Long time well-posdness of the Prandtl equations in Sobolev space

Chao-Jiang Xu, Xu Zhang

LONG TIME WELL-POSDNESS OF THE PRANDTL EQUATIONS IN SOBOLEV SPACE

CHAO-JIANG XU AND XU ZHANG

Abstract. In this paper, we study the long time well-posedness for the nonlinear Prandtl boundary layer equation on the half plane. While the initial data are small perturbations of some monotonic shear profile, we prove the existence, uniqueness and stability of solutions in weighted Sobolev space by energy methods. The key point is that the life span of the solution could be any large T as long as its initial date is a perturbation around the monotonic shear profile of small size like e^{-T}. The nonlinear cancellation properties of Prandtl equations under the monotonic assumption are the main ingredients to establish a new energy estimate.

Contents

1. Introduction

In this work, we study the initial-boundary value problem for the Prandtl boundary layer equation in two dimension, which reads

$$\begin{cases}\partial_t u + u\partial_x u + v\partial_y u + \partial_x p = \partial_y^2 u, \quad t>0,\, (x,y)\in\mathbb{R}^2_+,\\ \partial_x u + \partial_y v = 0,\\ u|_{y=0} = v|_{y=0} = 0,\ \lim\limits_{y\to+\infty} u = U(t,x),\\ u|_{t=0} = u_0(x,y)\,,\end{cases}$$

Date: May 8, 2016.
2010 *Mathematics Subject Classification.* 35M13, 35Q35, 76D10, 76D03, 76N20.
Key words and phrases. Prandtl boundary layer equation, energy method, well-posedness theory, monotonic condition, Sobolev space.

where $\mathbb{R}^2_+ = \{(x, y) \in \mathbb{R}^2; y > 0\}$, $u(t, x, y)$ represents the tangential velocity, $v(t, x, y)$ normal velocity. $p(t, x)$ and $U(t, x)$ are the values on the boundary of the Euler's pressure and Euler's tangential velocity and determined by the Bernoulli's law: $\partial_t U(t, x) + U(t, x)\partial_x U(t, x) + \partial_x p = 0$.

Prandtl equations is a major achievement in the progress of understanding the famous D'Alembert's paradox in fluid mechanics. In a word, D'Alembert's paradox can be stated as: while a solid body moves in an incompressible and inviscid potential flow, it undergoes neither drag or buoyancy. This of course disobeys our everyday experiences. In 1904, Prandtl said that, in fluid of small viscosity, the behavior of fluid near the boundary is completely different from that away from the boundary. Away from the boundary part can be almost considered as ideal fluid, but the near boundary part is deeply affected by the viscous force and is described by Prandtl boundary layer equation which was firstly derived formally by Prandtl in 1904 ([22]).

From the mathematical point of view, the well-posedness and justification of the Prandtl boundary layer theory don't have satisfactory theory yet, and remain open for general cases. During the past century, lots of mathematicians have investigated this problems. The Russian school has contributed a lot to the boundary layer theory and their works were collected in [21]. Up to now, the local existence theory for the Prandtl boundary layer equation has been achieved when the initial data belong to some special functional spaces: 1) the analytic space or analytic with respect to the tangential variable [15, 19, 24, 25]; 2) Sobolev spaces or Hölder spaces under monotonicity assumption [1, 17, 20, 21, 26]; 3) recently [7] in Gevrey class with non-degenerate critical point. See also [16] where the initial data is monotone on a number of intervals and analytic on the complement.

Except explaining the D'Alembert's Parabox, Prandtl equations play a vital role in the challenging problem: inviscid limit problem. In deed, as pointed out by Grenier-Guo-Nguyen [9, 10, 11], the long time behavior of the Prandtl equations is important to make progress towards the inviscid limit of the Navier-Stokes equations. We must understand behaviors of solutions to on a longer time interval than the one which causes the instability used to prove ill-posedness.

To the best of our knowledge, under the monotonic assumption, by using the Crocco transformation, Oleinik ([21]) obtained the long-time smooth solution in Hölder space for the Prandtl equation defined on the interval $0 \leq x \leq L$ with L very small. Xin-Zhang ([26]) proved the global existence of weak solutions if the pressure gradient has a favorable sign, that is $\partial_x p \leq 0$. See [18] for a similar work in 3-D case. The global existence of smooth solutions in the monotonic case remains open.

In the analytical frame, Ignatova-Vicol ([14]) recently get an almost global-in-time solution which is analytic with respect to the tangential variable, see also [27] for a same attempt work by using a refined Littlewood-Paley analysis. On the other side, without the monotonicity assumption, E and Engquist in [5] constructed finite time blowup solutions to the Prandtl equation. After this work, there are many un-stability or strong ill-posedness results. In particular, Gérard-Varet and Dormy [6] showed that the linearized Prandtl equation around the shear flow with a non-degenerate critical point is ill-posed in the sense of Hadamard in Sobolev spaces. See also [4, 8, 12, 13, 23] for the relative works.

Besides, Crocoo transformation can't be used to Navier-Stokes equations. The best choice left for us is to get the long time wellposedness by energy method, since energy method works well for both Navier-Stokes equations and Euler equations. Recently, there are two works[1, 20] where the local-in-time wellposedness is obtained by different kinds of energy methods. One is by Nash-Moser-Hörmander iteration. The other is by using uniform estimates of the regularized parabolic equation and Maximal Principle.

Motivated by above analysis, in this work, using directly energy method, we will prove the long time existence of smooth solutions of Prandtl equations in Sobolev space. In details, for any fixed $T > 0$, we will show that if the initial perturbation are size of e^{-T} small enough, then the life time of solutions to Prandtl equations could at least be T.

In what follows, we choose the uniform outflow $U(t, x) = 1$ which implies $p_x = 0$. In other words the following problem for the Prandtl equation is considered :

$$\begin{cases} \partial_t u + u\partial_x u + v\partial_y u = \partial_y^2 u, \ t > 0, \ (x, y) \in \mathbb{R}^2_+, \\ \partial_x u + \partial_y v = 0, \\ u|_{y=0} = v|_{y=0} = 0, \ \lim\limits_{y\to+\infty} u = 1, \\ u|_{t=0} = u_0(x, y). \end{cases} \tag{1.1}$$

The weighted Sobolev spaces (similar to [20]) are defined as follows:

$$\|f\|^2_{H^n_\lambda(\mathbb{R}^2_+)} = \sum_{|\alpha_1+\alpha_2|\le n} \int_{\mathbb{R}^2_+} \langle y\rangle^{2\lambda+2\alpha_2} |\partial_x^{\alpha_1}\partial_y^{\alpha_2} f|^2 dxdy\,,\ \lambda > 0,\ n \in \mathbb{N}^+.$$

Specially, $\|f\|_{L^2_\lambda(\mathbb{R}^2_+)} = \|f\|_{H^0_\lambda(\mathbb{R}^2_+)}$ and H^n stands for the usual Sobolev space.

Initial data of shear flow. Loosely speaking, shear flow is a solution to Prandtl equations and is independent of x. For more details, please check the *analysis of shear flow* part in Section 2 and Lemma 2.1. We denote shear flow as u^s. From now on, we consider solutions to Prandtl equations as their perturbations around some shear flow. That is to say,

$$u(t, x, y) = u^s(t, y) + \tilde{u}(t, x, y), t \ge 0.$$

Assume that u^s_0(initial datum of shear flow) satisfies the following conditions:

$$\begin{cases} u^s_0 \in C^{m+4}([0, +\infty[), \ \lim\limits_{y\to+\infty} u^s_0(y) = 1; \\ (\partial_y^{2p} u^s_0)(0) = 0, \ 0 \le 2p \le m + 4; \\ c_1\langle y\rangle^{-k} \le (\partial_y u^s_0)(y) \le c_2\langle y\rangle^{-k}, \ \forall\, y \ge 0, \\ |(\partial_y^p u^s_0)(y)| \le c_2\langle y\rangle^{-k-p+1}, \ \forall\ y \ge 0, \ 1 \le p \le m + 4, \end{cases} \tag{1.2}$$

for certain $c_1, c_2 > 0$ and even integer m.

We have the following long time wellposedness results.

Theorem 1.1. *Let $m \ge 6$ be an even integer, $k > 1$ and $-\frac{1}{2} < \nu < 0$. Assume that u^s_0 satisfies (1.2), the initial data $\tilde{u}_0 = (u_0 - u^s_0) \in H^{m+3}_{k+\nu}(\mathbb{R}^2_+)$, and $\tilde{u}_0$ satisfies the compatibility condition up to order $m + 2$. Then for any $T > 0$, there exists $\delta_0 > 0$ small enough such that if*

$$\|\tilde{u}_0\|_{H^{m+1}_{k+\nu}(\mathbb{R}^2_+)} \le \delta_0, \tag{1.3}$$

then the initial-boundary value problem (1.1) admits a unique solution (u, v) with

$$(u - u^s) \in L^\infty([0, T]; H^m_{k+\nu-\delta'}(\mathbb{R}^2_+)),\ v \in L^\infty([0, T]; L^\infty(\mathbb{R}_{y,+}; H^{m-1}(\mathbb{R}_x)),$$

where $\delta' > 0$ satisfying $\nu + \frac{1}{2} < \delta' < \nu + 1$ and $k + \nu - \delta' > \frac{1}{2}$.

Moreover, we have the stability with respect to the initial data in the following sense: given any two initial data

$$u_0^1 = u_0^s + \tilde{u}_0^1, \quad u_0^2 = u_0^s + \tilde{u}_0^2,$$

if u_0^s satisfies (1.2) and $\tilde{u}_0^1$, $\tilde{u}_0^2$ satisfies (1.3), then the solutions u^1 and u^2 to (1.1) satisfy,

$$\|u^1 - u^2\|_{L^\infty([0,T]; H^{m-3}_{k+\nu-\delta'}(\mathbb{R}^2_+))} \le C \|u_0^1 - u_0^2\|_{H^{m+1}_{k+\nu}(\mathbb{R}^2_+)},$$

where the constant C depends on the norm of $\partial_y u^1, \partial_y u^2$ in $L^\infty([0,T]; H^m_{k+\nu-\delta'+1}(\mathbb{R}^2_+))$.

Remark 1.2.

1. *We also can verify ,*

$$\partial_y(u - u^s) \in L^\infty([0,T]; H^m_{k+\nu-\delta'+1}(\mathbb{R}^2_+)),\ \partial_y v \in L^\infty([0,T]; H^{m-1}_{k+\nu-\delta'}(\mathbb{R}^2_+)).$$

2. *From (2.5) and (6.5), the relationship between the life span T and the size of initial data is:*

$$\delta_0 \ \approx \ e^{-T}.$$

3. *The results of main Theorem can be generated to the periodic case where x is in torus.*
4. *We find that the weight of solution $u(t) - u^s(t)$ is smaller than that of initial dates $u_0 - u_0^s$. There means that there exist decay loss of order $\delta' > 0$ which may be very small. It results from the term $v\,\partial_y u$ which is the major difficulty for the analysis of Prandtl equation.*

This article is arranged as follows. In Section 2, we explain the main difficulties for the study of the Prandtl equation and present an outline of our approach. In Section 3, we study the approximate solutions to (1.1) by a parabolic regularization. In Section 4, we prepare some technical tools and the formal transformation for the Prandtl equations. Sections 5 is dedicated to the uniform estimates of approximate solutions obtained in Section 3. We prove finally the main theorem in Section 6-7.

Notations: The letter C stands for various suitable constants, independent with functions and the special parameters, which may vary from line to line and step to step. When it depends on some crucial parameters in particular, we put a sub-index such as C_ϵ etc, which may also vary from line to line.

2. Preliminary

Difficulties and our approach. Now, we explain the main difficulties in proving Theorem 1.1, and present the strategies of our approach.

It is well-known that the major difficulty for the study of the Prandtl equation (1.1) is the term $v\,\partial_y u$, where the vertical velocity behaves like

$$v(t,x,y) = -\int_0^y \partial_x u(t,x,\tilde{y}) d\tilde{y},$$

by using the divergence free condition and boundary conditions. So it introduces a loss of x-derivative. The y-integration create also a loss of weights with respect to y-variable. Then the standard energy estimates do not work. This explains why there are few existence results in the literatures.

Recalling that in [1] (see also [20] for a similar transformation), under the monotonic assumption $\partial_y u > 0$, we divide the Prandtl equations by $\partial_y u$ and then take derivative with respect to y, to obtain an equation of the new unknown function $f = \left(\frac{u}{\partial_y u}\right)_y$. In the new equation, the term v disappears by using the divergence free condition. Here a little different from [1], we use $g_m = \left(\frac{\partial_x^m u}{\partial_y u}\right)_y$, where m stands for the highest derivative with x. From [20], we can observe that we only need to worry about the highest derivative with x. This is why we only define g_m.

In order to prove the existence of solutions, following the idea of Masmoudi-Wong ([20]), we will construct an approximate scheme and study the parabolic regularized Prandtl equation (3.1), which preserves the nonlinear structure of the original Prandlt equation (1.1), as well as the nonlinear cancellation properties. Then by uniform energy estimates of the approximate solutions, the existence of solutions to the original Prandlt equation (1.1) follows. This energy estimate also implies the uniqueness and the stability. The uniform energy estimate for the approximate solutions is the main duty of this paper.

Analysis of shear flow. We write the solution (u, v) of system (1.1) as

$$u(t, x, y) = u^s(t, y) + \tilde{u}(t, x, y),\ v(t, x, y) = \tilde{v}(t, x, y),$$

where $u^s(t, y)$ is the solution of the following heat equation

$$\begin{cases} \partial_t u^s - \partial_y^2 u^s = 0, \\ u^s|_{y=0} = 0, \lim\limits_{y\to+\infty} u^s(t, y) = 1, \\ u^s|_{t=0} = u_0^s(y). \end{cases} \tag{2.1}$$

Then (1.1) can be written as

$$\begin{cases} \partial_t \tilde{u} + (u^s + \tilde{u})\partial_x \tilde{u} + \tilde{v}(u_y^s + \partial_y \tilde{u}) = \partial_y^2 \tilde{u}, \\ \partial_x \tilde{u} + \partial_y \tilde{v} = 0, \\ \tilde{u}|_{y=0} = \tilde{v}|_{y=0} = 0,\ \lim\limits_{y\to+\infty} \tilde{u} = 0, \\ \tilde{u}|_{t=0} = \tilde{u}_0(x, y)\,. \end{cases} \tag{2.2}$$

We first study the shear flow,

Lemma 2.1. *Assume that the initial date u_0^s satisfy (1.2), then for any $T > 0$, there exist $\tilde{c}_1, \tilde{c}_2, \tilde{c}_3 > 0$ such that the solution $u^s(t, y)$ of the initial boundary value problem (2.1) satisfies*

$$\begin{cases} \tilde{c}_1 \langle y\rangle^{-k} \le \partial_y u^s(t, y) \le \tilde{c}_2 \langle y\rangle^{-k},\ \forall\, (t, y) \in [0, T] \times \bar{\mathbb{R}}_+, \\ |\partial_y^p u^s(t, y)| \le \tilde{c}_3 \langle y\rangle^{-k-p+1},\ \forall\ (t, y) \in [0, T] \times \bar{\mathbb{R}}_+,\ 1 \le p \le m + 4, \end{cases} \tag{2.3}$$

where $\tilde{c}_1, \tilde{c}_2, \tilde{c}_3$ depend on T.

Proof. Firstly, the solution of (2.1) can be written as

$$\begin{aligned} u^s(t, y) &= \frac{1}{2\sqrt{\pi t}} \int_0^{+\infty} \Big(e^{-\frac{(y-\tilde{y})^2}{4t}} - e^{-\frac{(y+\tilde{y})^2}{4t}}\Big) u_0^s(\tilde{y}) d\tilde{y} \\ &= \frac{1}{\sqrt{\pi}} \Big(\int_{-\frac{y}{2\sqrt{t}}}^{+\infty} e^{-\xi^2} u_0^s(2\sqrt{t}\xi + y) d\xi - \int_{\frac{y}{2\sqrt{t}}}^{+\infty} e^{-\xi^2} u_0^s(2\sqrt{t}\xi - y) d\xi \Big), \end{aligned}$$

which gives

$$\partial_t u^s(t, y) = \frac{1}{\sqrt{\pi t}} \Big(\int_{-\frac{y}{2\sqrt{t}}}^{+\infty} \xi\, e^{-\xi^2} (\partial_y u_0^s)(2\sqrt{t}\xi + y) d\xi$$

$$-\int_{\frac{y}{2\sqrt{t}}}^{+\infty}\xi\, e^{-\xi^2}(\partial_y u_0^s)(2\sqrt{t}\xi-y)d\xi\Big).$$

By using $(\partial_y^{2j}u_0^s)(0)=0$ for $0\le 2j\le m+4$, it follows

$$\begin{aligned}\partial_y^p u^s(t,y)=&\frac{1}{\sqrt{\pi}}\Big(\int_{-\frac{y}{2\sqrt{t}}}^{+\infty}e^{-\xi^2}(\partial_y^p u_0^s)(2\sqrt{t}\xi+y)d\xi\\&+(-1)^{p+1}\int_{\frac{y}{2\sqrt{t}}}^{+\infty}e^{-\xi^2}(\partial_y^p u_0^s)(2\sqrt{t}\xi-y)d\xi\Big)\\&=\frac{1}{2\sqrt{\pi t}}\int_0^{+\infty}\Big(e^{-\frac{(y-\tilde y)^2}{4t}}+(-1)^{p+1}e^{-\frac{(y+\tilde y)^2}{4t}}\Big)(\partial_y^p u_0^s)(\tilde y)d\tilde y,\end{aligned}\tag{2.4}$$

for all $1\le p\le m+4$.

For $p=1$, we have,

$$\begin{aligned}\partial_y u^s(t,y)=&\frac{1}{\sqrt{\pi}}\Big(\int_{-\frac{y}{2\sqrt{t}}}^{+\infty}e^{-\xi^2}(\partial_y u_0^s)(2\sqrt{t}\xi+y)d\xi\\&+\int_{\frac{y}{2\sqrt{t}}}^{+\infty}e^{-\xi^2}(\partial_y u_0^s)(2\sqrt{t}\xi-y)d\xi\Big)\\&=\frac{1}{2\sqrt{\pi t}}\int_0^{+\infty}\Big(e^{-\frac{(y-\tilde y)^2}{4t}}+e^{-\frac{(y+\tilde y)^2}{4t}}\Big)(\partial_y u_0^s)(\tilde y)d\tilde y\,.\end{aligned}$$

Thanks to the monotonic assumption (1.2), we have that

$$\begin{aligned}\partial_y u^s(t,y)&\approx\frac{1}{2\sqrt{\pi t}}\int_0^{+\infty}\Big(e^{-\frac{(y-\tilde y)^2}{4t}}+e^{-\frac{(y+\tilde y)^2}{4t}}\Big)\langle\tilde y\rangle^{-k}d\tilde y\\&\approx\frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{+\infty}e^{-\frac{(y+\tilde y)^2}{4t}}\langle\tilde y\rangle^{-k}d\tilde y\,.\end{aligned}$$

Recalling now Peetre's inequality, for any $\lambda\in\mathbb{R}$

$$\tilde c_0\langle y\rangle^{\lambda}\langle y+\tilde y\rangle^{-|\lambda|}\le\langle\tilde y\rangle^{\lambda}\le\tilde c_0^{-1}\langle y\rangle^{\lambda}\langle y+\tilde y\rangle^{|\lambda|},$$

then for $\lambda=-k$, we get the first estimate of (2.3) with

$$\tilde c_1=c_1\tilde c_0(1+T)^{-\frac{k}{2}},\ \ \tilde c_2=c_2\tilde c_0^{-1}(1+T)^{\frac{k}{2}}.\tag{2.5}$$

For the second estimate of (2.3), (2.4) implies

$$\begin{aligned}|\partial_y^p u^s(t,y)|&\le\frac{c_2}{2\sqrt{\pi t}}\int_0^{+\infty}\Big(e^{-\frac{(y-\tilde y)^2}{4t}}+e^{-\frac{(y+\tilde y)^2}{4t}}\Big)\langle\tilde y\rangle^{-k-p+1}d\tilde y\\&\le\frac{c_2}{2\sqrt{\pi t}}\int_{-\infty}^{+\infty}e^{-\frac{(y+\tilde y)^2}{4t}}\langle\tilde y\rangle^{-k-p+1}d\tilde y\,.\end{aligned}$$

Using now Peetre's inequality, with $\lambda=-k-p+1$, we get

$$|\partial_y^p u^s(t,y)|\le c_2\tilde c_0^{-1}(1+T)^{\frac{k+p-1}{2}}\langle y\rangle^{-k-p+1},$$

for any $(t,y)\in[0,T]\times\mathbb{R}_+$. □

Compatibility conditions and reduction of boundary data. We give now the precise version of the compatibility condition for the nonlinear system (2.2) and the reduction properties of boundary data.

Proposition 2.2. *Let $m \geq 6$ be an even integer, and assume that $\tilde{u}$ is a smooth solution of the system (2.2), then the initial data $\tilde{u}_0$ have to satisfy the following compatibility conditions up to order $m+2$:*

$$\begin{cases} \tilde{u}_0(x,0) = 0, \quad (\partial_y^2 \tilde{u}_0)(x,0) = 0, \ \forall x \in \mathbb{R}, \\ (\partial_y^4 \tilde{u}_0)(x,0) = \big(\partial_y u_0^s(0) + (\partial_y \tilde{u}_0)(x,0)\big)(\partial_y \partial_x \tilde{u}_0)(x,0), \forall x \in \mathbb{R}, \end{cases} \tag{2.6}$$

and for $4 \leq 2p \leq m$,

$$(\partial_y^{2(p+1)} \tilde{u}_0)(x,0) = \sum_{q=2}^{p} \sum_{(\alpha,\beta)\in\Lambda_q} C_{\alpha,\beta} \prod_{j=1}^{q} \partial_x^{\alpha_j} \partial_y^{\beta_j+1} \big(u_0^s + \tilde{u}_0\big)\big|_{y=0}, \ \ \forall x \in \mathbb{R}, \tag{2.7}$$

where

$$\begin{aligned} \Lambda_q = \Big\{ &(\alpha,\beta) = (\alpha_1, \cdots, \alpha_q; \beta_1, \cdots, \beta_q) \in \mathbb{N}^q \times \mathbb{N}^q; \\ &\alpha_j + \beta_j \leq 2p-1, \ \ 1 \leq j \leq q; \ \sum_{j=1}^{q} 3\alpha_j + \beta_j = 2p+1; \\ &\sum_{j=1}^{q} \beta_j \leq 2p-2, \ \ 0 < \sum_{j=1}^{q} \alpha_j \leq p-1 \Big\}. \end{aligned} \tag{2.8}$$

Remark that for $\alpha_j > 0$, we have $\partial_x^{\alpha_j} \partial_y^{\beta_j+1}\big(u^s + \tilde{u}\big) = \partial_x^{\alpha_j} \partial_y^{\beta_j+1} \tilde{u}$. So the condition $0 < \sum\limits_{j=1}^{q} \alpha_j$ implies that, for each terms of (2.7), there is at last one factor like $\partial_x^{\alpha_j} \partial_y^{\beta_j+1} \tilde{u}_0$.

Proof. By the assumption of this Proposition, $\tilde{u}$ is a smooth solution. If we need the existence of the trace of $\partial_y^{m+2} \tilde{u}$ on $y=0$, then we at least need to assume that $\tilde{u} \in L^\infty([0,T]; H^{m+3}_{k+\ell-1}(\mathbb{R}^2_+))$.

