'(\varphi) \in L^\infty(0, T; L^{2d/(d+2s)}(\Omega))$ and $G'(\psi) \in L^\infty(0, T; L^{q*}(\partial\Omega))$. Therefore, the weak formulation (3.5)-(3.6) of Definition 3.1 still makes sense if we set $\bar{p} = 2d/(d-2s)$ and $\bar{q} = q_*$ (and so $p = \bar{p}/(\bar{p}-1) = 2d/(d+2s)$ and $q = \bar{q}/(\bar{q}-1) = q'_*$).

Remark 3.5. Note that the double-well potential $F(x) = (x^2 - 1)^2$ satisfies (3.91) in all dimensions $d \leq 3$ in the range of parameters provided by (H4).

In view of Remark 3.4 and the proof of Theorem 3.5, we can state the following.

Theorem 3.6. *Let the assumptions (H1), (H4) hold and let $U_0 \in Y_m$. Then there exists a unique weak solution of problem (1.10)-(1.13) in the sense of Definition 3.1.*

4. FINITE DIMENSIONAL ATTRACTORS

In this section, we shall establish the existence of an exponential attractor for problem (1.10)-(1.13). Our first result is concerned with a dissipative estimate enjoyed by the (unique) weak solution of Theorem 3.5 (or Theorem 3.6). It will require an additional (sufficiently weak) assumption as follows.

(H6) There exist sufficiently large constants $C_1, C_3 > 0$, $C_2, C_4 \geq 0$ with $C_1 \geq c_F + C_\Omega$, $C_3 \geq c_G + 1$, such that

$$F(x) \geq C_1 x^2 - C_2, \quad G(x) \geq C_3 x^2 - C_4, \quad \forall x \in \mathbb{R}.$$

Here, $C_\Omega > 0$ is the best Poincaré-Sobolev constant in the inequality

$$\|\varphi - \langle \varphi \rangle\|_{L^2} \leq C_\Omega^{1/2} \|\nabla \varphi\|_{L^2}, \quad \varphi \in H^1(\Omega),$$

and $c_F, c_G > 0$ are the constants from hypothesis (H1).

Let us set

$$\mathcal{E}(U) := a_W(U, U) + \int_\Omega F(\varphi) dx + \int_{\partial\Omega} G(\psi) dS.$$

Theorem 4.1. *Let the assumptions of either Theorem 3.5 or Theorem 3.6 be satisfied, and further assume (H6). Then the following dissipative estimate holds:*

$$(4.1) \quad \mathcal{E}(U(t)) \leq \mathcal{E}(U_0) e^{-2t} + 2L(m), \quad \forall t \geq 0,$$

where $m > 0$ is such that $|\bar{\varphi}_0| \leq m$ and $L = L(m) > 0$ is a constant which is independent of the initial data and time. In particular, there exists a time $t_* = t_*(\|U_0\|_{Y_m}) > 0$, such that

$$(4.2) \quad \sup_{t \geq t_*} \mathcal{E}(U(t)) \leq 3L(m).$$

Proof. We first note that (4.2) is an immediate consequence of (4.1). To show (4.1), we shall proceed formally in the subsequent estimates, but recall that the arguments can be easily made rigorous by choosing to work with the (unique) strong solution of Theorem

3.3, followed by a standard passage to the limit in the estimates. In particular, each weak (and strong) solution satisfies the following energy identity

$$(4.3) \quad \frac{d}{dt} \mathcal{E}(U(t)) + \int_{\Omega} |\nabla \mu(t)|^2 dx + \int_{\partial\Omega} |\partial_t \psi(t)|^2 dS = 0,$$

for all $t \geq 0$. Next, let us test (3.6) with $U = (\varphi, \psi)$. We obtain

$$(4.4) \quad (\mu, \varphi) = a_W(U, U) + (F'(\varphi), \varphi) + (G'(\psi), \psi)_{\partial\Omega} + (\partial_t \psi, \psi)_{\partial\Omega}.$$

By the assumption (H1) owing to the fact that F, G are quadratic perturbations of strictly convex functions, for all $x \in \mathbb{R}$ we have

$$F'(x)x \geq F(x) - \frac{c_F x^2}{2} - F(0), \quad G'(x)x \geq G(x) - \frac{c_G x^2}{2} - G(0).$$

Therefore, from (4.4) we get

$$(4.5) \quad \begin{aligned} (\mu, \varphi) &\geq a_W(U, U) + \int_{\Omega} F(\varphi) dx + \int_{\partial\Omega} G(\psi) dS + (\partial_t \psi, \psi)_{\partial\Omega} \\ &\quad - \frac{c_F}{2} \|\varphi\|_{L^2}^2 - \frac{c_G}{2} \|\psi\|_{L^2(\partial\Omega)}^2 - (F(0)|\Omega| + G(0)|\partial\Omega|). \end{aligned}$$

On the other hand, we can exploit the Poincaré inequality $\|\mu - \bar{\mu}\|_{L^2}^2 \leq C_{\Omega} \|\nabla \mu\|_{L^2}^2$, and the conservation of mass $\langle \varphi \rangle = \langle \varphi_0 \rangle$, to observe that

$$(\mu, \varphi) = (\mu - \bar{\mu}, \varphi) \leq C_{\Omega}^{1/2} \|\nabla \mu\|_{L^2} \|\varphi\|_{L^2},$$

assuming for simplicity (for now) that $\langle \varphi_0 \rangle (= \langle \varphi \rangle) = 0$. Thus, by virtue of assumption (H6) we rewrite (4.5) and continue to estimate as follows:

$$\begin{aligned} C_{\Omega}^{1/2} \|\nabla \mu\|_{L^2} \|\varphi\|_{L^2} &\geq a_W(U, U) + \int_{\Omega} F(\varphi) dx + \int_{\partial\Omega} G(\psi) dS + (\partial_t \psi, \psi)_{\partial\Omega} \\ &\quad - \frac{c_F}{2} \|\varphi\|_{L^2}^2 - \frac{c_G}{2} \|\psi\|_{L^2(\partial\Omega)}^2 - (F(0)|\Omega| + G(0)|\partial\Omega|) \\ &\geq \frac{1}{2} \left(a_W(U, U) + \int_{\Omega} F(\varphi) dx + \int_{\partial\Omega} G(\psi) dS \right) \\ &\quad + \frac{1}{2} \int_{\Omega} F(\varphi) dx + \frac{1}{2} \int_{\partial\Omega} G(\psi) dS \\ &\quad - \frac{1}{2} \|\partial_t \psi\|_{L^2(\partial\Omega)}^2 - \frac{1}{2} \|\psi\|_{L^2(\partial\Omega)}^2 \\ &\quad - \frac{c_F}{2} \|\varphi\|_{L^2}^2 - \frac{c_G}{2} \|\psi\|_{L^2(\partial\Omega)}^2 - (F(0)|\Omega| + G(0)|\partial\Omega|) \\ &\geq \frac{1}{2} \mathcal{E}(U) - \frac{1}{2} \|\partial_t \psi\|_{L^2(\partial\Omega)}^2 - \frac{1}{2} \|\psi\|_{L^2(\partial\Omega)}^2 \\ &\quad + \frac{C_1}{2} \|\varphi\|_{L^2}^2 + \frac{C_3}{2} \|\psi\|_{L^2(\partial\Omega)}^2 \\ &\quad - \frac{c_F}{2} \|\varphi\|_{L^2}^2 - \frac{c_G}{2} \|\psi\|_{L^2(\partial\Omega)}^2 \\ &\quad - \left(F(0)|\Omega| + G(0)|\partial\Omega| + \frac{C_2}{2} |\Omega| + \frac{C_4}{2} |\partial\Omega| \right). \end{aligned}$$

