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Extending the parking space

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Abstract. The action of the symmetric group S_n on the set Park_n of parking functions of size n has received a great deal of attention in algebraic combinatorics. We prove that the action of S_n on Park_n extends to an action of S_{n+1}. More precisely, we construct a graded S_{n+1}-module V_n such that the restriction of V_n to S_n is isomorphic to Park_n. We describe the S_n-Frobenius characters of the module V_n in all degrees and describe the S_{n+1}-Frobenius characters of V_n in extreme degrees. We give a bivariate generalization $V_n^{(\ell,m)}$ of our module V_n whose representation theory is governed by a bivariate generalization of Dyck paths. A Fuss generalization of our results is a special case of this bivariate generalization.

Résumé. L'action du groupe symétrique S_n sur l'ensemble Park_n des fonctions de stationnement de longueur n a reçu beaucoup d'attention dans la combinatoire algébrique. Nous démontrons que l'action de S_n sur Park_n s'étend à une action de S_{n+1}. Plus précisément, nous construisons un gradué S_{n+1}-module V_n telles que la restriction de S_n est isomorphe à Park_n. Nous décrivons la S_n-Frobenius caractères des modules V_n à tous les degrés et décrivent le S_{n+1}-Frobenius caractères de V_n en degrés extrêmes. Nous donnons une généralisation bivariée $V_n^{(\ell,m)}$ de notre module V_n dont la représentation théorie est régi par une généralisation bivariée des chemins de Dyck. Une généralisation Fuss de nos résultats est un cas particulier de cette généralisation bivariée.

Keywords: parking functions, symmetric group, Dyck paths, representation, matriod

1 Introduction

This paper is about extending the visible permutation action of S_n on the space Park_n spanned by parking functions of size n to a hidden action of the larger symmetric group S_{n+1}. The S_{n+1}-module we construct will be a subspace of the coordinate ring of the reflection representation of type A_n and will inherit the polynomial grading of this coordinate ring. Using statistics on Dyck paths, we will give an explicit combinatorial formula for the graded S_n-Frobenius character of our module and will describe the extended S_{n+1}-Frobenius character in extreme degrees.

As far as the authors know, this is the first example and proof of an extension of Park_n to S_{n+1} an S_{n+1}-module.

We remark that our result is the 'best possible' in two senses. First, it is not always possible to extend Park_n to an S_{n+2}-module; for example, the action of S_4 on Park_4 does not extend to an action of S_6.

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Also, from a combinatorial point of view, one may be interested in extending the action of S_n on Park_n to a *permutation* action of the larger symmetric group S_{n+1}. Our extended module is graded, but is not a permutation module. However, it is impossible to extend the action of S_4 on Park_4 to a permutation action of S_5.

2 Background and Main Results

A length n sequence $(a_1, \ldots, a_n)$ of positive integers is called a *parking function of size* n if its nondecreasing rearrangement $(b_1 \leq \cdots \leq b_n)$ satisfies $b_i \leq i$ for all i. Parking functions were introduced by Konheim and Weiss [KW66] in the context of computer science, but have seen much application in algebraic combinatorics with connections to Catalan combinatorics, Shi hyperplane arrangements, diagonal coinvariant rings, and rational Cherednik algebras. The set of parking functions of size n is famously counted by $(n+1)^{n-1}$. The $\mathbb{C}$-vector space Park_n spanned by the set of parking functions of size n carries a natural permutation action of the symmetric group S_n on n letters:

$$w.(a_1, \ldots, a_n) = (a_{w(1)}, \ldots, a_{w(n)}) \tag{1}$$

for $w \in S_n$ and $(a_1, \ldots, a_n) \in \mathrm{Park}_n$.

A *partition* λ of a positive integer n is a weakly decreasing sequence $\lambda = (\lambda_1 \geq \cdots \geq \lambda_k)$ of nonnegative integers which sum to n. We write $\lambda \vdash n$ to mean that λ is a partition of n and define $|\lambda| := n$. We call k the *length* of the partition λ. The *Ferrers diagram* of λ consists of λ_i left justified boxes in the i^{th} row from the top ('English notation'). If λ is a partition, we define a new partition $\mathrm{mult}(\lambda)$ whose parts are obtained by listing the (positive) part multiplicities in λ in weakly decreasing order. For example, we have that $\mathrm{mult}(4,4,3,3,3,1,0,0) = (3,2,2,1)$.

We will make use of two partial orders on partitions in this paper. The first partial order is *Young's lattice* with relations given by $\lambda \subseteq \mu$ if $\lambda_i \leq \mu_i$ for all $i \geq 1$ (where we append an infinite string of zeros to the ends of λ and μ so that these inequalities make sense). Equivalently, we have that $\lambda \subseteq \mu$ if and only if the Ferrers diagram of λ fits inside the Ferrers diagram of μ. *Dominance order* on partitions is defined by $\lambda \preceq \mu$ if for all $i \geq 1$ we have the inequality of partial sums $\lambda_1 + \cdots + \lambda_i \leq \mu_1 + \cdots + \mu_i$ (where we again append an infinite string of zeros to the ends of λ and μ). Observe that either of the relations $\lambda \subseteq \mu$ or $\lambda \preceq \mu$ imply that $|\lambda| \leq |\mu|$.

For a partition $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash n$, we let S_λ denote the *Young subgroup* $S_{\lambda_1} \times \cdots \times S_{\lambda_k}$ of S_n. We denote by M^λ the coset representation of S_n given by $M^\lambda := \mathrm{Ind}_{S_\lambda}^{S_n}(\mathbf{1}_{S_\lambda}) \cong_{S_n} \mathbb{C}S_n/S_\lambda$ and we denote by S^λ the irreducible representation of S_n labeled by the partition λ.

