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HAL Id: hal-01227218
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Submitted on 10 Nov 2015

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Geometric Weil representation: local field case

Vincent Lafforgue, Sergey Lysenko

Abstract. Let \( k \) be an algebraically closed field of characteristic \( > 2 \), \( F = k((t)) \) and \( G = \text{Sp}_{2d} \). In this paper we propose a geometric analog of the Weil representation of the metaplectic group \( \tilde{G}(F) \). This is a category of certain perverse sheaves on some stack, on which \( \tilde{G}(F) \) acts by functors. This construction will be used in [11] (and subsequent publications) for the proof of the geometric Langlands functoriality for some dual reductive pairs.

1. Introduction

1.1 This paper followed by [11] form a series, where we prove the geometric Langlands functoriality for the dual reductive pair \( \text{Sp}_{2n}, \text{SO}_{2m} \) (in the everywhere nonramified case).

Let \( k = \mathbb{F}_q \) with \( q \) odd, set \( \mathcal{O} = k[[t]] \subset F = k((t)) \). Write \( \Omega \) for the completed module of relative differentials of \( \mathcal{O} \) over \( k \). Let \( M \) be a free \( \mathcal{O} \)-module of rank 2 with symplectic form \( \wedge^2 M \to \Omega \), set \( G = \text{Sp}(M) \). The group \( G(F) \) admits a nontrivial metaplectic extension

\[
1 \to \{\pm 1\} \to \tilde{G}(F) \to G(F) \to 1
\]

(defined up to a unique isomorphism). Let \( \psi : k \to \bar{\mathbb{Q}}_l^* \) be a nontrivial additive character, let \( \chi : \Omega(F) \to \bar{\mathbb{Q}}_l^* \) be given by \( \chi(\omega) = \psi(\text{Res} \omega) \). Write \( H = M \oplus \Omega \) for the Heisenberg group of \( M \) with operation

\[
(m_1,a_1)(m_2,a_2) = (m_1 + m_2, a_1 + a_2 + \frac{1}{2}\omega(m_1,m_2)) \quad m_i \in M, a_i \in \Omega
\]

Denote by \( S_\psi \) the Weil representation of \( H(M)(F) \) with central character \( \chi \). As a representation of \( \tilde{G}(F) \), it decomposes \( S_\psi \to S_\psi,\text{odd} \oplus S_\psi,\text{even} \) into a direct sum of two irreducible smooth representations, where the even (resp., the odd) part is unramified (resp., ramified).

The discovery of this representation by A. Weil in [14] had a major influence on the theory of automorphic forms (among numerous developments and applications are Howe duality for reductive dual pairs, particular cases of classical Langlands functoriality, Siegel-Weil formulas, relation with \( L \)-functions, representation-theoretic approach to the theory of theta-series. We refer the reader to [3], [9], [7], [12], [13] for history and details).

In this paper we introduce a geometric analog of the Weil representation \( S_\psi \). The pioneering work in this direction is due to P. Deligne [2], where a geometric approach to the Weil
representation of a symplectic group over a finite field was set up. It was further extended by Gurevich-Hadani in [4, 5]. The point of this paper is to develop the geometric theory in the case when a finite field is replaced by a local non-archimedian field.

First, we introduce a $k$-scheme $\tilde{L}_d(M(F))$ of discrete lagrangian lattices in $M(F)$ and a certain $\mu_2$-gerb $\tilde{L}_d(M(F))$ over it. We view the metaplectic group $\tilde{G}(F)$ as a group stack over $k$. We construct a category

$$W(\tilde{L}_d(M(F)))$$

of certain perverse sheaves on $\tilde{L}_d(M(F))$, which provides a geometric analog of $S_{\psi,\text{even}}$. The metaplectic group $\tilde{G}(F)$ acts on the category $W(\tilde{L}_d(M(F)))$ by functors. This action is geometric in the sense that it comes from a natural action of $\tilde{G}(F)$ on $\tilde{L}_d(M(F))$ (cf. Theorem 2).