Application of Young's inequality yields

$$\begin{aligned} &\frac{1}{2} \mathcal{E}(U) + \frac{C_1}{2} \|\varphi\|_{L^2}^2 + \frac{C_3}{2} \|\psi\|_{L^2(\partial\Omega)}^2 - \frac{c_F}{2} \|\varphi\|_{L^2}^2 - \frac{c_G + 1}{2} \|\psi\|_{L^2(\partial\Omega)}^2 \\ &\quad - \left(F(0)|\Omega| + G(0)|\partial\Omega| + \frac{C_2}{2} |\Omega| + \frac{C_4}{2} |\partial\Omega| \right) \\ &\leq \frac{1}{2} \|\nabla \mu\|_{L^2}^2 + \frac{C_{\Omega}}{2} \|\varphi\|_{L^2}^2 + \frac{1}{2} \|\partial_t \psi\|_{L^2(\partial\Omega)}^2 \end{aligned}$$

and so we easily deduce

$$(4.6) \quad \frac{1}{2} \mathcal{E}(U) \leq \frac{1}{2} \left(\|\nabla \mu\|_{L^2}^2 + \|\partial_t \psi\|_{L^2(\partial\Omega)}^2 \right) + L_*,$$

for some constant $L_* > 0$ which depends only on $C_2, C_4, F(0), G(0)$ and $|\Omega|, |\partial\Omega|$. It follows by virtue of (4.6) and the energy identity (4.3) that we have

$$(4.7) \quad \frac{d}{dt} \mathcal{E}(U(t)) + \mathcal{E}(U(t)) \leq 2L_*,$$

for all $t \geq 0$. By means of Gronwall's inequality we then obtain

$$(4.8) \quad \mathcal{E}(U(t)) \leq \mathcal{E}(U(0)) e^{-t} + 2L_*.$$

If $\langle \varphi_0 \rangle \neq 0$ is such that $\langle \varphi \rangle = \langle \varphi_0 \rangle \in [-m, m]$ for some $m > 0$, observe that if U is a weak solution with initial datum U_0 for the problem (1.10)-(1.13) with potentials F, G , then for $\tilde{\varphi} = \varphi - \langle \varphi_0 \rangle, \tilde{\psi} = \psi - \langle \varphi_0 \rangle, \tilde{U} = (\tilde{\varphi}, \tilde{\psi})$ is a weak solution with initial datum $\tilde{U}(0) = (\varphi_0 - \langle \varphi_0 \rangle, \psi_0 - \langle \varphi_0 \rangle)$ for the same problem with potentials

$$\tilde{F}(x) := F(x + \langle \varphi_0 \rangle) - F(\langle \varphi_0 \rangle), \quad \tilde{G}(x) := G(x + \langle \varphi_0 \rangle) - G(\langle \varphi_0 \rangle).$$

Since now $\langle \tilde{\varphi} \rangle = 0$, we can employ the dissipative estimate (4.8) for the solution $\tilde{U}$ and easily arrive at the final inequality (4.1), for some $L(m) > 0$, owing to the fact that $|\langle \varphi_0 \rangle| \leq m$. The proof is complete. ■

Theorem 4.2. *Let the assumptions of Theorem 4.1 be satisfied. There exists a constant $R = R(m) > 0$, independent of time and the initial data, such that*

$$(4.9) \quad \sup_{t \geq t_*+1} \left(\|U(t)\|_{Z_m}^2 + \int_t^{t+1} \|(\partial_t \varphi(s), \partial_t \psi(s))\|_{V^{s,l}}^2 ds \right) \leq R.$$

Proof. As usual, in this proof $C, Q > 0$ will denote a constant and function which are independent of U , time and the initial data. Let $t_* > 0$ be the entering time from (4.2). For $t \geq t_*$, from (4.3) it follows that

$$\frac{d}{dt} [(t - t_*) \mathcal{E}(U(t))] + (t - t_*) \left(\|\nabla \mu(t)\|_{L^2}^2 + \|\partial_t \psi(t)\|_{L^2(\partial\Omega)}^2 \right) = \mathcal{E}(U(t)).$$

Integrating the foregoing identity over (t_*, T) , with any $T > t_*$, in light of (4.2) we obtain

$$(4.10) \quad \int_{t_*}^T (t - t_*) \left(\|\nabla \mu(t)\|_{L^2}^2 + \|\partial_t \psi(t)\|_{L^2(\partial\Omega)}^2 \right) dt \leq L(m) (T - t_*).$$

Moreover, the energy identity (4.3) also gives in light of (4.2) that

$$(4.11) \quad \int_{t_*}^T \|\nabla \mu(t)\|_{L^2}^2 + \|\partial_t \psi(t)\|_{L^2(\partial\Omega)}^2 dt \leq L(m).$$

Next, we recall the identity (3.38) contained in the proof of Theorem 3.3 which actually holds also by our strong solution (of course, now $\alpha = 0$). In particular, it reads

$$(4.12) \quad \begin{aligned} & \frac{d}{dt} \left(\|\partial_t \psi(t)\|_{L^2(\partial\Omega)}^2 + \|\partial_t \varphi(t)\|_{H^{-1}}^2 \right) \\ & + 2 \|(\partial_t \varphi(t), \partial_t \psi(t))\|_{V^{s,l}}^2 \\ & = -2(f'(\varphi(t)) \partial_t \varphi(t), \partial_t \varphi(t)) - 2(g'(\psi(t)) \partial_t \psi(t), \partial_t \psi(t))_{\partial\Omega}. \end{aligned}$$

By assumption (H1) and the interpolation inequality (3.39), (4.12) yields

$$(4.13) \quad \begin{aligned} & \frac{d}{dt} \left(\|\partial_t \psi(t)\|_{L^2(\partial\Omega)}^2 + \|\partial_t \varphi(t)\|_{H^{-1}}^2 \right) + \|(\partial_t \varphi(t), \partial_t \psi(t))\|_{V^{s,l}}^2 \\ & \leq C \left(\|\partial_t \psi(t)\|_{L^2(\partial\Omega)}^2 + \|\partial_t \varphi(t)\|_{H^{-1}}^2 \right), \end{aligned}$$

for all $t \geq t_*$, for some constant $C > 0$ which depends only on $s \in (1/2, 1)$ and the constants $c_F, c_G > 0$ from (H1). Multiplying both sides of (4.13) by $(t - t_*)$ and integrating the resulting inequality over (T, t_*) , with any $T \geq t_* + 1$, we deduce

$$(4.14) \quad \begin{aligned} & (T - t_*) \left(\|\partial_t \psi(T)\|_{L^2(\partial\Omega)}^2 + \|\partial_t \varphi(T)\|_{H^{-1}}^2 \right) \\ & + \int_{t_*}^T (t - t_*) \|(\partial_t \varphi(t), \partial_t \psi(t))\|_{V^{s,l}}^2 dt \end{aligned}$$

$$\begin{aligned}
&\leq C \int_{t_*}^T [(t - t_*) + 1] \left(\|\nabla \mu(t)\|_{L^2}^2 + \|\partial_t \psi(t)\|_{L^2(\partial\Omega)}^2 \right) dt \\
&\leq L(m) (T - t_* + 1),
\end{aligned}$$

on account of estimates (4.10)-(4.11) and the fact that $\|\partial_t \varphi\|_{H^{-1}} = \|\nabla \mu\|_{L^2}$. In particular, (4.14) yields that

$$(4.15) \quad \sup_{t \geq t_*+1} \left(\|\partial_t \psi(t)\|_{L^2(\partial\Omega)}^2 + \|\partial_t \varphi(t)\|_{H^{-1}}^2 + \|\nabla \mu(t)\|_{L^2}^2 \right) \leq 2L(m).$$

It remains to note that integrating (4.13) over $(t, t+1)$, and using (4.15) we also obtain

$$(4.16) \quad \sup_{t \geq t_*+1} \int_t^{t+1} \|(\partial_t \varphi(s), \partial_t \psi(s))\|_{V^{s,l}}^2 ds \leq 4L(m).$$

Next, we exploit the fact that $(\varphi, \psi) \in L^\infty(t_*, \infty; V^{s,l})$ uniformly with respect to time and the initial data (see (4.2)) to observe that $f(\varphi) = F'(\varphi) \in L^\infty(t_*, \infty; L^1(\Omega))$, as well as $g(\psi) = G'(\psi) \in L^\infty(t_*, \infty; L^1(\Omega))$ uniformly in time and the initial data, by virtue of the assumptions on F, G (see (H2)-(H3), and (H4), respectively). To this end, recalling from (3.47) that

$$\langle \mu(t) \rangle = \langle f(\varphi(t)) \rangle + |\Omega|^{-1} ((g(\psi(t)) + \beta\psi(t) - \partial_t \psi(t), 1)_{\partial\Omega})$$

by application of (4.15), we find that $\langle \mu \rangle \in L^\infty(t_*+1, \infty)$ uniformly in time and the initial data. More precisely, together with (4.15) it yields

$$(4.17) \quad \sup_{t \geq t_*+1} \|\mu(t)\|_{H^1}^2 \leq Q(L(m)),$$

for some function $Q > 0$ independent of (U, μ) , time and the initial data. In order to deduce the uniform $L^\infty(\Omega) \times L^\infty(\partial\Omega)$ -estimate for $(\varphi(t), \psi(t))$, we exploit once again the "elliptic" regularity result of Theorem 2.13, (i), to the following problem

$$(4.18) \quad \begin{cases} (-\Delta)_\Omega^s \varphi(t) + f(\varphi(t)) = h_1(t) := \mu(t), \\ D_l \psi(t) + C_s N^{2-2s} \varphi(t) + \beta\psi(t) + g(\psi(t)) = h_2(t) := -\partial_t \psi(t). \end{cases}$$