Let R_n denote the $\mathbb{C}$-vector space of class functions $S_n \to \mathbb{C}$. Identifying modules with their characters, the set $\{S^\lambda : \lambda \vdash n\}$ forms a basis of R_n. The graded vector space $R := \bigoplus_{n \geq 0} R_n$ attains the structure of a $\mathbb{C}$-algebra via the induction product $S^\lambda \circ S^\mu := \mathrm{Ind}_{S_n \times S_m}^{S_{n+m}}(S^\lambda \otimes_{\mathbb{C}} S^\mu)$, where $\lambda \vdash n$ and $\mu \vdash m$.

We denote by Λ the ring of symmetric functions (in an infinite set of variables $X_1, X_2, \ldots$, with coefficients in $\mathbb{C}$). The $\mathbb{C}$-algebra Λ is graded and we denote by Λ_n the homogeneous piece of degree n. Given a partition λ, we denote the corresponding Schur function by s_λ and the corresponding complete homogeneous symmetric function by h_λ.

The *Frobenius character* is the graded $\mathbb{C}$-algebra isomorphism $\mathrm{Frob} : R \to \Lambda$ induced by setting $\mathrm{Frob}(S^\lambda) = s_\lambda$. It is well known that we have $\mathrm{Frob}(M^\lambda) = h_\lambda$. Generalizing slightly, if $V = \bigoplus_{k \geq 0} V(k)$ is a graded S_n-module, define $\mathrm{grFrob}(V; q) \in \Lambda \otimes_{\mathbb{C}} \mathbb{C}[[q]]$ to be the formal power series in q with coefficients in Λ given by $\mathrm{grFrob}(V; q) := \sum_{k \geq 0} \mathrm{Frob}(V(k)) q^k$.

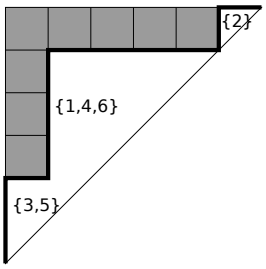

Fig. 1: A Dyck path of size 6.

A *Dyck path of size* n is a lattice path D in $\mathbb{Z}^2$ consisting of vertical steps $(0,1)$ and horizontal steps $(1,0)$ which starts at $(0,0)$, ends at (n,n), and stays weakly above the line $y=x$. A maximal continguous sequence of vertical steps in D is called a *vertical run* of D.

We will associate two partitions to a Dyck path D of size n. The *vertical run partition* $\lambda(D) \vdash n$ is obtained by listing the (positive) lengths of the vertical runs of D in weakly decreasing order. For example, if D is the Dyck path in Figure 1, then $\lambda(D) = (3,2,1)$. The *area partition* $\mu(D)$ is the partition of length n whose Ferrers diagram is the set of boxes to the upper left of D in the $n \times n$ square with lower left coordinate at the origin. For example, if D is the Dyck path of size 6 in Figure 1, then $\mu(D) = (5,1,1,1,0,0)$. The boxes in the Ferrers diagram of $\mu(D)$ are shaded. We define the *area* statistic[(i)] on Dyck paths by $\mathrm{area}(D) = |\mu(D)|$. For the Dyck path in our running exampe, $\mathrm{area}(D) = 8$. By construction, we have that $\mathrm{mult}(\mu(D)) = \lambda(D)$ for any Dyck path D of size n.

Dyck paths of size n can be used to obtain a decomposition of Park_n as a direct sum of coset modules M^λ. In particular, let D be a Dyck path of size n. A *labeling* of D assigns each vertical run of D to a subset of $[n] := \{1,2,\ldots,n\}$ of size equal to the length of that vertical run such that every letter in $[n]$ appears exactly once as a label of a vertical run. Figure 1 shows an example of a labeled Dyck path of size 6, where the subsets labeling the vertical runs are placed just to the right of the runs.

The set of labeled Dyck paths of size n carries an action of S_n given by label permutation. There is an S_n-equivariant bijection from the set of labeled Dyck paths D of size n to parking functions $(a_1,\ldots,a_n)$ of size n given by letting a_i be one greater than the x-coordinate of the vertical run of D labeled by i. For example, the labeled Dyck path in Figure 1 corresponds to the parking function $(2,6,1,2,1,2) \in \mathrm{Park}_6$. Since any fixed labeled Dyck path of size D generates a cyclic S_n-module isomorphic to $M^{\lambda(D)}$, it is immediate that the parking space Park_n decomposes into coset representations as

$$\mathrm{Park}_n \cong_{S_n} \bigoplus_D M^{\lambda(D)}, \tag{2}$$

where the direct sum is over all Dyck paths D of size n. Equivalently, we have that the Frobenius character of Park_n is given by $\mathrm{Frob}(\mathrm{Park}_n) = \sum_D h_{\lambda(D)}$. For example, the 5 Dyck paths of size 3 shown in Figure 3 lead to the Frobenius character

$$\mathrm{Frob}(\mathrm{Park}_3) = h_{(3)} + 3h_{(2,1)} + h_{(1,1,1)}. \tag{3}$$

[(i)] Many authors instead define the area of a Dyck path D to be the number of complete lattice squares between D and the line $y=x$, so that our statistic would be the 'coarea'.

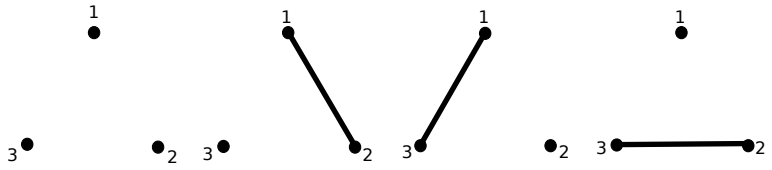

Fig. 2: The four slim subgraphs of K_3.