Recalling the boundary condition in (2.2):

$$\tilde{u}(t,x,0) = 0, \quad \tilde{v}(t,x,0) = 0, \quad (t,x) \in [0,T] \times \mathbb{R},$$

then the following is obvious:

$$(\partial_t \partial_x^n \tilde{u})(t,x,0) = 0, \quad (\partial_t \partial_x^n \tilde{v})(t,x,0) = 0, \quad (t,x) \in [0,T] \times \mathbb{R}, \ 0 \leq n \leq m.$$

Thus the first result of (2.6) is exactly the compatibility of the solution with the initial data at $t = 0$. For the second result of (2.6), using the equation of (2.2), we find that, fro $0 \leq n \leq m$

$$(\partial_y^2 \partial_x^n \tilde{u})(t,x,0) = 0, \quad (\partial_t \partial_y^2 \partial_x^n \tilde{u})(t,x,0) = 0, \quad (t,x) \in [0,T] \times \mathbb{R}.$$

Derivating the equation of (2.2) with y,

$$\partial_t \partial_y \tilde{u} + \partial_y\big((u^s + \tilde{u})\partial_x \tilde{u}\big) + \partial_y\big(\tilde{v}(u_y^s + \partial_y \tilde{u})\big) = \partial_y^3 \tilde{u},$$

observing

$$\left(\partial_y\big((u^s + \tilde{u})\partial_x \tilde{u}\big) + \partial_y\Big(\tilde{v}(u_y^s + \partial_y \tilde{u})\Big)\right)\Big|_{y=0} = 0,$$

then we get

$$(\partial_t \partial_y \tilde{u}))|_{y=0} = (\partial_y^3 \tilde{u}_\epsilon)|_{y=0}.$$

Derivating again the equation of (2.2) with y,

$$\partial_t\partial_y^2\tilde{u} + \partial_y^2\Big((u^s+\tilde{u})\partial_x\tilde{u}\Big) + \partial_y^2\Big(\tilde{v}(u^s_y+\partial_y\tilde{u})\Big) = \partial_y^4\tilde{u},$$

using Leibniz formula

$$\begin{aligned}&\partial_y^2\Big((u^s+\tilde{u})\partial_x\tilde{u}\Big) + \partial_y^2\Big(\tilde{v}(u^s_y+\partial_y\tilde{u})\Big)\\ &= (\partial_y^2(u^s+\tilde{u}))\partial_x\tilde{u} + (\partial_y^2\tilde{v})(u^s_y+\partial_y\tilde{u})\\ &\quad + (u^s+\tilde{u})\partial_y^2\partial_x\tilde{u} + \tilde{v}\partial_y^2(u^s_y+\partial_y\tilde{u})\\ &\qquad + 2(\partial_y(u^s+\tilde{u}))\partial_y\partial_x\tilde{u} + 2(\partial_y\tilde{v})\partial_y(u^s_y+\partial_y\tilde{u}),\end{aligned}$$

thus,

$$(\partial_y^4\tilde{u})(t,x,0) = \Big(u^s_y(t,0) + (\partial_y\tilde{u})(t,x,0)\Big)(\partial_y\partial_x\tilde{u})(t,x,0),$$

and

$$\begin{aligned}(\partial_t\partial_y^4\tilde{u})(t,x,0) =&\Big(\partial_y u^s(t,0) + (\partial_y\tilde{u})(t,x,0)\Big)\Big((\partial_y^3\partial_x\tilde{u})(t,x,0)\Big)\\ &+\Big(\partial_y^3 u^s(t,0) + (\partial_y^3\tilde{u})(t,x,0)\Big)\Big((\partial_y\partial_x\tilde{u})(t,x,0)\Big).\end{aligned}\tag{2.9}$$

For $p=2$, we have

$$\partial_t\partial_y^4\tilde{u} + \partial_y^4\Big((u^s+\tilde{u})\partial_x\tilde{u}\Big) + \partial_y^4\Big(\tilde{v}(u^s_y+\partial_y\tilde{u})\Big) = \partial_y^6\tilde{u},$$

using Leibniz formula

$$\begin{aligned}&\partial_y^4\Big((u^s+\tilde{u})\partial_x\tilde{u}\Big) + \partial_y^4\Big(\tilde{v}(u^s_y+\partial_y\tilde{u})\Big)\\ &= (\partial_y^4(u^s+\tilde{u}))\partial_x\tilde{u} + (\partial_y^4\tilde{v})(u^s_y+\partial_y\tilde{u}) + (u^s+\tilde{u})\partial_y^4\partial_x\tilde{u} + \tilde{v}\partial_y^4(u^s_y+\partial_y\tilde{u})\\ &\quad + \sum_{1\le j\le 3} C_j^4\Big((\partial_y^j(u^s+\tilde{u}))\partial_y^{4-j}\partial_x\tilde{u} + (\partial_y^j\tilde{v})\partial_y^{4-j}(u^s_y+\partial_y\tilde{u})\Big),\end{aligned}$$

thus, by (2.9)

$$\begin{aligned}(\partial_y^6\tilde{u})(t,x,0) &= (\partial_t\partial_y^4\tilde{u})(t,x,0) - (\partial_y^3\partial_x u)(u^s_y+\partial_y\tilde{u})(t,x,0)\\ &\quad + \sum_{1\le j\le 3} C_j^4\Big((\partial_y^j(u^s+\tilde{u}))\partial_y^{4-j}\partial_x\tilde{u} + (\partial_y^j\tilde{v})\partial_y^{4-j}(u^s_y+\partial_y\tilde{u})\Big)(t,x,0)\\ &= \quad\Big(\partial_y^3 u^s(t,0) + (\partial_y^3\tilde{u})(t,x,0)\Big)\Big((\partial_y\partial_x\tilde{u})(t,x,0)\Big)\\ &\quad + \sum_{1\le j\le 3} C_j^4\Big((\partial_y^j(u^s+\tilde{u}))\partial_y^{4-j}\partial_x\tilde{u} - (\partial_y^{j-1}\partial_x\tilde{u})\partial_y^{4-j}(u^s_y+\partial_y\tilde{u})\Big)(t,x,0).\end{aligned}\tag{2.10}$$

Taking the values at $t=0$, we have proven (2.7) for $p=2$. The case of $p\ge 3$ is then by induction. $\square$

Remark 2.3. *By the similar methods, we can prove that if $\tilde{u}$ is a smooth solution of the system (2.2), then we have*

$$\begin{cases} \tilde{u}(t,x,0)=0,\ (\partial_y^2\tilde{u})(t,x,0)=0,\ \forall(t,x)\in[0,T]\times\mathbb{R}, \\ (\partial_y^4\tilde{u})(t,x,0)=\big(u_y^s(t,0)+(\partial_y\tilde{u})(t,x,0)\big)(\partial_y\partial_x\tilde{u})(t,x,0), \forall(t,x)\in[0,T]\times\mathbb{R}, \end{cases}$$

and for $4\le 2p\le m$,

$$(\partial_y^{2(p+1)}\tilde{u})(t,x,0)=\sum_{q=2}^{p}\sum_{(\alpha,\beta)\in\Lambda_q}C_{\alpha,\beta}\prod_{j=1}^{q}\partial_x^{\alpha_j}\partial_y^{\beta_j+1}\Big(u^s(t,0)+\tilde{u}(t,x,0)\Big), \tag{2.11}$$

for all $(t,x)\in[0,T]\times\mathbb{R}$, where Λ_q is defined in (2.8).

See Lemma 5.9 of [20] and Lemma 4 of [7] for the similar results.

Remark that the condition $0<\sum_{j=1}^{q}\alpha_j$ implies that, for each terms of (2.11), there is at last one factor like $\partial_x^{\alpha_j}\partial_y^{\beta_j+1}\tilde{u}(t,x,0)$.

3. The approximate solutions

To prove the existence of solution of the Prandtl equation, we study a parabolic regularized equation for which we can get the existence by using the classical energy method.

Nonlinear regularized Prandtl equation. We study the following nonlinear regularized Prandtl equation, for $0<\epsilon\le 1$,

$$\begin{cases} \partial_t\tilde{u}_\epsilon+(u^s+\tilde{u}_\epsilon)\partial_x\tilde{u}_\epsilon+v_\epsilon(u_y^s+\partial_y\tilde{u}_\epsilon)=\partial_y^2\tilde{u}_\epsilon+\epsilon\partial_x^2\tilde{u}_\epsilon, \\ \partial_x\tilde{u}_\epsilon+\partial_y v_\epsilon=0, \\ \tilde{u}_\epsilon|_{y=0}=v_\epsilon|_{y=0}=0,\ \lim\limits_{y\to+\infty}\tilde{u}_\epsilon=0, \\ \tilde{u}_\epsilon|_{t=0}=\tilde{u}_{0,\epsilon}=\tilde{u}_0+\epsilon\mu_\epsilon\,, \end{cases} \tag{3.1}$$

where we choose the corrector $\epsilon\mu_\epsilon$ such that $\tilde{u}_0+\epsilon\mu_\epsilon$ satisfies the compatibility condition up to order $m+2$ for the regularized system (3.1).

We study now the boundary data of the solution for the regularized nonlinear system (3.1) which give also the precise version of the compatibility condition for the system (3.1), see [2, 3] for the Prandtl equation with non-compatible data.

Proposition 3.1. *Let $m\ge 6$ be an even integer $1<k, 0<\ell<\frac{1}{2}$ and $k+\ell>\frac{3}{2}$, and assume that $\tilde{u}_0$ satisfies the compatibility conditions (2.6) and (2.7) for the system (2.2), and $\mu_\epsilon\in H^{m+3}_{k+\ell'-1}(\mathbb{R}^2_+)$ for some $\frac{1}{2}<\ell'<\ell+\frac{1}{2}$ such that $\tilde{u}_0+\epsilon\mu_\epsilon$ satisfies the compatibility conditions up to order $m+2$ for the regularized system (3.1). If $\tilde{u}_\epsilon\in L^\infty([0,T];H^{m+3}_{k+\ell}(\mathbb{R}^2_+))\cap Lip([0,T];H^{m+1}_{k+\ell}(\mathbb{R}^2_+))$ is a solution of the system (3.1), then we have*

$$\begin{cases} \tilde{u}_\epsilon(t,x,0)=0,\ (\partial_y^2\tilde{u}_\epsilon)(t,x,0)=0,\ \forall(t,x)\in[0,T]\times\mathbb{R}, \\ (\partial_y^4\tilde{u}_\epsilon)(t,x,0)=\big(u_y^s(t,0)+(\partial_y\tilde{u}_\epsilon)(t,x,0)\big)(\partial_y\partial_x\tilde{u}_\epsilon)(t,x,0), \forall(t,x)\in[0,T]\times\mathbb{R}, \end{cases}$$

and for $4 \le 2p \le m$,

$$
\begin{aligned}
(\partial_y^{2(p+1)}\tilde{u}_\epsilon)(t,x,0) = \sum_{q=2}^{p}\sum_{l=0}^{q-1}\epsilon^l \sum_{(\alpha^l,\beta^l)\in\Lambda_q^l} & C_{\alpha^l,\beta^l} \\
& \times \prod_{j=1}^{q}\partial_x^{\alpha_j^l}\partial_y^{\beta_j^l+1}\big(u^s(t,0)+\tilde{u}_\epsilon(t,x,0)\big),
\end{aligned} \tag{3.2}
$$

for all $(t,x)\in[0,T]\times\mathbb{R}$, where

$$
\begin{aligned}
\Lambda_q^l = \Big\{ & (\alpha,\beta) = (\alpha_1,\cdots,\alpha_p;\beta_1,\cdots,\beta_p)\in\mathbb{N}^q\times\mathbb{N}^q; \\
& \alpha_j+\beta_j \le 2p-1, ,\ 1\le j\le q;\ \sum_{j=1}^{q} 3\alpha_j+\beta_j = 2p+4l+1; \\
& \sum_{j=1}^{q}\beta_j \le 2p-2l-2,\ 0<\sum_{j=1}^{q}\alpha_j \le p+2l-1 \Big\}.
\end{aligned}
$$

Remark 3.2. .

1. *Remark that the condition $0<\sum_{j=1}^{q}\alpha_j^l$ implies that, for each terms of (3.2), there are at last one factor like $\partial_x^{\alpha_j^l}\partial_y^{\beta_j^l+1}\tilde{u}_\epsilon(t,x,0)$.*
2. *Here we change the notation for the wighted index of function space, in fact, using the notations of Theorem 1.1, we have*

$$\ell = \nu-\delta'+1, \quad \ell' = \nu+1.$$

Proof. Firstly, for $p\le\frac{m}{2}$, we have $\partial_y^{2p+2}\tilde{u}_\epsilon \in L^\infty([0,T];H^1_{k+\ell+2p+1}(\mathbb{R}^2_+))$. So the trace of $\partial_y^{2p+2}\tilde{u}_\epsilon$ exists on $y=0$.

Using the boundary condition of (3.1), we have, for $0\le n\le m+2$,

$$\partial_x^n\tilde{u}_\epsilon(t,x,0)=0, \quad \partial_x^n v_\epsilon(t,x,0)=0, \quad (t,x)\in[0,T]\times\mathbb{R},$$

and for $0\le n\le m$

$$(\partial_t\partial_x^n\tilde{u}_\epsilon)(t,x,0)=0, \quad (\partial_t\partial_x^n v_\epsilon)(t,x,0)=0, \quad (t,x)\in[0,T]\times\mathbb{R}.$$

From the equation of (3.1), we get also

$$(\partial_y^2\partial_x^n\tilde{u}_\epsilon)(t,x,0)=0, \quad (\partial_t\partial_y^2\partial_x^n\tilde{u}_\epsilon)(t,x,0)=0, \quad (t,x)\in[0,T]\times\mathbb{R}. \tag{3.3}$$

On the other hand,

$$\partial_t\partial_y\tilde{u}_\epsilon+\partial_y\big((u^s+\tilde{u}_\epsilon)\partial_x\tilde{u}_\epsilon\big)+\partial_y\big(v_\epsilon(u^s_y+\partial_y\tilde{u}_\epsilon)\big)=\partial_y^3\tilde{u}_\epsilon+\epsilon\partial_x^2\partial_y\tilde{u}_\epsilon,$$

observing

$$\big[\partial_y\big((u^s+\tilde{u}_\epsilon)\partial_x\tilde{u}_\epsilon\big)+\partial_y\Big(v_\epsilon(u^s_y+\partial_y\tilde{u}_\epsilon)\big)\big]\big|_{y=0}=0,$$

we get

$$(\partial_t\partial_y\tilde{u}_\epsilon)|_{y=0}=(\partial_y^3\tilde{u}_\epsilon)|_{y=0}+\epsilon(\partial_x^2\partial_y\tilde{u}_\epsilon)|_{y=0}.$$

We have also

$$\partial_t\partial_y^2\tilde{u}_\epsilon+\partial_y^2\big((u^s+\tilde{u}_\epsilon)\partial_x\tilde{u}_\epsilon\big)+\partial_y^2\big(v_\epsilon(u^s_y+\partial_y\tilde{u}_\epsilon)\big)=\partial_y^4\tilde{u}_\epsilon+\epsilon\partial_x^2\partial_y^2\tilde{u}_\epsilon,$$

using Leibniz formula

$$
\begin{aligned}
&\partial_y^2\big((u^s+\tilde u_\epsilon)\partial_x\tilde u_\epsilon\big)+\partial_y^2\big(v_\epsilon(u^s_y+\partial_y\tilde u_\epsilon)\big)\\
&=(\partial_y^2(u^s+\tilde u_\epsilon))\partial_x\tilde u_\epsilon+(\partial_y^2 v_\epsilon)(u^s_y+\partial_y\tilde u_\epsilon)\\
&\quad+(u^s+\tilde u_\epsilon)\partial_y^2\partial_x\tilde u_\epsilon+v_\epsilon\partial_y^2(u^s_y+\partial_y\tilde u_\epsilon)\\
&\qquad+2(\partial_y(u^s+\tilde u_\epsilon))\partial_y\partial_x\tilde u_\epsilon+2(\partial_y v_\epsilon)\partial_y(u^s_y+\partial_y\tilde u_\epsilon),
\end{aligned}
$$

thus,

$$
(\partial_y^4\tilde u_\epsilon)(t,x,0)=\big(u^s_y(t,0)+(\partial_y\tilde u_\epsilon)(t,x,0)\big)\,(\partial_y\partial_x\tilde u_\epsilon)(t,x,0). \tag{3.4}
$$

Applying ∂_t to (3.4), we have

$$
\begin{aligned}
&(\partial_t\partial_y^4\tilde u_\epsilon)(t,x,0)=\big(\partial_y^3 u^s(t,0)+(\partial_y^3\tilde u_\epsilon)(t,x,0)+\epsilon(\partial_x^2\partial_y\tilde u_\epsilon)(t,x,0)\big)\,(\partial_y\partial_x\tilde u_\epsilon)(t,x,0)\\
&+\big(u^s_y(t,0)+(\partial_y\tilde u_\epsilon)(t,x,0)\big)\,\big((\partial_y^3\partial_x\tilde u_\epsilon)(t,x,0)+\epsilon(\partial_x^3\partial_y\tilde u_\epsilon)(t,x,0)\big)\,.
\end{aligned}
$$

On the other hand, we have

$$
\partial_t\partial_y^4\tilde u_\epsilon+\partial_y^4\big((u^s+\tilde u_\epsilon)\partial_x\tilde u_\epsilon\big)+\partial_y^4\big(v_\epsilon(u^s_y+\partial_y\tilde u_\epsilon)\big)=\partial_y^6\tilde u_\epsilon+\epsilon\partial_x^2\partial_y^4\tilde u_\epsilon,
$$

using Leibniz formula

$$
\begin{aligned}
&\partial_y^4\big((u^s+\tilde u_\epsilon)\partial_x\tilde u_\epsilon\big)+\partial_y^4\big(v_\epsilon(u^s_y+\partial_y\tilde u_\epsilon)\big)\\
&=(\partial_y^4(u^s+\tilde u_\epsilon))\partial_x\tilde u_\epsilon+(\partial_y^4 v_\epsilon)(u^s_y+\partial_y\tilde u_\epsilon)\\
&\quad+(u^s+\tilde u_\epsilon)\partial_y^4\partial_x\tilde u_\epsilon+v_\epsilon\partial_y^4(u^s_y+\partial_y\tilde u_\epsilon)\\
&\qquad+\sum_{1\le j\le 3}C^4_j\big((\partial_y^j(u^s+\tilde u_\epsilon))\partial_y^{4-j}\partial_x\tilde u_\epsilon+(\partial_y^j v_\epsilon)\partial_y^{4-j}(u^s_y+\partial_y\tilde u_\epsilon)\big),
\end{aligned}
$$

thus,

$$
\begin{aligned}
(\partial_y^6\tilde u_\epsilon)(t,x,0)&=(\partial_t\partial_y^4\tilde u_\epsilon)(t,x,0)-(\partial_y^3\partial_x u_\epsilon)(u^s_y+\partial_y\tilde u_\epsilon)(t,x,0)\\
&\quad+\sum_{1\le j\le 3}C^4_j\big[(\partial_y^j(u^s+\tilde u_\epsilon))\partial_y^{4-j}\partial_x\tilde u_\epsilon+(\partial_y^j v_\epsilon)\partial_y^{4-j}(u^s_y+\partial_y\tilde u_\epsilon)\big](t,x,0)\\
&\qquad-\underline{\epsilon\partial_x^2\partial_y^4\tilde u_\epsilon(t,x,0)}.
\end{aligned}
$$

Using (3.4), we get then

$$
\begin{aligned}
(\partial_y^6\tilde u_\epsilon)(t,x,0)&=\big(\partial_y^3u^s(t,0)+\partial_y^3\tilde u_\epsilon(t,x,0)\big)\partial_y\partial_x\tilde u_\epsilon(t,x,0)\\
&\qquad-\underline{2\epsilon\partial_x\partial_y\tilde u_\epsilon(t,x,0)(\partial_y\partial_x^2\tilde u_\epsilon)(t,x,0)}\\
&+\sum_{1\le j\le 3}C^4_j\big[(\partial_y^j(u^s+\tilde u_\epsilon))\partial_y^{4-j}\partial_x\tilde u_\epsilon-\partial_y^{j-1}\partial_x\tilde u_\epsilon\partial_y^{4-j}(u^s_y+\partial_y\tilde u_\epsilon)\big](t,x,0),
\end{aligned} \tag{3.5}
$$

Compared to (2.10), the underlined term is the new term.

This is the Proposition 3.1 for $p=2$. We can complete the proof of Proposition 3.1 by induction. □

The proof of the above Proposition implies also the following result.

Corollary 3.3. *Let $m\ge 6$ be an even integer, assume that $\tilde u_0$ satisfies the compatibility conditions (2.6) - (2.7) for the system (2.2) and $\partial_y\tilde u_0\in H^{m+2}_{k+\ell'}(\mathbb{R}^2_+)$, then there exists $\epsilon_0>0$, and for any $0<\epsilon\le\epsilon_0$ there exists $\mu_\epsilon\in H^{m+3}_{k+\ell'-1}(\mathbb{R}^2_+)$ such that*

$\tilde u_0+\epsilon\mu_\epsilon$ satisfies the compatibility condition up to order $m+2$ for the regularized system (3.1). Moreover, for any $m\le\tilde m\le m+2$

$$\|\partial_y\tilde u_{0,\epsilon}\|_{H^{\tilde m}_{k+\ell'}(\mathbb{R}^2_+)}\le\frac{3}{2}\|\partial_y\tilde u_0\|_{H^{\tilde m}_{k+\ell'}(\mathbb{R}^2_+)},$$

and

$$\lim_{\epsilon\to0}\|\partial_y\tilde u_{0,\epsilon}-\partial_y\tilde u_0\|_{H^{\tilde m}_{k+\ell'}(\mathbb{R}^2_+)}=0.$$

Proof. We use the proof of the Proposition 3.1.

Taking the values at $t=0$ for (3.3), then (2.6) implies that the function μ_ϵ satisfies

$$(\partial_x^n\mu_\epsilon)(x,0)=0,\quad(\partial_y^2\partial_x^n\mu_\epsilon)(x,0)=0,\quad x\in\mathbb{R}.$$

Taking $t=0$ for (3.4), we have

$$\begin{aligned}(\partial_y^4\tilde u_0)(x,0)+\epsilon(\partial_y^4\mu_\epsilon)(x,0))=&\big[\partial_yu_0^s(0)+(\partial_y\tilde u_0)(x,0)+\epsilon(\partial_y\mu_\epsilon)(x,0)\big]\\&\times\big[(\partial_y\partial_x\tilde u_0)(x,0)+\epsilon(\partial_y\partial_x\mu_\epsilon)(x,0)\big],\end{aligned}$$

using (2.6), we have that μ_ϵ satisfies

$$\begin{aligned}(\partial_y^4\mu_\epsilon)(x,0))=&\big(\partial_yu_0^s(0)+(\partial_y\tilde u_0)(x,0)\big)(\partial_y\partial_x\mu_\epsilon)(x,0)\\&+(\partial_y\mu_\epsilon)(x,0)(\partial_y\partial_x\tilde u_0)(x,0)\\&+\epsilon(\partial_y\partial_x\mu_\epsilon)(x,0)(\partial_y\partial_x\mu_\epsilon)(x,0).\end{aligned}$$

We have also

$$\begin{aligned}(\partial_t\partial_y^4\tilde u_\epsilon)(0,x,0)=&\big(\partial_y^3u_0^s(0)+(\partial_y^3\tilde u_\epsilon)(0,x,0)+\epsilon(\partial_x^2\partial_y\tilde u_\epsilon)(0,x,0)\big)\\&\times\big((\partial_y^3\partial_x\tilde u_\epsilon)(0,x,0)+\epsilon(\partial_x^3\partial_y\tilde u_\epsilon)(0,x,0)\big).\end{aligned}$$

Taking the values at $t=0$ for (3.5), we obtain a restraint condition for $(\partial_y^6\mu_\epsilon)(x,0)$,

$$\begin{aligned}\partial_y^6\mu_\epsilon(x,0)=&((\partial_y^3u_0^s+\partial_y^3\tilde u_0)\partial_y\partial_x\mu_\epsilon)|_{y=0}+\partial_y^3\mu_\epsilon\partial_y\partial_x\tilde u_0|_{y=0}+\epsilon\partial_y^3\mu_\epsilon\partial_y\partial_x\mu_\epsilon|_{y=0}\\&-\underline{2\partial_x\partial_y\tilde u_0(x,0)(\partial_y\partial_x^2\tilde u_0)(x,0)}-2\epsilon\partial_x\partial_y\tilde u_0(x,0)(\partial_y\partial_x^2\mu_\epsilon)(t,x,0)\\&-2\epsilon\partial_x\partial_y\mu_\epsilon(t,x,0)(\partial_y\partial_x^2\tilde u_0)(t,x,0)-2\epsilon^2\partial_x\partial_y\mu_\epsilon(x,0)(\partial_y\partial_x^2\mu_\epsilon)(x,0)\\&+\sum_{1\le j\le3}C_j^4\big[\partial_y^j\big(u_0^s+\tilde u_0\big)\partial_y^{4-j}\partial_x\mu_\epsilon+\partial_y^j\mu\partial_y^{4-j}\partial_x\tilde u_0+\epsilon\partial_y^j\mu\partial_y^{4-j}\partial_x\mu_\epsilon\big]\big|_{y=0}\\&-\sum_{1\le j\le3}C_j^4\big[\partial_y^{j-1}\partial_x\tilde u_0\partial_y^{4-j}\mu_\epsilon+\epsilon\partial_y^{j-1}\partial_x\mu_\epsilon\partial_y^{4-j}\partial_y\mu_\epsilon\big]\big|_{y=0}\\&-\sum_{1\le j\le3}C_j^4\partial_y^{j-1}\partial_x\mu_\epsilon\partial_y^{4-j}(\partial_yu_0^s+\partial_y\tilde u_0)\big|_{y=0},\end{aligned}$$

thus

$$\begin{aligned}\partial_y^6\mu_\epsilon(x,0)=&-\underline{2\partial_x\partial_y\tilde u_0(x,0)(\partial_y\partial_x^2\tilde u_0)(x,0)}\\&+\sum_{\alpha_1,\beta_1;\alpha_2,\beta_2}C_{\alpha_1,\beta_1;\alpha_2,\beta_2}\partial_x^{\alpha_1}\partial_y^{\beta_1+1}(u_0^s+\tilde u_0)\partial_x^{\alpha_1}\partial_y^{\beta_1+1}\mu_\epsilon(x,0)\\&+\sum_{\alpha_1,\beta_1;\alpha_2,\beta_2}C_{\alpha_1,\beta_1;\alpha_2,\beta_2}\partial_x^{\alpha_1}\partial_y^{\beta_1+1}\mu_\epsilon\partial_x^{\alpha_1}\partial_y^{\beta_1+1}\mu_\epsilon(x,0),\end{aligned}\tag{3.6}$$

where the summation is for the index $\alpha_2+\beta_2\le3$; $\alpha_1+\beta_1+\alpha_2+\beta_2\le3$. The underlined term in the above equality is deduced from the underlined term in (3.5). All these underlined terms are from the added regularizing term $\epsilon\partial_x^2\tilde u$ in the equation (3.1). This means that the regularizing term $\epsilon\partial_x^2\tilde{u}$ has an affect on the boundary. This is why we add a corrector term.