Indeed, according to (4.15)-(4.17) and assumption (H1) (which necessarily implies that $f(x)x \geq -C|x| - C$, $g(x)x \geq -C|x| - C$), we have

$$(4.19) \quad \begin{aligned} \|\varphi(t)\|_{L^\infty}^2 + \|\psi(t)\|_{L^\infty(\partial\Omega)}^2 &\leq C \left(1 + \|h_1(t)\|_{L^2}^2 + \|h_2(t)\|_{L^2(\partial\Omega)}^2 \right) \\ &\leq Q(L(m)), \end{aligned}$$

for all $t \geq t_* + 1$. It remains to exploit (4.19) and part (ii) of Theorem 2.13, when the nonlinearities in (4.18) are to be interpreted as bounded external forces. We easily find

$$(4.20) \quad \sup_{t \geq t_*+1} \|(\varphi(t), \psi(t))\|_{D(A_W^{s,l})}^2 \leq Q(L(m)).$$

Finally, collecting (4.16), (4.20) and (4.17) we have obtained (4.9), which completes the proof. ■

According to Theorem 3.5 (or Theorem 3.6), and in view of Theorem 4.1, problem (1.10)-(1.13) generates a dissipative semigroup $S(t) : Y_m \rightarrow Y_m$, given by

$$S(t)(\varphi_0, \psi_0) = (\varphi(t), \psi(t)),$$

where $(\varphi(t), \psi(t))$ is the unique weak solution of (1.10)-(1.13). Our main result of this section is the following.

Theorem 4.3. *Let the assumptions of Theorem 4.2 be satisfied for some $F, G \in C^3(\mathbb{R})$. Then the dynamical system $(S(t), Y_m)$ admits an exponential attractor $\mathcal{M}$ bounded in Z_m in the following sense:*

- (i) *The sets $\mathcal{M}$ are positively invariant with respect to the semigroup $S(t)$, that is, $S(t)\mathcal{M} \subseteq \mathcal{M}$, for all $t \geq 0$.*

(ii) The fractal dimension of $\mathcal{M}$ is finite, that is,

$$\dim_F(\mathcal{M}, V^{s,l}) \leq c < \infty,$$

where $c > 0$ can be computed explicitly.

(iii) Each $\mathcal{M}$ attracts exponentially any bounded subset of Y_m , that is, there exist a positive constant ρ and a monotone nonnegative function Q , such that, for every bounded subset B of Y_m , we have

$$\text{dist}_{V^{s,l}}(S(t)B, \mathcal{M}) \leq Q(\|B\|_{Y_m})e^{-\rho t},$$

where $\text{dist}_{V^{s,l}}(X, Y) := \sup_{x \in X} \inf_{y \in Y} \|x - y\|_{V^{s,l}}$ is the Hausdorff semidistance.

In order to prove this result we report the following basic abstract result on the existence of exponential attractors for discrete maps (see, for instance, [28]).

Theorem 4.4. *Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be two Banach spaces such that $\mathcal{X}_2$ is compactly embedded in $\mathcal{X}_1$. Let X_0 be a bounded subset of $\mathcal{X}_2$ and consider a nonlinear map $\Sigma : X_0 \rightarrow X_0$ satisfying the smoothing property*

$$(4.21) \quad \|\Sigma(x_1) - \Sigma(x_2)\|_{\mathcal{X}_2} \leq d \|x_1 - x_2\|_{\mathcal{X}_1},$$

for all $x_1, x_2 \in X_0$, where $d > 0$ depends on X_0 . Then the discrete dynamical system (X_0, Σ^n) possesses a discrete exponential attractor $\mathcal{E}_M^* \subset \mathcal{X}_2$, that is, a compact set in $\mathcal{X}_1$ with finite fractal dimension such that

$$(4.22) \quad \Sigma(\mathcal{E}_M^*) \subset \mathcal{E}_M^*,$$

$$(4.23) \quad \text{dist}_{\mathcal{X}_1}(\Sigma^n(X_0), \mathcal{E}_M^*) \leq d_X e^{-\rho_* n}, \quad n \in \mathbb{N},$$

where d_X and ρ_* are positive constants independent of n , with the former depending on X_0 .

Let us now introduce a ball $\mathbb{B}_R \subset D(A_W^{s,l})$ with sufficiently large radius $R > 0$ such that

$$\mathbb{B}_R := \left\{ U = (\varphi, \psi) \in D(A_W^{s,l}) : \|U\|_{D(A_W^{s,l})} \leq R, \quad |\langle \varphi \rangle| \leq m \right\}.$$

By Theorem 4.2, there exists a sufficiently large $R_m > 0$ such that the ball $\mathbb{B} := \mathbb{B}_{R_m}$ will be absorbing for $S(t)$ and $S(t)\mathbb{B} \subset \mathbb{B}$ for all $t \geq t_* + 2$. Thus it suffices to construct the exponential attractor on the set $\mathbb{B}$. Next, we prove a series of elementary lemmas with help from Theorem 4.2. The first lemma is concerned with the Lipschitz-in-time regularity of the semigroup $S(t)$.

Lemma 4.5. *Let the assumptions of Theorem 4.3 be satisfied. There exists a constant $C > 0$ such that*

$$(4.24) \quad \|S(t)U_0 - S(\tilde{t})U_0\|_{H^{-1} \times L^2(\partial\Omega)} \leq C|t - \tilde{t}|,$$

for all $t, \tilde{t} > 0$ and any $U_0 = (\varphi_0, \psi_0) \in \mathbb{B}$.

Proof. The claim (4.24) follows from the basic equality

$$S(t)U_0 - S(\tilde{t})U_0 = \int_t^{\tilde{t}} \partial_y(S(y))U_0 dy$$

and estimate (4.15). ■

The second result states the validity of the smoothing property for the semigroup $S(t)$.

Lemma 4.6. *Let the assumptions of Theorem 4.3 be satisfied. Indicate by $U_i = (\varphi_i, \psi_i)$ the solution to problem (1.10)-(1.13) which corresponds to the initial datum $U_{i0} = (\varphi_{i0}, \psi_{i0}) \in \mathbb{B}$, where $i = 1, 2$. Then the following estimate holds:*

$$(4.25) \quad \begin{aligned} & \|\varphi_1(t) - \varphi_2(t)\|_{H^s}^2 + \|\psi_1(t) - \psi_2(t)\|_{H^l(\partial\Omega)}^2 \\ & \leq Ce^{Ct} \left(\|\varphi_1(0) - \varphi_2(0)\|_{L^2}^2 + \|\psi_1(0) - \psi_2(0)\|_{L^2(\partial\Omega)}^2 \right), \end{aligned}$$

for all $t \geq 1$, and some positive constant C which only depends on $\mathbb{B}$.

Proof. First, we note that due to estimate (4.9) we have

$$(4.26) \quad \|U_i(t)\|_{D(A_W^{s,l})} + \|U_i(t)\|_{L^\infty \times L^\infty(\partial\Omega)} \leq R_0, \quad i = 1, 2, \quad t \geq 0$$

and

$$(4.27) \quad \int_0^t \|(\partial_t \varphi_i(s), \partial_t \psi_i(s))\|_{V^{s,l}}^2 ds \leq R_0(1+t), \quad i = 1, 2, \quad t \geq 0.$$

Here the constant $R_0 > 0$ is independent of time and depends only on $\mathbb{B}$. In this proof, the constant $C = C(R_0) > 0$ will be independent of time and the initial data. As in the proof of Theorem 3.4, set $\bar{\mu} := \mu_1 - \mu_2$, $\bar{\varphi} := \varphi_1 - \varphi_2$ and $\bar{\psi} := \psi_1 - \psi_2$. Then $(\bar{\varphi}, \bar{\psi})$ satisfies the problem

$$(4.28) \quad \partial_t \bar{\varphi} + A_N \bar{\mu} = 0,$$

$$(4.29) \quad \begin{pmatrix} 0 \\ \partial_t \bar{\psi} \end{pmatrix} + A_W^{s,l} \begin{pmatrix} \bar{\varphi} \\ \bar{\psi} \end{pmatrix} + \begin{pmatrix} f(\varphi_1) - f(\varphi_2) \\ g(\psi_1) - g(\psi_2) \end{pmatrix} = \begin{pmatrix} \bar{\mu} \\ 0 \end{pmatrix},$$

subject to the initial conditions

$$\bar{\varphi}(0) = \varphi_{10} - \varphi_{20}, \quad \bar{\psi}(0) = \psi_{10} - \psi_{20}.$$

In contrast to the proof of Theorem 3.4, we generally have that $\langle \bar{\varphi}(t) \rangle = \langle \bar{\varphi}(0) \rangle \neq 0$.