The vector space underlying the S_{n+1}-module which will extend Park_n is a subspace of the polynomial ring $\mathbb{C}[x_1, \ldots, x_{n+1}]$ in $n+1$ variables and first studied in the work of Postnikov and Shapiro [PS04]. Let K_{n+1} denote the complete graph on the vertex set $[n+1]$. Given an edge $e = (i < j)$ in K_{n+1}, we associate the polynomial weight $p(e) := x_i - x_j \in \mathbb{C}[x_1, \ldots, x_{n+1}]$. A subgraph $G \subseteq K_{n+1}$ (identified with its edge set) gives rise to the polynomial weight $p(G) := \prod_{e \in G} p(e)$. Following Postnikov and Shapiro, we call a subgraph $G \subseteq K_{n+1}$ *slim* if the complement edge set $K_{n+1} - G$ is a connected graph on the vertex set $[n+1]$.

Definition 1 *Denote by V_n the $\mathbb{C}$-linear subspace of $\mathbb{C}[x_1, \ldots, x_{n+1}]$ given by*

$$V_n := \mathrm{span}\{p(G) \,:\, G \text{ is a slim subgraph of } K_{n+1}\}. \tag{4}$$

Let $V_n(k)$ denote the homogeneous piece of V_n of polynomial degree k; the space $V_n(k)$ is spanned by those polynomials $p(G)$ corresponding to slim subgraphs G of K_{n+1} with k edges.

In the case $n = 2$, Figure 2 shows that four slim subgraphs of the complete graph K_3. From left to right, the corresponding polynomials are $1, x_1 - x_2, x_1 - x_3$, and $x_2 - x_3$. It follows that $V_2(0) = \mathrm{span}\{1\}$ and $V_2(1) = \mathrm{span}\{x_1 - x_2, x_1 - x_3, x_2 - x_3\}$. Observe that the graded Frobenius character of V_2 is $\mathrm{grFrob}(V_2; q) = s_{(3)}q^0 + s_{(2,1)}q^1$. By the branching rule for symmetric groups (see [Sag01]), we have that $\mathrm{grFrob}(\mathrm{Res}^{S_3}_{S_2}(V_2); q) = s_{(2)}q^0 + (s_{(2)} + s_{(1,1)})q^1$. Setting $q = 1$ yields $\mathrm{Frob}(\mathrm{Res}^{S_3}_{S_2}(V_2)) = 2s_{(2)} + s_{(1,1)}$, which agrees with the Frobenius character of Park_2.

While the set of polynomials $\{p(G) \,:\, G \text{ is a slim subgraph of } K_{n+1}\}$ is linearly dependent in general, a basis for V_n can be constructed using standard matroid theoretic results. Fix a total order on the edge set of K_{n+1}. Given a spanning tree T of K_{n+1}, the *external activity* $\mathrm{ex}(T)$ of T is the set of edges $e \in K_{n+1}$ such that e is the minimal edge of the unique cycle in $T \cup \{e\}$. A basis of V_n is given by $\{p(K_{n+1} - (T \cup \mathrm{ex}(T))) \,:\, T \text{ is a spanning tree of } K_{n+1}\}$. It follows immediately from Cayley's theorem that $\dim V_n = (n+1)^{n-1}$.

Since the slimness of a subgraph is preserved under the action of S_{n+1} on the vertex set $[n+1]$ and $p(G)$ is homogeneous of degree equal to the number of edges in G, it follows that $V_n = \bigoplus_{k \geq 0} V_n(k)$ is a graded S_{n+1}-submodule of the polynomial ring $\mathbb{C}[x_1, \ldots, x_{n+1}]$. In fact, the space V_n sits inside the copy of the coordinate ring of the reflection representation of type A_n sitting inside $\mathbb{C}[x_1, \ldots, x_{n+1}]$ generated by $x_i - x_{i+1}$ for $1 \leq i \leq n$.

The following result was conjectured by the first author. We postpone its proof, along with the proofs of the other results in this section, to Section 3.

Theorem 2 *Embed S_n into S_{n+1} by letting S_n act on the first n letters. We have that*

$$\mathrm{Res}^{S_{n+1}}_{S_n}(V_n(k)) \cong_{S_n} \bigoplus_D M^{\lambda(D)}, \tag{5}$$

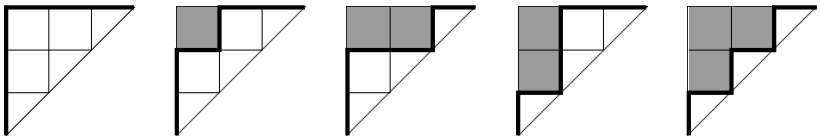

Fig. 3: The 5 Dyck paths of size 3. From left to right, their contributions to the graded Frobenius character $\mathrm{grFrob}(\mathrm{Res}^{S_4}_{S_3}(V_3); q)$ are $h_{(3)}q^0, h_{(2,1)}q^1, h_{(2,1)}q^2, h_{(2,1)}q^2$, and $h_{(1,1,1)}q^3$.

where the direct sum is over all Dyck paths of size n and area k. In particular, by Equation 2 we have that

$$\mathrm{Res}^{S_{n+1}}_{S_n}(V_n) \cong_{S_n} \mathrm{Park}_n. \tag{6}$$

Equivalently, we have that $\mathrm{grFrob}(\mathrm{Res}^{S_{n+1}}_{S_n}(V_n); q) = \sum_D q^{\mathrm{area}(D)} h_{\lambda(D)}$, where the sum is over all Dyck paths D of size n. For example, computing the area and run partitions of the 5 Dyck paths of size 3 shown in Figure 3 shows that

$$\mathrm{grFrob}(\mathrm{Res}^{S_4}_{S_3}(V_3); q) = h_{(3)}q^0 + h_{(2,1)}q^1 + 2h_{(2,1)}q^2 + h_{(1,1,1)}q^3. \tag{7}$$

Postnikov and Shapiro showed that the dimension of the vector space V_n is equal to $(n+1)^{n-1}$, however the S_n-module structure of V_n has remained unstudied. Indeed, Theorem 2 is the first description of the S_n-module structure of V_n.