More generally, for $6 \le 2p \le m$, we have that $(\partial_y^{2(p+1)}\mu_\epsilon)(x,0)$ is a linear combination of the terms of the form

$$\prod_{j=1}^{q_1}\left(\partial_x^{\alpha_j^1}\partial_y^{\beta_j^1+1}(u_0^s+\tilde{u}_0)\right)\Big|_{y=0}, \quad \prod_{i=1}^{q_2}\left(\partial_x^{\alpha_i^2}\partial_y^{\beta_i^2+1}\mu_\epsilon\right)\Big|_{y=0},$$

and

$$\prod_{j=1}^{q_1}\left(\partial_x^{\alpha_j^1}\partial_y^{\beta_j^1+1}(u_0^s+\tilde{u}_0)\right)\Big|_{y=0} \times \prod_{i=1}^{q_2}\left(\partial_x^{\alpha_i^2}\partial_y^{\beta_i^2+1}\mu_\epsilon\right)\Big|_{y=0},$$

where the coefficients of the combination can be depends on ϵ but with a non-negative power. We have also $\alpha_j^l + \beta_j^l + 1 \le 2p, l = 1, 2$, thus $(\partial_y^{2(p+1)}\mu_\epsilon)(x,0)$ is determined by the low order derivatives of μ_ϵ and these of $\tilde{u}_0$.

We now construct a polynomial function $\tilde{\mu}_\epsilon$ on y by the following Taylor expansion,

$$\tilde{\mu}_\epsilon(x,y) = \sum_{p=3}^{\frac{m}{2}+1}\tilde{\mu}_\epsilon^{2p}(x)\frac{y^{2p}}{(2p)!},$$

where

$$\tilde{\mu}_\epsilon^6(x) = -2(\partial_x\partial_y\tilde{u}_0)(x,0)(\partial_y\partial_x^2\tilde{u}_0)(x,0),$$

and $\tilde{\mu}_\epsilon^{2p}(x)$ will give successively by $(\partial_y^{2q}\mu_\epsilon)(x,0)$ with $(\partial_y^{2q+1}\mu_\epsilon)(x,0) = 0, q = 0, \cdots, m$, and it is then determined by $(\partial_x^\alpha\partial_y^\beta\tilde{u}_0)|_{y=0}$. Finally we take $\mu_\epsilon = \chi(y)\tilde{\mu}_\epsilon$ with $\chi \in C^\infty([0,+\infty[)$; $\chi(y) = 1$, $0 \le y \le 1$; $\chi(y) = 0$, $y \ge 2$. Thus we complete the proof of the Corollary. □

Remark 3.4. *Suppose that $\tilde{u}_0$ satisfies the compatibility conditions up to order $m+2$ for the system (2.2) with $m \ge 4$, then for the regularized system (3.1), if we want to obtain the smooth solution $\tilde{w}_\epsilon$, we have to add a non-trivial corrector μ_ϵ to the initial data such that $\tilde{u}_0 + \epsilon\mu_\epsilon$ satisfies the compatibility conditions up to order $m+2$ for the system (3.1). In fact, if we take μ_ϵ with*

$$(\partial_y^j\mu_\epsilon)(x,0) = 0, \quad 0 \le j \le 5,$$

then (3.6) implies

$$(\partial_y^6\mu_\epsilon)(x,0) = -2(\partial_x\partial_y\tilde{u}_0)(x,0)(\partial_y\partial_x^2\tilde{u}_0)(x,0),$$

which is not equal to 0. So added a corrector is necessary for the initial data of the regularized system.

We will prove the the existence of the approximate solutions of the system (3.1) by using the following equation of vorticity $\tilde{w}_\epsilon = \partial_y\tilde{u}_\epsilon$, it reads

$$\begin{cases} \partial_t\tilde{w}_\epsilon + (u^s+\tilde{u}_\epsilon)\partial_x\tilde{w}_\epsilon + v_\epsilon(u_{yy}^s + \partial_y\tilde{w}_\epsilon) = \partial_y^2\tilde{w}_\epsilon + \epsilon\partial_x^2\tilde{w}_\epsilon, \\ \partial_y\tilde{w}_\epsilon|_{y=0} = 0, \\ \tilde{w}_\epsilon|_{t=0} = \tilde{w}_{0,\epsilon} = \tilde{w}_0 + \epsilon\partial_y\mu_\epsilon, \end{cases} \tag{3.7}$$

where

$$\tilde{u}_\epsilon(t,x,y) = -\int_y^{+\infty}\tilde{w}_\epsilon(t,x,\tilde{y})d\tilde{y}, \quad \tilde{v}_\epsilon(t,x,y) = -\int_0^y \partial_x\tilde{u}_\epsilon(t,x,\tilde{y})d\tilde{y}. \tag{3.8}$$

We have the following theorem for the existence of approximate solutions

Theorem 3.5. *Let $\partial_y\tilde u_0\in H^{m+2}_{k+\ell}(\mathbb{R}^2_+)$, and $m\ge 6$ be an even integer, $k>1, 0\le \ell<\frac12, k+\ell>\frac32$, assume that $\tilde u_0$ satisfies the compatibility conditions of order $m+2$ for the system (2.2). Suppose that the shear flow satisfies*

$$|\partial_y^{p+1}u^s(t,y)|\le C\langle y\rangle^{-k-p},\quad (t,y)\in[0,T_1]\times\mathbb{R}_+,\ 0\le p\le m+2.$$

Then, for any $0<\epsilon\le\epsilon_0$ and $0<\bar\zeta$, there exits $T_\epsilon>0$ which depends on ϵ and $\bar\zeta$, such that if

$$\|\tilde w_0\|_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)}\le\bar\zeta,$$

then the system (3.7)-(3.8) admits a unique solution

$$\tilde w_\epsilon\in L^\infty([0,T_\epsilon];H^{m+2}_{k+\ell}(\mathbb{R}^2_+)),$$

which satisfies

$$\|\tilde w_\epsilon\|_{L^\infty([0,T_\epsilon];H^m_{k+\ell}(\mathbb{R}^2_+))}\le\frac43\|\tilde w_{0,\epsilon}\|_{H^m_{k+\ell}(\mathbb{R}^2_+)}\le 2\|\tilde w_0\|_{H^m_{k+\ell}(\mathbb{R}^2_+)}.\tag{3.9}$$

Remark 3.6. *.*

(1) *Remark that T_ϵ depends on ϵ and $\bar\zeta$, and $T_\epsilon\to0$ as $\epsilon\to0$. So this is not a bounded estimate for the approximate solution sequences $\{u^s+\tilde u_\epsilon; 0<\epsilon\le\epsilon_0\}$ where $\epsilon_0>0$ is given in Corollary 3.3. When the initial data $\tilde u_0$ is small enough, we observe that $u^s+\tilde u_\epsilon$ preserves the monotonicity and convexity of the shear flow on $[0,T_\epsilon]$.*

(2) *In this theorem, for the regularized Prandtl equation, there are not constrain conditions on the initial date, meaning that we don't need the monotonicity or convexcity of shear flow u^s, and $\bar\zeta$ is also arbitrary.*

If $\tilde w_\epsilon$ is a solution of the system (3.7)-(3.8), then (A.1) with $\lim_{y\to+\infty}\tilde u_\epsilon=0$ imply

$$\tilde u_\epsilon\in L^\infty([0,T_\epsilon];H^{m+2}_{k+\ell-1}(\mathbb{R}^2_+)),$$

and

$$\tilde v_\epsilon\in L^\infty([0,T_\epsilon];L^\infty(\mathbb{R}_{y,+};H^{m+1}(\mathbb{R}_x)).$$

Integrating the equation of (3.7) over $[y,+\infty[$ imply that $(\tilde u_\epsilon,\tilde v_\epsilon)$ is a solution of the system (3.1), except the boundary condition to check:

$$\tilde u_\epsilon(t,x,0)=-\int_0^{+\infty}\tilde w_\epsilon(t,x,\tilde y)d\tilde y=0,\quad (t,x)\in[0,T_\epsilon]\times\mathbb{R}.\tag{3.10}$$

In fact, noting $f(t,x)=-\int_0^{+\infty}\tilde w_\epsilon(t,x,\tilde y)d\tilde y=\tilde u_\epsilon(t,x,0)$, a direct calculate give

$$\begin{cases}\partial_t f+f\partial_x f=\epsilon\partial_x^2 f,\quad (t,x)\in]0,T_\epsilon]\times\mathbb{R};\\ f|_{t=0}=0,\end{cases}\tag{3.11}$$

here we use

$$\begin{aligned}\int_0^{+\infty}v_\epsilon(u^s_{yy}+\partial_y\tilde w_\epsilon)dy&=\big[v_\epsilon(u^s_y+\tilde w_\epsilon)\big]_0^{+\infty}-\int_0^\infty(\partial_y v_\epsilon)(u^s_y+\tilde w_\epsilon)dy\\&=\int_0^\infty(\partial_x u_\epsilon)\partial_y(u^s+\tilde u_\epsilon)dy\\&=\big[(\partial_x u_\epsilon)(u^s+\tilde u_\epsilon)\big]\big|_0^{+\infty}-\int_0^\infty(\partial_x w_\epsilon)(u^s+\tilde u_\epsilon)dy\\&=-f\partial_x f-\int_0^\infty(\partial_x w_\epsilon)(u^s+\tilde u_\epsilon)dy.\end{aligned}$$

Since $f \in L^\infty([0,T_\epsilon], H^{m+2}(\mathbb{R}))$, the uniqueness of solution for equation (3.11) imply that $f=0$ on $[0,T_\epsilon]\times\mathbb{R}$. (3.10) imply also

$$\tilde{u}_\epsilon(t,x,y) = -\int_y^{+\infty}\tilde{w}_\epsilon(t,x,\tilde{y})d\tilde{y} = \int_0^y \tilde{w}_\epsilon(t,x,\tilde{y})d\tilde{y}, \quad (t,x,y)\in[0,T_\epsilon]\times\mathbb{R}^2_+.$$

We will prove Theorem 3.5 by the following three Propositions, where the first one is devoted to the local existence of approximate solution $\tilde{w}_\epsilon$ of (3.7).

Proposition 3.7. *Let $\tilde{w}_{0,\epsilon}\in H^{m+2}_{k+\ell}(\mathbb{R}^2_+)$, $m\geq 6$ be an even integer, $k>1, 0\leq \ell<\frac{1}{2}, k+\ell>\frac{3}{2}$, and satisfy the compatibility conditions up to order $m+2$ for (3.7). Suppose that the shear flow satisfies*

$$|\partial_y^{p+1}u^s(t,y)|\leq C\langle y\rangle^{-k-p}, \quad (t,y)\in[0,T_1]\times\mathbb{R}_+,\ 0\leq p\leq m+2.$$

Then, for any $0<\epsilon\leq 1$ and $\bar{\zeta}>0$, there exits $T_\epsilon>0$ such that if

$$\|\tilde{w}_{0,\epsilon}\|_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)}\leq\bar{\zeta},$$

then the system (3.7) admits a unique solution

$$\tilde{w}_\epsilon\in L^\infty([0,T_\epsilon]; H^{m+2}_{k+\ell}(\mathbb{R}^2_+)).$$

Remark 3.8. *If $\tilde{w}_0\in H^{m+2}_{k+\ell}(\mathbb{R}^2_+)$ is the initial data in Theorem 3.5, using Corollary 3.3, there exists $\epsilon_0>0$, and for any $0<\epsilon\leq\epsilon_0$, there exists $\mu_\epsilon\in H^{m+3}_{k+\ell}(\mathbb{R}^2_+)$ such that $\tilde{w}_{0,\epsilon}=\tilde{w}_0+\epsilon\partial_y\mu_\epsilon$ satisfies the compatibility conditions up to order $m+2$ for the system (3.7), and*

$$\|\tilde{w}_{0,\epsilon}\|_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)}\leq\frac{3}{2}\|\tilde{w}_0\|_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)}.$$

Then, using Proposition 3.7, we obtain also the existence of the approximate solution under the assumption of Theorem 3.5.

The proof of this Proposition is standard since the equation in (3.7) is a parabolic type equation. Firstly, we establish the *à priori* estimate and then prove the existence of solution by the standard iteration and weak convergence methods. Because we work in the weighted Sobolev space and the computation is not so trivial, we give a detailed proof in the Appendix B, to make the paper self-contained. So the rest of this section is devoted to proving the estimate (3.9).

Uniform estimate with loss of x-derivative In the proof of the Proposition 3.7 (see Lemma B.2), we already get the *à priori* estimate for $\tilde{w}_\epsilon$. Now we try to prove the estimate (3.9) in a new way, and our object is to establish an uniform estimate with respect to $\epsilon>0$. We first treat the easy part in this subsection.

We define the non-isotropic Sobolev norm,

$$\|f\|^2_{H^{m,m-1}_{k+\ell}(\mathbb{R}^2_+)}=\sum_{|\alpha_1+\alpha_2|\leq m,\alpha_1\leq m-1}\|\langle y\rangle^{k+\ell+\alpha_2}\partial_x^{\alpha_1}\partial_y^{\alpha_2}f\|^2_{L^2(\mathbb{R}^2_+)}, \tag{3.12}$$

where we don't have the m-order derivative with respect to x-variable. Then

$$\|f\|^2_{H^m_{k+\ell}(\mathbb{R}^2_+)}=\|f\|^2_{H^{m,m-1}_{k+\ell}(\mathbb{R}^2_+)}+\|\partial_x^m f\|^2_{L^2_{k+\ell}(\mathbb{R}^2_+)}.$$

Proposition 3.9. *Let $m \geq 6$ be an even integer, $k > 1, 0 < \ell < \frac{1}{2}, k + \ell > \frac{3}{2}$, and assume that $\tilde{w}_\epsilon \in L^\infty([0, T_\epsilon]; H^{m+2}_{k+\ell}(\mathbb{R}^2_+))$ is a solution to (3.7), then we have*

$$\begin{aligned}\frac{d}{dt}\|\tilde{w}_\epsilon\|^2_{H^{m,m-1}_{k+\ell}(\mathbb{R}^2_+)} &+ \|\partial_y \tilde{w}_\epsilon\|^2_{H^{m,m-1}_{k+\ell}(\mathbb{R}^2_+)} \\ &+ \epsilon\|\partial_x \tilde{w}_\epsilon\|^2_{H^{m,m-1}_{k+\ell}(\mathbb{R}^2_+)} \leq C_1\Big(\|\tilde{w}_\epsilon\|^2_{H^m_{k+\ell}(\mathbb{R}^2_+)} + \|\tilde{w}_\epsilon\|^m_{H^m_{k+\ell}(\mathbb{R}^2_+)}\Big),\end{aligned} \tag{3.13}$$

where $C_1 > 0$ is independent of ϵ.

Remark. The above estimate is uniform with respect to $\epsilon > 0$, but on the left hand of (3.13), we missing the terms $\|\partial_x^m \tilde{w}_\epsilon\|^2_{L^2_{k+\ell}}$. This is because that we can't control the term

$$\partial_x^m \tilde{v}_\epsilon(t, x, y) = -\int_0^y \partial_x^{m+1}\tilde{u}_\epsilon(t, x, \tilde{y})d\tilde{y},$$

which is the major difficulty in the study of the Prandtl equation. We will study this term in the next Proposition with a non-uniform estimate firstly, and then focus on proving the uniform estimate in the rest part of this paper.

Proof. For $|\alpha| = \alpha_1 + \alpha_2 \leq m, \alpha_1 \leq m-1$, we have

$$\begin{aligned}&\partial_t \partial^\alpha \tilde{w}_\epsilon - \epsilon\partial_x^2\partial^\alpha \tilde{w}_\epsilon - \partial_y^2 \partial^\alpha \partial \tilde{w}_\epsilon \\ &= -\partial^\alpha\big((u^s + \tilde{u}_\epsilon)\partial_x \tilde{w}_\epsilon\big) - \partial^\alpha\big(\tilde{v}_\epsilon(u^s_{yy} + \partial_y \tilde{w}_\epsilon)\big).\end{aligned} \tag{3.14}$$

Multiplying the (3.14) with $\langle y\rangle^{2(k+\ell+\alpha_2)}\partial^\alpha \tilde{w}_\epsilon$, and integrating over $\mathbb{R}^2_+$,

$$\begin{aligned}&\int_{\mathbb{R}^2_+}(\partial_t\partial^\alpha\tilde{w}_\epsilon)\langle y\rangle^{2(k+\ell)+2\alpha_2}\partial^\alpha\tilde{w}_\epsilon dxdy - \epsilon\int_{\mathbb{R}^2_+}(\partial_x^2\partial^\alpha\tilde{w}_\epsilon)\langle y\rangle^{2(k+\ell)+2\alpha_2}\partial^\alpha\tilde{w}_\epsilon dxdy \\ &\qquad - \int_{\mathbb{R}^2_+}(\partial_y^2\partial^\alpha\tilde{w}_\epsilon)\langle y\rangle^{2(k+\ell)+2\alpha_2}\partial^\alpha\tilde{w}_\epsilon dxdy \\ &= -\int_{\mathbb{R}^2_+}\partial^\alpha\big((u^s+\tilde{u}_\epsilon)\partial_x\tilde{w}_\epsilon - \tilde{v}_\epsilon(u^s_{yy}+\partial_y\tilde{w}_\epsilon)\big)\langle y\rangle^{2(k+\ell)+2\alpha_2}\partial^\alpha\tilde{w}_\epsilon dxdy.\end{aligned}$$

Remark that for $\tilde{w}_\epsilon \in L^\infty([0, T_\epsilon]; H^{m+2}_{k+\ell}(\mathbb{R}^2_+))$, all above integrations are in the classical sense. We deal with each term on the left hand respectively. After integration by part, we have

$$\begin{aligned}&\int_{\mathbb{R}^2_+}(\partial_t\partial^\alpha\tilde{w}_\epsilon)\langle y\rangle^{2(k+\ell)+2\alpha_2(\mathbb{R}^2_+)}\partial^\alpha\tilde{w}_\epsilon dxdy = \frac{1}{2}\frac{d}{dt}\|\partial^\alpha\tilde{w}_\epsilon\|^2_{L^2_{k+\ell+\alpha_2}(\mathbb{R}^2_+)}, \\ &-\epsilon\int_{\mathbb{R}^2_+}(\partial_x^2\partial^\alpha\tilde{w}_\epsilon)\langle y\rangle^{2(k+\ell)+2\alpha_2(\mathbb{R}^2_+)}\partial^\alpha\tilde{w}_\epsilon dxdy = \epsilon\|\partial_x\partial^\alpha\tilde{w}_\epsilon\|^2_{L^2_{k+\ell+\alpha_2}(\mathbb{R}^2_+)},\end{aligned}$$

and

$$\begin{aligned}&-\int_{\mathbb{R}^2_+}\partial_y^2\partial^\alpha\tilde{w}_\epsilon\langle y\rangle^{2(k+\ell)+2\alpha_2}\partial^\alpha\tilde{w}_\epsilon dxdy \\ &= \|\partial_y\partial^\alpha\tilde{w}_\epsilon\|^2_{L^2_{k+\ell+\alpha_2}(\mathbb{R}^2_+)} + \int_{\mathbb{R}^2_+}\partial^\alpha\partial_y\tilde{w}_\epsilon(\langle y\rangle^{2(k+\ell)+2\alpha_2})'\partial^\alpha\tilde{w}_\epsilon dxdy \\ &\qquad + \int_{\mathbb{R}}(\partial^\alpha\partial_y\tilde{w}_\epsilon\partial^\alpha\tilde{w}_\epsilon)\big|_{y=0}dx.\end{aligned}$$

Cauchy-Schwarz inequality implies

$$\begin{aligned}&\left|\int_{\mathbb{R}^2_+}\partial^\alpha\partial_y\tilde{w}_\epsilon(\langle y\rangle^{2(k+\ell)+2\alpha_2})'\partial^\alpha\tilde{w}_\epsilon dxdy\right|\\&\qquad\le\frac{1}{16}\|\partial_y\partial^\alpha\tilde{w}_\epsilon\|^2_{L^2_{k+\ell+\alpha_2}(\mathbb{R}^2_+)}+C\|\partial^\alpha\tilde{w}_\epsilon\|^2_{L^2_{k+\ell+\alpha_2-1}(\mathbb{R}^2_+)}.\end{aligned}$$

We study now the term

$$\int_{\mathbb{R}}\left(\partial^\alpha\partial_y\tilde{w}_\epsilon\partial^\alpha\tilde{w}_\epsilon\right)\big|_{y=0}dx.$$

Case : $|\alpha|\le m-1$, using the trace Lemma A.2, we have

$$\begin{aligned}\left|\int_{\mathbb{R}}\left(\partial^\alpha\partial_y\tilde{w}_\epsilon\partial^\alpha\tilde{w}_\epsilon\right)\big|_{y=0}dx\right|&\le\|(\partial^\alpha\partial_y\tilde{w}_\epsilon)|_{y=0}\|_{L^2(\mathbb{R})}\|(\partial^\alpha\tilde{w}_\epsilon)|_{y=0}\|_{L^2(\mathbb{R})}\\&\le C\|\partial^\alpha\partial_y^2\tilde{w}_\epsilon\|_{L^2_{k+\ell}(\mathbb{R}^2_+)}\|\partial^\alpha\partial_y\tilde{w}_\epsilon\|_{L^2_{k+\ell}(\mathbb{R}^2_+)}\\&\le C\|\partial_y\tilde{w}_\epsilon\|_{H^{m,m-1}_{k+\ell}(\mathbb{R}^2_+)}\|\tilde{w}_\epsilon\|_{H^m_{k+\ell}(\mathbb{R}^2_+)}\\&\le\frac{1}{16}\|\partial_y\tilde{w}_\epsilon\|^2_{H^{m,m-1}_{k+\ell}(\mathbb{R}^2_+)}+C\|\tilde{w}_\epsilon\|^2_{H^m_{k+\ell}(\mathbb{R}^2_+)}.\end{aligned}$$

Case : $\alpha_1=m-1,\alpha_2=1$, using (3.3), we have

$$(\partial^\alpha\tilde{w}_\epsilon)|_{y=0}=(\partial_x^{\alpha_1}\partial_y^2\tilde{u}_\epsilon)|_{y=0}=0,$$

thus

$$\int_{\mathbb{R}}\left(\partial^\alpha\partial_y\tilde{w}_\epsilon\partial^\alpha\tilde{w}_\epsilon\right)|_{y=0}dx=0.$$

Case : $\alpha_1=0,\alpha_2=m$. Only in this case, we need to suppose that m is even. Using again the trace Lemma A.2, we have

$$\begin{aligned}\left|\int_{\mathbb{R}}\left(\partial_y^{m+1}\tilde{w}_\epsilon\partial_y^m\tilde{w}_\epsilon\right)|_{y=0}dx\right|&\le\|(\partial_y^{m+2}\tilde{u}_\epsilon)|_{y=0}\|_{L^2(\mathbb{R})}\|(\partial_y^m\tilde{w}_\epsilon)|_{y=0}\|_{L^2(\mathbb{R})}\\&\le C\|(\partial_y^{m+2}\tilde{u}_\epsilon)|_{y=0}\|_{L^2(\mathbb{R})}\|\partial_y^{m+1}\tilde{w}_\epsilon\|_{L^2_{k+\ell}(\mathbb{R}^2_+)}\\&\le\frac{1}{16}\|\partial_y\tilde{w}_\epsilon\|^2_{H^{m,m-1}_{k+\ell}(\mathbb{R}^2_+)}+C\|(\partial_y^{m+2}\tilde{u}_\epsilon)|_{y=0}\|^2_{L^2(\mathbb{R})}.\end{aligned}$$

Using Proposition 3.1 and the trace Lemma A.2, we can estimate the above last term $\|(\partial_y^{m+2}\tilde{u}_\epsilon)|_{y=0}\|^2_{L^2(\mathbb{R})}$ by a finite summation of the following forms

$$\|\prod_{j=1}^p(\partial_x^{\alpha_j}\partial_y^{\beta_j+1}(u^s+\tilde{u}_\epsilon))|_{y=0}\|^2_{L^2(\mathbb{R})}\le C\|\partial_y\prod_{j=1}^p(\partial_x^{\alpha_j}\partial_y^{\beta_j+1}(u^s+\tilde{u}_\epsilon))\|^2_{L^2_{\frac12+\delta}(\mathbb{R}^2_+)}$$

with $2\le p\le\frac{m}{2}$, $\alpha_j+\beta_j\le m-1$ and $\{j;\alpha_j>0\}\ne\emptyset$. Then using Sobolev inequality and $m\ge6$, we get

$$\|(\partial_y^{m+2}\tilde{u}_\epsilon)|_{y=0}\|_{L^2(\mathbb{R})}\le C\|\tilde{w}_\epsilon\|^{m/2}_{H^m_{k+\ell}(\mathbb{R}^2_+)}.$$