Thus, letting $\tilde{U} = (\tilde{\varphi}, \tilde{\psi}) := (\bar{\varphi} - N_0, \bar{\psi} - N_0)$ where $N_0 := \langle \bar{\varphi}(0) \rangle$ and then testing

(4.28) by $A_N^{-1}(\tilde{\varphi})$ and (4.29) by $\tilde{U}$, we derive

$$(4.30) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} \left(\|\tilde{\varphi}(t)\|_{H^{-1}}^2 + \|\tilde{\psi}(t)\|_{L^2(\partial\Omega)}^2 \right) + a_W(\tilde{U}(t), \tilde{U}(t)) + \beta N_0(\tilde{\psi}(t), 1)_{\partial\Omega} \\ & = - (f(\varphi_1(t)) - f(\varphi_2(t)), \tilde{\varphi}(t)) - (g(\psi_1(t)) - g(\psi_2(t)), \tilde{\psi}(t))_{\partial\Omega}, \end{aligned}$$

which is the analogue of (3.73) in the case when $N_0 = \langle \bar{\varphi}(0) \rangle \neq 0$. Since by (4.26) we have a uniform estimate on the L^∞ -norms of the solutions U_i , we can estimate both terms on the right-hand side of (4.30) as follows:

$$(4.31) \quad \begin{aligned} & \left| (f(\varphi_1) - f(\varphi_2), \tilde{\varphi}) + (g(\psi_1) - g(\psi_2), \tilde{\psi})_{\partial\Omega} \right| \\ & \leq C(R_0) \left(N_0^2 + \|\tilde{\varphi}\|_{L^2}^2 + \|\tilde{\psi}\|_{L^2(\partial\Omega)}^2 \right). \end{aligned}$$

Inserting this estimate on the right-hand side of (4.30) and appealing to the interpolation inequality (3.39), and subsequently to Gronwall's inequality, it follows that

$$(4.32) \quad \begin{aligned} & \|\tilde{\varphi}(t)\|_{H^{-1}}^2 + \|\tilde{\psi}(t)\|_{L^2(\partial\Omega)}^2 + \int_0^t a_W(\tilde{U}(s), \tilde{U}(s)) ds \\ & \leq C e^{Ct} \left(N_0^2 + \|\tilde{\varphi}(0)\|_{H^{-1}}^2 + \|\tilde{\psi}(0)\|_{L^2(\partial\Omega)}^2 \right). \end{aligned}$$

Consequently, from (4.32) we can derive

$$(4.33) \quad \begin{aligned} & \|\bar{\varphi}(t)\|_{H^{-1}}^2 + \|\bar{\psi}(t)\|_{L^2(\partial\Omega)}^2 + \int_0^t \|(\bar{\varphi}(s), \bar{\psi}(s))\|_{V^{s,l}}^2 ds \\ & \leq C e^{Ct} \left(\|\bar{\varphi}(0)\|_{L^2}^2 + \|\bar{\psi}(0)\|_{L^2(\partial\Omega)}^2 \right). \end{aligned}$$

The preliminary estimate (4.33) turns out to become crucial for (4.25). To this end, we test (4.28) by $A_N^{-1}(\partial_t \tilde{\varphi})$ and (4.29) by $(\partial_t \tilde{\varphi}, \partial_t \tilde{\psi})$, and obtain after standard transformations, that

$$(4.34) \quad \begin{aligned} & \frac{d}{dt} \left[a_W(\tilde{U}(t), \tilde{U}(t)) \right] + \|\partial_t \tilde{\varphi}(t)\|_{H^{-1}}^2 + \|\partial_t \tilde{\psi}(t)\|_{L^2(\partial\Omega)}^2 \\ & = (f(\varphi_1(t)) - f(\varphi_2(t)), \partial_t \tilde{\varphi}(t)) \\ & + (g(\psi_1(t)) - g(\psi_2(t)), \partial_t \tilde{\psi}(t))_{\partial\Omega}. \end{aligned}$$

Estimating the last two terms in a standard fashion (using (4.26) and (3.39)), we find

$$\begin{aligned}
& (f(\varphi_1) - f(\varphi_2), \partial_t \tilde{\varphi}) + (g(\psi_1) - g(\psi_2), \partial_t \tilde{\psi})_{\partial\Omega} \\
& \leq \frac{1}{2} \left(\|\partial_t \tilde{\psi}\|_{L^2(\partial\Omega)}^2 + C(R_0) \|\tilde{\psi}\|_{L^2(\partial\Omega)}^2 \right) + \frac{1}{2} \left(\|\partial_t \tilde{\varphi}\|_{L^2}^2 + C(R_0) \|\tilde{\varphi}\|_{L^2}^2 \right) \\
& \leq \frac{1}{2} \left(\|\partial_t \tilde{\psi}\|_{L^2(\partial\Omega)}^2 + C \|\tilde{\psi}\|_{L^2(\partial\Omega)}^2 \right) \\
& + \frac{1}{2} \left(C_{\varepsilon,s} \|\partial_t \tilde{\varphi}\|_{H^{-1}}^2 + \varepsilon \|\partial_t \tilde{\varphi}\|_{H^s}^2 \right) + \frac{C}{2} \|\tilde{\varphi}\|_{L^2}^2,
\end{aligned}$$

for any $\varepsilon > 0$. Inserting now this estimate on the right-hand side of (4.34) we arrive at the following inequality:

$$\begin{aligned}
(4.35) \quad & \frac{d}{dt} \left[a_W \left(\tilde{U}(t), \tilde{U}(t) \right) \right] + \|\partial_t \tilde{\varphi}(t)\|_{H^{-1}}^2 + \frac{1}{2} \|\partial_t \tilde{\psi}(t)\|_{L^2(\partial\Omega)}^2 \\
& \leq C \left(\|\tilde{\varphi}(t)\|_{L^2}^2 + \|\tilde{\psi}(t)\|_{L^2(\partial\Omega)}^2 \right) + \left(C_{\varepsilon,s} \|\partial_t \tilde{\varphi}(t)\|_{H^{-1}}^2 + \varepsilon \|\partial_t \tilde{\varphi}(t)\|_{H^s}^2 \right).
\end{aligned}$$

We now need to control the last term on the right-hand side of (4.35). To this end, we differentiate both equations of (4.28)-(4.29) with respect to time, and set $\bar{u} := \partial_t \tilde{\varphi} (= \partial_t \tilde{\varphi})$, $\bar{v} := \partial_t \tilde{\psi} (= \partial_t \tilde{\psi})$, $\bar{w} := \partial_t \tilde{\mu}$. Then $(\bar{u}, \bar{v}, \bar{w})$ solves the problem

$$(4.36) \quad \partial_t \bar{u} + A_N \bar{w} = 0,$$

$$(4.37) \quad \begin{pmatrix} 0 \\ \partial_t \bar{v} \end{pmatrix} + A_W^{s,l} \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix} + \begin{pmatrix} \partial_t (f(\varphi_1) - f(\varphi_2)) \\ \partial_t (g(\psi_1) - g(\psi_2)) \end{pmatrix} = \begin{pmatrix} \bar{w} \\ 0 \end{pmatrix}.$$