It is natural to ask for an explicit description of the S_{n+1}-structure of V_n or of its graded pieces $V_n(k)$. This problem is open in general, but we can describe the extended structure of $V_n(k)$ in the extreme degrees $k = 0, 1, \ldots, n$ as well as $k = \binom{n}{2}$. Let C_{n+1} be the cyclic subgroup of S_{n+1} generated by the long cycle $c := (1, 2, \ldots, n+1)$ and let ζ be the linear representation of C_{n+1} which sends c to $e^{\frac{2\pi i}{n+1}}$. Mackey's Theorem can be used to prove that the *Lie representation* $\mathrm{Lie}_n := \mathrm{Ind}^{S_{n+1}}_{C_{n+1}}(\zeta)$ of S_{n+1} satisfies $\mathrm{Res}^{S_{n+1}}_{S_n}(\mathrm{Lie}_n) \cong_{S_n} \mathbb{C}[S_n]$. Stanley proved that the Lie representation arises as the action of S_{n+1} on the top poset cohomology of the lattice of set partitions of $[n+1]$, tensored with the sign representation [Sta82].

Proposition 3 *The module $V_n(0)$ carries the trivial representation of S_{n+1}, the module $V_n(1)$ carries the reflection representation of S_{n+1}, and in general $V_n(k) = \mathrm{Sym}^k(V_n(1))$ for $k < n$. The module $V_n(\binom{n}{2}) = V_n(\mathrm{top})$ carries the Lie representation of S_{n+1} tensor the sign representation.*

The first part of this result is optimal in the sense that if $k \geq n$ then $V_n(k)$ is a proper subspace of $\mathrm{Sym}^k(V_n(1))$.

We will prove a bivariate generalization of Theorem 2 which includes a 'Fuss generalization' as a special case. Given $\ell, m, n > 0$, define a (ℓ, m)-*Dyck path of size* n to be a lattice path D in $\mathbb{Z}^2$ consisting of vertical steps $(0, 1)$ and horizontal steps $(1, 0)$ which starts at $(-\ell + 1, 0)$, ends at (mn, n), and stays weakly above the line $y = \frac{x}{m}$. Taking $\ell = m = 1$, we recover the classical notion of a Dyck path of size n. Taking $\ell = 1$ and m general, the $(1, m)$-Dyck paths are the natural Fuss extension of Dyck paths. As before, we define the *vertical run partition* $\lambda(D) \vdash n$ of an (ℓ, m)-Dyck path D of size n to be the partition obtained by listing the lengths of the vertical runs of D in weakly decreasing order. We also define the *area partition* $\mu(D)$ to be the length n partition whose Ferrers diagram fits between D and a

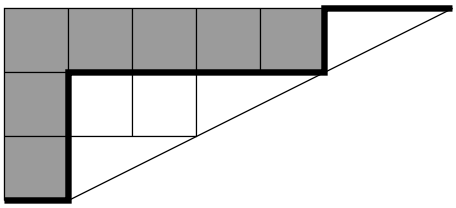

Fig. 4: A $(2,2)$-Dyck path of size 3.

$(\ell - 1 + mn) \times n)$ rectangle with lower left hand coordinate $(-\ell + 1, 0)$. The *area* of D is defined by $\mathrm{area}(D) := |\mu(D)|$. We have that $\mathrm{mult}(\mu(D)) = \lambda(D)$.

Figure 4 shows an example of a $(2,2)$-Dyck path of size 3. The path D starts at $(-1,0)$, ends at $(6,3)$, and stays above the line $y = \frac{x}{2}$. We have that $\lambda(D) = (2,1) \vdash 3$, $\mu(D) = (5,1,1)$, and $\mathrm{area}(D) = 7$.

Let $K_{n+1}^{(\ell,m)}$ be the multigraph on the vertex set $[n+1]$ with m edges between i and j for all $1 \leq i < j \leq n$ and ℓ edges between i and $n+1$ for all $1 \leq i \leq n$. We call a sub-multigraph G of $K_{n+1}^{(\ell,m)}$ *slim* if the multi-edge set difference $K_{n+1}^{(\ell,m)} - G$ is a connected multigraph on $[n+1]$. We extend the polynomial weight $p(G) \in \mathbb{C}[x_1, \ldots, x_{n+1}]$ to multigraphs G in the obvious way.

Definition 4 *Let $V_n^{(\ell,m)}$ be the $\mathbb{C}$-linear subspace of $\mathbb{C}[x_1, \ldots, x_{n+1}]$ given by the span*

$$V_n^{(\ell,m)} := \mathrm{span}\{p(G) \,:\, G \text{ is a slim sub-multigraph of } K_{n+1}^{(\ell,m)}\}. \tag{8}$$

As in the case $m = \ell = 1$, the space $V_n^{(\ell,m)}$ is stable under the action of S_n and is a graded S_n-representation with respect to the standard polynomial degree. When $\ell = m$, $V_n^{(\ell,m)}$ also caries an action of S_{n+1}. Postnikov and Shapiro showed that the dimension of $V_n^{(\ell,m)}$ is $(mn + \ell)^{n-1}$ [PS04]. Let $V_n^{(\ell,m)}(k)$ be the degree k piece of $V_n^{(\ell,m)}$.