Case : $1\le\alpha_1\le m-2,\alpha_1+\alpha_2=m,\alpha_2$ **even**, using the same argument to the precedent case, we have

$$\begin{aligned}\left|\int_{\mathbb{R}}(\partial^\alpha\partial_y\tilde{w}_\epsilon\partial^\alpha\tilde{w}_\epsilon)|_{y=0}dx\right|&=\left|\int_{\mathbb{R}}(\partial_x^{\alpha_1}\partial_y^{\alpha_2+1}\tilde{w}_\epsilon\partial_x^{\alpha_1}\partial_y^{\alpha_2}\tilde{w}_\epsilon)|_{y=0}dx\right|\\&\le\|(\partial_x^{\alpha_1}\partial_y^{\alpha_2+1}\tilde{w}_\epsilon)|_{y=0}\|_{L^2(\mathbb{R})}\|(\partial_x^{\alpha_1}\partial_y^{\alpha_2}\tilde{w}_\epsilon)|_{y=0}\|_{L^2(\mathbb{R})}\end{aligned}$$

$$\le \frac{1}{16}\|\partial_y\tilde w_\epsilon\|^2_{H^{m,m-1}_{k+\ell}(\mathbb{R}^2_+)} + C\|(\partial_x^{\alpha_1}\partial_y^{\alpha_2+2}\tilde u_\epsilon)|_{y=0}\|^2_{L^2(\mathbb{R})}$$
$$\le \frac{1}{16}\|\partial_y\tilde w_\epsilon\|^2_{H^{m,m-1}_{k+\ell}(\mathbb{R}^2_+)} + C\|\tilde w_\epsilon\|^{\alpha_2}_{H^{m}_{k+\ell}(\mathbb{R}^2_+)}.$$

Case : $1\le \alpha_1\le m-2, \alpha_1+\alpha_2=m, \alpha_2$ **odd**, integration by part with respect to x variable implies

$$\left|\int_{\mathbb{R}}(\partial_x^{\alpha_1}\partial_y^{\alpha_2+1}\tilde w_\epsilon\partial_x^{\alpha_1}\partial_y^{\alpha_2}\tilde w_\epsilon)|_{y=0}dx\right| = \left|\int_{\mathbb{R}}(\partial_x^{\alpha_1-1}\partial_y^{\alpha_2+1}\tilde w_\epsilon\partial_x^{\alpha_1+1}\partial_y^{\alpha_2}\tilde w_\epsilon)|_{y=0}dx\right|$$
$$\le \|(\partial_x^{\alpha_1-1}\partial_y^{\alpha_2+1}\tilde w_\epsilon)|_{y=0}\|_{L^2(\mathbb{R})}\|(\partial_x^{\alpha_1+1}\partial_y^{\alpha_2}\tilde w_\epsilon)|_{y=0}\|_{L^2(\mathbb{R})}$$
$$\le \frac{1}{16}\|\partial_y\tilde w_\epsilon\|^2_{H^{m,m-1}_{k+\ell}(\mathbb{R}^2_+)} + C\|(\partial_x^{\alpha_1+1}\partial_y^{\alpha_2+1}\tilde u_\epsilon)|_{y=0}\|^2_{L^2(\mathbb{R})}$$
$$\le \frac{1}{16}\|\partial_y\tilde w_\epsilon\|^2_{H^{m,m-1}_{k+\ell}(\mathbb{R}^2_+)} + C\|\tilde w_\epsilon\|^{\alpha_2-1}_{H^{m}_{k+\ell}(\mathbb{R}^2_+)}.$$

Finally, we have proven

$$\int_{\mathbb{R}^2_+}\big(\partial_t\partial^\alpha\tilde w_\epsilon-\partial_y^2\partial^\alpha\tilde w_\epsilon-\epsilon\partial_x^2\partial^\alpha\tilde w_\epsilon\big)\langle y\rangle^{2(k+\ell+\alpha_2)}\partial^\alpha\tilde w_\epsilon dxdy$$
$$\ge \frac{1}{2}\frac{d}{dt}\|\partial^\alpha\tilde w_\epsilon\|^2_{L^2_{k+\ell+\alpha_2}}+\epsilon\|\partial_x\partial^\alpha\tilde w_\epsilon\|^2_{L^2_{k+\ell+\alpha_2}}+\|\partial_y\partial^\alpha\tilde w_\epsilon\|^2_{L^2_{k+\ell+\alpha_2}}$$
$$-\frac{1}{4}\|\partial_y\tilde w_\epsilon\|^2_{H^{m,m-1}_{k+\ell}(\mathbb{R}^2_+)}-C\|\tilde w_\epsilon\|^m_{H^m_{k+\ell}(\mathbb{R}^2_+)}.$$

We estimate now the right hand of (3.14). For the first item, we need to split it into two parts

$$-\partial^\alpha\big((u^s+\tilde u_\epsilon)\partial_x\tilde w_\epsilon\big)=-(u^s+\tilde u_\epsilon)\partial_x\partial^\alpha\tilde w_\epsilon+[(u^s+\tilde u_\epsilon),\partial^\alpha]\partial_x\tilde w_\epsilon.$$

Firstly, we have

$$\int_{\mathbb{R}^2_+}\big((u^s+\tilde u_\epsilon)\partial_x\partial^\alpha\tilde w_\epsilon\big)\langle y\rangle^{2(k+\ell+\alpha_2)}\partial^\alpha\tilde w_\epsilon dxdy\le\|\partial_x\tilde u_\epsilon\|_{L^\infty}\|\partial^\alpha\tilde w_\epsilon\|^2_{L^2_{k+\ell+\alpha_2}},$$

then using (A.2), we get

$$\left|\int_{\mathbb{R}^2_+}\big((u^s+\tilde u_\epsilon)\partial_x\partial^\alpha\tilde w_\epsilon\big)\langle y\rangle^{2(\ell+\alpha_2)}\partial^\alpha\tilde w_\epsilon dxdy\right|\le\|\tilde w_\epsilon\|_{H^3_1}\|\partial^\alpha\tilde w_\epsilon\|^2_{L^2_{k+\ell+\alpha_2}}.$$

For the commutator operator, in fact, it can be written as

$$[(u^s+\tilde u_\epsilon),\partial^\alpha]\partial_x\tilde w_\epsilon=\sum_{\beta\le\alpha,\,1\le|\beta|}C^\beta_\alpha\,\partial^\beta(u^s+\tilde u_\epsilon)\partial^{\alpha-\beta}\partial_x\tilde w_\epsilon.$$

Then for $|\alpha|\le m, m\ge 4$, using the Sobolev inequality again and Lemma A.1,

$$\|[(u^s+\tilde u),\partial^\alpha]\partial_x\tilde w_\epsilon\|_{L^2_{k+\ell+\alpha_2}}\le C(\|\tilde w_\epsilon\|_{H^m_{k+\ell}}+\|\tilde w_\epsilon\|^2_{H^m_{k+\ell}}).$$

Thus

$$\left|\int_{\mathbb{R}^2_+}\langle y\rangle^{2(k+\ell+\alpha_2)}\big([(u^s+\tilde u_\epsilon),\partial^\alpha]\partial_x\tilde w_\epsilon\big)\cdot\partial^\alpha\tilde w_\epsilon dxdy\right|\le C\big(\|\tilde w_\epsilon\|^2_{H^m_{k+\ell}}+\|\tilde w_\epsilon\|^3_{H^m_{k+\ell}}\big),$$

and

$$\left|\int_{\mathbb{R}^2_+}\langle y\rangle^{2(k+\ell+\alpha_2)}\big(\partial^\alpha\big((u^s+\tilde u_\epsilon)\partial_x\tilde w_\epsilon\big)\big)\partial^\alpha\tilde w_\epsilon dxdy\right|\le C\big(\|\tilde w_\epsilon\|^2_{H^m_{k+\ell}}+\|\tilde w_\epsilon\|^3_{H^m_{k+\ell}}\big),$$

where C is independent of ϵ.

For the next one, similar to the first term in (3.14), we have

$$\partial^\alpha\big(\tilde v_\epsilon(u^s_{yy}+\partial_y\tilde w_\epsilon)\big)=\tilde v_\epsilon\partial_y\partial^\alpha\tilde w_\epsilon-[\tilde v_\epsilon,\partial^\alpha]\partial_y\tilde w_\epsilon+\partial^\alpha(\tilde v_\epsilon u^s_{yy}).$$

Then

$$\begin{aligned}\left|\int_{\mathbb{R}^2_+}\tilde v_\epsilon\langle y\rangle^{2(k+\ell+\alpha_2)}(\partial_y\partial^\alpha\tilde w_\epsilon)\cdot\partial^\alpha\tilde w_\epsilon dxdy\right|&\le\|\tilde v_\epsilon\|_{L^\infty(\mathbb{R}^2_+)}\|\partial_y\tilde w_\epsilon\|_{H^m_{k+\ell}}\|\tilde w_\epsilon\|_{H^m_{k+\ell}}\\&\le\frac14\|\partial_y\tilde w_\epsilon\|^2_{H^m_{k+\ell}(\mathbb{R}^2_+)}+C\|\tilde w_\epsilon\|^4_{H^m_{k+\ell}(\mathbb{R}^2_+)}\end{aligned}$$

where we have used

$$\begin{aligned}\|\tilde v_\epsilon\|_{L^\infty(\mathbb{R}^2_+)}&\le C\|\partial_x\tilde u_\epsilon\|_{L^\infty(\mathbb{R}_x;L^2_{\frac12+\delta}(\mathbb{R}_{y,+}))}\\&\le C\int_{\mathbb{R}^2_+}\langle y\rangle^{1+2\delta}(|\partial_x\tilde u_\epsilon|^2+|\partial_x^2\tilde u_\epsilon|^2)dxdy\\&\le C\int_{\mathbb{R}^2_+}\langle y\rangle^{3+2\delta}(|\partial_x\tilde w_\epsilon|^2+|\partial_x^2\tilde w_\epsilon|^2)dxdy\le C\|\tilde w_\epsilon\|_{H^2_{\frac32+\delta}},\end{aligned}$$

where $\delta>0$ is small.

Noticing that

$$[\tilde v_\epsilon,\partial^\alpha]\partial_y\tilde w_\epsilon=\sum_{\beta\le\alpha,1\le|\beta|}C^\beta_\alpha\ \partial^\beta\tilde v_\epsilon\ \partial^{\alpha-\beta}\partial_y\tilde w_\epsilon.$$

Since H^m_ℓ is an algebra for $m\ge6$, we only need to pay attention to the order of derivative in the above formula. Firstly for $|\beta|\ge1$, we have for $|\alpha-\beta|+1\le m$,

$$-\partial^\beta\tilde v_\epsilon=\partial_x^{\beta_1}\partial_y^{\beta_2}\int_0^y\tilde u_{\epsilon,x}d\tilde y=\begin{cases}\partial_x^{\beta_1+1}\partial_y^{\beta_2-1}\tilde u_\epsilon, & \beta_2\ge1,\\ \int_0^y\partial_x^{\beta_1+1}\tilde u_\epsilon d\tilde y, & \beta_2=0.\end{cases}$$

Now using the hypothesis $\beta\le\alpha,1\le|\beta|$ and $\beta_1\le\alpha_1\le m-1$, using Lemma A.1, we get

$$\|[\tilde v_\epsilon,\partial^\alpha]\partial_y\tilde w_\epsilon\|_{L^2_{k+\ell+\alpha_2}}\le C\|\tilde w_\epsilon\|^2_{H^m_{k+\ell}}.$$

On the other hand, if $\alpha_2=0$, using $-1+\ell<-\frac12$, we can get

$$\|\partial_x^{m-1}(\tilde v_\epsilon u^s_{yy})\|_{L^2_{k+\ell}}\le C\|\partial_x^m\tilde u_\epsilon\|_{L^2_{\frac12+\delta}(\mathbb{R}^2_+)}\|u^s_{yy}\|_{L^2_{k+\ell}(\mathbb{R}_+)}\le C\|\tilde w_\epsilon\|_{H^m_{\frac32+\delta}}.$$

Similar computation for other cases, we can get, for $\alpha_2>0,\alpha_1+\alpha_2\le m$,

$$\|\partial^\alpha(\tilde v_\epsilon u^s_{yy})\|_{L^2_{k+\ell+\alpha_2}}\le C\|\tilde w_\epsilon\|_{H^m_{k+\ell}}.$$

Combining the above estimates, we have finished the proof of the Proposition 3.9. □

Smallness of approximate solutions. To close the energy estimate, we still need to estimate the term $\partial_x^m\tilde w_\epsilon$.

Proposition 3.10. *Under the hypothesis of Theorem 3.5, and with the same notations as in Proposition 3.9, we have*

$$\begin{aligned}&\frac12\frac{d}{dt}\|\partial_x^m\tilde w_\epsilon\|^2_{L^2_{k+\ell}}+\frac{3\epsilon}{4}\|\partial_x^{m+1}\tilde w_\epsilon\|^2_{L^2_{k+\ell}}+\frac34\|\partial_y\partial_x^m\tilde w_\epsilon\|^2_{L^2_{k+\ell}}\\&\le C\big(\|\tilde w_\epsilon\|^2_{H^m_{k+\ell}}+\|\tilde w_\epsilon\|^3_{H^m_{k+\ell}}\big)+\frac{32}{\epsilon}\big(\|\tilde w_\epsilon\|^4_{H^m_{k+\ell}}+\|\tilde w_\epsilon\|^2_{H^m_{k+\ell}}\big).\end{aligned}\tag{3.15}$$

Proof. We have

$$\partial_t\partial_x^m\tilde{w}_\epsilon-\partial_y^2\partial_x^m\tilde{w}_\epsilon-\epsilon\partial_x^m\partial_x^2\tilde{w}_\epsilon=-\partial_x^m\big((u^s+\tilde{u}_\epsilon)\partial_x\tilde{w}_\epsilon\big)-\partial_x^m\big(\tilde{v}_\epsilon(\partial_y\tilde{w}_\epsilon+u^s_{yy})\big),$$

then the same computations as in Proposition 3.9 give

$$\begin{aligned}&\frac{d}{2dt}\|\partial_x^m\tilde{w}_\epsilon\|^2_{L^2_{k+\ell}}+\epsilon\|\partial_x^{m+1}\tilde{w}_\epsilon\|^2_{L^2_{k+\ell}}+\frac{3}{4}\|\partial_y\partial_x^m\tilde{w}_\epsilon\|^2_{L^2_{k+\ell}}\\&\le C(\|\tilde{w}_\epsilon\|^2_{H^m_{k+\ell}}+\|\tilde{w}_\epsilon\|^3_{H^m_{k+\ell}})\\&+\left|\int_{\mathbb{R}^2_+}\partial_x^m\big(\tilde{v}_\epsilon(\partial_y\tilde{w}_\epsilon+u^s_{yy})\big)\langle y\rangle^{2(k+\ell)}\partial_x^m\tilde{w}_\epsilon dxdy\right|,\end{aligned}\tag{3.16}$$

where the boundary terms is more easy to control, since

$$(\partial_y\partial_x^m\tilde{w}_\epsilon)(t,x,0)=(\partial_y^2\partial_x^m\tilde{u}_\epsilon)(t,x,0)=0,\ \ (t,x)\in[0,T]\times\mathbb{R}.$$

The estimate of the last term on right hand is the main obstacle for the study of the Prandtl equations.

$$\begin{aligned}\partial_x^m\big(\tilde{v}_\epsilon(\partial_y\tilde{w}_\epsilon+u^s_{yy})\big)&=\tilde{v}_\epsilon\partial_x^m\partial_y\tilde{w}_\epsilon+(\partial_x^m\tilde{v}_\epsilon)(\partial_y\tilde{w}_\epsilon+u^s_{yy})\\&\quad+\sum_{1\le j\le m-1}C^j_m\,\partial_x^j\tilde{v}_\epsilon\partial_x^{m-j}\partial_y\tilde{w}_\epsilon.\end{aligned}$$

For the first term

$$\begin{aligned}\int_{\mathbb{R}^2_+}\tilde{v}_\epsilon(\partial_x^m\partial_y\tilde{w}_\epsilon)\langle y\rangle^{2(k+\ell)}(\partial_x^m\tilde{w}_\epsilon)dxdy&=\frac12\int\tilde{v}_\epsilon\langle y\rangle^{2(k+\ell)}\partial_y(\partial_x^m\tilde{w}_\epsilon)^2dxdy\\&=\frac12\int\tilde{u}_{\epsilon,x}\langle y\rangle^{2(k+\ell)}(\partial_x^m\tilde{w}_\epsilon)^2dxdy\\&-\ell\int\tilde{v}_\epsilon\langle y\rangle^{2(k+\ell)-1}(\partial_x^m\tilde{w}_\epsilon)^2dxdy\\&\le C\|\tilde{w}_\epsilon\|^3_{H^m_{k+\ell}},\end{aligned}$$

where we have used $\tilde{v}_\epsilon|_{y=0}=0$, and

$$\left|\int_{\mathbb{R}^2_+}\Big(\sum_{1\le j\le m-1}C^j_m\,\partial_x^j\tilde{v}_\epsilon\partial_x^{m-j}\partial_y\tilde{w}_\epsilon)\langle y\rangle^{2(k+\ell)}(\partial_x^m\tilde{w}_\epsilon)dxdy\right|\le C\|\tilde{w}_\epsilon\|^3_{H^m_{k+\ell}}.$$

Finally for the worst term, we have

$$\begin{aligned}&\left|\int_{\mathbb{R}^2_+}(\partial_x^m\tilde{v}_\epsilon)(\partial_y\tilde{w}_\epsilon+u^s_{yy})\langle y\rangle^{2(k+\ell)}(\partial_x^m\tilde{w}_\epsilon)dxdy\right|\\&\le C\|\partial_x^m\tilde{v}_\epsilon\|_{L^2(\mathbb{R}_x;L^\infty(\mathbb{R}_+))}\|\partial_y\tilde{w}_\epsilon\|_{L^\infty(\mathbb{R}_x;L^2_{k+\ell}(\mathbb{R}_+))}\|\tilde{w}_\epsilon\|_{H^m_{k+\ell}}\\&\qquad+\|\partial_x^m\tilde{v}_\epsilon u^s_{yy}\|_{L^2_{k+\ell}(\mathbb{R}^2_+)}\|\tilde{w}_\epsilon\|_{H^m_{k+\ell}}.\end{aligned}$$

On the other hand, observing

$$\partial_x^m\tilde{v}_\epsilon(t,x,y)=-\int_0^y\partial_x^{m+1}\tilde{u}_\epsilon(t,x,\tilde{y})d\tilde{y},$$

then using Sobolev inequality and Lemma A.1, for $\delta>0$ small,

$$\|\partial_x^m\tilde{v}_\epsilon\|_{L^2(\mathbb{R}_x;L^\infty(\mathbb{R}_+))}\le C\|\partial_x^{m+1}\tilde{u}_\epsilon\|_{L^2_{\frac12+\delta}(\mathbb{R}^2_+)}\le C\|\partial_x^{m+1}\tilde{w}_\epsilon\|_{L^2_{\frac32+\delta}(\mathbb{R}^2_+)},$$

we get

$$\|\partial_x^m \tilde v_\epsilon\|_{L^2(\mathbb{R}_x;L^\infty(\mathbb{R}_+))} \le C\|\partial_x^{m+1}\tilde w_\epsilon\|_{L^2_{\frac32+\delta}(\mathbb{R}^2_+)}.$$

Using the hypothesis for the shear flow u^s and $\ell-1<-\frac12$,

$$\begin{aligned}\|\partial_x^m(\tilde v_\epsilon u^s_{yy})\|_{L^2_{k+\ell}(\mathbb{R}^2_+)} &\le \|\partial_x^m\tilde v_\epsilon\|_{L^2(\mathbb{R}_x;L^\infty(\mathbb{R}_+))}\|u^s_{yy}\|_{L^2_{k+\ell}(\mathbb{R}_+)}\\ &\le C\|\partial_x^{m+1}\tilde w_\epsilon\|_{L^2_{\frac32+\delta}(\mathbb{R}^2_+)},\end{aligned}$$

and for $k+\ell\ge\frac32+\delta$,

$$\|\partial_y\tilde w_\epsilon\|_{L^\infty(\mathbb{R}_x;L^2_{k+\ell}(\mathbb{R}_+))}\le C\|\partial_y\tilde w_\epsilon\|_{H^1(\mathbb{R}_x;L^2_{k+\ell}(\mathbb{R}_+))}\le C\|\tilde w_\epsilon\|_{H^m_{k+\ell}(\mathbb{R}^2_+)}.$$

Thus, we have

$$\begin{aligned}&\int\big(\partial_x^m\big(\tilde v_\epsilon(\partial_y\tilde w_\epsilon+u^s_{yy})\big)\big)\langle y\rangle^{2(k+\ell)}\partial_x^m\tilde w_\epsilon dxdy\\ &\le C\|\tilde w_\epsilon\|^3_{H^m_{k+\ell}}+\frac{32}{\epsilon}\big(\|\tilde w_\epsilon\|^4_{H^m_{k+\ell}}+\|\tilde w_\epsilon\|^2_{H^m_{k+\ell}}\big)+\frac{\epsilon}{4}\|\partial_x^{m+1}\tilde w_\epsilon\|^2_{L^2_{\frac32+\delta}}.\end{aligned}\tag{3.17}$$

From (3.16) and (3.17), we have, if $k+\ell>\frac32$,

$$\begin{aligned}&\frac12\frac{d}{dt}\|\partial_x^m\tilde w_\epsilon\|^2_{L^2_{k+\ell}}+\frac{3\epsilon}{4}\|\partial_x^{m+1}\tilde w_\epsilon\|^2_{L^2_{k+\ell}}+\frac34\|\partial_y\partial_x^m\tilde w_\epsilon\|^2_{L^2_{k+\ell}}\\ &\le C\big(\|\tilde w_\epsilon\|^2_{H^m_{k+\ell}}+\|\tilde w_\epsilon\|^3_{H^m_{k+\ell}}\big)+\frac{32}{\epsilon}\big(\|\tilde w_\epsilon\|^4_{H^m_{k+\ell}}+\|\tilde w_\epsilon\|^2_{H^m_{k+\ell}}\big).\end{aligned}$$

□

End of proof of Theorem 3.5. Combining (3.13) and (3.15), for $m\ge6, k>1, \frac32-k<\ell<\frac12$ and $0<\epsilon\le1$, we get

$$\frac{d}{dt}\|\tilde w_\epsilon\|^2_{H^m_{k+\ell}(\mathbb{R}^2_+)}\le\frac{C}{\epsilon}\big(\|\tilde w_\epsilon\|^2_{H^m_{k+\ell}(\mathbb{R}^2_+)}+\|\tilde w_\epsilon\|^m_{H^m_{k+\ell}(\mathbb{R}^2_+)}\big),\tag{3.18}$$

with $C>0$ independent of ϵ.

From (3.18), by the nonlinear Gronwall's inequality, we have

$$\|\tilde w_\epsilon(t)\|^{m-2}_{H^m_{k+\ell}(\mathbb{R}^2_+)}\le\frac{\|\tilde w_\epsilon(0)\|^{m-2}_{H^m_{k+\ell}}}{e^{-\frac{C}{\epsilon}t(\frac m2-1)}-(\frac m2-1)\frac{C}{\epsilon}t\|\tilde w_\epsilon(0)\|^{m-2}_{H^m_{k+\ell}}},\ \ 0<t\le T_\epsilon,$$

where we choose $T_\epsilon>0$ such that

$$\Big(e^{-\frac{C}{\epsilon}T_\epsilon(\frac m2-1)}-(\frac m2-1)\frac{C}{\epsilon}T_\epsilon\bar\zeta^{\,m-2}\Big)^{-1}=\Big(\frac43\Big)^{m-2}.\tag{3.19}$$

Finally, we get for any $\|\tilde w_\epsilon(0)\|_{H^m_{k+\ell}}\le\bar\zeta$, and $0<\epsilon\le\epsilon_0$,

$$\|\tilde w_\epsilon(t)\|_{H^m_{k+\ell}(\mathbb{R}^2_+)}\le\frac43\|\tilde w_\epsilon(0)\|_{H^m_{k+\ell}(\mathbb{R}^2_+)}\le2\|\tilde w_0\|_{H^m_{k+\ell}(\mathbb{R}^2_+)},\ 0<t\le T_\epsilon.$$

□

The rest of this paper is dedicated to improve the results of Proposition 3.10, and try to get an uniform estimate with respect to ϵ. Of course, we have to recall the assumption on the shear flow in the main Theorem 1.1.

