A short proof of Cartan’s Nullstellensatz for entire functions in C^n
Raymond Mortini

To cite this version: Raymond Mortini. A short proof of Cartan’s Nullstellensatz for entire functions in C^n. Archiv der Mathematik, Springer Verlag, 2015, 105 (2), pp.149-152. <10.1007/s00013-015-0786-x>. <hal-01229964>
A SHORT PROOF OF CARTAN’S NULLSTELLENSATZ
FOR ENTIRE FUNCTIONS IN $\mathbb{C}^n$

RAYMOND MORTINI

Abstract. Using the fact that the maximal ideals in the polydisk algebra are given by the kernels of point evaluations, we derive a simple formula that gives a solution to the Bézout equation in the space of all entire functions of several complex variables. Thus a short and easy analytic proof of Cartan’s Nullstellensatz is obtained.

1. Introduction

The aim of this note is to give a short and easy proof of Cartan’s Nullstellensatz:

Theorem 1.1. Let $H(\mathbb{C}^n)$ be the space of functions holomorphic in $\mathbb{C}^n$. Given $f_j \in H(\mathbb{C}^n)$, the Bézout equation $\sum_{j=1}^N g_j f_j = 1$ admits a solution $(g_1, \ldots, g_N) \in H(\mathbb{C}^n)^N$ if and only if the functions $f_j$ have no common zero in $\mathbb{C}^n$.

The usual proofs use a lot of machinery from sheaf theory, cohomology, (see for example [2]), or are based on the Hörmander-Wolff method by solving higher order $\bar{\partial}$-equations using the Koszul complex, a tool from homological algebra (see [6, p. 128-131]). The one-dimensional case, first done by Wedderburn, is very easy (see for example [4, p. 118-120] for the classical approach or [1, p. 130] for the $\bar{\partial}$-approach). For our proof to work, we shall only use a standard fact from an introductory course to functional analysis, namely Gelfand’s main theorem: the maximal ideals in a commutative unital complex Banach algebra coincide with the kernels of the multiplicative linear functionals (see for instance [5]). The idea is to apply this result to the polydisk algebras on an increasing sequence of polydisks $D_k$ and to glue together the solutions to the Bézout equations $\sum_{j=1}^N g_j f_j = 1$ on $D_k$ by using a Mittag-Leffler type trick. The major hurdle to overcome was of course to find suitable summands that guarantee at the end the holomorphy.

Date: June 1, 2015.

1991 Mathematics Subject Classification. Primary 32A15; Secondary 46J15.

Key words and phrases. Entire functions; Cartan’s Nullstellensatz; polydisk algebra; maximal ideals.
2. The general solution to the Bézout equation

Let $\mathbb{D}$ be the unit disk and let $A(D^n)$ be the polydisk algebra; that is the algebra of those continuous functions on the closed polydisk

$$D^n = \{(z_1, \ldots, z_n) \in \mathbb{C}^n : |z_j| \leq 1\}$$

which are holomorphic in $\mathbb{D}^n$. Endowed with the supremum norm, $A(D^n)$ becomes a uniform algebra and coincides with the closure on $D^n$ of the polynomials in $\mathbb{C}[z_1, \ldots, z_n]$. We actually only need that $A(D^n)$ is the uniform algebra generated by the coordinate functions $Z_j$, $j = 1, \ldots, n$ on $D^n$. It is now straightforward to show that an ideal $I$ in $A(D^n)$ is maximal if and only if it coincides with $M(a_0) = \{f \in A(D^n) : f(a_0) = 0\}$ for some $a_0 \in D^n$ (just take a character $m$ on $A(D^n)$ and put $a_0 = (m(Z_1), \ldots, m(Z_n))$).

Hence the Bézout equation $\sum_{j=1}^n x_j f_j$ in $A(D^n)$ has a solution if and only if $\bigcap_{j=1}^n Z_{D^n}(f_j) = \emptyset$, where $Z_{D^n}(f)$ is the zero set of $f$ on $D^n$. We will use the following well-known elementary result. For the reader’s convenience we reproduce the proof here (see [3]), because its understanding is fundamental for our construction.

**Lemma 2.1.** Let $R$ be a commutative unital ring. Suppose that $a = (a_1, \ldots, a_N)$ is an invertible $N$-tuple in $R^N$ and let $x = (x_1, \ldots, x_N)$ satisfy $\sum_{j=1}^N x_j a_j = 1$; that is $xa^t = 1$. Then every other representation $1 = \sum_{j=1}^N y_j a_j$ of $1$ can be deduced from the former by letting $y = x + aH$, where $H$ is an antisymmetric $(N \times N)$—matrix over $R$; that is $H = -H^t$, where $H^t$ is the transpose of $H$.