We note that $\langle \bar{u}(t) \rangle = 0$. Next, we test (4.36) with $A_N^{-1}(\bar{u})$ and (4.37) with $\bar{U} = (\bar{u}, \bar{v})$. Adding the resulting equations, we obtain

$$\begin{aligned}
(4.38) \quad & \frac{1}{2} \frac{d}{dt} \left(\|\bar{u}(t)\|_{H^{-1}}^2 + \|\bar{v}(t)\|_{L^2(\partial\Omega)}^2 \right) + a_W(\bar{U}(t), \bar{U}(t)) \\
& = -(\partial_t (f(\varphi_1(t)) - f(\varphi_2(t))), \bar{u}(t)) - (\partial_t (g(\psi_1(t)) - g(\psi_2(t))), \bar{v}(t))_{\partial\Omega}.
\end{aligned}$$

We estimate

$$\begin{aligned}
(4.39) \quad & -(\partial_t (f(\varphi_1) - f(\varphi_2)), \bar{u}) \\
& = -\left(f'(\varphi_1) \bar{u}, \bar{u} \right) - \left((f'(\varphi_1) - f'(\varphi_2)) \partial_t \varphi_2, \bar{u} \right) \\
& \leq c_F \|\bar{u}\|_{L^2}^2 + C(R_0) \|\bar{u}\|_{H^s} \|\tilde{\varphi}\|_{L^2} \|\partial_t \varphi_2\|_{H^s} \\
& \leq \left(C_{F,s,\varepsilon} \|\bar{u}\|_{H^{-1}}^2 + C_\varepsilon(R_0) \|\tilde{\varphi}\|_{L^2}^2 \|\partial_t \varphi_2\|_{H^s}^2 \right) + 2\varepsilon \|\bar{u}\|_{H^s}^2,
\end{aligned}$$

owing to assumption (H1), the interpolation inequality (3.39) and (4.26) to control the L^∞ -norms. We have also exploited the fact that $|\tilde{\varphi}| |\partial_t \varphi_2| \in H^{-s}(\Omega)$ as a product of functions in $L^2(\Omega) \times H^s(\Omega)$ whenever $s \in (1/2, 1) \cap (d/4, 1)$. Similarly, we derive

$$\begin{aligned}
(4.40) \quad & -(\partial_t (g(\psi_1) - g(\psi_2)), \bar{v})_{\partial\Omega} \\
& = -\left(g'(\psi_1) \bar{v}, \bar{v} \right)_{\partial\Omega} - \left((g'(\psi_1) - g'(\psi_2)) \partial_t \psi_2, \bar{v} \right)_{\partial\Omega} \\
& \leq c_G \|\bar{v}\|_{L^2(\partial\Omega)}^2 + C(R_0) \|\bar{v}\|_{H^l(\partial\Omega)} \|\tilde{\psi}\|_{L^2(\partial\Omega)} \|\partial_t \psi_2\|_{H^l(\partial\Omega)} \\
& \leq C \|\bar{v}\|_{L^2(\partial\Omega)}^2 + \left(\varepsilon \|\bar{v}\|_{H^l(\partial\Omega)}^2 + C_\varepsilon(R_0) \|\tilde{\psi}\|_{L^2(\partial\Omega)}^2 \|\partial_t \psi_2\|_{H^l(\partial\Omega)}^2 \right),
\end{aligned}$$

for any $\varepsilon > 0$. Again we have used (4.26) and the fact that $|\tilde{\psi}| |\partial_t \psi_2| \in H^{-l}(\partial\Omega)$ as a product of functions in $L^2(\partial\Omega) \times H^l(\partial\Omega)$ provided that $l \in ((d-1)/4, 1)$. Inserting now (4.39)-(4.40) into the right-hand side of (4.38) we can infer the existence of a sufficiently small $\varepsilon_0 > 0$ such that

$$\begin{aligned}
(4.41) \quad & \frac{d}{dt} \left(\|\bar{u}(t)\|_{H^{-1}}^2 + \|\bar{v}(t)\|_{L^2(\partial\Omega)}^2 \right) + \varepsilon_0 a_W(\bar{U}(t), \bar{U}(t)) \\
& \leq C \left(\|\bar{u}(t)\|_{H^{-1}}^2 + \|\bar{v}(t)\|_{L^2(\partial\Omega)}^2 \right)
\end{aligned}$$

$$+ C \left(\|\bar{\varphi}(t)\|_{L^2}^2 \|\partial_t \varphi_2(t)\|_{H^s}^2 + \|\bar{\psi}(t)\|_{L^2(\partial\Omega)}^2 \|\partial_t \psi_2(t)\|_{H^l(\partial\Omega)}^2 \right).$$

Let us define

$$\bar{E}(t) := \|\bar{u}(t)\|_{H^{-1}}^2 + \|\bar{v}(t)\|_{L^2(\partial\Omega)}^2 + a_W \left(\tilde{U}(t), \tilde{U}(t) \right).$$

Note that

$$\begin{aligned} (4.42) \quad a_W(\tilde{U}, \tilde{U}) &= \frac{C_{d,s}}{2} \int_{\Omega} \int_{\Omega} \frac{(\bar{\varphi}(x) - \bar{\varphi}(y))^2}{|x - y|^{d+2s}} dx dy + \int_{\partial\Omega} \beta (\bar{\psi} - N_0)^2 dS \\ &\quad + \frac{C_{d-1,l}}{2} \int_{\partial\Omega} \int_{\partial\Omega} \frac{(\bar{\psi}(x) - \bar{\psi}(y))^2}{|x - y|^{d-1+2l}} dS_x dS_y \\ &= a_W \left(\begin{pmatrix} \bar{\varphi} \\ \bar{\psi} \end{pmatrix}, \begin{pmatrix} \bar{\varphi} \\ \bar{\psi} \end{pmatrix} \right) - 2\beta \int_{\partial\Omega} \bar{\psi} N_0 dS + \beta N_0^2 |\partial\Omega| \\ &\geq \|(\bar{\varphi}, \bar{\psi})\|_{V^{s,l}}^2 - \beta \|\bar{\psi}\|_{L^2(\partial\Omega)}^2, \end{aligned}$$

owing to the fact that

$$\begin{aligned} 2\beta \int_{\partial\Omega} \bar{\psi} N_0 dS &= \beta \left(2|\partial\Omega|^{1/2} N_0 \int_{\partial\Omega} \bar{\psi} dS |\partial\Omega|^{-1/2} \right) \\ &\leq \beta \left(N_0^2 |\partial\Omega| + |\partial\Omega|^{-1} \left(\int_{\partial\Omega} \bar{\psi} dS \right)^2 \right) \\ &\leq \beta N_0^2 |\partial\Omega| + \beta \int_{\partial\Omega} (\bar{\psi})^2 dS. \end{aligned}$$

Adding now the resulting inequality (4.41) to (4.35) and choosing another sufficiently small number $\varepsilon < \varepsilon_0$, we infer

$$\frac{d}{dt} \bar{E}(t) \leq C \bar{E}(t) + C \|(\bar{\varphi}(t), \bar{\psi}(t))\|_{L^2 \times L^2(\partial\Omega)}^2 \|(\partial_t \varphi_2(t), \partial_t \psi_2(t))\|_{V^{s,l}}^2,$$

for all $t > 0$. Recalling that $V^{s,l} \subset L^2(\Omega) \times L^2(\partial\Omega)$, we can further exploit the inequality in (4.42) to get

$$\begin{aligned} (4.43) \quad \frac{d}{dt} \bar{E}(t) &\leq C \bar{E}(t) + C a_W(\tilde{U}(t), \tilde{U}(t)) \|(\partial_t \varphi_2(t), \partial_t \psi_2(t))\|_{V^{s,l}}^2 \\ &\quad + C \beta \|\bar{\psi}(t)\|_{L^2(\partial\Omega)}^2 \|(\partial_t \varphi_2(t), \partial_t \psi_2(t))\|_{V^{s,l}}^2 \\ &\leq C \bar{E}(t) \left(1 + \|(\partial_t \varphi_2(t), \partial_t \psi_2(t))\|_{V^{s,l}}^2 \right) \\ &\quad + C \beta \|\bar{\psi}(t)\|_{L^2(\partial\Omega)}^2 \|(\partial_t \varphi_2(t), \partial_t \psi_2(t))\|_{V^{s,l}}^2. \end{aligned}$$