4. Formal transformations

Since the estimate (3.13) is independent of ϵ, we only need to treat (3.15) in a new way to get an estimate which is also independent of ϵ. To simplify the notations, from now on, we drop the notation tilde and sub-index ϵ, that is, with no confusion, we take

$$u = \tilde{u}_\epsilon, \quad v = \tilde{v}_\epsilon, \quad w = \tilde{w}_\epsilon.$$

Let $w \in L^\infty([0,T]; H^m_{k+\ell}(\mathbb{R}^2_+), m \geq 6, k > 1, 0 < \ell < \frac{1}{2},\ \frac{1}{2} < \ell' < \ell + \frac{1}{2},\ k + \ell > \frac{3}{2}$ be a classical solution of (3.7) which satisfies the following *à priori* condition

$$\|w\|_{L^\infty([0,T]; H^m_{k+\ell}(\mathbb{R}^2_+))} \leq \zeta. \tag{4.1}$$

Then (A.2) gives

$$\begin{aligned}
\|\langle y\rangle^{k+\ell} w\|_{L^\infty([0,T]\times\mathbb{R}^2_+)} \leq & C(\|\langle y\rangle^{\frac{1}{2}+\delta}(\langle y\rangle^{k+\ell} w)_y\|_{L^\infty([0,T];L^2(\mathbb{R}^2_+))} \\
& + \|\langle y\rangle^{\frac{1}{2}+\delta}(\langle y\rangle^{k+\ell} w)_{xy}\|_{L^\infty([0,T];L^2(\mathbb{R}^2_+))}) \\
& \leq C_m \|w\|_{L^\infty([0,T]; H^m_{k+\ell}(\mathbb{R}^2_+))},
\end{aligned}$$

which implies

$$|\partial_y u(t,x,y)| = |w(t,x,y)| \leq C_m\, \zeta\, \langle y\rangle^{-k-\ell}, \quad (t,x,y) \in [0,T]\times\mathbb{R}^2_+.$$

We assume that ζ **is small enough** such that

$$C_m\, \zeta \leq \frac{\tilde{c}_1}{4}, \tag{4.2}$$

where C_m is the above Sobolev embedding constant. Then we have for $\ell \geq 0$,

$$\frac{\tilde{c}_1}{4}\langle y\rangle^{-k} \leq |u^s_y + u_y| \leq 4\tilde{c}_2 \langle y\rangle^{-k}, \quad (t,x,y) \in [0,T]\times\mathbb{R}\times\mathbb{R}^+. \tag{4.3}$$

The formal transformation of equations. Under the conditions (4.2) and (4.3), in this subsection, we will introduce the following formal transformations of system (3.1). Set, for $0 \leq n \leq m$

$$g_n = \left(\frac{\partial^n_x u}{u^s_y + u_y}\right)_y,\ \eta_1 = \frac{u_{xy}}{u^s_y + u_y},\ \eta_2 = \frac{u^s_{yy} + u_{yy}}{u^s_y + u_y}, \forall (t,x,y) \in [0,T]\times\mathbb{R}^2_+.$$

Formally, we will use the following notations

$$\partial_y^{-1} g_n(t,x,y) = \frac{\partial^n_x u}{u^s_y + u_y}(t,x,y),\ \partial_y \partial_y^{-1} g_n = g_n,\ \forall (t,x,y) \in [0,T]\times\mathbb{R}^2_+$$

Applying ∂^n_x to (3.1), we have

$$\begin{aligned}
\partial_t \partial^n_x u + (u^s + u)\partial_x \partial^n_x u + (\partial^n_x v)(u^s_y + \partial_y u)& \\
= \partial^2_y \partial^n_x u + \epsilon \partial^2_x \partial^n_x u + A^1_n + A^2_n&,
\end{aligned} \tag{4.4}$$

where

$$A^1_n = -[\partial^n_x,\ (u^s + u)]\partial_x u = -\sum_{i=1}^n C^i_n \partial^i_x u\, \partial^{n+1-i}_x u,$$

$$A^2_n = -[\partial^n_x,\ (u^s_y + \partial_y u)]v = -\sum_{i=1}^n C^i_n \partial^i_x w\, \partial^{n-i}_x v.$$

Dividing (4.4) with $(u^s_y + u_y)$ and performing ∂_y on the resulting equation, observing

$$\partial_x \partial^n_x u + \partial_y \partial^n_x v = \partial^n_x(\partial_x u + \partial_y v) = 0,$$

we have for $j=1,2$,

$$\partial_y\left(\frac{\partial_t\partial_x^n u}{u_y^s+u_y}\right)+(u^s+u)\partial_y\left(\frac{\partial_x\partial_x^n u}{u_y^s+u_y}\right)$$
$$=\partial_y\left(\frac{\partial_y^2\partial_x^n u+\epsilon\partial_x^2\partial_x^n u}{u_y^s+u_y}\right)+\partial_y\left(\frac{A_n^1+A_n^2}{u_y^s+u_y}\right).$$

We compute each term on the support of ,

$$\partial_y\left(\frac{\partial_t\partial_x^n u}{u_y^s+u_y}\right)=\partial_y\biggl(\partial_t\frac{\partial_x^n u}{u_y^s+u_y}+\partial_y^{-1}g_n\,\frac{\partial_t u_y+\partial_t u_y^s}{u_y^s+u_y}\biggr)$$
$$=\partial_t g_n+\partial_y\biggl(\partial_y^{-1}g_n\,\frac{\partial_t u_y^s+\partial_t u_y}{u_y^s+\tilde u_y}\biggr),$$

$$(u^s+u)\partial_y\left(\frac{\partial_x\partial_x^n u}{u_y^s+u_y}\right)=(u^s+u)\biggl\{\partial_x\partial_y\left(\frac{\partial_x^n u}{u_y^s+u_y}\right)+\partial_y\left(\frac{\partial_x^n u}{u_y^s+u_y}\right)\frac{u_{xy}}{u_y^s+u_y}$$
$$+\left(\frac{\partial_x^n u}{u_y^s+u_y}\right)\partial_y\left(\frac{u_{xy}}{u_y^s+u_y}\right)\biggr\}$$
$$=(u^s+u)(\partial_x g_n+g_n\,\eta_1+\partial_y^{-1}g_n\,\partial_y\eta_1),$$

$$\frac{\partial_y^2\partial_x^n u}{u_y^s+u_y}=\partial_y^2\left(\frac{\partial_x^n u}{u_y^s+u_y}\right)+2\left(\frac{\partial_y u}{u_y^s+u_y}\right)\frac{u_{yy}^s+u_{yy}}{u_y^s+u_y}-\partial_x^n u\,\partial_y^2\left(\frac{1}{u_y^s+u_y}\right),$$

$$\partial_y^2\left(\frac{1}{u_y^s+u_y}\right)=-\partial_y\left(\frac{u_{yy}^s+u_{yy}}{(u_y^s+u_y)^2}\right)=-\frac{u_{yyy}^s+u_{yyy}}{(u_y^s+u_y)^2}+2\left(\frac{u_{yy}^s+u_{yy}}{(u_y^s+u_y)}\right)^2\frac{1}{u_y^s+u_y},$$

$$\frac{\partial_y\partial_x^n u}{u_y^s+u_y}\frac{u_{yy}^s+u_{yy}}{u_y^s+u_y}=\left(\frac{\partial_x^n u}{u_y^s+u_y}\right)_y\frac{u_{yy}^s+u_{yy}}{u_y^s+u_y}-\frac{\partial_x^n u}{u_y^s+u_y}\left(\frac{u_{yy}^s+u_{yy}}{(u_y^s+u_y)}\right)^2.$$

So

$$\frac{\partial_y^2\partial_x^n u}{u_y^s+u_y}=\partial_y g_n+2(g_n\eta_2-2\partial_y^{-1}g_n\eta_2^2)+\partial_y^{-1}g_n\left(\frac{u_{yyy}^s+u_{yyy}}{(u_y^s+\tilde u_y)}\right),$$

$$\partial_y\left(\frac{\partial_y^2\partial_x^n u}{u_y^s+u_y}\right)=\partial_y^2 g_n+2(\partial_y g_n)\eta_2+2g_n\partial_y\eta_2-4g_n\eta_2^2$$
$$-8\partial_y^{-1}g_n\eta_2\partial_y\eta_2+\partial_y\biggl(\partial_y^{-1}g_n\,\frac{u_{yyy}^s+u_{yyy}}{u_y^s+u_y}\biggr).$$

Similarly, we have

$$\frac{\partial_x^2\partial_x^n u}{u_y^s+u_y}=\partial_x^2\left(\frac{\partial_x^n u}{u_y^s+u_y}\right)+2\left(\frac{\partial_x^n u}{u_y^s+u_y}\right)_x\frac{u_{xy}}{u_y^s+u_y}$$
$$-2\frac{\partial_x^n u}{u_y^s+u_y}\left(\frac{u_{xy}}{(u_y^s+u_y)}\right)^2+\frac{\partial_x^n u}{u_y^s+u_y}\frac{u_{xxy}}{(u_y^s+u_y)},$$

$$\partial_y\left(\frac{\partial_x^2\partial_x^n u}{u_y^s+u_y}\right)=\partial_x^2 g_n+2\partial_x g_n\eta_1+2\partial_x\partial_y^{-1}g_n\partial_y\eta_1$$

$$- 2g_n\eta_1^2 - 4\partial_y^{-1}g_n\eta_1\partial_y\eta_1 + \partial_y\left(\partial_y^{-1}g_n\,\frac{u_{xxy}}{u_y^s+u_y}\right).$$

For the boundary condition, we only need to pay attention to $j=1$. From (4.4) and the boundary condition for (u,v) in (3.1), we observe

$$\partial_x^n u|_{y=0}=0,\ \partial_y^2\partial_x^n u|_{y=0}=0,\ (u_y^s+u_y)|_{y=0}\neq 0.$$

At the same time,

$$\begin{aligned}0=\frac{\partial_y^2\partial_x^n u}{u_y^s+u_y}\Big|_{y=0} &= \partial_y g_n|_{y=0}+2(g_n\eta_2-2(\partial_y^{-1}g_n)\eta_2^2)|_{y=0}\\ &\quad+\partial_y^{-1}g_n\left(\frac{u^s_{yyy}+u_{yyy}}{(u_y^s+\tilde u_y)}\right)\Big|_{y=0},\end{aligned}$$

and

$$\eta_2|_{y=0}=\frac{u^s_{yy}+u_{yy}}{u_y^s+u_y}\Big|_{y=0}=0,\quad \partial_y^{-1}g_n(t,x,y)|_{y=0}=\frac{\partial_x^n u}{u_y^s+u_y}(t,x,y)\Big|_{y=0}=0,$$

we get then

$$(\partial_y g_n)|_{y=0}=0,\quad 0\le n\le m.$$

Finally, we have, for $j=1,2$,

$$\begin{cases}\partial_t g_n+(u^s+u)\partial_x g_n-\partial_y^2 g_n-\epsilon\partial_x^2 g_n\\ \qquad\qquad -\epsilon\,2\,(\partial_x\partial_y^{-1}g_n)\partial_y\eta_1=M_n,\\ (\partial_y g_n)|_{y=0}=0,\\ g_n|_{t=0}=g_{n,0},\end{cases}\tag{4.5}$$

with $M_n=\sum_{j=1}^6 M_j^n$,

$$\begin{aligned}M_1^n&=-(u^s+u)(g_n\eta_1+(\partial_y^{-1}g_n)\partial_y\eta_1),\\ M_2^n&=2(\partial_y g_n)\eta_2+2g_n(\partial_y\eta_2-2\eta_2^2)-8(\partial_y^{-1}g_n)\,\eta_2\partial_y\eta_2,\\ M_3^n&=\epsilon\big(2(\partial_x g_n)\eta_1-2g_n\,\eta_1^2-4(\partial_y^{-1}g_n)\eta_1\partial_y\eta_1\big),\\ M_4^n&=\partial_y\left(\partial_y^{-1}g_n\,\frac{(u^s+u)w_x+v(w_y+u^s_{yy})}{u_y^s+u_y}\right),\\ M_5^n&=-\partial_y\left(\frac{\sum_{i=1}^n C_n^i\partial_x^i u\cdot\partial_x^{n+1-i}u}{u_y^s+u_y}\right),\\ M_6^n&=-\partial_y\left(\frac{\sum_{i=1}^n C_n^i\partial_x^i w\cdot\partial_x^{n-i}v}{u_y^s+u_y}\right),\end{aligned}$$

where we have used the relation,

$$\partial_t u_y^s+\partial_t u_y-(u^s_{yyy}+u_{yyy})-\epsilon u_{xxy}=-(u^s+u)w_x+v(u^s_{yy}+w_y).$$

5. Uniform estimate

In the future application(see Lemma 6.3), we need that the weight of g_m big then $\frac12$, but from the definition, $w\in H^{m+2}_{k+\ell}(\mathbb{R}^2_+)$ imply only $g_m\in H^2_\ell(\mathbb{R}^2_+)$ with $0<\ell<\frac12$. So the first step is to improve this weights if the weight of the initial data is more big. We first have

Lemma 5.1. *If* $\tilde{w}_0 \in H^{m+2}_{k+\ell'}(\mathbb{R}^2_+), m \geq 6, k > 1, 0 < \ell < \frac{1}{2},\ \frac{1}{2} < \ell' < \ell+\frac{1}{2},\ k+\ell > \frac{3}{2}$ *which satisfies* (4.1)-(4.2) *with* $0 < \zeta \leq 1$, *then* $(g_m)(0) \in H^2_{k+\ell}(\mathbb{R}^2_+)$, *and we have*

$$\|(g_m)(0)\|_{H^2_{\ell'}(\mathbb{R}^2_+)} \leq C\|\tilde{w}_0\|_{H^{m+2}_{k+\ell''}(\mathbb{R}^2_+)}.$$

Remark. In fact, observing

$$g_m(0) = \left(\frac{\partial^m_x \tilde{u}_0}{u^s_{0,y} + \tilde{u}_{0,y}}\right)_y = \frac{\partial_y \partial^m_x \tilde{u}_0}{u^s_{0,y} + \tilde{u}_{0,y}} - \frac{\partial^m_x \tilde{u}_0}{u^s_{0,y} + \tilde{u}_{0,y}}\eta_2(0),$$

then (4.3) implies

$$\langle y\rangle^{k+\ell}|g_m(0)| \leq C\langle y\rangle^{k+\ell'}|\partial^m_x \tilde{w}_0| + C\langle y\rangle^{k+\ell'-1}|\partial^m_x \tilde{u}_0|,$$

which finishes the proof of this Lemma.

Proposition 5.2. *Let* $w \in L^\infty([0,T]; H^{m+2}_{k+\ell}(\mathbb{R}^2_+)), m \geq 6, k > 1, 0 \leq \ell < \frac{1}{2},\ \ell' > \frac{1}{2},\ \ell' - \ell < \frac{1}{2},\ k+\ell > \frac{3}{2}$, *satisfy* (4.1)-(4.2) *with* $0 < \zeta \leq 1$. *Assume that the shear flow* u^s *verifies the conclusion of Lemma 2.1, and* g_n *satisfies the equation* (4.5) *for* $1 \leq n \leq m$, *then we have the following estimates, for* $t \in [0,T]$

$$\begin{aligned}\frac{d}{dt}\sum_{n=1}^m \|g_n\|^2_{L^2_{\ell'}(\mathbb{R}^2_+)} + \sum_{n=1}^m \|\partial_y g_n\|^2_{L^2_{\ell'}(\mathbb{R}^2_+)} + \epsilon\sum_{n=1}^m \|\partial_x g_n\|^2_{L^2_{\ell'}(\mathbb{R}^2_+)}& \\ \leq C_2\Big(\sum_{n=1}^m \|g_n\|^2_{L^2_{\ell'}(\mathbb{R}^2_+)} + \|w\|^2_{H^m_{k+\ell}(\mathbb{R}^2_+)}\Big),&\end{aligned} \tag{5.1}$$

where C_2 *is independent of* ϵ.

Approach of the proof for the Proposition 5.2: We can't prove (5.1) directly, since the approximate solution w_ϵ obtained in Theorem 3.5 is belongs to $L^\infty([0,T_\epsilon]; H^{m+2}_{k+\ell}(\mathbb{R}^2_+))$, which implies only $g_n \in L^\infty([0,T_\epsilon]; H^2_\ell(\mathbb{R}^2_+))$. Then we can't use $\langle y\rangle^{2\ell'}g_n \in L^\infty([0,T_\epsilon]; H^2_{\ell-2\ell'}(\mathbb{R}^2_+))$ as the test function to the equation (4.5). To overcome this difficulty, we consider that (4.5) as a linear system for $g_n, n = 1, \cdots, m$ with the coefficients and the source terms depends on w and their derivatives up to order m, we will clarify this confirmation in the following proof of the the Proposition 5.2. We prove now the estimate (5.1) by the following approach: For the linear system (4.5), we prove firstly (5.1) as *à priori* estimate. Lemma 5.1 imply that $g_n(0) \in H^2_{\ell'}(\mathbb{R}^2_+), n = 1, \cdots, m$, then by using Hahn-Banach theorem, this *à priori* estimate imply the existence of solutions

$$g_n \in L^\infty([0,T]; H^2_{\ell'}(\mathbb{R}^2_+)), \quad n = 1, \cdots, m.$$

Finally, by uniqueness, we can prove the estimate (5.1) by proving it as *à priori* estimate. So that the proof of the Proposition 5.2 is reduced to the proof of the *à priori* estimate (5.1).

Proof of the *à priori* estimate (5.1). Multiplying the linear system (4.5) by $\langle y\rangle^{2\ell'}g_n \in L^\infty([0,T]; H^2_{-\ell'}(\mathbb{R}^2_+))$ and integrating over $\mathbb{R} \times \mathbb{R}^+$. We start to deal with the left hand of (4.5) first, we have

$$\int_{\mathbb{R}^2_+} \partial_t g_n\, \langle y\rangle^{2\ell'} g_n dxdy = \frac{1}{2}\frac{d}{dt}\|g_n\|^2_{L^2_{\ell'}(\mathbb{R}^2_+)},$$

and

$$\begin{aligned}\int_{\mathbb{R}^2_+}(u^s+u)\partial_x g_n\,\langle y\rangle^{2\ell'}g_n dxdy =&\frac{1}{2}\int_{\mathbb{R}^2_+}(u^s+u)\cdot\partial_x(\langle y\rangle^{2\ell'}g_n^2)dxdy\\ &\le\frac{1}{2}\|u_x\|_{L^\infty(\mathbb{R}^2_+)}\|g_n\|^2_{L^2_{\ell'}(\mathbb{R}^2_+)}\\ &\le C\|w\|_{H^2_1(\mathbb{R}^2_+)}\|g_n\|^2_{L^2_{\ell'}(\mathbb{R}^2_+)}.\end{aligned}$$

Integrating by part, where the boundary value is vanish,

$$\begin{aligned}-\int_{\mathbb{R}^2_+}\partial_y^2 g_n\,\langle y\rangle^{2\ell'}g_n dxdy &= \|\partial_y g_n\|^2_{L^2_{\ell'}(\mathbb{R}^2_+)}+\int_{\mathbb{R}^2_+}\partial_y g_n(\langle y\rangle^{2\ell'})' g_n dxdy\\ &\ge\frac{3}{4}\|\partial_y g_n\|^2_{L^2_{\ell'}(\mathbb{R}^2_+)}-4\|g_n\|^2_{L^2(\mathbb{R}^2_+)},\end{aligned}$$

and

$$-\epsilon\int_{\mathbb{R}^2_+}\partial_x^2 g_n\,\langle y\rangle^{2\ell'}g_n dxdy=\epsilon\|\partial_x g_n\|^2_{L^2_{\ell'}(\mathbb{R}^2_+)}.$$

We have also

$$\begin{aligned}&-\epsilon\int_{\mathbb{R}^2_+}\big(\partial_x\partial_y^{-1}g_n\big)\partial_y\eta_1\langle y\rangle^{2\ell'}g_n dxdy\\ &=\epsilon\int_{\mathbb{R}^2_+}\partial_y^{-1}g_n\partial_y\eta_1\langle y\rangle^{2\ell'}\partial_x g_n dxdy\\ &\quad+\epsilon\int_{\mathbb{R}^2_+}\partial_y^{-1}g_n(\partial_y\partial_x\eta_1)\langle y\rangle^{2\ell'}g_n dxdy\\ &\le\epsilon\|\partial_y^{-1}g_n\partial_y\eta_1\|^2_{L^2_{\ell'}(\mathbb{R}^2_+)}+\frac{\epsilon}{8}\|\partial_x g_n\|^2_{L^2_{\ell'}(\mathbb{R}^2_+)}\\ &\quad+\epsilon\|\partial_y^{-1}g_n\partial_y\partial_x\eta_1\|^2_{L^2(\mathbb{R}^2_+)}+\epsilon\|g_n\|^2_{L^2_{\ell'}(\mathbb{R}^2_+)}.\end{aligned}$$

So by (4.5) and $0<\epsilon\le 1$, we obtain

$$\begin{aligned}&\frac{d}{dt}\|g_n\|^2_{L^2_{\ell'}(\mathbb{R}^2_+)}+\|\partial_y g_n\|^2_{L^2_{\ell'}(\mathbb{R}^2_+)}+\epsilon\|\partial_x g_n\|^2_{L^2_{\ell'}(\mathbb{R}^2_+)}\\ &\le C\|g_n\|^2_{L^2_{\ell'}(\mathbb{R}^2_+)}+\|(\partial_y^{-1}g_n)\partial_y\eta_1\|^2_{L^2_{\ell'}(\mathbb{R}^2_+)}\\ &\qquad+\|(\partial_y^{-1}g_n)\partial_y\partial_x\eta_1\|^2_{L^2(\mathbb{R}^2_+)}+2\sum_{j=1}^{6}\Big|\int_{\mathbb{R}^2_+}M_j\,\langle y\rangle^{2\ell'}g_n dxdy\Big|.\end{aligned}$$

Then we can finish the proof of the *à priori* estimate (5.1) by the following four Lemmas. □

Lemma 5.3. *Under the assumption of Proposition 5.2, we have*

$$\|\partial_y^{-1}g_n\partial_y\eta_1\|^2_{L^2_{\ell'}(\mathbb{R}^2_+)}+\|\partial_y^{-1}g_n\partial_y\partial_x\eta_1\|^2_{L^2(\mathbb{R}^2_+)}\le C\|g_n\|^2_{L^2_{\ell'}(\mathbb{R}^2_+)}.$$

where $\tilde{C}$ is independent of ϵ.

Proof. Notice that (4.1) and (4.2) imply

$$\begin{aligned}&|\eta_1|\le C\langle y\rangle^{-\ell},\quad|\partial_x\eta_1|\le C\langle y\rangle^{-\ell},\\ &|\partial_y\eta_1|\le C\langle y\rangle^{-\ell-1},\quad|\partial_y\partial_x\eta_1|\le C\langle y\rangle^{-\ell-1}.\end{aligned}$$

Then $\ell' > \frac{1}{2}, \ell' - \ell < \frac{1}{2}$, imply

$$\begin{aligned}\|\partial_y^{-1} g_n(\partial_y\partial_x\eta_1)\|^2_{L^2_{\ell'}(\mathbb{R}^2_+)} &\leq C\int_{\mathbb{R}^2_+} \langle y\rangle^{2(\ell'-\ell-1)}\Big(\int_0^y g_n(t,x,\tilde{y})d\tilde{y}\Big)^2 dxdy\\ &\leq C\|g_n\|^2_{L^2_{\ell'}(\mathbb{R}^2_+)}.\end{aligned}$$

Similarly, we also obtain

$$\|\partial_y^{-1} g_n\partial_y\eta_1\|^2_{L^2_{\ell'}(\mathbb{R}^2_+)} \leq C\|g_n\|^2_{L^2_{\ell'}(\mathbb{R}^2_+)}.$$

□

Lemma 5.4. *Under the assumption of Proposition 5.2, we have*

$$\begin{aligned}\left|\int_{\mathbb{R}^2_+}\sum_{j=0}^4 \tilde{M}^n_j\,\langle y\rangle^{2\ell'} g_n dxdy\right| \leq &\frac{1}{8}\|\partial_y g_n\|^2_{L^2_{\ell'}(\mathbb{R}^2_+)} + \frac{\epsilon}{8}\|\partial_x g_n\|^2_{L^2_{\ell'}(\mathbb{R}^2_+)}\\ &+ \tilde{C}(\|g_n\|^2_{L^2_{\ell'}(\mathbb{R}^2_+)} + \|w\|^2_{H^m_{k+\ell}(\mathbb{R}^2_+)}),\end{aligned}$$

where $\tilde{C}$ is independent of ϵ.