The category $W(\tilde{L}_d(M(F)))$ has a distinguished object $S_{M(F)}$ corresponding to the unique non-ramified vector of $S_{\psi,\text{even}}$.

Our category $W(\tilde{L}_d(M(F)))$ is obtained from Weil representations of symplectic groups $\text{Sp}_{2n}(k)$ by some limit procedure. This uses a construction of geometric canonical intertwining operators for such representations. A similar result has been announced by Gurevich and Hadani in [4] and proved for $d=1$ in [5]. We give a proof for any $d$ (cf. Theorem 3). When this paper has already been written we learned about a new preprint [6], where a result similar to our Theorem 1 is claimed to be proved for all $d$. However, the sheaves of canonical intertwining operators constructed in loc.cit. and in this paper live on different bases.

Finally, in Section 7 we give a global application. Let $X$ be a smooth projective curve. Write $\Omega_X$ for the canonical line bundle on $X$. Let $G$ denote the sheaf of automorphisms of $O_X^d \oplus \Omega_X^d$ preserving the natural symplectic form $\Lambda^2(O_X^d \oplus \Omega_X^d) \to \Omega_X$.

Our Theorem 3 relates $S_{M(F)}$ with the theta-sheaf $\text{Aut}$ on the moduli stack $\text{Bun}_G$ of metaplectic bundles on $X$. This result will play an important role in [11].

1.2 Notation In Section 2 we let $k = \mathbb{F}_q$ of characteristic $p > 2$. Starting from Section 3 we assume $k$ either finite as above or algebraically closed with a fixed inclusion $\mathbb{F}_q \hookrightarrow k$. All the schemes (or stacks) we consider are defined over $k$.

Fix a prime $\ell \neq p$. For a scheme (or stack) $S$ write $D(S)$ for the bounded derived category of $\ell$-adic étale sheaves on $S$, and $P(S) \subset D(S)$ for the category of perverse sheaves.

Fix a nontrivial character $\psi : \mathbb{F}_p \to \mathbb{Q}_\ell^\times$, write $L_\psi$ for the corresponding Artin-Shreier sheaf on $A^1$. Fix a square root $\mathbb{Q}_\ell(\frac{1}{2})$ of the sheaf $\mathbb{Q}_\ell(1)$ on $\text{Spec} \mathbb{F}_q$. Isomorphism classes of such correspond to square roots of $q$ in $\mathbb{Q}_\ell$.

If $V \to S$ and $V^* \to S$ are dual rank $n$ vector bundles over a stack $S$, we normalize the Fourier transform $\text{Four}_\psi : D(V) \to D(V^*)$ by $\text{Four}_\psi(K) = (p_{V^*})!(\xi^* L_\psi \otimes p_V^* K)[n](\frac{1}{2})$, where $p_V, p_{V^*}$ are the projections, and $\xi : V \times_S V^* \to A^1$ is the pairing.

Our conventions about $\mathbb{Z}/2\mathbb{Z}$-gradings are those of [10].
2. Canonical interwining operators: finite field case

2.1 Let $M$ be a symplectic $k$-vector space of dimension $2d$. The symplectic form on $M$ is denoted $\omega(\cdot, \cdot)$. The Heisenberg group $H = M \times \mathbb{A}^1$ with operation

$$(m_1, a_1)(m_2, a_2) = (m_1 + m_2, a_1 + a_2 + \frac{1}{2}\omega(m_1, m_2)) \quad m_i \in M, a_i \in \mathbb{A}^1$$

is algebraic over $k$. Set $G = \text{Sp}(M)$. Write $\mathcal{L}(M)$ for the variety of lagrangian subspaces in $M$. Fix a one-dimensional $k$-vector space $J$ (purely of degree $d$ mod 2 as $\mathbb{Z}/2\mathbb{Z}$-graded). Let $\mathcal{A}$ be the (purely of degree zero as $\mathbb{Z}/2\mathbb{Z}$-graded) line bundle over $\mathcal{L}(M)$ with fibre $J \otimes \det L$ at $L \in \mathcal{L}(M)$. Write $\tilde{\mathcal{L}}(M)$ for the gerb of square roots of $\mathcal{A}$. The line bundle $\mathcal{A}$ is $G$-equivariant, so $G$ acts naturally on $\tilde{\mathcal{L}}(M)$.