Multiplying now both sides of (4.43) by $t > 0$ and integrating the resulting inequality over $(0, T)$ with any $T \geq 1$, we derive

$$\begin{aligned} T \bar{E}(T) &\leq C \int_0^T t \left(1 + \|(\partial_t \varphi_2(t), \partial_t \psi_2(t))\|_{V^{s,l}}^2 \right) \bar{E}(t) dt + \int_0^T \bar{E}(t) dt \\ &\quad + C \beta \int_0^T t \|\bar{\psi}(t)\|_{L^2(\partial\Omega)}^2 \|(\partial_t \varphi_2(t), \partial_t \psi_2(t))\|_{V^{s,l}}^2 dt \\ &\leq CT \int_0^T \left(1 + \|(\partial_t \varphi_2(t), \partial_t \psi_2(t))\|_{V^{s,l}}^2 \right) \bar{E}(t) dt \\ &\quad + C \beta T \int_0^T \|\bar{\psi}(t)\|_{L^2(\partial\Omega)}^2 \|(\partial_t \varphi_2(t), \partial_t \psi_2(t))\|_{V^{s,l}}^2 dt. \end{aligned}$$

This entails for any $T \geq 1$,

$$\begin{aligned} (4.44) \quad \bar{E}(T) &\leq C \int_0^T \left(1 + \|(\partial_t \varphi_2(t), \partial_t \psi_2(t))\|_{V^{s,l}}^2 \right) \bar{E}(t) dt \\ &\quad + C \beta \int_0^T \|\bar{\psi}(t)\|_{L^2(\partial\Omega)}^2 \|(\partial_t \varphi_2(t), \partial_t \psi_2(t))\|_{V^{s,l}}^2 dt. \end{aligned}$$

Setting

$$\begin{aligned}\kappa(T) &:= C\beta \int_0^T \|\bar{\psi}(t)\|_{L^2(\partial\Omega)}^2 \|(\partial_t \varphi_2(t), \partial_t \psi_2(t))\|_{V^{s,l}}^2 dt, \\ \alpha(T) &:= \int_0^T \left(1 + \|(\partial_t \varphi_2(t), \partial_t \psi_2(t))\|_{V^{s,l}}^2\right) dt\end{aligned}$$

we observe preliminarily that κ, α are non-negative, well-defined for all $T \geq 1$ (i.e., their integrands belong to $L^1(0, T)$, owing to (4.27) and (4.33)), but also κ is a non-decreasing function of T . Therefore, the application of Gronwall's inequality to (4.44) yields

$$\begin{aligned}(4.45) \quad \bar{E}(T) &\leq \kappa(T) e^{\alpha(T)} \leq C e^{CT} \int_0^T \|\bar{\psi}(t)\|_{L^2(\partial\Omega)}^2 \|(\partial_t \varphi_2(t), \partial_t \psi_2(t))\|_{V^{s,l}}^2 dt \\ &\leq C e^{CT} \left(\|\bar{\varphi}(0)\|_{L^2}^2 + \|\bar{\psi}(0)\|_{L^2(\partial\Omega)}^2 \right),\end{aligned}$$

owing once again to (4.27) and (4.33). We can exploit (4.33) together with $a_W(\tilde{U}, \tilde{U}) \leq \bar{E}$ and (4.45) to infer that

$$(4.46) \quad \|(\bar{\varphi}(T), \bar{\psi}(T))\|_{V^{s,l}}^2 \leq C e^{CT} \left(\|\bar{\varphi}(0)\|_{L^2}^2 + \|\bar{\psi}(0)\|_{L^2(\partial\Omega)}^2 \right),$$

for any $T \geq 1$. Henceforth, the claim (4.25) follows from (4.46) and this completes the proof. ■

Proof of Theorem 4.3. We shall now exploit the statements of Lemmas 4.5 and 4.6 to finish the proof. First, we recall once again that $\mathbb{B} := \mathbb{B}_{R_m}$ is an absorbing set for $S(t)$ and $S(t)\mathbb{B} \subset \mathbb{B}$ for $t \geq t^\sharp$, for some $t^\sharp \geq 1$. Thus we can apply the statement of Theorem 4.4 to the map $\Sigma = S(t^\sharp)$. Indeed, setting $X_0 = \mathbb{B}$ the mapping $\Sigma : X_0 \rightarrow X_0$ enjoys the smoothing property (4.25) owing to the fact that the embeddings $H^s(\Omega) \subset L^2(\Omega)$, $H^l(\partial\Omega) \subset L^2(\partial\Omega)$ are compact. Therefore Theorem 4.4 applies to Σ and there exists a compact set $\mathcal{M}^* \in X_0$ with finite fractal dimension (with respect to the metric topology of $L^2(\Omega) \times L^2(\partial\Omega)$) that satisfies (4.22) and (4.23). Moreover, since $\mathbb{B}$ is an absorbing set for these semigroups in Y_m then these (discrete) attractors attract exponentially fast all the bounded subsets of Y_m . In order to construct the attractor for the semigroup with continuous time, as usual we use the formula

$$\mathcal{M} = \cup_{t \in [t^\sharp, 2t^\sharp]} S(t) \mathcal{M}^*.$$

We now recall that the semigroup $S(t)$ is Lipschitz continuous on $[t^\sharp, 2t^\sharp] \times \mathbb{B}$ in the norm of $H^{-1}(\Omega) \times L^2(\partial\Omega)$. This was verified in Lemma 4.5 as well as in Theorem 3.4 (cf. also (4.32)). Thus, we can verify in a standard way that the exponential attractor $\mathcal{M}$ satisfies (i), (ii) and (iii) of Theorem 4.3, but with the norm of $V^{s,l}$ being replaced by that of $H^{-1}(\Omega) \times L^2(\partial\Omega)$. In order to extend these properties to the required $V^{s,l}$ -norm it suffices to recall that $\mathcal{M}$ is bounded in $\mathbb{B}$, the semigroup possesses a smoothing property from Y_m into Z_m and to exploit basic interpolation inequalities between $D(A_W^{s,l}) \subset V^{s,l} \subset L^2(\Omega) \times L^2(\partial\Omega)$ and $H^s(\Omega) \subset L^2(\Omega) \subset H^{-1}(\Omega)$, respectively. The proof of Theorem 4.3 is finished. ■

As a consequence of Theorem 4.3, we may conclude this section with the following.

Corollary 4.7. *Under the assumptions of Theorem 4.3, there exist the global attractor $\mathcal{A}$, bounded in Z_m , of fractal dimension such that*

$$\dim_F(\mathcal{A}, V^{s,l}) < \infty.$$

5. APPENDIX: THE ISING SPIN EXCHANGE MODEL

Here we wish to present a simple example of an interacting many-particle system with a sufficiently accessible microscopic description, whose hydrodynamic description of the collective behaviour of the particles may eventually lead to the macroscopic evolution (1.10)-(1.13). Following [5], we start from a simple lattice model for a mixture of two-phase constituents A, B in an semi-infinite system with a planar free surface. That is, from

a microscopic viewpoint one may consider a d -dimensional (periodic) lattice for the bulk and $(d - 1)$ -dimensional lattice for the planar surface (cf. [5, Figure 1]): each lattice site i carries a spin variable which represents the degree of freedom that a site is taken by an A -atom (spin is $+1$) or by a B -atom (spin is -1). If $p(i)$ is the probability that the spin at site i is $+1$, then the expected value of the spin there is $\varphi(i) = (+1)p(i) + (-1)(1 - p(i)) = 2p(i) - 1 \in [-1, 1]$. In this Ising model description of the two-phase mixture, the Helmholtz free energy of a distribution φ over the (bulk plus planar surface) lattice is given by $E(\varphi) = H(\varphi) - \theta S(\varphi)$, where $H(u)$ is the sum of an internal and surface interaction energies, θ is the absolute temperature and $S = S_b + S_s$ is the total bulk plus surface entropy. In the Bragg-Williams approximation (see, e.g., [24]) the entropy S_b in the bulk takes the form

$$S_b(\varphi) = - \sum_{\text{bulk: } i} ((1 + \varphi(i)) \log(1 + \varphi(i)) + (1 - \varphi(i)) \log(1 - \varphi(i))).$$