Theorem 5 *We have that*

$$(V_n^{(\ell,m)}(k)) \cong_{S_n} \bigoplus_D M^{\lambda(D)}, \tag{9}$$

where the direct sum is over all (ℓ,m)-Dyck paths of size n and area k. For $k < n$, $V_n^{(\ell,m)}(k) = \mathrm{Sym}^k(V_n^{(\ell,m)}(1))$.

When $\ell = m$, the top degree piece of $V_n^{(\ell,m)}$ is isomorphic, as an S_{n+1}-module, to $\mathrm{Lie}_n \otimes \mathrm{sign}^{\otimes \ell}$.

While the degree 0 and 1 pieces of $V_n^{(\ell,m)}$ have S_{n+1}-structure given by the trivial representation and the reflection representation, respectively, the authors do not know of a nice expression for the extended Frobenius character in other degrees.

3 Proofs

While Theorem 5 implies Theorem 2, the proof of Theorem 5 is a straightforward extension of the proof of Theorem 2 and it will be instructive to prove Theorem 2 first.

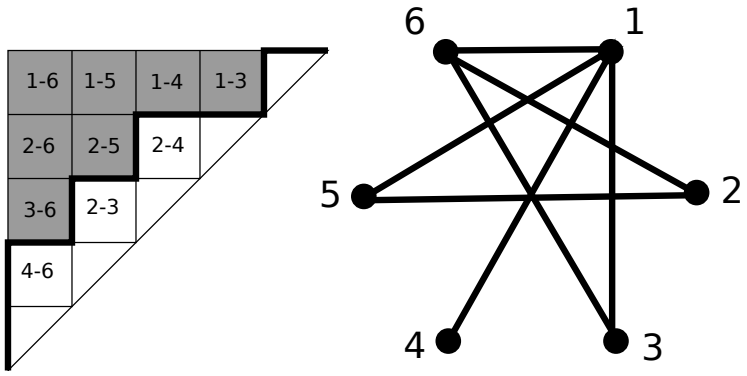

Fig. 5: A Dyck path D of size 5 and the associated subgraph $G(D)$ of K_6.

The first step in the proof of Theorem 2 is to relate the modules on both sides of the claimed isomorphism by associating a subgraph $G(D)$ of K_{n+1} and a polynomial $p(D) \in \mathbb{C}[x_1, \ldots, x_{n+1}]$ to any Dyck path D of size n. We start by labeling the 1×1 box b which is completely above the line $y = x$ with the edge $e(b) = (n - j, n - i)$ in K_{n+1}, where (i, j) is the upper left coordinate of b. See Figure 5 for an example of this labeling in the case $n = 5$. We let $G(D)$ be the subgraph of K_{n+1} consisting of those edges $e(b)$ for which the box b is to the upper left of the path D. In Figure 5, the shaded boxes above the path D each contribute an edge to the subgraph $G(D)$ and we have that $G(D) = \{1-6, 1-5, 1-4, 1-3, 2-6, 2-5, 3-6\}$.

Lemma 6 *The subgraph $G(D)$ is slim for any Dyck path D.*

Proof: The subgraph $G(D)$ contains none of the edges in the path $1 - 2 - \cdots - (n + 1)$. □

By Lemma 6, the polynomial $p(D) := p(G(D))$ is contained in V_n. For example, if $n = 5$ and D is the Dyck path shown in Figure 5, we have that

$$p(D) = (x_1 - x_6)(x_1 - x_5)(x_1 - x_4)(x_1 - x_3)(x_2 - x_6)(x_2 - x_5)(x_3 - x_6) \in V_5. \tag{10}$$

By construction, for any Dyck path D the polynomial $p(D)$ is homogeneous with degree equal to $\mathrm{area}(D)$.

In order to prove the direct sum decomposition in Theorem 2, we will show that the polynomials $p(D)$ project nicely onto a certain subspace of $\mathbb{C}[x_1, \ldots, x_{n+1}]$. Since Theorem 2 only concerns the restriction of V_n to S_n, it is natural to consider a subspace of $\mathbb{C}[x_1, \ldots, x_{n+1}]$ which is closed under the action of S_n but not of S_{n+1}.

Let $st_n := (n - 1, n - 2, \ldots, 1)$ be the *staircase partition* of length $n - 1$. We call a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ *sub-staircase* if $\lambda \subseteq st_n$ (observe that this definition has tacit dependence on n). For any Dyck path D of size n, the partition $\mu(D)$ is sub-staircase.

For a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$, we use the shorthand $x^\lambda := x_1^{\lambda_1} \cdots x_n^{\lambda_n} \in \mathbb{C}[x_1, \ldots, x_n]$. We call a monomial $x_1^{d_1} \cdots x_{n+1}^{d_{n+1}}$ in the variables $x_1, \ldots, x_{n+1}$ *sub-staircase* if there exists a permutation $w \in S_n$ and a sub-staircase partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n) \vdash n$

$$x_1^{d_1} \cdots x_{n+1}^{d_{n+1}} = w.x^\lambda. \tag{11}$$

In particular, the variable x_{n+1} does not appear in any sub-staircase monomial. If the monomial $x_1^{d_1} \cdots x_{n+1}^{d_{n+1}}$ is sub-staircase, the partition λ is uniquely determined from the monomial; call this the *exponent partition* of the monomial. Let $W_n \subset \mathbb{C}[x_1, \ldots, x_{n+1}]$ be the $\mathbb{C}$-linear span of all sub-staircase monomials. The subspace W_n is closed under the action of S_n, but not under the action of S_{n+1}.

In the case $n = 3$, the S_3-orbits of the 16 staircase monomials in $\mathbb{C}[x_1, \ldots, x_4]$ are shown in the following table, where the left column shows a representative from each orbit.