**Proof.** Suppose that $1 = xa^t$ and $1 = ya^t$. For $k = 1, \ldots, N$, multiply the first equation by $y_k$ and the second by $x_k$. Then

$$x_k - y_k = \sum_{j \neq k} a_j(y_j x_k - y_k x_j).$$

Thus $y = x + aH$ for some antisymmetric matrix $H$.

To prove the converse, let $1 = xa^t$. Since $H$ is antisymmetric we have (due to the transitivity of matrix multiplication and $xy^t = yx^t$)

$$(aH)a^t = a(Ha^t) = a(aH^t)^t = a(-aH)^t = (-aH)a^t.$$

Thus $(aH)a^t = 0$. Hence

$$ya^t = (x + aH)a^t = xa^t + (aH)a^t = 1 + 0 = 1.$$

\[\square\]

3. Proof of Theorem 1.1

**Proof.** Let $f_j \in H(C^n)$ and put $f = (f_1, \ldots, f_N)$. Suppose that $\bigcap_{j=1}^N Z(f_j) = \emptyset$. For $k \in \mathbb{N}^*$, let $D_k = (kD)^n$ be the closed polydisk

$$D_k = \{(z_1, \ldots, z_n) \in \mathbb{C}^n : |z_j| \leq k\}.$$
Note that $D_k \subseteq D_{k+1}$. Let $a_k \in A(D_{k+1})^N$ be a solution to the Bézout equation $a_k \cdot f^t = 1$ on $D_{k+1}$. Using Tietze’s extension theorem $^1$, we may assume that the tuples $a_k$ have been continuously extended to $\mathbb{C}^n$. By Lemma 2.1, there is an antisymmetric matrix $H_k$ over $A(D_{k+1})$ such that

$$a_{k+1} = a_k + f \cdot H_k.$$

Put $a_0 = 0$ and $H_0 = 0$. For $k = 0, 1, \ldots$, let $P_k$ be an antisymmetric $N \times N$-matrix of polynomials in $\mathbb{C}[z_1, \ldots, z_n]$ such that

$$\max_{D_{k+1}} ||f \cdot H_k - f \cdot P_k||_N < 2^{-k}.$$

We claim that the $N$-tuple

$$g := \sum_{k=0}^{\infty} (a_{k+1} - a_k - f \cdot P_k)$$

belongs to $H(\mathbb{C}^n)^N$ and is a solution to the Bézout equation $g \cdot f^t = 1$ in $H(\mathbb{C}^n)$. In fact, let $D_m$ be fixed. Then the series defining $g$ is uniformly convergent on $D_m$ since

$$g = \sum_{k=0}^{m} (a_{k+1} - a_k - f \cdot P_k) + \sum_{k=m+1}^{\infty} (a_{k+1} - a_k - f \cdot P_k)$$

$$= a_{m+1} - f \cdot (\sum_{k=0}^{m} P_k) + \sum_{k=m+1}^{\infty} (a_{k+1} - a_k - f \cdot P_k)$$

and the tail can be majorated on $D_m$ by

$$\sum_{k=m+1}^{\infty} ||f \cdot H_k - f \cdot P_k||_N < 2^{-m}.$$

Moreover, on $D_m$, $a_{m+1}$ and all the summands in the series $\sum_{k=m+1}^{\infty}$ are holomorphic. Since $m$ was arbitrarily chosen, we conclude that $g \in H(\mathbb{C}^n)^N$.

Moving again to $D_m$ we see that, due to the antisymmetry of the matrices $H_k$ and $P_k$,

$$g \cdot f^t = a_{m+1} \cdot f^t - f \cdot (\sum_{k=0}^{m} P_k) \cdot f^t + \sum_{k=m+1}^{\infty} f \cdot (H_k - P_k) \cdot f^t$$

$$= 1 - 0 + 0 = 1.$$

\[\square\]

$^1$ Since we will consider a telescoping series $\sum T_j$, where the domains of definition of the summands $T_j$ are strictly increasing, even an application of Tietze’s theorem is not necessary.

$^2$ Of course, outside $D_k$ the Bézout equation is not necessarily satisfied.
Acknowledgements
I thank the referee of the journal “American Math. Monthly” for some useful comments concerning the introductory section of a previous version of the paper.

REFERENCES