Proof. Recalling $M^n_1 = -(u^s+u)\big(g_n\eta_1 + (\partial_y^{-1}g_n)\partial_y\eta_1\big)$, by Lemma 5.3,

$$\left|\int_{\mathbb{R}^2_+}(u^s+u)g_n\eta_1\,\langle y\rangle^{2\ell'} g_n dxdy\right| \leq C\|g_n\|^2_{L^2_{\ell'}(\mathbb{R}^2_+)},$$

$$\int_{\mathbb{R}^2_+}|(u^s+u)(\partial_y^{-1}g_n)\partial_y\eta_1\,\langle y\rangle^{2\ell'} g_n|dydx \leq C\|w\|^2_{H^n_{k+\ell}} + C\|g_n\|^2_{L^2_{\ell'}}.$$

Besides, we have

$$\left|\int_{\mathbb{R}^2_+} M^n_1\,\langle y\rangle^{2\ell'} g_n dxdy\right| \leq C(\|g_n\|^2_{L^2_{\ell'}(\mathbb{R}^2_+)} + \|w\|^2_{H^n_{\ell'}(\mathbb{R}^2_+)}).$$

The estimates of M^n_2 and M^n_3 needs the following decay rate of η_2:

$$\begin{aligned}&|\eta_2| \leq C\langle y\rangle^{-1}, \quad |\partial_x\eta_2| \leq C\langle y\rangle^{-\ell-1},\\ &|\partial_y\eta_2| \leq C\langle y\rangle^{-2}, \quad |\partial_y\partial_x\eta_2| \leq C\langle y\rangle^{-\ell-2}.\end{aligned}$$

Recall $M^n_2 = 2\partial_y g_n\eta_2 + 2g_n(\partial_y\eta_2 - 2\eta_2^2) - 8\partial_y^{-1}g_n\,\eta_2\partial_y\eta_2$. We have

$$\left|\int_{\mathbb{R}^2_+} g_n(\partial_y\eta_2 - \eta_2^2)\,\langle y\rangle^{2\ell'} g_n dxdy\right| \leq C\|g_n\|^2_{L^2_{\ell'}(\mathbb{R}^2_+)},$$

$$\left|\int_{\mathbb{R}^2_+}(\partial_y g_n)\eta_2\,\langle y\rangle^{2\ell'} g_n dxdy\right| \leq C\|g_n\|^2_{L^2_{\ell'}(\mathbb{R}^2_+)} + \frac{1}{8}\|\partial_y g_n\|^2_{L^2_{\ell'}(\mathbb{R}^2_+)},$$

$$\begin{aligned}&\left|2\int_{\mathbb{R}^2_+}\partial_y^{-1}g_n\eta_2\,\partial_y\eta_2\langle y\rangle^{2\ell'} g_n dxdy\right|\\ &\qquad \leq C\|\langle y\rangle^{\ell'-3}\partial_y^{-1}g_n\|^2_{L^2} + \|g_n\|^2_{L^2_{\ell'}(\mathbb{R}^2_+)} \leq C\|g_n\|^2_{L^2_{\ell'}(\mathbb{R}^2_+)}.\end{aligned}$$

All together, we conclude

$$\left|\int_{\mathbb{R}^2_+} M^n_2\langle y\rangle^{2\ell'} g_n dxdy\right| \leq C(\|g_n\|^2_{L^2_{\ell'}(\mathbb{R}^2_+)} + \|w\|^2_{H^n_{\ell'}(\mathbb{R}^2_+)}) + \frac{1}{8}\|\partial_y g_n\|^2_{L^2_{\ell'}(\mathbb{R}^2_+)},$$

and exactly same computation gives also

$$\left|\int_{\mathbb{R}^2_+} M_3^n \langle y\rangle^{2\ell'} g_n dxdy\right| \le C(\|g_n\|^2_{L^2_{\ell'}(\mathbb{R}^2_+)} + \|w\|^2_{H^n_{k+\ell}(\mathbb{R}^2_+)}) + \frac{\epsilon}{8}\|\partial_x g_n\|^2_{L^2_{\ell'}(\mathbb{R}^2_+)}.$$

Now using (4.1)-(4.2) and $m \ge 6$, with the same computation as above, we can get

$$\left|\int_{\mathbb{R}^2_+} M_4^n \langle y\rangle^{2\ell'} g_n dxdy\right| \le C\left(\|g_n\|^2_{L^2_{\ell'}(\mathbb{R}^2_+)} + \|w\|^2_{H^n_{k+\ell}(\mathbb{R}^2_+)}\right).$$

which finishes the proof of Lemma 5.4. □

Lemma 5.5. *Under the assumption of Proposition 5.2, we have*

$$\left|\int_{\mathbb{R}^2_+} M_5^n \,\langle y\rangle^{2\ell'} g_n dxdy\right| \le \tilde{C}\left(\sum_{p=1}^n \|\tilde{g}_p\|^2_{L^2_{\ell'}(\mathbb{R}^2_+)} + \|w\|^2_{H^n_{k+\ell}(\mathbb{R}^2_+)}\right),$$

where $\tilde{C}$ is independent of ϵ.

Proof. Recall,

$$\begin{aligned}\tilde{M}_5^n &= \sum_{i\ge 4} C_n^i g_i\, \partial_x^{n+1-i} u + \sum_{1\le i\le 3} C_n^i \partial_x^i u\, g_{n+1-i} \\ &\quad + \sum_{i\ge 4} C_n^i \partial_y^{-1} g_n \partial_x^{n+1-i} w + \sum_{1\le i\le 3} C_n^i \partial_x^i w\, \partial_y^{-1} g_{n+1-i},\end{aligned}$$

here if $n \le 3$, we have only the last term. Then, for $\|w\|_{H^m_{k+\ell}} \le \zeta \le 1, m \ge 6$,

$$\begin{aligned}&\sum_{i\ge 4} C_n^i \|g_i\, \partial_x^{n+1-i} u\|_{L^2_{\ell'}(\mathbb{R}^2_+)} + \sum_{1\le i\le 3} \|\partial_x^i u\, g_{n+1-i}\|_{L^2_{\ell'}(\mathbb{R}^2_+)} \\ &\le \sum_{i\ge 4} C_n^i \|g_i\|_{L^2_{\ell'}(\mathbb{R}^2_+)} \|\partial_x^{n+1-i} u\|_{L^\infty(\mathbb{R}^2_+)} \\ &\quad + \sum_{1\le i\le 3} C_n^i \|\partial_x^i u\|_{L^\infty(\mathbb{R}^2_+)} \|\tilde{g}_{n+1-i}\|_{L^2_{\ell'}(\mathbb{R}^2_+)} \\ &\le C \sum_{i\ge 4} C_n^i \|g_i\|_{L^2_{\ell'}(\mathbb{R}^2_+)} \|w\|_{H_1^{n+3-i}} \\ &\quad + C \sum_{1\le i\le 3} C_n^i \|w\|_{H_1^{i+3}} \|g_{n+1-i}\|_{L^2_{\ell'}(\mathbb{R}^2_+)} \\ &\le C \sum_{i=1}^n \|g_i\|_{L^2_{\ell'}}.\end{aligned}$$

Similarly, for the second line in M_5, by Lemma 5.3, we have

$$\begin{aligned}\sum_{i\ge 4} C_n^i \|(\partial_y^{-1} g_i)\partial_x^{n+1-i} w\|_{L^2_{\ell'}(\mathbb{R}^2_+)} &\le \sum_{i\ge 4} C_n^i \|\langle y\rangle^{\ell'-\ell-1} (\partial_y^{-1} g_i)\|_{L^2(\mathbb{R}^2_+)} \|\langle y\rangle^{\ell+1} \partial_x^{n+1-i} w\|_{L^\infty} \\ &\le C \sum_{i=1}^n \|g_i\|_{L^2_{\ell'}(\mathbb{R}^2_+)}.\end{aligned}$$

We have proven Lemma 5.5. □

Lemma 5.6. *Under the assumption of Proposition 5.2, we have*

$$\left|\int_{\mathbb{R}^2_+} M_6^n \langle y\rangle^{2\ell'} g_n dxdy\right|$$
$$\leq \frac{1}{8m}\sum_{p=1}^{n}\|\partial_y g_p\|^2_{L^2_{\ell'}(\mathbb{R}^2_+)} + \tilde{C}\left(\sum_{p=1}^{n}\|g_p\|^2_{L^2_{\ell'}(\mathbb{R}^2_+)} + \|w\|^2_{H^n_{k+\ell}(\mathbb{R}^2_+)}\right),$$

where $\tilde{C}$ is independents of ϵ.

Proof. Recall

$$M_6 = \sum_{i=1}^{n} C_n^i g_i \eta_2 \partial_x^{n-i} v + \sum_{i=1}^{n} C_n^i g_i \partial_x^{n+1-i} u + \sum_{i=1}^{n} C_n^i \partial_y g_i \partial_x^{n-i} v$$
$$+ \sum_{i=1}^{n} \partial_y^{-1} g_i \left(C_n^i \partial_x^{n-i} v \partial_y \eta_2 + C_n^i \partial_x^{n+1-i} u \eta_2\right).$$

In M_6^n, we just study the term $\partial_y g_1 \partial_x^{n-1} v$ as an example, the others terms are similar,

$$\int_{\mathbb{R}^2_+} \partial_y g_1 \partial_x^{n-1} v \langle y\rangle^{2\ell'} g_n = -\int_{\mathbb{R}^2_+} g_1 \partial_x^{n-1} v \langle y\rangle^{2\ell'} \partial_y g_n$$
$$+ \int_{\mathbb{R}^2_+} g_1 \partial_x^{n} u \langle y\rangle^{2\ell'} g_n dxdy,$$

$$\int_{\mathbb{R}^2_+} g_1 \partial_x^{n-1} v \langle y\rangle^{2\ell'} \partial_y g_n dxdy \leq \frac{1}{8m}\|\partial_y g_n\|^2_{L^2_{\ell'}} + C\|g_1 \partial_x^{n-1} v\|^2_{L^2_{\ell'}},$$

$$\begin{aligned}
\|g_1 \partial_x^{n-1} v\|^2_{L^2_{\ell'}} &\leq \sup_{x\in\mathbb{R}} \int_0^{+\infty} \langle y\rangle^{2\ell'} g_1^2 dy \ \sup_{y\in\mathbb{R}_+} \int_{-\infty}^{+\infty} \left|\int_0^y \partial_x^n u dz\right|^2 dy \\
&\leq \left(\|g_1\|^2_{L^2_{\ell'}(\mathbb{R}^2_+)} + \|\partial_x g_1\|^2_{L^2_{\ell'}(\mathbb{R}^2_+)}\right) \int_{-\infty}^{+\infty} \left|\int_0^{+\infty} |\partial_x^n u| dz\right|^2 dy \\
&\leq C\Big(\|g_1\|^2_{L^2_{\ell'}(\mathbb{R}^2_+)} + \|\partial_x g_1\|^2_{L^2_{\ell'}(\mathbb{R}^2_+)}\Big) \\
&\qquad \times \int_{-\infty}^{+\infty} \left|\int_0^{+\infty} \langle y\rangle^{-k-\ell+1} \langle y\rangle^{k+\ell-1} |\partial_x^n u| dz\right|^2 dy \\
&\leq C\Big(\|g_1\|^2_{L^2_{\ell'}(\mathbb{R}^2_+)} + \|g_2\|^2_{L^2_{\ell'}(\mathbb{R}^2_+)} + \|w\|^2_{H^m_{k+\ell}}\Big) \\
&\qquad \times \int_{-\infty}^{+\infty} \left|\int_0^{+\infty} \langle y\rangle^{-k-\ell+1} \langle y\rangle^{k+\ell-1} |\partial_x^n u| dz\right|^2 dy \\
&\leq C\sum_{i=1}^{2}\|g_i\|^2_{L^2_{\ell'}} + C\|w\|^2_{H^m_{k+\ell}}.
\end{aligned}$$

Here we have used Lemma 5.3 and

$$k+\ell-1 > \frac{1}{2},\ \ \|w\|_{H^m_{k+\ell}} \leq 1,$$

and

$$\partial_x g_j = g_{j+1} - g_j \eta_1 - \partial_y^{-1} g_n \cdot \partial_y \eta_1.$$

By the similar trick, we have completed the proof of this lemma. □

6. Existence of the solution

Now, we can conclude the following energy estimate for the sequence of approximate solutions.

Theorem 6.1. *Assume u^s satisfies Lemma 2.1. Let $m \geq 6$ be an even integer, $k + \ell > \frac{3}{2}, 0 < \ell < \frac{1}{2}, \quad \frac{1}{2} < \ell' < \ell + \frac{1}{2}, \,$, and $\tilde{u}_0 \in H^{m+3}_{k+\ell'-1}(\mathbb{R}^2_+)$ which satisfies the compatibility conditions (2.6)-(2.7). Suppose that $\tilde{w}_\epsilon \in L^\infty([0,T]; H^{m+2}_{k+\ell}(\mathbb{R}^2_+))$ is a solution to (3.7) such that*

$$\|\tilde{w}_\epsilon\|_{L^\infty([0,T]; H^m_{k+\ell}(\mathbb{R}^2_+)} \leq \zeta$$

with

$$0 < \zeta \leq 1, \quad C_m \zeta \leq \frac{\tilde{c}_1}{2},$$

where $0 < T \leq T_1$ and T_1 is the lifespan of shear flow u^s in the Lemma 2.1, C_m is the Sobolev embedding constant in (4.2). Then there exists $C_T > 0, \tilde{C}_T > 0$ such that,

$$\|\tilde{w}_\epsilon\|_{L^\infty([0,T]; H^m_{k+\ell}(\mathbb{R}^2_+))} \leq C_T \|\tilde{u}_0\|_{H^{m+1}_{k+\ell'-1}(\mathbb{R}^2_+)}, \tag{6.1}$$

where $C_T > 0$ is increasing with respect to $0 < T \leq T_1$ and independent of $0 < \epsilon \leq 1$.

Firstly, we collect some results to be used from Section 3 - 5. We come back to the notations with tilde and the sub-index ϵ. Then $g^\epsilon_m, h^\epsilon_m$ are the the functions defined by $\tilde{u}_\epsilon$. Under the hypothesis of Theorem 6.1, we have proven the estimates (3.13) and (5.1)

$$\begin{aligned} \frac{d}{dt} \|\tilde{w}_\epsilon\|^2_{H^{m,m-1}_{k+\ell}(\mathbb{R}^2_+)} &+ \|\partial_y \tilde{w}_\epsilon\|^2_{H^{m,m-1}_{k+\ell}(\mathbb{R}^2_+)} \\ &+ \epsilon \|\partial_x \tilde{w}_\epsilon\|^2_{H^{m,m-1}_{k+\ell}(\mathbb{R}^2_+)} \leq C_1 \|\tilde{w}_\epsilon\|^2_{H^m_{k+\ell}(\mathbb{R}^2_+)}, \end{aligned} \tag{6.2}$$

$$\begin{aligned} \frac{d}{dt} \sum_{n=1}^m \|g^\epsilon_n\|^2_{L^2_{\ell'}(\mathbb{R}^2_+)} &+ \sum_{n=1}^m \|\partial_y g^\epsilon_n\|^2_{L^2_{\ell'}(\mathbb{R}^2_+)} + \epsilon \sum_{n=1}^m \|\partial_x g^\epsilon_n\|^2_{L^2_{\ell'}(\mathbb{R}^2_+)} \\ &\leq C_2 \big(\sum_{n=1}^m \|g^\epsilon_n\|^2_{L^2_{\ell'}(\mathbb{R}^2_+)} + \|\tilde{w}_\epsilon\|^2_{H^m_{k+\ell}(\mathbb{R}^2_+)} \big) \,, \end{aligned} \tag{6.3}$$

Lemma 6.2. *For the inital date, we have*

$$\begin{aligned} T^\epsilon_m(g,w)(0) &= \sum_{n=1}^m \|g^\epsilon_n(0)\|^2_{L^2_{\ell'}(\mathbb{R}^2_+)} + \|\tilde{w}_\epsilon(0)\|^2_{H^{m,m-1}_{k+\ell}(\mathbb{R}^2_+)} \\ &\leq C \|\tilde{u}_0\|^2_{H^{m+1}_{k+\ell'-1}(\mathbb{R}^2_+)}, \end{aligned}$$

where C is independent of ϵ.

Proof. Notice for any $1 \leq n \leq m$,

$$g^\epsilon_n = \big(\frac{\partial^n_x \tilde{u}_\epsilon}{u^s_y + \tilde{w}_\epsilon} \big)_y = \frac{\partial^n_x \partial_y \tilde{u}_\epsilon}{u^s_y + \tilde{w}_\epsilon} - \frac{\partial^n_x \tilde{u}_\epsilon}{u^s_y + \tilde{w}_\epsilon} \eta_2,$$

and $\tilde{u}_\epsilon(0)=\tilde{u}_0$, then we deduce, for any $1\le n\le m$,

$$
\begin{aligned}
\|g_n^\epsilon(0)\|^2_{L^2_{\ell'}(\mathbb{R}^2_+)} &\le 2\left\|\frac{\partial_x^n\partial_y\tilde{u}_0}{u^s_{0,y}+\tilde{w}_0}\right\|^2_{L^2_{\ell'}(\mathbb{R}^2_+)} + 2\left\|\frac{\partial_x^n\tilde{u}_0}{u^s_{0,y}+\tilde{w}_0}\eta_2(0)\right\|^2_{L^2_{\ell'}(\mathbb{R}^2_+)}\\
&\le C\big(\|\partial_x^n\partial_y\tilde{u}_0\|^2_{L^2_{k+\ell'}(\mathbb{R}^2_+)}+\|\partial_x^n\tilde{u}_0\|^2_{L^2_{k+\ell'-1}(\mathbb{R}^2_+)}\big)\le C\|\tilde{u}_0\|^2_{H^{m+1}_{k+\ell'-1}(\mathbb{R}^2_+)}.
\end{aligned}
$$

□

From (6.2) and (6.3), we have

$$
\begin{aligned}
&\|g_m^\epsilon\|^2_{L^2_{\ell'}(\mathbb{R}^2_+)}+\|\tilde{w}_\epsilon\|^2_{H^{m,m-1}_{k+\ell}(\mathbb{R}^2_+)}\\
&\le C_8e^{C_2t}\int_0^t e^{-C_2\tau}\|\tilde{w}_\epsilon(\tau)\|^2_{H^m_{k+\ell}(\mathbb{R}^2_+)}d\tau+C_9e^{C_2t}\|\tilde{u}_0\|^2_{H^{m+1}_{k+\ell'-1}(\mathbb{R}^2_+)}.
\end{aligned}
\tag{6.4}
$$

Lemma 6.3. *We have also the following estimate :*

$$
\|\partial_x^m\tilde{w}_\epsilon\|^2_{L^2_{k+\ell}(\mathbb{R}^2_+)}\le\tilde{C}\|g_m^\epsilon\|^2_{L^2_{\ell'}}.
$$

where $\tilde{C}$ is independent of ϵ.

Proof. By the definition,

$$
\partial_x^m\tilde{u}_\epsilon(t,x,y)=(u^s_y+\tilde{w}_\epsilon)\int_0^y g_m^\epsilon(t,x,\tilde{y})d\tilde{y},\quad y\in\mathbb{R}_+,
$$

Therefore,

$$
\partial_x^m\tilde{w}=(u^s_{yy}+(\tilde{w}_\epsilon)_y)\int_0^y g_m^\epsilon(t,x,\tilde{y})d\tilde{y}-(u^s_y+\tilde{w}_\epsilon)g_m^\epsilon(t,x,y),\ \ y\ge 0,
$$

and

$$
\begin{aligned}
\|\partial_x^m\tilde{w}\|^2_{L^2_{k+\ell}}&\le C\int_{\mathbb{R}^2_+}\langle y\rangle^{2\ell-2}\left(\int_0^y g_m^\epsilon(t,x,z)dz\right)^2dxdy+\|g_m^\epsilon(t)\|^2_{L^2_{\ell'}(\mathbb{R}^2_+)}\\
&\le C\|g_m^\epsilon(t)\|^2_{L^2_{\ell'}(\mathbb{R}^2_+)},
\end{aligned}
$$

where we have used $\ell-1<-\frac{1}{2}$ and $\frac{1}{2}<\ell'$.

□

End of proof of Theorem 6.1. Combining (6.4), Lemma 6.2 and Lemma 6.3, we get, for any $t\in]0,T]$,

$$
\begin{aligned}
\|\tilde{w}_\epsilon(t)\|^2_{H^m_{k+\ell}(\mathbb{R}^2_+)}\le&\tilde{C}_8e^{C_2t}\int_0^t e^{-C_2\tau}\|\tilde{w}_\epsilon(\tau)\|^2_{H^m_{k+\ell}(\mathbb{R}^2_+)}d\tau\\
&+\tilde{C}_9e^{C_2t}\|\tilde{u}_0\|^2_{H^{m+1}_{k+\ell'-1}(\mathbb{R}^2_+)},
\end{aligned}
$$

with $\tilde{C}_8,\tilde{C}_9$ independent of $0<\epsilon\le 1$. We have by Gronwell's inequality that, for any $t\in]0,T]$,

$$
\|\tilde{w}_\epsilon(t)\|^2_{H^m_{k+\ell}(\mathbb{R}^2_+)}\le\tilde{C}_9e^{(C_2+\tilde{C}_8)t}\|\tilde{u}_0\|^2_{H^{m+1}_{k+\ell'-1}(\mathbb{R}^2_+)}.
$$

So it is enough to take

$$
C_T^2=\tilde{C}_9e^{(C_2+\tilde{C}_8)T}\tag{6.5}
$$

which gives (6.1), and C_T is increasing with respect to T. We finish the proof of Theorem 6.1. □

Theorem 6.4. *Assume u^s satisfies Lemma 2.1, and let $\tilde{u}_0 \in H^{m+3}_{k+\ell'-1}(\mathbb{R}^2_+)$, $m \geq 6$ be an even integer, $k > 1, 0 < \ell < \frac{1}{2},\ \frac{1}{2} < \ell' < \ell + \frac{1}{2},\ k + \ell > \frac{3}{2}$, and*

$$0 < \zeta \leq 1 \ \text{ with } C_m \zeta \leq \frac{\tilde{c}_1}{2},$$

where C_m is the Sobolev embedding constant. If there exists $0 < \zeta_0$ small enough such that,

$$\|\tilde{u}_0\|_{H^{m+1}_{k+\ell'-1}(\mathbb{R}^2_+)} \leq \zeta_0,$$

then, there exists $\epsilon_0 > 0$ and for any $0 < \epsilon \leq \epsilon_0$, the system (3.7) admits a unique solution $\tilde{w}_\epsilon$ such that

$$\|\tilde{w}_\epsilon\|_{L^\infty([0,T_1];H^m_{k+\ell}(\mathbb{R}^2_+))} \leq \zeta,$$

where T_1 is the lifespan of shear flow u^s in the Lemma 2.1.