1	
x_1	x_2, x_3
x_1^2	x_2^2, x_3^2
x_1x_2	x_1x_3, x_2x_3
$x_1^2x_2$	$x_1^2x_3, x_2^2x_1, x_2^2x_3, x_3^2x_1, x_3^2x_2$

The S_3-orbits are parametrized by sub-staircase partitions $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ and each orbit contains a unique representative of the form x^λ. The staircase monomials form a linear basis of W_3 and the cyclic S_3-submodule of W_3 generated by x^λ is isomorphic to $M^{\mathrm{mult}(\lambda)}$. The natural bijection between exponent vectors and parking functions affords an isomorphism $W_3 \cong_{S_3} \mathrm{Park}_3$. These observations generalize in a straightforward way to the following lemma, whose proof is left to the reader.

Lemma 7 *The set of sub-staircase monomials forms a linear basis for W_n and is closed under the action of S_n. The S_n-orbits are parametrized by sub-staircase partitions λ, and the orbit labeled by λ has a unique monomial of the form x^λ. The cyclic S_n-submodule of W_n generated by x^λ is isomorphic to $M^{\mathrm{mult}(\lambda)}$ and we have that $W_n \cong_{S_n} \mathrm{Park}_n$.*

With Lemma 7 in mind, we will construct a graded S_n-module isomorphism $V_n \xrightarrow{\sim} W_n$. We define a graded S_n-module homomorphism $\phi : V_n \to W_n$ by the following composition:

$$\phi : V_n \hookrightarrow \mathbb{C}[x_1, \ldots, x_{n+1}] \twoheadrightarrow \mathbb{C}[x_1, \ldots, x_n] \twoheadrightarrow W_n, \tag{12}$$

where the first map is inclusion, the second is the specialization $x_{n+1} = 0$, and the third linear map fixes the space W_n pointwise and sends monomials which are not sub-staircase to zero.

We want to show that ϕ is an isomorphism. Postnikov and Shapiro showed that $\dim(W_n) = \dim(V_n) = (n+1)^{n-1}$ [PS04], so it is enough to show that ϕ is surjective. We will do this by analyzing the polynomials $\phi(p(D))$, where D is a Dyck path of size n.

The set of sub-staircase partitions forms an order ideal in dominance order. The next lemma states that the transition matrix between the set $\{\phi(p(D)) : D \text{ a Dyck path of size } n\}$ expands in the monomial basis of W_n given by $\{x^\lambda : \lambda \text{ sub-staircase}\}$ in a unitriangular way with respect to any linear extension of dominance order (where we associate $\phi(p(D))$ with the partition $\mu(D)$).

Lemma 8 *Let D be a Dyck path of size n. There exist integers $c_{\lambda,w} \in \mathbb{Z}$ such that*

$$\phi(p(D)) = x^{\mu(D)} + \sum_{\substack{\lambda \prec \mu(D) \\ |\lambda| = |\mu(D)| \\ w \in S_n}} c_{\lambda,w} w.x^\lambda. \tag{13}$$

Proof: By definition, we have that

$$p(D) = \prod_{e=(i<j)\in G(D)} (x_i - x_j), \tag{14}$$

so (up to sign) a typical monomial in the expansion of $p(D)$ is obtained by choosing an endpoint of every edge in $G(D)$ and multiplying the corresponding variables. The map ϕ kills any monomial which contains the variable x_{n+1}, so up to sign a typical monomial in $\phi(p(D))$ is obtained by choosing an endpoint of each edge in $G(D)$ and multiplying the corresponding variables such that whenever $G(D)$ has an edge of the form $(i < n+1)$, we choose the smaller endpoint i. The result follows from the construction of $G(D)$ and the definition of dominance order. □

As an example of Lemma 8, consider the case $n = 5$ and let the Dyck path D be shown in Figure 5 with $\mu(D) = (4, 2, 1)$. To calculate $\phi(p(D))$, we set $x_6 = 0$ in the product formula for $p(D)$ given in Equation 10 and expand. The resulting polynomial is

$$\phi(p(D)) = x_1(x_1 - x_5)(x_1 - x_4)(x_1 - x_3)x_2(x_2 - x_5)x_3 \tag{15}$$

$$= x_1^4 x_2^2 x_3 + \text{terms involving sub-staircase monomials with exponent partition} \prec (4, 2, 1). \tag{16}$$

We are ready to complete the proof of Theorem 2.

Proof of Theorem 2: By Lemma 7, the set of sub-staircase monomials forms a linear basis of W_n, so Lemma 8 implies that the S_n-module homomorphism $\phi : V_n \to W_n$ is surjective. Since $\dim(V_n) = \dim(W_n)$, this implies that ϕ is also injective and gives an isomorphism $\mathrm{Res}^{S_{n+1}}_{S_n}(V_n) \cong_{S_n} \mathrm{Park}_n$. To prove the graded isomorphism in Theorem 2, it is enough to observe that $\mathrm{mult}(\mu(D)) = \lambda(D)$ for any Dyck path D and apply Lemmas 7 and 8 together with the fact that ϕ is graded. □

It may be tempting to guess that $p(D)$ generates a cyclic S_n-submodule of V_n isomorphic to $M^{\lambda(D)}$, but this is false in general. The reason for this is that while the 'leading term' in the expansion of $\phi(p(D))$ in Lemma 8 generates the submodule $M^{\lambda(D)}$ under the action of S_n, the other terms in this expansion can cause $\phi(p(D))$ to generate a different cyclic submodule.

We are ready to prove the claimed S_{n+1}-structure of the extreme degrees of the graded module $V_n(k)$.