Clearly, S_b has two local minima at -1 and $+1$, respectively, and this form of entropy naturally penalizes values of φ outside the interval $[-1, 1]$. By analogy the surface entropy S_s may take the form of a sum over the lattice sites in the plane surface only, of some function $e_s(\varphi(i))$, that describes the possible preferential (or repulsion) of one of the atoms A, B with the surface plane. Following [5] (cf. also [25]), the total Hamiltonian is given by

$$(5.1) \quad H(\varphi) = -\frac{1}{2} \sum_{\text{bulk: } i, j} J(i - j) \varphi(i) \varphi(j) - \frac{1}{2} \sum_{\text{surface: } i, j} J_s(i - j) \varphi(i) \varphi(j)$$

where the first sum is over the lattice sites in the bulk, excluding the surface plane, while the second term runs over lattice sites within the surface plane only. Here $J(i - j) = J(j - i)$ and $J_s(i - j) = J_s(j - i)$ denote the pairwise interaction energies in the bulk and surface plane, respectively, and we have allowed for the possibility that the particle interaction in the surface plane differs from the interaction in the bulk. This is quite similar to the Ising model considered in [5] where only nearest-neighbour interactions give a contribution to the energy. Here, we allow for long-range interactions between different sites i, j both in the bulk and the surface plane, that is, in contrast to [5, 13] in our Ising model the horizon for "spin-to-spin" interactions may be arbitrarily large up to the size of the system. We can further extend this framework to a formulation with continuous spins for a basic lattice that is made up into relatively small blocks each of which has a large number of sites, $\varphi(i)$ is then the average spin at a site within the block centred at position i . Thus, our lattice could also be a lattice (i.e., cubic for the bulk and square for the surface plane) of blocks with "block spins" which take values in $[-1, 1]$. More precisely, all values are being possible in the limit as the number of particles in a block tends to infinity, in which case the interaction between blocks both in the bulk and the surface plane may arise from long-range like Kac potentials.

Next, we may consider a rearrangement of the sums in (5.1), without changing the Helmholtz free energy $E = H - \theta S$, by replacing H with

$$(5.2) \quad H(\varphi) = \frac{1}{4} \sum_{\text{bulk: } i, j} J(i - j) (\varphi(i) - \varphi(j))^2 + \frac{1}{4} \sum_{\text{surface: } i, j} J_s(i - j) (\varphi(i) - \varphi(j))^2$$

such that $S = S_b + S_s$ can be rewritten as

$$(5.3) \quad \begin{aligned} & -\theta S(\varphi) \\ &= \sum_{\text{bulk: } i} \left[\theta ((1 + \varphi(i)) \log(1 + \varphi(i)) + (1 - \varphi(i)) \log(1 - \varphi(i))) - C_J (\varphi(i))^2 \right] \\ &+ \sum_{\text{surface: } i} \left[\theta e_s(\varphi(i)) - C_{J_s} (\varphi(i))^2 \right], \end{aligned}$$

where $C_J = (1/2) \sum_{bulk: i} J(i)$, $C_{J_s} = (1/2) \sum_{surface: i} J_s(i)$, assuming they are both finite. We recall that quite often both terms in (5.3) can be replaced by

$$(5.4) \quad \sum_{bulk: i} F(\varphi(i)) + \sum_{surface: i} G(\varphi(i)),$$

with both $F(\varphi)$, $G(\varphi)$ being defined everywhere on $\mathbb{R}$, and with F assuming a double-well structure of double potential which has two local minima at $+1$ and -1 , respectively. For instance, $F(\varphi) = C_F (\varphi^2 - 1)^2$, for some constant $C_F > 0$ which depends on $\theta > 0$. As in the classical case, also in the Ising model, phase transitions are only possible for low temperatures. Indeed, note that when $\theta > \theta_c$ (=critical temperature) is large enough the first term in (5.3) is large and is the bulk sum over the lattice of a function which has a unique minimum. Therefore in this case, no phase transition occurs and the binary mixture is stable. However, at lower temperatures $0 < \theta < \theta_c$ this function has two wells of equal depth, each minimizing the energy. With (5.1) and (5.2) in mind, we can now rewrite the free energy E associated with this Ising model, as follows:

$$(5.5) \quad \begin{aligned} E(\varphi) = & \frac{1}{4} \sum_{bulk: i, j} J(i-j) (\varphi(i) - \varphi(j))^2 \\ & + \frac{1}{4} \sum_{surface: i, j} J_s(i-j) (\varphi(i) - \varphi(j))^2 \\ & + \sum_{bulk: i} F(\varphi(i)) + \sum_{surface: i} G(\varphi(i)). \end{aligned}$$

We can now take the hydrodynamic limit in (5.5) to arrive at the corresponding continuous version

$$(5.6) \quad \begin{aligned} E(\varphi) = & \frac{1}{4} \int_{T^d} \int_{T^d} J(x-y) |\varphi(x) - \varphi(y)|^2 dx dy + \int_{T^d} F(\varphi) dx \\ & + \frac{1}{4} \int_{T^{d-1}} \int_{T^{d-1}} J_s(x-y) |\varphi(x) - \varphi(y)|^2 dS_x dS_y + \int_{T^{d-1}} G(\varphi) dS_x. \end{aligned}$$

The passage to the continuum limit to deduce (5.6) from (5.5) can be made rigorous using the theory of Gibb's measures, as performed by Giacomini and Lebowitz in [21] for the nonlocal Cahn-Hilliard equation (1.6). Although we don't perform such a limit here we further wish to discuss the possible choices for the bulk and surface interactions kernels J, J_s . We will further assume that they are given by attractive and radially symmetric potentials $J, J_s \geq 0$, of the form

$$(5.7) \quad J(r) = \frac{j_{\sigma, \infty}}{|r|^{d+\sigma}}, \quad J_s(r) = \frac{j_{l, \infty}}{|r|^{d-1+2l}},$$

for some $\sigma \in (0, 2)$, $l \in (0, 1)$ and constants $j_{\sigma, \infty}, j_{l, \infty} > 0$. Interestingly, such examples in (5.7) are quite common in "long-range" spin interactions in uniaxial Ising models that describe important classes of material interactions in fluids, alloys, etc (see [10, Table IV, (8)]). For a proper choice of the normalizing constants $j_{\sigma, \infty} = (1/2) C_{d, \sigma/2}$, $j_{l, \infty} = (1/2) C_{d-1, l}$, and taking $\sigma = 2s$, $s \in (0, 1)$, we also observe that the energy from (5.6) reduces to the one in (3.27). More precisely, replacing T^d also by a general domain $\Omega \subset \mathbb{R}^d$, it becomes

$$(5.8) \quad \begin{aligned} E(\varphi) = & \frac{C_{d, s}}{2} \int_{\Omega} \int_{\Omega} \frac{(\varphi(x) - \varphi(y))^2}{|x - y|^{d+2s}} dx dy \\ & + \frac{C_{d-1, l}}{2} \int_{\partial\Omega} \int_{\partial\Omega} \frac{(\varphi(x) - \varphi(y))^2}{|x - y|^{d-1+2l}} dS_x dS_y \\ & + \int_{\Omega} F(\varphi) dx + \int_{\partial\Omega} \left(G(\varphi) + \frac{\beta \varphi^2}{2} \right) dS. \end{aligned}$$

Here, we have also added another quadratic term $(\beta/2)\varphi^2$ for the sake of mathematical convenience (this term does not affect the analysis but only the assumptions on G slightly). One can now obtain the nonlocal system (1.10)-(1.13) by performing a standard derivation

in which one first defines two generalized forces (the so-called chemical potentials), one in the bulk μ and one on the surface $\mu_{\partial\Omega}$, respectively. They are defined as the variational derivatives of $E(\varphi)$ with respect to φ , restricted to Ω and $\partial\Omega$, respectively, by making use of the integration by parts formula (see Definition 2.11 and (2.76)) associated with the regional fractional Laplacian $(-\Delta)_\Omega^s$ and the fractional Laplace-Beltrami operator $(-\Delta)_{\partial\Omega}^l$, respectively. More precisely, they are given by

$$\mu = (-\Delta)_\Omega^s \varphi + F'(\varphi), \text{ in } \Omega$$

and

$$(5.9) \quad \mu_{\partial\Omega} = (-\Delta)_{\partial\Omega}^l \varphi + C_s N^{2-2s} \varphi + \beta \varphi + G'(\varphi) \text{ on } \partial\Omega.$$

Then the first equation of (1.10)-(1.13) can be derived by means of the conservation law $\partial_t \varphi = \operatorname{div}(m(\varphi) \nabla \mu)$, assuming a constant mobility coefficient ($m(\varphi) \equiv 1$), while the third equation can be simply deduced by assuming that φ obeys the following relaxation equation

$$(5.10) \quad \partial_t \varphi = -\mu_{\partial\Omega} \text{ on } \partial\Omega.$$

As usual, a further no-flux boundary condition (see (1.11)) on the bulk chemical potential μ is assumed in order to conserve mass in the bulk Ω . This concludes our brief justification for a nonlocal macroscopic model of phase-transitions in a domain Ω in the presence of a wall $\partial\Omega$.