For a $k$-point $L \in \mathcal{L}(M)$ write $L^0$ for a $k$-point of $\tilde{\mathcal{L}}(M)$ over $L$. Write $\bar{L} = L \oplus k$, this is a subgroup of $H(k)$ equipped with the character $\chi_L : \bar{L} \to \bar{\mathbb{Q}}_\ell^*$ given by $\chi_L(l, a) = \psi(a)$, $l \in L, a \in k$. Write

$$\mathcal{H}_L = \{ f : H(k) \to \bar{\mathbb{Q}}_\ell | f(\bar{l}h) = \chi_L(\bar{l})f(h), \text{ for } \bar{l} \in \bar{L}, h \in H \}$$

This is a representation of $H(k)$ by right translations. Write $\mathcal{S}(H)$ for the space of all $\bar{\mathbb{Q}}_\ell$-valued functions on $H(k)$. The group $G$ acts naturally in $\mathcal{S}(H)$. For $L \in \mathcal{L}(M), g \in G$ we have an isomorphism $\mathcal{H}_L \to \mathcal{H}_{gL}$ sending $f$ to $gf$.

The purpose of Sections 2 and 3 is to study the canonical interwining operators (and their geometric analogs) between various models $H_L$ of the Weil representation. The corresponding results for a finite field were formulated by Gurevich and Hadani [4] without a proof (we give all proofs for the sake of completeness). Besides, our setting is a bit different from loc.cit, we work with gerbs instead of the total space of the corresponding line bundles.

2.2 For $k$-points $L^0, N^0 \in \tilde{\mathcal{L}}(M)$ we will define a canonical interwining operator

$$F_{N^0, L^0} : \mathcal{H}_L \to \mathcal{H}_N$$

They will satisfy the properties

- $F_{L^0, L^0} = \text{id}$
- $F_{R^0, N^0} \circ F_{N^0, L^0} = F_{R^0, L^0}$ for any $R^0, N^0, L^0 \in \tilde{\mathcal{L}}(M)$
- for any $g \in G$ we have $g \circ F_{N^0, L^0} \circ g^{-1} = F_{gN^0, gL^0}$.
- under the natural action of $\mu_2$ on the set $\tilde{\mathcal{L}}(M)(k)$ of (isomorphism classes of) $k$-points, $F_{N^0, L^0}$ is odd as a function of $N^0$ and of $L^0$. 

3
In (Remark 2, Section 3.1) we will define a function $F^{cl}$ on the set of $k$-points of $\tilde{L}(M) \times \tilde{L}(M) \times H$, which we denote $F_{N^0,L^0}(h)$ for $h \in H$. It will realize the operator $F_{N^0,L^0}$ by

$$(F_{N^0,L^0}f)(h_1) = \int_{h_2 \in H} F_{N^0,L^0}(h_1 h_2^{-1}) f(h_2) dh_2$$

All our measures on finite sets are normalized by requiring the volume of a point to be one.

Given two functions $f_1, f_2 : H \rightarrow \bar{Q}_\ell$ their convolution $f_1 * f_2 : H \rightarrow \bar{Q}_\ell$ is defined by

$$(f_1 * f_2)(h) = \int_{v \in H} f_1(hv^{-1}) f_2(v) dv \quad h \in H$$

The function $F_{N^0,L^0}$ will satisfy the following:

- $F_{N^0,L^0}(\bar{n}h) = \chi_N(\bar{n})\chi_L(\bar{l}) F_{N^0,L^0}(h)$ for $\bar{l} \in \bar{L}, \bar{n} \in \bar{N}, h \in H$.
- $F_{gN^0,gL^0}(gh) = F_{N^0,L^0}(h)$ for $g \in G, h \in H$.
- Convolution property: $F_{R^0,L^0} = F_{R^0,N^0} * F_{N^0,L^0}$ for any $R^0, N^0, L^0 \in \tilde{L}(M)$.