Proof of Proposition 3: It is clear from the definitions that $V_n(0)$ carries the trivial representation of S_{n+1}. The space $V_n(1)$ has basis given by the polynomials $x_1 - x_2, x_2 - x_3, \ldots, x_n - x_{n+1}$ and hence carries the reflection representation of S_{n+1} (i.e., the irreducible S_{n+1}-module corresponding to the partition $(n, 1)$). Since $V_n \subseteq \mathrm{Sym}(V_n(1))$ we are claiming that in degree $k < n$ this is an equality. The Hilbert series of V_n is the Tutte polynomial evaluation $q^{\binom{n+1}{2}-n} T_{K_{n+1}}(1, 1/q)$ and so we must prove that the first $n-1$ terms of this sum are the binomial coefficents $\binom{n+k-1}{k}$. There is nothing special about K_{n+1} in this claim and we will prove a more general statement in Lemma 9.

To prove that $V_n(\mathrm{top})$ is isomorphic to $\mathrm{Lie}_{n+1} \otimes \mathrm{sign}$ we reason as follows. The space $V_n(\mathrm{top})$ is spanned by those $p(G)$ where the complementary subgraph $K_{n+1} \setminus G$ is connected and has n edges.

Let $\mathcal{A}_n$ denote the braid arrangementin $\mathbb{C}^{n+1}$, which is the union of those hyperplanes with at least two coordinates equal. Let $H^*(\mathbb{C}^{n+1} \setminus \mathcal{A}_n; \mathbb{C})$ denote the (complexified) de Rham cohomology ring of its

complement. Consider, now, the linear map $c : V_n(\text{top}) \to H^n(\mathbb{C}^{n+1} \setminus \mathcal{A}_n)$ that sends

$$p(G) \mapsto p(G) \cdot d(x_1 - x_2) \wedge d(x_2 - x_3) \wedge \cdots \wedge d(x_n - x_{n+1}) / \prod_{1 \leq i < j \leq n} (x_i - x_j).$$

This is an isomorphism of vector spaces, since it is division by the Vandermond product, followed by multiplication by the n-form. To see that c is equivariant notice that $\bigwedge^n V_n(1)$ carries the sign representation of S_{n+1}, because it is 1 dimensional and non-trivial. Likewise does the one dimensional representation spanned by the Vadnermond product. It follows that the signs introduced by multiplication by the n-form and division by the Vandermond cancel, and c is equivariant.

Finally, the top degree cohomology of the complement $\mathbb{C}^{n+1} \setminus \mathcal{A}_n$ is known to be S_{n+1}-isomorphic to the top degreee *Whitney homology* of its lattice of flats. The lattice of flats of $\mathcal{A}_n$ is the partition lattice Π_{n+1} and by a result of Stanley [Sta82] (beautifuly recounted by Wachs in [Wac07]), the top degree Whitney homology of the partition lattice Π_{n+1} is $\text{Lie}_{n+1} \otimes \text{sign}$. □

Lemma 9 *Let G be a connected graph on v vertices with e edges. Denote the Tutte polynomial of G by $T_G(x, y)$. Then, the polynomial $q^{e-v+1}T_G(1, 1/q)$ takes the form,*

$$1 + (v-1)q + \binom{v}{2}q^2 + \binom{v+1}{3}q^3 + \cdots + \binom{(v-1)+(v-2)-1}{v-2}q^{v-2} + O(q^{v-1}).$$

Proof: We write $T_G(x, y)$ in terms of the two variable *coboundary polynomial*, $\overline{\chi}_G(\lambda, \nu)$. This is the sum

$$\overline{\chi}_G(\lambda, \nu) = \frac{1}{\lambda} \sum_{i=0}^{e} c_i(G; \lambda)\nu^i$$

where $c_i(G; \lambda)$ is the number of ways to color the vertices of G with λ colors and exactly i monochromatic edges. It is a fact that this is a polynomial in λ and ν. Now by [Whi92, Theorem 6.3.26],

$$q^{e-v+1}T_G(1, 1/q) = \frac{q^e}{(1-q)^{v-1}}\overline{\chi}_G(0, 1/q).$$

Thus, to prove the first part of the lemma we will show that $c_i(G; \lambda) = 0$ for $e - v + 1 < i < e$, and that $c_e(G; \lambda) = \lambda$. Suppose that we have colored the vertices of G and we have more than $e - v + 1$ monochromatic edges. Then the collection of monochromatic edges forms a connected subgraph of G. It follows that all vertices of G are colored the same and hence all edges of G are monochromatic. This means that $c_i(G; \lambda) = 0$ unless $i = e$. That $c_e(G; \lambda) = \lambda$ is clear. □

The proof of Theorem 5 is a straight-forward extension of the proof of Theorem 2 and is only sketched.

Proof of Theorem 5, sketch: Given any (ℓ, m)-Dyck path D of size n we associate a sub-multigraph $G(D)$ of $K_{n+1}^{(\ell,m)}$ by letting every box which contributes to $\text{area}(D)$ correspond to a single edge in the multigraph $G(D)$; the labeling which accomplishes this is shown in Figure 6 in the case $(\ell, m) = (3, 2)$ and $n = 4$. For general ℓ, m, and n, we label the boxes in the i^{th} row from the top from left to right with $(\ell+m-2)$ copies of the edge $i-(n+1)$, m copies of the edge $i-n$, m copies of the edge $i-(n-1), \ldots,$ m copies of the edge $i-(i+2)$, and $(m-1)$ copies of the edge $i-(i+1)$.

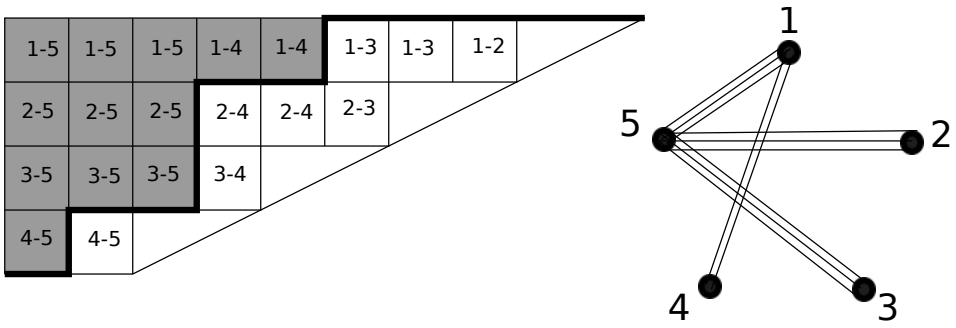

Fig. 6: A $(3,2)$-Dyck path D of size 4 and the associated sub-multigraph $G(D)$ of $K_4^{(3,2)}$.