6. FINAL REMARKS

In the system that we have just solved, we have exploited a formulation based on a fractional dynamic boundary condition associated with the regional fractional Laplacian $(-\Delta)_\Omega^s$ and a double-well (polynomial like) potential F . We recall that such dynamic boundary conditions are purely "local" in nature [19] (namely, they are satisfied only on $\partial\Omega$), exactly as in the case of the dynamic boundary condition associated with the classical Laplace operator Δ . If one wishes to consider dynamic boundary conditions for other families (or versions) of fractional operators, say the restricted fractional Laplacian $(-\Delta)^s$ (see, e.g., [18]), this may still be possible but these conditions are expected to be "nonlocal" in nature in the sense that they need to be satisfied over $\mathbb{R}^d \setminus \Omega$ (in particular, that this would require to specify the initial datum $\psi(0) = \psi_0$ over all of $\mathbb{R}^d \setminus \Omega$). This, unfortunately, may prove to be quite impractical from the point of view of applications (see [19, Introduction] for further discussions). We point out that another interesting problem is when the operator in the definition of the chemical potential μ is of fractional order (say, either $(-\Delta)_\Omega^s$ or $(-\Delta)^s$, for $s \in (0, 1)$) and the corresponding Dirichlet or Neumann boundary condition associated with this fractional operator is considered for φ instead of (5.9)-(5.10). This case was fully addressed recently in [15], in the case when F is a polynomial potential, and for more general operators that cover a substantially number of other cases. In view of the derivation and aforementioned discussion from the Appendix, it is our opinion that the case of a singular potential is an interesting question worth exploring. The question of convergence to a single steady state of any given weak solution to (1.10)-(1.13) is presently also an open question.

Acknowledgement 6.1. *I wish to acknowledge the reviewers whose important comments have improved the presentation of the article.*

REFERENCES

- [1] H. Abels, S. Bosia, M. Grasselli, Cahn-Hilliard equation with nonlocal singular free energies, Ann. Mat. Pura Appl. 194 (2015), 1071–1106.
- [2] K. Bogdan, K. Burdzy, Z.Q. Chen, Censored stable processes, Probab. Theory Related Fields 127 (2003), 89–152.
- [3] P.W. Bates, J. Han, The Neumann boundary problem for a nonlocal Cahn-Hilliard equation, J. Differential Equations 212 (2005), 235–277.
- [4] P.W. Bates, J. Han, The Dirichlet boundary problem for a nonlocal Cahn-Hilliard equation, J. Math. Anal. Appl. 311 (2005), 289–312.

- [5] K. Binder, H. L. Frisch, Dynamics of surface enrichment: A theory based on the Kawasaki spin-exchange model in the presence of a wall, *Z.Phys.B* 84 (1991), 403–418.
- [6] J.W. Cahn, J.E. Hilliard, Free energy of a nonuniform system. I. Interfacial free energy, *J. Chem. Phys.* 28 (1958), 258–267.
- [7] L. Cherfils, A. Miranville, S. Zelik, The Cahn-Hilliard equation with logarithmic potentials, *Milan J. Math.* 79 (2011), 561–596.
- [8] P. Colli, S. Frigeri, M. Grasselli, Global existence of weak solutions to a nonlocal Cahn-Hilliard-Navier-Stokes system, *J. Math. Anal. Appl.* 386 (2012), 428–444.
- [9] K. Deimling, *Nonlinear Functional Analysis*, Springer-Verlag, 1984.
- [10] M.E. Fischer, The renormalization group in the theory of critical behavior, *Rev. Mod. Phys.* 46 (1974), 597–616.
- [11] S. Frigeri, M. Grasselli, Global and trajectory attractors for a nonlocal Cahn-Hilliard system, *J. Dyn. Diff. Eqns.* 24 (2012), 827–856.
- [12] S. Frigeri, M. Grasselli, Nonlocal Cahn-Hilliard-Navier-Stokes systems with singular potentials, *Dyn. Partial Differ. Equ.* 9 (2012), 273–304.
- [13] H.P. Fischer, J. Reinhard, W. Dieterich, J.-F. Gouyet, P. Maass, A. Majhofer, D. Reinel, Time-dependent density functional theory and the kinetics of lattice gas systems in contact with a wall, *J.Chem.Phys.* 108 (1998), 3028–3037.
- [14] H. Gajewski, K. Zacharias, On a nonlocal phase separation model, *J. Math. Anal. Appl.* 286 (2003), 11–31.
- [15] C.G. Gal, On the strong-to-strong interaction case for doubly nonlocal Cahn-Hilliard equations, *Discrete and Continuous Dynamical Systems-A* 37 (2017), 1–37.
- [16] C.G. Gal, M. Grasselli, Longtime behavior of nonlocal Cahn-Hilliard equations, *Discrete Contin. Dyn. Syst.* 34 (2014), no. 1, 145–179.
- [17] C.G. Gal, M. Grasselli, Singular limit of viscous Cahn-Hilliard equations with memory and dynamic boundary conditions, *Discrete Contin. Dyn. Syst. Ser. B* 18 (2013), no. 6, 1581–1610.
- [18] C.G. Gal, M. Warma, Reaction-diffusion equations with fractional diffusion on non-smooth domains with various boundary conditions, *Discrete Contin. Dyn. Syst.* 36 (2016), 1279–1319.
- [19] C.G. Gal and M. Warma, Elliptic and parabolic equations with fractional diffusion and dynamic boundary conditions, *Evolution Equations and Control Theory* 5 (2016), 61–103.
- [20] C.G. Gal, M. Warma, Bounded solutions for nonlocal boundary valued problems on Lipschitz manifolds with boundary, *Advanced Nonlinear Studies* 16 (2016), 529–550.
- [21] G. Giacomin, J.L. Lebowitz, Phase segregation dynamics in particle systems with long range interactions, I. Macroscopic limits, *J. Statist. Phys.* 87 (1997), 37–61.
- [22] Q.Y. Guan, Integration by parts formula for regional fractional Laplacian, *Comm. Math. Phys.* 266 (2006), 289–329.
- [23] Q.Y. Guan, Z.M. Ma, Reflected symmetric α -stable processes and regional fractional Laplacian, *Probab. Theory Related Fields* 134 (2006), 649–694.
- [24] P. Haasen, *Physical Metallurgy*, Cambridge Univ. Press, Cambridge, 1978.
- [25] A.G. Khachaturyan, *Theory of Structural Transformations in Solids*, Wiley-Interscience Publications, New York, 1983.
- [26] S.-O. Londen, H. Petzeltová, Convergence of solutions of a non-local phase-field system, *Discrete Contin. Dyn. Syst. Ser. S* 4 (2011), 653–670.
- [27] S.-O. Londen, H. Petzeltová, Regularity and separation from potential barriers for a non-local phase-field system, *J. Math. Anal. Appl.* 379 (2011), 724–735.
- [28] A. Miranville, S. Zelik, Attractors for dissipative partial differential equations in bounded and unbounded domains. *Handbook of differential equations: evolutionary equations*. Vol. IV, 103–200, *Handb. Differ. Equ.*, Elsevier/North-Holland, Amsterdam, 2008.
- [29] A. Novick-Cohen, The Cahn-Hilliard equation, *Evolutionary equations*, Vol. IV, 201–228, *Handb. Differ. Equ.*, Elsevier/North-Holland, Amsterdam, 2008.
- [30] E. Di Nezza, G. Palatucci, E. Valdinoci, Hitchhiker’s guide to the fractional Sobolev spaces, *Bull. Sci. Math.* 136 (2012), 521–573.
- [31] S.P. Neumana, D.M. Tartakovsky, Perspective on theories of non-Fickian transport in heterogeneous media, *Advances in Water Resources* 32 (2009), 670–680.
- [32] L. Vlahos, H. Isliker, Y. Kominis, K. Hizonidis, Normal and Anomalous Diffusion: A Tutorial, in *Order and Chaos*, Vol. 10 (ed. T. Bountis), Patras University Press, 2008.
- [33] M. Warma, The fractional relative capacity and the fractional Laplacian with Neumann and Robin boundary conditions on open sets, *Potential Anal.* 42 (2015), 499–547.

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