Remark 6.5. *Under the uniform monotonic assumption (1.2), some results of above theorem holds for any fixed $T > 0$. But ζ_0 decreases as T increases, according to the (2.5).*

Proof. We fix $0 < \epsilon \leq 1$, then for any $\tilde{w}_0 \in H^{m+2}_{k+\ell}(\mathbb{R}^2_+)$, Theorem 3.5 ensures that, there exists $\epsilon_0 > 0$ and for any $0 < \epsilon \leq \epsilon_0$, there exits $T_\epsilon > 0$ such that the system (3.7) admits a unique solution $\tilde{w}_\epsilon \in L^\infty([0, T_\epsilon]; H^{m+2}_{k+\ell}(\mathbb{R}^2_+))$ which satisfies

$$\|\tilde{w}_\epsilon\|_{L^\infty([0,T_\epsilon];H^m_{k+\ell}(\mathbb{R}^2_+))} \leq \frac{4}{3}\|\tilde{w}_\epsilon(0)\|_{H^m_{k+\ell}(\mathbb{R}^2_+)} \leq 2\|\tilde{u}_0\|_{H^{m+1}_{k+\ell-1}(\mathbb{R}^2_+)}.$$

Now choose ζ_0 such that

$$\max\{2, C_{T_1}\}\zeta_0 \leq \frac{\zeta}{2}.$$

On the other hand, taking $\tilde{w}_\epsilon(T_\epsilon)$ as initial data for the system (3.7), Theorem 3.5 ensures that there exits $T'_\epsilon > 0$, which is defined by (3.19) with $\bar{\zeta} = \frac{\zeta}{2}$, such that the system (3.7) admits a unique solution $\tilde{w}'_\epsilon \in L^\infty([T_\epsilon, T_\epsilon + T'_\epsilon]; H^m_{k+\ell}(\mathbb{R}^2_+))$ which satisfies

$$\|\tilde{w}'_\epsilon\|_{L^\infty([T_\epsilon,T_\epsilon+T'_\epsilon];H^m_{k+\ell}(\mathbb{R}^2_+))} \leq \frac{4}{3}\|\tilde{w}_\epsilon(T_\epsilon)\|_{H^m_{k+\ell}(\mathbb{R}^2_+)} \leq \zeta.$$

Now, we extend $\tilde{w}_\epsilon$ to $[0, T_\epsilon + T'_\epsilon]$ by $\tilde{w}'_\epsilon$, then we get a solution $\tilde{w}_\epsilon \in L^\infty([0, T_\epsilon + T'_\epsilon]; H^m_{k+\ell}(\mathbb{R}^2_+))$ which satisfies

$$\|\tilde{w}_\epsilon\|_{L^\infty([0,T_\epsilon+T'_\epsilon];H^m_{k+\ell}(\mathbb{R}^2_+))} \leq \zeta.$$

So if $T_\epsilon + T'_\epsilon < T_1$, we can apply Theorem 6.1 to $\tilde{w}_\epsilon$ with $T = T_\epsilon + T'_\epsilon$, and use (6.1), this gives

$$\|\tilde{w}_\epsilon\|_{L^\infty([0,T_\epsilon+T'_\epsilon];H^m_{k+\ell}(\mathbb{R}^2_+))} \leq C_{T_1}\|\tilde{u}_0\|_{H^{m+1}_{k+\ell-1}(\mathbb{R}^2_+)} \leq \frac{\zeta}{2}.$$

Now taking $\tilde{w}_\epsilon(T_\epsilon + T'_\epsilon)$ as initial data for the system (3.7), applying again Theorem 3.5, for the same $T'_\epsilon > 0$, the system (3.7) admits a unique solution $\tilde{w}'_\epsilon \in L^\infty([T_\epsilon + T'_\epsilon, T_\epsilon + 2T'_\epsilon]; H^m_{k+\ell}(\mathbb{R}^2_+))$ which satisfies

$$\|\tilde{w}'_\epsilon\|_{L^\infty([T_\epsilon+T'_\epsilon,T_\epsilon+2T'_\epsilon];H^m_{k+\ell}(\mathbb{R}^2_+))} \leq \frac{4}{3}\|\tilde{w}_\epsilon(T_\epsilon + T'_\epsilon)\|_{H^m_{k+\ell}(\mathbb{R}^2_+)} \leq \zeta.$$

Now, we extend $\tilde{w}_\epsilon$ to $[0, T_\epsilon + 2T'_\epsilon]$ by $\tilde{w}'_\epsilon$, then we get a solution $\tilde{w}_\epsilon \in L^\infty([0, T_\epsilon + 2T'_\epsilon]; H^m_{k+\ell}(\mathbb{R}^2_+))$ which satisfies

$$\|\tilde{w}_\epsilon\|_{L^\infty([0,T_\epsilon+2T'_\epsilon];H^m_{k+\ell}(\mathbb{R}^2_+))} \leq \zeta.$$

So if $T_\epsilon + 2T'_\epsilon < T_1$, we can apply Theorem 6.1 to $\tilde{w}_\epsilon$ with $T = T_\epsilon + 2T'_\epsilon$, and use (6.1), this gives again

$$\|\tilde{w}_\epsilon\|_{L^\infty([0,T_\epsilon+2T'_\epsilon];H^m_{k+\ell}(\mathbb{R}^2_+))} \leq C_{T_1}\|\tilde{u}_0\|_{H^{m+1}_{k+\ell-1}(\mathbb{R}^2_+)} \leq \frac{\zeta}{2}.$$

Then by recurrence, we can extend the solution $\tilde{w}_\epsilon$ to $[0, T_1]$, and then the lifespan of approximate solution is equal to that of shear flow if the initial date $\tilde{u}_0$ is small enough. □

We have obtained the following estimate, for $m \geq 6$ and $0 < \epsilon \leq \epsilon_0$,

$$\|\tilde{w}_\epsilon(t)\|_{H^m_{k+\ell}(\mathbb{R}^2_+)} \leq \zeta, \quad t \in [0, T_1].$$

By using the equation (3.7) and the Sobolev inequality, we get, for $0 < \delta < 1$

$$\|\tilde{w}_\epsilon\|_{Lip([0,T_1];C^{2,\delta}(\mathbb{R}^2_+))} \leq M < +\infty.$$

Then taking a subsequence, we have, for $0 < \delta' < \delta$,

$$\tilde{w}_\epsilon \to \tilde{w}\ (\epsilon \to 0),\ \text{locally strong in}\ \ C^0([0,T_1]; C^{2,\delta'}(\mathbb{R}^2_+))\,,$$

and

$$\partial_t\tilde{w} \in L^\infty([0,T_1]; H^{m-2}_{k+\ell}(\mathbb{R}^2_+)), \quad \tilde{w} \in L^\infty([0,T_1]; H^m_{k+\ell}(\mathbb{R}^2_+)),$$

with

$$\|\tilde{w}\|_{L^\infty([0,T_1];H^m_{k+\ell}(\mathbb{R}^2_+))} \leq \zeta.$$

Then we have

$$\tilde{u} = \partial_y^{-1} w \in L^\infty([0,T_1]; H^m_{k+\ell-1}(\mathbb{R}^2_+)),$$

where we use the Hardy inequality (A.1), since

$$\lim_{y\to+\infty} \tilde{u}(t,x,y) = -\lim_{y\to+\infty}\int_y^{+\infty} \tilde{w}(t,x,\tilde{y})d\tilde{y} = 0.$$

In fact, we also have

$$\lim_{y\to 0} \tilde{u}(t,x,y) = \lim_{y\to 0}\int_0^y \tilde{w}(t,x,\tilde{y})d\tilde{y} = 0.$$

Using the condition $k + \ell - 1 > \frac{1}{2}$, we have also

$$\tilde{v} = -\int_0^y \tilde{u}_x\, d\tilde{y} \in L^\infty([0,T_1]; L^\infty(\mathbb{R}_{+,y}); H^{m-1}(\mathbb{R}_x)).$$

We have proven that, $\tilde{w}$ is a classical solution to the following vorticity Prandtl equation

$$\begin{cases} \partial_t\tilde{w} + (u^s + \tilde{u})\partial_x\tilde{w} + \tilde{v}\partial_y(u^s_y + \tilde{w}) = \partial_y^2\tilde{w}, \\ \partial_y\tilde{w}|_{y=0} = 0, \\ \tilde{w}|_{t=0} = \tilde{w}_0, \end{cases}$$

and $(\tilde{u}, \tilde{v})$ is a classical solution to (2.2). Finally, $(u, v) = (u^s + \tilde{u}, \tilde{v})$ is a classical solution to (1.1), and satisfies (6.6). In conclusion, we have proved the following theorem which is the existence part of main Theorem 1.1.

Theorem 6.6. *Let $m \geq 6$ be an even integer, $k > 1, 0 < \ell < \frac{1}{2}$, $\frac{1}{2} < \ell' < \ell + \frac{1}{2}$, $k + \ell > \frac{3}{2}$, assume that u_0^s satisfies (1.2), the initial date $\tilde{u}_0 \in H^{m+3}_{k+\ell'-1}(\mathbb{R}^2_+)$ and $\tilde{u}_0$ satisfies the compatibility condition (2.6)-(2.7) up to order $m+2$. Then there exists $T > 0$ such that if*

$$\|\tilde{u}_0\|_{H^{m+1}_{k+\ell'-1}(\mathbb{R}^2_+)} \leq \delta_0,$$

for some $\delta_0 > 0$ small enough, then the initial-boundary value problem (2.2) admits a solution $(\tilde{u}, \tilde{v})$ with

$$\tilde{u} \in L^\infty([0,T]; H^m_{k+\ell-1}(\mathbb{R}^2_+)), \quad \partial_y \tilde{u} \in L^\infty([0,T]; H^m_{k+\ell}(\mathbb{R}^2_+)).$$

Moreover, we have the following energy estimate,

$$\|\partial_y \tilde{u}\|_{L^\infty([0,T]; H^m_{k+\ell}(\mathbb{R}^2_+))} \leq C \|\tilde{u}_0\|^2_{H^{m+1}_{k+\ell'-1}(\mathbb{R}^2_+)}. \tag{6.6}$$

7. Uniqueness and stability

Now, we study the stability of solutions which implies immediately the uniqueness of solution.

Let $\tilde{u}^1, \tilde{u}^2$ be two solutions obtained in Theorem 6.6 with respect to the initial date $\tilde{u}_0^1, \tilde{u}_0^2$ respectively. Denote $\bar{u} = \tilde{u}^1 - \tilde{u}^2$ and $\bar{v} = \tilde{v}^1 - \tilde{v}^2$, then

$$\begin{cases} \partial_t \bar{u} + (u^s + \tilde{u}_1)\partial_x \bar{u} + (u^s_y + \tilde{u}_{1,y})\bar{v} = \partial_y^2 \bar{u} - \tilde{v}_2 \partial_y \bar{u} - (\partial_x \tilde{u}_2)\bar{u}, \\ \partial_x \bar{u} + \partial_y \bar{v} = 0, \\ \bar{u}|_{y=0} = \bar{v}|_{y=0} = 0, \\ \bar{u}|_{t=0} = \tilde{u}_0^1 - \tilde{u}_0^2. \end{cases}$$

So it is a linear equation for $\bar{u}$. We also have for the vorticity $\bar{w} = \partial_y \bar{u}$,

$$\begin{cases} \partial_t \bar{w} + (u^s + \tilde{u}_1)\partial_x \bar{w} + (u^s_{yy} + \tilde{w}_{1,y})\bar{v} = \partial_y^2 \bar{w} - \tilde{v}_2 \partial_y \bar{w} - (\partial_x \tilde{w}_2)\bar{u}, \\ \partial_y \bar{w}|_{y=0} = 0, \\ \bar{w}|_{t=0} = \tilde{w}_0^1 - \tilde{w}_0^2. \end{cases} \tag{7.1}$$

Estimate with a loss of x-derivative. Firstly, for the vorticity $\bar{w} = \partial_y \bar{u}$, we deduce an energy estimate with a loss of x-derivative with the anisotropic norm defined by (3.12).

Proposition 7.1. *Let $\tilde{u}^1, \tilde{u}^2$ be two solutions obtained in Theorem 6.6 with respect to the initial date $\tilde{u}_0^1, \tilde{u}_0^2$, then we have*

$$\frac{d}{dt}\|\bar{w}\|^2_{H^{m-2,m-3}_{k+\ell}(\mathbb{R}^2_+)} + \|\partial_y \bar{w}\|^2_{H^{m-2,m-3}_{k+\ell}(\mathbb{R}^2_+)} \leq \bar{C}_1 \|\bar{w}\|^2_{H^{m-2}_{k+\ell}}, \tag{7.2}$$

where the constant $\bar{C}_1$ depends on the norm of $\tilde{w}^1, \tilde{w}^2$ in $L^\infty([0,T]; H^m_{k+\ell}(\mathbb{R}^2_+))$.

Proof. The proof of this Proposition is similar to the proof of the Proposition 3.9, and we need to use that $m-2$ is even. We only give the calculation for the terms which need a different argument. Moreover we also explain why we only get the estimate on $\|\bar{w}\|^2_{H^{m-2}_{k+\ell}}$ but require the norm of $\tilde{w}^1, \tilde{w}^2$ in $L^\infty([0,T]; H^m_{k+\ell}(\mathbb{R}^2_+))$. With out loss of the generality, we suppose that $\|\bar{w}\|_{H^{m-2}_{k+\ell}} \leq 1$, $\|\tilde{w}^1\|_{H^m_{k+\ell}} \leq 1$ and $\|\tilde{w}^2\|_{H^m_{k+\ell}} \leq 1$.

Derivating the equation of (7.1) with $\partial^\alpha=\partial_x^{\alpha}\partial_y^{\alpha_2}$, for $|\alpha|=\alpha_1+\alpha_2\le m-2, \alpha_1\le m-3$,

$$\begin{aligned}\partial_t\partial^\alpha\bar w-\partial_y^2\partial^\alpha\partial\bar w=&-\partial^\alpha\big((u^s+\tilde u_1)\partial_x\bar w+\tilde v_2\partial_y\bar w\\&+(u^s_{yy}+\tilde w_{1,y})\bar v+(\partial_x\tilde w_2)\bar u\big).\end{aligned}\tag{7.3}$$

Multiplying the above equation with $\langle y\rangle^{k+\ell'+\alpha_2}\partial^\alpha\bar w$, the same computation as in the proof of the Proposition 3.9, in particular, the reduction of the boundary-data are the same, gives

$$\begin{aligned}&\int_{\mathbb{R}^2_+}\Big(\partial_t\partial^\alpha\bar w-\partial_y^2\partial^\alpha\bar w\Big)\langle y\rangle^{2(k+\ell+\alpha_2)}\partial^\alpha\bar w dxdy\\&\ge\frac12\frac{d}{dt}\|\partial^\alpha\bar w\|^2_{L^2_{k+\ell+\alpha_2}}+\frac34\|\partial_y\bar w\|^2_{H^{m-2,m-3}_{k+\ell}}-C\|\bar w\|^2_{H^{m-2}_{k+\ell}}.\end{aligned}$$

As for the right hand of (7.3), for the first item, we split it into two parts

$$-\partial^\alpha\Big((u^s+\tilde u_1)\partial_x\bar w\Big)=-(u^s+\tilde u_1)\partial_x\partial^\alpha\bar w+[(u^s+\tilde u_1),\partial^\alpha]\partial_x\bar w.$$

Firstly, we have

$$\left|\int_{\mathbb{R}^2_+}\big((u^s+\tilde u_1)\partial_x\partial^\alpha\bar w\big)\langle y\rangle^{2(\ell+\alpha_2)}\partial^\alpha\bar w dxdy\right|\le\|\tilde w_1\|_{H^3_1}\|\partial^\alpha\bar w\|^2_{L^2_{k+\ell+\alpha_2}}.$$

For the commutator operator, we have,

$$\|[(u^s+\tilde u_1),\partial^\alpha]\partial_x\tilde w_\epsilon\|_{L^2_{k+\ell+\alpha_2}}\le C\|\tilde w_1\|_{H^{m-2}_{k+\ell}(\mathbb{R}^2_+)}\|\bar w\|_{H^{m-2,m-3}_{k+\ell}(\mathbb{R}^2_+)}.$$

Notice that for this term, we don't have the loss of x-derivative.

With the similar method for the terms $\tilde v_2\partial_y\bar w$, we get

$$\left|\int_{\mathbb{R}^2_+}\tilde v_2\partial_y\bar w\langle y\rangle^{2(\ell+\alpha_2)}\partial^\alpha\bar w dxdy\right|\le\|\tilde w_2\|_{H^{m-2}_{k+\ell}(\mathbb{R}^2_+)}\|\bar w\|^2_{H^{m-2,m-3}_{k+\ell}(\mathbb{R}^2_+)}.$$

For the next one, we have

$$\partial^\alpha\Big((u^s_{yy}+\partial_y\tilde w_1)\bar v\Big)=\sum_{\beta\le\alpha}C^\alpha_\beta\,\partial^\beta(u^s_{yy}+\partial_y\tilde w_1)\partial^{\alpha-\beta}\bar v,$$

and thus

$$\begin{aligned}&\left\|\sum_{\beta\le\alpha,1\le|\beta|<|\alpha|}C^\alpha_\beta\,\partial^\beta(u^s_{yy}+\partial_y\tilde w_1)\partial^{\alpha-\beta}\bar v\right\|_{L^2_{k+\ell+\alpha_2}}\\&\qquad\le C\|\tilde w_1\|_{H^{m-2}_{k+\ell}(\mathbb{R}^2_+)}\|\bar w\|_{H^{m-2,m-3}_{k+\ell}(\mathbb{R}^2_+)}.\end{aligned}$$

On the other hand, using Lemma A.1 and $\frac32-k<\ell<\frac12$,

$$\begin{aligned}&\big\|\big(\partial^\alpha(u^s_{yy}+\partial_y\tilde w_1)\big)\bar v\big\|_{L^2_{k+\ell+\alpha_2}}\le\big\|\big(\partial^\alpha u^s_{yy}\big)\bar v\big\|_{L^2_{k+\ell+\alpha_2}}+\big\|\big(\partial^\alpha\partial_y\tilde w_1\big)\bar v\big\|_{L^2_{k+\ell+\alpha_2}}\\&\quad\le C\,\|\bar v\|_{L^2(\mathbb{R}_x;L^\infty(\mathbb{R}_+))}+C\|\tilde w_1\|_{H^m_{k+\ell}(\mathbb{R}^2_+)}\,\|\bar v\|_{L^\infty(\mathbb{R}^2_+)}\\&\quad\le C\,\|\bar u_x\|_{L^2_{\frac12+\delta}(\mathbb{R}^2_+)}+C\|\tilde w_1\|_{H^m_{k+\ell}(\mathbb{R}^2_+)}(\|\bar u_x\|_{L^2_{\frac12+\delta}(\mathbb{R}^2_+)}+\|\bar u_{xx}\|_{L^2_{\frac12+\delta}(\mathbb{R}^2_+)})\\&\quad\le C(1+\|\tilde w_1\|_{H^m_{k+\ell}(\mathbb{R}^2_+)})\,\|\bar w\|_{H^2_{\frac12+\delta}(\mathbb{R}^2_+)}\\&\quad\le C(1+\|\tilde w_1\|_{H^m_{k+\ell}(\mathbb{R}^2_+)})\,\|\bar w\|_{H^2_{k+\ell}(\mathbb{R}^2_+)}\,.\end{aligned}$$

So this term requires the norms $\|\tilde{w}_1\|_{H^m_{k+\ell}(\mathbb{R}^2_+)})$.

Moreover, if $\alpha_2 \neq 0$

$$\begin{aligned}\big\|(u^s_{yy} + \partial_y \tilde{w}_1)\partial^\alpha \bar{v}\big\|_{L^2_{k+\ell+\alpha_2}} &= \big\|(u^s_{yy} + \partial_y \tilde{w}_1)\partial^{\alpha_1}_x \partial^{\alpha_2 - 1} \bar{u}_x\big\|_{L^2_{k+\ell+\alpha_2}} \\ &\le C(1 + \|\tilde{w}_1\|_{H^{m-1}_{k+\ell}(\mathbb{R}^2_+)})\, \|\bar{w}\|_{H^{m-2}_{k+\ell}(\mathbb{R}^2_+)}\,,\end{aligned}$$

and also if $\alpha_2 = 0$

$$\begin{aligned}\big\|(u^s_{yy} + \partial_y \tilde{w}_1)\partial^{\alpha_1}_x \bar{v}\big\|_{L^2_{k+\ell}} &= \big\|(u^s_{yy} + \partial_y \tilde{w}_1)\partial^{-1}_y \partial^{\alpha_1}_x \bar{u}_x\big\|_{L^2_{k+\ell}} \\ &\le C(1 + \|\tilde{w}_1\|_{H^{m-1}_{k+\ell}(\mathbb{R}^2_+)})\, \big\|\partial^{\alpha_1+1}_x \bar{w}\big\|_{L^2_{\frac{3}{2}+\delta}(\mathbb{R}^2_+)}\,.\end{aligned}$$

These two cases imply the loss of x-derivative.

Similar argument also gives

$$\left|\int_{\mathbb{R}^2_+} \big(\partial^\alpha(\partial_x \tilde{w}_2)\bar{u}\big)\langle y\rangle^{2(\ell+\alpha_2)}\partial^\alpha \bar{w}dxdy\right| \le C\|\tilde{w}_2\|_{H^m_{k+\ell}(\mathbb{R}^2_+)}\|\bar{w}\|^2_{H^{m-2}_{k+\ell}(\mathbb{R}^2_+)},$$

which finishes the proof of the Proposition 7.1. □

Estimate on the loss term. To close the estimate (6.6), we need to study the terms $\|\partial^{m-2}_x \bar{w}\|_{L^2_{k+\ell}(\mathbb{R}^2_+)}$ which is missing in the left hand side of (7.2).

Similar to the argument in Section 6, we will recover this term by the estimate of functions

$$\bar{g}_n = \left(\frac{\partial^n_x \bar{u}}{u^s_y + \tilde{u}_{1,y}}\right)_y, \quad \forall (t,x,y) \in [0,T]\times\mathbb{R}\times\mathbb{R}^+.$$

Proposition 7.2. *Let $\tilde{u}^1, \tilde{u}^2$ be two solutions obtained in Theorem 6.6 with respect to the initial date $\tilde{u}^1_0, \tilde{u}^2_0$, then we have*

$$\begin{aligned}\frac{d}{dt}\sum_{n=1}^{m-2}\|\bar{g}_n\|^2_{L^2_{\ell'}(\mathbb{R}^2_+)} + \sum_{n=1}^{m-2}\|\partial_y \bar{g}_n\|^2_{L^2_{\ell'}(\mathbb{R}^2_+)} \\ \le C_2\Big(\sum_{n=1}^{m-2}\|\bar{g}_n\|^2_{L^2_{\ell'}(\mathbb{R}^2_+)} + \|\bar{w}\|^2_{H^{m-2}_{k+\ell}}\Big),\end{aligned}$$

where the constant $\bar{C}_2$ depends on the norm of $\tilde{w}^1, \tilde{w}^2$ in $L^\infty([0,T]; H^m_{k+\ell}(\mathbb{R}^2_+))$.

These Propositions can be proven by using exactly the same calculation as in Section 5. The only difference is that when we use the Leibniz formula, for the term where the order of derivatives is $|\alpha| = m-2$, it acts on the coefficient which depends on $\tilde{u}^1, \tilde{u}^2$. Therefore, we need their norm in the order of $(m-2)+1$. So we omit the proof of this Proposition here.

With the similar argument to the proof of Theorem 6.1, we get

$$\|\bar{w}\|_{L^\infty([0,T];H^{m-2}_{k+\ell}(\mathbb{R}^2_+))} \le C\|\bar{u}_0\|_{H^{m+1}_{k+\ell'-1}(\mathbb{R}^2_+)},$$

which finishes the proof of Theorem 1.1.

Appendix A. Some inequalities

We will use the following Hardy type inequalities.

Lemma A.1. *Let $f:\mathbb{R}\times\mathbb{R}^+\to\mathbb{R}$. Then*

(i) *if $\lambda>-\frac{1}{2}$ and $\lim\limits_{y\to\infty} f(x,y)=0$, then*

$$\|\langle y\rangle^{\lambda} f\|_{L^2(\mathbb{R}^2_+)}\le C_\lambda\|\langle y\rangle^{\lambda+1}\partial_y f\|_{L^2(\mathbb{R}^2_+)};\tag{A.1}$$

(ii) *if $-1\le\lambda<-\frac{1}{2}$ and $f(x,0)=0$, then*

$$\|\langle y\rangle^{\lambda} f\|_{L^2(\mathbb{R}^2_+)}\le C_\lambda\|\langle y\rangle^{\lambda+1}\partial_y f\|_{L^2(\mathbb{R}^2_+)}.$$

Here $C_\lambda\to+\infty$, as $\lambda\to-\frac{1}{2}$.

We need the following trace theorem in the weighted Sobolev space.

Lemma A.2. *Let $\lambda>\frac{1}{2}$, then there exists $C>0$ such that for any function f defined on $\mathbb{R}^2_+$, if $\partial_y f\in L^2_\lambda(\mathbb{R}^2_+)$, it admits a trace on $\mathbb{R}_x\times\{0\}$, and satisfies*

$$\|\gamma_0(f)\|_{L^2(\mathbb{R}_x)}\le C\|\partial_y f\|_{L^2_\lambda(\mathbb{R}^2_+)},$$

where $\gamma_0(f)(x)=f(x,0)$ is the trace operator.

The proof of the above two Lemmas is elementary, so we leave it to the reader.
We use also the following Sobolev inequality and algebraic properties of $H^m_{k+\ell}(\mathbb{R}^2_+)$,

Lemma A.3. *For the suitable functions f,g, we have:*
1) If the function f satisfies $f(x,0)=0$ or $\lim_{y\to+\infty} f(x,y)=0$, then for any small $\delta>0$,

$$\|f\|_{L^\infty(\mathbb{R}^2_+)}\le C(\|f_y\|_{L^2_{\frac{1}{2}+\delta}(\mathbb{R}^2_+)}+\|f_{xy}\|_{L^2_{\frac{1}{2}+\delta}(\mathbb{R}^2_+)}).\tag{A.2}$$

2) For $m\ge 6, k+\ell>\frac{3}{2}$, and any $\alpha,\beta\in\mathbb{N}^2$ with $|\alpha|+|\beta|\le m$, we have

$$\|(\partial^\alpha f)(\partial^\beta g)\|_{L^2_{k+\ell+\alpha_2+\beta_2}(\mathbb{R}^2_+)}\le C\|f\|_{H^m_{k+\ell}(\mathbb{R}^2_+)}\|g\|_{H^m_{k+\ell}(\mathbb{R}^2_+)}.\tag{A.3}$$

3) For $m\ge 6, k+\ell>\frac{3}{2}$, and any $\alpha\in\mathbb{N}^2, p\in\mathbb{N}$ with $|\alpha|+p\le m$, we have,

$$\|(\partial^\alpha f)(\partial^p_x(\partial_y^{-1} g))\|_{L^2_{k+\ell+\alpha_2}(\mathbb{R}^2_+)}\le C\|f\|_{H^m_{k+\ell}(\mathbb{R}^2_+)}\|g\|_{H^m_{\frac{1}{2}+\delta}(\mathbb{R}^2_+)},$$

where ∂_y^{-1} is the inverse of derivative ∂_y, meaning, $\partial_y^{-1}g=\int_0^y g(x,\tilde{y})\,d\tilde{y}$.