For any (ℓ, m)-Dyck path D of size n, the multigraph complement of $G(D)$ within $K_n^{(\ell,m)}$ contains each of the edges in the path $1-2-\cdots-n-(n+1)$ with multiplicity at least one. Therefore, the sub-multigraph $G(D)$ is slim and the polynomial $p(D) := p(G(D))$ is contained in $V_n^{(\ell,m)}$.

We say that a partition λ with n parts is *sub-(ℓ, m)-staircase* if in Young's lattice we have the relation $\lambda \subseteq (\ell-1+m(n-1), \ell-1+m(n-2), \ldots, \ell-1)$. A monomial $x_1^{d_1}\cdots x_{n+1}^{d_{n+1}}$ is *sub-(ℓ, m)-staircase* if there exists $w \in S_n$ and a sub-(ℓ, m)-staircase partition λ such that

$$x_1^{d_1}\cdots x_{n+1}^{d_{n+1}} = x_{w(1)}^{\lambda_1}\cdots x_{w(n)}^{\lambda_n}. \tag{17}$$

Let $W_n^{(\ell,m)}$ be the subspace of $\mathbb{C}[x_1, \ldots, x_{n+1}]$ spanned by the set of all sub-(ℓ, m)-staircase monomials. We have that $W_n^{(\ell,m)}$ is closed under the action of S_n and the degree k homogeneous piece of $W_n^{(\ell,m)}$ is isomorphic as an S_n-module to the direct sum on the right hand side of the isomorphism asserted in Theorem 5.

The isomorphism in Theorem 5 is proven by showing that the graded S_n-module homomorphism $\phi^{(\ell,m)} : V_n^{(\ell,m)} \to W_n^{(\ell,m)}$ given by the composite

$$\phi^{(\ell,m)} : V_n^{(\ell,m)} \hookrightarrow \mathbb{C}[x_1, \ldots, x_{n+1}] \twoheadrightarrow \mathbb{C}[x_1, \ldots, x_n] \twoheadrightarrow W_n^{(\ell,m)} \tag{18}$$

is an isomorphism, where the first map is inclusion, the second is the evaluation $x_{n+1} = 0$, and the third fixes $W_n^{(\ell,m)}$ pointwise and sends every monomial which is not sub-(ℓ, m)-staircase to zero.

Postnikov and Shapiro proved that the vector space $V_n^{(\ell,m)}$ has dimension $(\ell + mn)^{n-1}$ [PS04]. A standard counting argument shows that there are $(\ell + mn)^{n-1}$ sub-(ℓ, m)-staircase monomials, so we have that $\dim(V_n^{(\ell,m)}) = \dim(W_n^{(\ell,m)})$. Therefore, to show that $\phi^{(\ell,m)}$ is a graded isomorphism of S_n-modules, it is enough to show that $\phi^{(\ell,m)}$ is surjective.

To show that $\phi^{(\ell,m)}$ is surjective, we prove a generalization of Lemma 8 which states that for any (ℓ, m)-Dyck path D of size n, the monomial expansion of $\phi^{(\ell,m)}(p(D))$ has the form

$$\phi^{(\ell,m)}(p(D)) = x^{\mu(D)} + \text{terms involving monomials whose exponent partitions are} \prec \mu(D), \tag{19}$$

where we extend the definition of $\mu(D)$ to (ℓ, m)-Dyck paths of size n in the obvious way. This triangularity result implies that $\phi^{(\ell,m)}$ is surjective, and dimension counting implies that $\phi^{(\ell,m)}$ is a graded S_n-module isomorphism. Theorem 5 follows.

The claim about $V_n^{(\ell,m)}$ in low degree follows since $V_n \subset V_n^{(\ell,m)}$. The claim about $V_n^{(\ell,\ell)}(\text{top})$ follows since this space is isomorphic to $V_n(\text{top})$, the isomorphism being division by $\prod_{i<j}(x_i - x_j)^{\ell-1}$. □

4 Concluding Remarks

In this paper we constructed a graded S_{n+1}-module V_n which satisfies $\text{Res}_{S_n}^{S_{n+1}}(V_n) \cong_{S_n} \text{Park}_n$. As we mentioned in Section 1, there does not exist an S_6-module M such that $\text{Res}_{S_4}^{S_6}(M) \cong_{S_4} \text{Park}_4$, so we cannot hope for an extension of Park_n to a symmetric group of higher rank than $n+1$ in general.

On the other hand, we identified the top degree $V_n(\text{top})$ of V_n with the Lie representation Lie_n of S_{n+1} tensor the sign representation. Whitehouse [Whi97] proved that the representation Lie_n extends to S_{n+2}. This suggests the following problem.

Problem 10 *For which values of n and k does $V_n(k)$ extend to a representation of S_{n+2}?*

By Whitehouse's result, for any $n > 0$, the k-value $k = \binom{n-1}{2}$ leads to an extension as in Problem 10. Also, since $V_n(0)$ is the trivial representation of S_{n+1}, one can take $k = 0$ and n arbitrary. On the other hand, if $k = 1$ we have that $V_n(1)$ is the reflection representation of S_{n+1}. For $n > 3$, this representation is not the restriction of any S_{n+2}-module.

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