Proof. For (1), using $f(x,0)=0$, we have

$$\begin{aligned}\|f\|_{L^\infty(\mathbb{R}^2_+)}&=\left\|\int_0^y(\partial_y f)(x,\tilde{y})\,d\tilde{y}\right\|_{L^\infty(\mathbb{R}^2_+)}\le C\|\partial_y f\|_{L^\infty(\mathbb{R}_x;L^2_{\frac{1}{2}+\delta}(\mathbb{R}_+))}\\&\le C(\|\partial_y f\|_{L^2_{\frac{1}{2}+\delta}(\mathbb{R}^2_+)}+\|\partial_x\partial_y f\|_{L^2_{\frac{1}{2}+\delta}(\mathbb{R}^2_+)}).\end{aligned}$$

If $\lim_{y\to+\infty} f(x,y)=0$, we use

$$f(x,y)=-\int_y^\infty(\partial_y f)(x,\tilde{y})\,d\tilde{y}.$$

For (2), firstly, $m \geq 6$ and $|\alpha| + |\beta| \leq m$ imply $|\alpha| \leq m-2$ or $|\beta| \leq m-2$, without loss of generality, we suppose that $|\alpha| \leq m-2$. Then, using the conclusion of (1), we have

$$\begin{aligned}\|(\partial^\alpha f)(\partial^\beta g)\|_{L^2_{k+\ell+\alpha_2+\beta_2}(\mathbb{R}^2_+)} &\leq \|\langle y\rangle^{\alpha_2}(\partial^\alpha f)\|_{L^\infty(\mathbb{R}^2_+)}\|\partial^\beta g\|_{L^2_{k+\ell+\beta_2}(\mathbb{R}^2_+)}\\ &\leq C\|f\|_{H^{|\alpha|+2}_{\frac12+\delta}(\mathbb{R}^2_+)}\|\partial^\beta g\|_{L^2_{k+\ell+\beta_2}(\mathbb{R}^2_+)},\end{aligned}$$

which give (A.3).

For (3), if $|\alpha| \leq m-2$, we have

$$\begin{aligned}&\|(\partial^\alpha f)(\partial^p_x(\partial_y^{-1}g))\|_{L^2_{k+\ell+\alpha_2}(\mathbb{R}^2_+)}\\ &\qquad\leq \|\langle y\rangle^{k+\ell+\alpha_2}(\partial^\alpha f)\|_{L^2(\mathbb{R}_{y,+};L^\infty(\mathbb{R}_x))}\|\partial^p_x(\partial_y^{-1}g)\|_{L^\infty(\mathbb{R}_{y,+};L^2(\mathbb{R}_x))}\\ &\qquad\leq C\|f\|_{H^{|\alpha|+2}_{k+\ell}(\mathbb{R}^2_+)}\|\partial^p_x g\|_{L^2_{\frac12+\delta}(\mathbb{R}^2_+)}.\end{aligned}$$

If $p \leq m-2$, we have

$$\begin{aligned}&\|(\partial^\alpha f)(\partial^p_x(\partial_y^{-1}g))\|_{L^2_{k+\ell+\alpha_2}(\mathbb{R}^2_+)}\\ &\qquad\leq \|\langle y\rangle^{k+\ell+\alpha_2}(\partial^\alpha f)\|_{L^2(\mathbb{R}^2_+)}\|\partial^p_x(\partial_y^{-1}g)\|_{L^\infty(\mathbb{R}^2_+)}\\ &\qquad\leq C\|f\|_{H^{|\alpha|}_{k+\ell}(\mathbb{R}^2_+)}\|\partial^p_x g\|_{L^\infty(\mathbb{R}_x;L^2_{\frac12+\delta}(\mathbb{R}_{y,+}))}\\ &\qquad\leq C\|f\|_{H^{|\alpha|}_{k+\ell}(\mathbb{R}^2_+)}\|g\|_{H^m_{\frac12+\delta}(\mathbb{R}^2_+)}.\end{aligned}$$

We have completed the proof of the Lemma. □

Appendix B. The existence of approximate solutions

Now, we prove the Proposition 3.7, the existence of solution to the vorticity equation $\tilde w_\epsilon = \partial_y \tilde u_\epsilon$ and suppose that m, k, ℓ and $u^s(t,y)$ satisfy the assumption of Proposition 3.7,

$$\begin{cases}\partial_t \tilde w_\epsilon + (u^s + \tilde u_\epsilon)\partial_x \tilde w_\epsilon + v_\epsilon(u^s_{yy} + \partial_y \tilde w_\epsilon) = \partial^2_y \tilde w_\epsilon + \epsilon\partial^2_x \tilde w_\epsilon,\\ \partial_y \tilde w_\epsilon|_{y=0} = 0\\ \tilde w_\epsilon|_{t=0} = \tilde w_{0,\epsilon},\end{cases} \tag{B.1}$$

where

$$\tilde u_\epsilon(t,x,y) = -\int_y^{+\infty}\tilde w_\epsilon(t,x,\tilde y)d\tilde y,\quad \tilde v_\epsilon(t,x,y) = -\int_0^y \partial_x \tilde u_\epsilon(t,x,\tilde y)d\tilde y.$$

We will use the following iteration process to prove the existence of solution, where $w^0 = \tilde w_{0,\epsilon}$,

$$\begin{cases}\partial_t w^n + (u^s + u^{n-1})\partial_x w^n + (u^s_{yy} + \partial_y w^{n-1})v^n = \partial^2_y w^n + \epsilon\partial^2_x w^n,\\ \partial_y w^n|_{y=0} = 0\\ w^n|_{t=0} = \tilde w_{0,\epsilon},\end{cases} \tag{B.2}$$

with

$$u^{n-1}(t,x,y) = -\int_y^{+\infty} w^{n-1}(t,x,\tilde y)d\tilde y,$$

and

$$v^n(t,x,y) = -\int_0^y \partial_x u^n(t,x,\tilde y)d\tilde y$$

$$= \int_0^y \int_{\tilde{y}}^{+\infty} \partial_x w^n(t,x,z) dz d\tilde{y}.$$

Here for the boundary data, we have

$$\partial_y^3 w^n|_{y=0} = ((u_y^s + w^{n-1})\partial_x w^n)|_{y=0},$$

$$\begin{aligned}
&(\partial_y^5 w^n)(t,x,0)\\
&= \left(\partial_y^3 u^s(t,0) + \partial_y^2 w^{n-1}(t,x,0) + \epsilon(\partial_x^2 w^{n-1})(t,x,0)\right)(\partial_x w^n)(t,x,0)\\
&\quad + \left(u_y^s(t,0) + (w^{n-1})(t,x,0)\right)\left((\partial_y^2\partial_x w^n)(t,x,0) + \epsilon(\partial_x^3 w^n)(t,x,0)\right)\\
&\qquad\qquad - (\partial_y\partial_x w^n)(u_y^s + w^{n-1})(t,x,0)\\
&\quad + \sum_{1\le j\le 3} C_j^4\Big((\partial_y^j(u^s + u^{n-1}))\partial_y^{4-j}\partial_x u^n - (\partial_y^{j-1}\partial_x \tilde{u}^n)\partial_y^{4-j}(u_y^s + w^{n-1})\Big)(t,x,0)\\
&\qquad\qquad - \epsilon\partial_x^2\Big(\left(u_y^s(t,0) + (w^{n-1})(t,x,0)\right)(\partial_x w^n)(t,x,0)\Big).
\end{aligned}$$

and also for $3 \le p \le \frac{m}{2}+1$, $\partial_y^{2p+1} w^n|_{y=0}$ is a linear combination of the terms of the form:

$$\prod_{j=1}^{q_1}\left(\partial_x^{\alpha_j}\partial_y^{\beta_j+1}\left(u^s + u^n\right)\right)\Big|_{y=0} \times \prod_{l=1}^{q_2}\left(\partial_x^{\tilde{\alpha}_l}\partial_y^{\tilde{\beta}_l+1}\left(u^s + u^{n-i}\right)\right)\Big|_{y=0}, \tag{B.3}$$

where $2 \le q_1 + q_2 \le p,\ 1 \le i \le \min\{n,\, p\}$ and

$$\begin{aligned}
&\alpha_j + \beta_j \le 2p-1,\ 1 \le j \le q_1;\ \tilde{\alpha}_l + \tilde{\beta}_l \le 2p-1,\ 1 \le l \le q_2;\\
&\sum_{j=1}^{q_1}(3\alpha_j + \beta_j) + \sum_{l=1}^{q_2}(3\tilde{\alpha}_l + \tilde{\beta}_l) = 2p+1\,;\\
&\sum_{j=1}^{q_1}\beta_j + \sum_{l=1}^{q_2}\tilde{\beta}_l \le 2p-2;\ \sum_{j=1}^{q_1}\alpha_j + \sum_{l=1}^{q_2}\tilde{\alpha}_l \le p-1,\ \ 0 < \sum_{j=1}^{q_1}\alpha_j.
\end{aligned}$$

Remark that the condition $0 < \sum_{j=1}^{q_1}\alpha_j$ implies that, in (B.3), there are at last one factor like $\partial_x^{\alpha_j}\partial_y^{\beta_j+1}u^n(t,x,0)$.

For given w^{n-1}, we have $u^{n-1} = \partial_y^{-1}w^{n-1}$ and $v^n = -\partial_y^{-1}u_x^n$. We will prove the existence and boundness of the sequence $\{w^n, n\in\mathbb{N}\}$ in $L^\infty([0,T_\epsilon]; H^{m+2}_{k+\ell}(\mathbb{R}^2_+))$ to the linear equation (B.2) firstly, then the existence of solution to (B.1) is guaranteed by using the standard weak convergence methods.

Lemma B.1. *Assume that $w^{n-i} \in L^\infty([0,T]; H^{m+2}_{k+\ell}(\mathbb{R}^2_+)), 1 \le i \le \min\{n, \frac{m}{2}+1\}$ and $\tilde{w}_{0,\epsilon}$ satisfies the compatibility condition up to order $m+2$ for the system (B.1), then the initial-boundary value problem (B.2) admit a unique solution w^n such that, for any $t\in[0,T]$,*

$$\frac{d}{dt}\|w^n(t)\|^2_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)} \le B_T^{n-1}\|w^n(t)\|^2_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)} + D_T^{n-1}\|w^n\|^{m+2}_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)}, \tag{B.4}$$

where

$$B_T^{n-1} = C\Big(1 + \sum_{i=1}^{\min\{n,m/2+1\}} \|w^{n-i}\|_{L^\infty([0,T]; H^{m+2}_{k+\ell}(\mathbb{R}^2_+))}$$

$$+ (1+\frac{1}{\epsilon})\sum_{i=1}^{\min\{n,m/2+1\}}\|w^{n-i}\|^2_{L^\infty([0,T];H^{m+2}_{k+\ell}(\mathbb{R}^2_+))}\Big),$$

and

$$D_T^{n-1} = C\sum_{i=1}^{\min\{n,m/2+1\}}\|w^{n-i}\|^{m+2}_{L^\infty([0,T];H^{m+2}_{k+\ell}(\mathbb{R}^2_+))}.$$

Proof. Once we get *à priori* estimate for this linear problem, the existence of solution is guaranteed by the Hahn-Banach theorem. So we only prove the *à priori* estimate of the smooth solutions.

For any $\alpha \in \mathbb{N}^2, |\alpha| \le m+2$, taking the equation (B.2) with derivative ∂^α, multiplying the resulting equation by $\langle y\rangle^{2k+2\ell+2\alpha_2}\partial^\alpha w^n$ and integrating by part over $\mathbb{R}^2_+$, one obtains that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|w^n\|^2_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)} + \|\partial_y w^n\|^2_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)} + \epsilon\|\partial_x w^n\|^2_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)}\\
&= \sum_{|\alpha|\le m+2}\int_{\mathbb{R}^2_+}\langle y\rangle^{2k+2\ell+2\alpha_2}\partial^\alpha\big((u^s+u^{n-1})\partial_x w^n\\
&\qquad - (\partial_y^{-1}u^n_x)(u^s_{yy}+\partial_y w^{n-1})\big)\partial^\alpha w^n dxdy\\
&\quad + \sum_{|\alpha|\le m+2}\int_{\mathbb{R}^2_+}(\langle y\rangle^{2k+2\ell+2\alpha_2})'\partial^\alpha\partial_y w^n\partial^\alpha\partial_y w^n dxdy\\
&\qquad + \sum_{|\alpha|\le m+2}\int_{\mathbb{R}}(\partial^\alpha\partial_y w^n\partial^\alpha\partial_y w^n)\big|_{y=0}dx,
\end{aligned}\tag{B.5}$$

With similar analysis to Section 5, we have

$$\begin{aligned}
&\left|\int_{\mathbb{R}^2_+}\langle y\rangle^{2k+2\ell+2\alpha_2}(u^s+u^{n-1})\partial_x\partial^\alpha w^n\partial^\alpha w^n dxdy\right|\\
&= \left|-\frac{1}{2}\int_{\mathbb{R}^2_+}\langle y\rangle^{2k+2\ell+2\alpha_2}\partial_x(u^s+u^{n-1})\partial^\alpha w^n\partial^\alpha w^n dxdy\right|\\
&\le C\|u^{n-1}\|_{L^\infty(\mathbb{R}^2_+)}\|w^n\|^2_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)},
\end{aligned}$$

and

$$\begin{aligned}
&\left|\int_{\mathbb{R}^2_+}\langle y\rangle^{2k+2\ell+2\alpha_2}[\partial^\alpha,(u^s+u^{n-1})]\partial_x w^n\partial^\alpha w^n dxdy\right|\\
&\le C(1+\|w^{n-1}\|_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)})\|w^n\|^2_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)}.
\end{aligned}$$

For the second term on the right hand of (B.5), by using the Leibniz formula, we need to pay more attention to the following two terms

$$\begin{aligned}
&\left|\int_{\mathbb{R}^2_+}\langle y\rangle^{2k+2\ell+2\alpha_2}\big(\partial^\alpha\partial_y^{-1}u^n_x\big)(u^s_{yy}+\partial_y w^{n-1})\partial^\alpha w^n dxdy\right|\\
&\le C(1+\|w^{n-1}\|_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)})\|\partial_x w^n\|_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)}\|w^n\|_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)}\\
&\le \frac{\epsilon}{2}\|\partial_x w^n\|^2_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)} + \frac{C}{\epsilon}(1+\|w^{n-1}\|_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)})^2\|w^n\|^2_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)},
\end{aligned}$$

and

$$\int_{\mathbb{R}^2_+} \langle y\rangle^{2k+2\ell+2\alpha_2} v^n\big)\big(\partial^\alpha\partial_y w^{n-1}\big)\partial^\alpha w^n dxdy$$
$$= -\int_{\mathbb{R}^2_+} \partial_y\big(\langle y\rangle^{2k+2\ell+2\alpha_2}(\partial_y^{-1}u_x^n)\big)\big(\partial^\alpha w^{n-1}\big)\partial^\alpha w^n dxdy$$
$$-\int_{\mathbb{R}^2_+} \big(\langle y\rangle^{2k+2\ell+2\alpha_2}(\partial_y^{-1}u_x^n)\big)\big(\partial^\alpha w^{n-1}\big)\partial_y\partial^\alpha w^n dxdy,$$

here we have used $v^n|_{y=0} = 0$, thus

$$\left|\int_{\mathbb{R}^2_+} \langle y\rangle^{2k+2\ell+2\alpha_2} v^n\big)\big(\partial^\alpha\partial_y w^{n-1}\big)\partial^\alpha w^n dxdy\right|$$
$$\le C\|w^{n-1}\|_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)}\big(\|w^n\|^2_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)} + \|\partial_y w^n\|_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)}\|w^n\|_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)}\big).$$

For the boundary term, similar to the proof of Proposition 3.9, we can get

$$\sum_{|\alpha|\le m+2}\left|\int_{\mathbb{R}}\big(\partial^\alpha\partial_y w^n\partial^\alpha\partial_y w^n\big)\big|_{y=0}dx\right|$$
$$\le \frac{1}{16}\|\partial_y w^n\|^2_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)} + C\|w^{n-1}\|^{m+2}_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)}\|w^n\|^{m+2}_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)}.$$

We get finally

$$\frac{d}{dt}\|w^n(t)\|^2_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)} + \|\partial_y w^n(t)\|^2_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)} + \epsilon\|\partial_x w^n(t)\|^2_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)}$$
$$\le B_T^{n-1}\|w^n(t)\|^2_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)} + D_T^{n-1}\|w^n\|^{m+2}_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)}.$$

□

Lemma B.2. *Suppose that m, k, ℓ and $u^s(t,y)$ satisfy the assumption of Proposition 3.7, $\bar\zeta > 0$, then for any $0 < \epsilon \le 1$, there exists $T_\epsilon > 0$ such that for any $\tilde w_{0,\epsilon} \in H^{m+2}_{k+\ell}(\mathbb{R}^2_+)$ with*

$$\|\tilde w_{0,\epsilon}\|_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)} \le \bar\zeta,$$

the iteration equations (B.2) admit a sequence of solution $\{w^n, n\in\mathbb{N}\}$ such that, for any $t\in[0,T_\epsilon]$,

$$\|w^n(t)\|_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)} \le \frac{4}{3}\|\tilde w_{0,\epsilon}\|_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)}, \quad \forall n\in\mathbb{N}.$$

Remark. Here $\bar\zeta$ is aribitary.

Proof. Integrating (B.4) over $[0,t]$, for $0 < t \le T$ and $T > 0$ small,

$$\|w^n(t)\|^m_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)} \le \frac{\|\tilde w_{0,\epsilon}\|^m_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)}}{e^{-\frac{m}{2}B_T^{n-1}t} - \frac{m}{2}D_T^{n-1}t\|\tilde w_{0,\epsilon}\|^m_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)}}.$$

We prove the Lemma by induction. For $n = 1$, we have

$$B_T^0 = C\left(1 + \|\tilde w_{0,\epsilon}\|_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)} + (1+\frac{1}{\epsilon})\|\tilde w_{0,\epsilon}\|^2_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)}\right)$$
$$\le C\left(1 + \bar\zeta + (1+\frac{1}{\epsilon})\bar\zeta^2\right),$$

and

$$D_T^{n-1} = C\|\tilde{w}_{0,\epsilon}\|^{m+2}_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)} \le C\bar{\zeta}^{m+2}.$$

Choose $T_\epsilon > 0$ small such that

$$\left(e^{-\frac{m}{2}C\left(1+2\bar{\zeta}+4(1+\frac{1}{\epsilon})\bar{\zeta}^2\right)T_\epsilon} - \frac{m}{2}C(2\bar{\zeta})^{m+2}T_\epsilon(2\bar{\zeta})^m\right)^{-1} = \left(\frac{4}{3}\right)^m,$$

we get

$$\|w^1(t)\|_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)} \le \frac{4}{3}\|\tilde{w}_{0,\epsilon}\|_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)}.$$

Now the induction hypothesis is: for $0 \le t \le T_\epsilon$,

$$\|w^{n-1}(t)\|_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)} \le \frac{4}{3}\|\tilde{w}_{0,\epsilon}\|_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)},$$

thanks to the choose of T_ϵ, we have also

$$\left(e^{-\frac{m}{2}B^{n-1}_{T_\epsilon}T_\epsilon} - \frac{m}{2}D^{n-1}_{T_\epsilon}T_\epsilon\|\tilde{w}_{0,\epsilon}\|^m_{H^{m+2}_{k+\ell}(\mathbb{R}^2_+)}\right)^1 \le \left(\frac{4}{3}\right)^m$$

for any $t \in [0, T_\epsilon]$, then we finish the proof of the Lemma B.2. □

Acknowledgments

The first author was partially supported by " the Fundamental Research Funds for the Central Universities" and the NSF of China (No. 11171261). The second author was supported by a period of sixteen months scholarship from the State Scholarship Fund of China, and he would thank "Laboratoire de mathématiques Raphaël Salem de l'Université de Rouen" for their hospitality.

References

[1] R. Alexandre, Y.-G. Wang, C.-J. Xu, and T. Yang. Well-posedness of the Prandtl equation in sobolev spaces. *J. Amer. Math. Soc.*, 28(2015), 745-784.

[2] M. Cannone1, M. C. Lombardo, and M. Sammartino. Well-posedness of Prandtl equations with non-compatible data. *Nonlinearity* 26 (2013), 3077-3100.

[3] M. Cannone, M. C. Lombardo, and M. Sammartino. On the Prandtl boundary layer equations in presence of corner singularities. *Acta Appl. Math.*, 132 (2014), 139-149.

[4] W. E. Boundary layer theory and the zero-viscosity limit of the Navier-Stokes equation. *Acta Math. Sin. (Engl. Ser.)* 16(2000), 207-218.

[5] W. E and B. Enquist. Blow up of solutions of the unsteady Prandtl's equation, *Comm. Pure Appl. Math.*, 50(1997), 1287-1293.

[6] D. Gérard-Varet and E. Dormy. On the ill-posedness of the Prandtl equation. *J. Amer. Math. Soc.*, 23(2010), 591-609.

[7] D. Gerard-Varet and N. Masmoudi. Well-posedness for the Prandtl system without analyticity or monotonicity. *to appear "Ann. Eco. Norm. Sup. Paris", preprint arXiv:1305.0221.*

[8] D. Gérard-Varet and T. Nguyen. Remarks on the ill-posedness of the Prandtl equation. *Asymptot. Anal.*, 77 (2012), 71-88.

[9] E. Grenier, Y. Guo, and T. Nguyen. Spectral instability of characteristic boundary layer flows. *Preprint arXiv:1406.3862, 2014.*

[10] E. Grenier, Y. Guo, and T. Nguyen. Spectral instability of symmetric shear flows in a two-dimensional channel. *Preprint arXiv:1402.1395, 2014.*

[11] E. Grenier, Y. Guo, and T. Nguyen. Spectral stability of Prandtl boundary layers: an overview. *Analysis (Berlin)* 35 (2015), 343355.

[12] E. Grenier. On the nonlinear instability of Euler and Prandtl equations. *Comm. Pure Appl. Math.* 53(2000), 1067-1091.

[13] Y. Guo and T. Nguyen. A note on Prandtl boundary layers. *Comm. Pure Appl. Math.*, 64 (2011), 1416–1438.

[14] M. Ignatova and V. Vicol. Almost global existence for the Prandtl boundary layer equations. *Arch. Ration. Mech. Anal.*, 220(2016), 809-848.
[15] I. Kukavica and V. Vicol. On the local existence of analytical solution to the Prandtl boundary layer equations. *Commun. Math. Sci.* 11(2013), 269-292.
[16] I. Kukavica , N. Masmoudi, V. Vicol and T. K. Wong. On the local well-posedness of the Prandtl and the hydrostatic Euler equations with multiple monotonicity regions. *SIAM Journal of Mathematical Analysis* 46(2014), 3865-3890.
[17] C.-J. Liu, Y.-G. Wang, and T. Yang. A well-posedness theory for the Prandtl equations in three space variables. *preprint arXiv:1405.5308*, 2014.
[18] C.-J. Liu, Y.-G. Wang, and T. Yang. Global existence of weak solutions to the three-dimensional Prandtl equations with A special structure. *Preprint arXiv:1509.03856*, 2015.
[19] M. C. Lombardo, M. Cannone, and M. Sammartino. Well-posedness of the boundary layer equations. *SIAM J. Math. Anal.*, 35(2003), 987-1004.
[20] N. Masmoudi and T. K. Wong. Local-in-time existence and uniqueness of solutions to the Prandtl equations by energy methods. *Comm. Pure Appl. Math.*, 68(2015),1683-1741.
[21] O. A. Oleinik and V. N. Samokhin. *Mathematical models in boundary layer theory*, volume 15. CRC Press, 1999.
[22] L. Prandtl. Über flüssigkeitsbewegungen bei sehr kleiner reibung. *Verhaldlg III Int. Math. Kong*, pages 484-491, 1904.
[23] M. Renardy. Ill-posedness of the hydrostatic Euler and Navier-Stokes equations. *Arch. Ration. Mech. Anal.*, 194(2009), 877-886.
[24] M. Sammartino and R. E. Caflisch. Zero viscosity limit for analytic solutions of the Navier-Stokes equation on a half-space. I. existence for Euler and Prandtl equations. *Comm. Math. Phy.*, 192(1998), 433-461.
[25] M. Sammartino and R. E. Caflisch. Zero viscosity limit for analytic solutions of the Navier-Stokes equation on a half-space. II. construction of the Navier-Stokes solution. *Comm. Math. Phy.*, 192(1998), 463-491.
[26] Z. Xin and L. Zhang. On the global existence of solutions to the Prandtl's system. *Adv. Math.*, 181(2004), 88-133.
[27] P. Zhang and Z. Zhang. Long time well-posdness of Prandtl system with small and analytic initial data. *Preprint arXiv:1409.1648*, 2014.

Chao-Jiang XU
School of Mathematics and Statistics, Wuhan University
430072, Wuhan, P. R. China
and
Université de Rouen, CNRS, UMR 6085-Laboratoire de Mathématiques
76801 Saint-Etienne du Rouvray, France
E-mail address: chao-jiang.xu@univ-rouen.fr

Xu ZHANG
School of Mathematical Sciences, Xiamen University, Xiamen, Fujian 361005, China
and
Université de Rouen, CNRS, UMR 6085-Laboratoire de Mathématiques
76801 Saint-Etienne du Rouvray, France
E-mail address: xu.zhang1@etu.univ-rouen.fr