2.3 First, we define the non-normalized function $\tilde{F}_{N,L} : H \rightarrow \bar{Q}_\ell$, it will depend only on $N, L \in \tilde{L}(M)$, not on their enhanced structure.

Given $N, L \in \tilde{L}(M)$ let $\chi_{NL} : \bar{N} \bar{L} \rightarrow \bar{Q}_\ell$ be the function given by

$$\chi_{NL}(\bar{n}) = \chi_N(\bar{n})\chi_L(\bar{l}),$$

it is correctly defined. Note that $\bar{N} \bar{L} = \bar{L}\bar{N}$ but $\chi_{NL} \neq \chi_{LN}$ in general. Set

$$\tilde{F}_{N,L}(h) = \begin{cases} \chi_{NL}(h), & \text{if } h \in \bar{N} \bar{L} \\ 0, & \text{otherwise} \end{cases}$$

Note that $\chi_{LL} = \chi_L$.

Given $L, R, N \in \tilde{L}(M)$ with $N \cap L = N \cap R = 0$, define $\theta(R, N, L) \in \bar{Q}_\ell$ as follows. There is a unique map $b : L \rightarrow N$ such that $R = \{l + b(l) \in L \oplus N \mid l \in L\}$. Set

$$\theta(R, N, L) = \int_{l \in L} \psi(\frac{1}{2} \omega(l, b(l))) dl$$

This expression has been considered in ([10], Appendix B).

Lemma 1. 1) Let $L, N \in \tilde{L}(M)$. If $L \cap N = 0$ then $\tilde{F}_{L,N} \ast \tilde{F}_{N,L} = q^{2d+1} \tilde{F}_{L,L}$.

2) Let $L, R, N \in \tilde{L}(M)$ with $N \cap L = N \cap R = 0$. Then $\tilde{F}_{R,N} \ast \tilde{F}_{N,L} = q^{d+1} \theta(R, N, L) \tilde{F}_{R,L}$
Proof 2) Using $L \oplus N = N \oplus R = M$, for $h \in H$ we get

$$(\tilde{F}_{R,N} \ast \tilde{F}_{N,L})(h) = q^{d+1} \int_{v \in N \setminus H} \chi_{RN}(hv^{-1})\chi_{NL}(v)dv = q^{d+1} \int_{r \in R} \chi_{RN}((n,0)(-r,0))\chi_{NL}(r,0)dr$$

Because of the equivariance property of $\tilde{F}_{R,N} \ast \tilde{F}_{N,L}$, we may assume $h = (n,0), n \in N$. We get

$$(\tilde{F}_{R,N} \ast \tilde{F}_{N,L})(h) = q^{d+1} \int_{r \in R} \chi_{RN}((n,0)(-r,0))\chi_{NL}(r,0)dr$$

$$= q^{d+1} \int_{r \in R} \psi(\omega(r,n))\chi_{NL}(r,0)dr \quad (1)$$

The latter formula essentially says that the resulting function on $N$ is the Fourier transform of some local system on $R$ (the symplectic form on $M$ induces an isomorphism $R \sim N^*$). This will be used for geometrization in Lemma 2.

There is a unique map $b : L \to N$ such that $R = \{l + b(l) \in L \oplus N \mid l \in L\}$. So, the above integral rewrites

$$(\tilde{F}_{R,N} \ast \tilde{F}_{N,L})(h) = q^{d+1} \int_{l \in L} \psi(\omega(l,n))\chi_{NL}((l+b(l),0))dl =$$

$$= q^{d+1} \int_{l \in L} \psi(\omega(l,n))\chi_{NL}(b(l))((l,0))\omega(l,b(l)))dl = q^{d+1} \int_{l \in L} \psi(\omega(l,n) + \frac{1}{2} \omega(l,b(l)))dl \quad (2)$$

Note that if $R = L$ then $b = 0$ and the latter formula yields 1).

Let us identify $N \sim L^*$ via the map sending $n \in N$ to the linear functional $l \mapsto \omega(l,n)$. Denote by $\langle \cdot, \cdot \rangle$ the symmetric pairing between $L$ and $L^*$. By Sublemma 1 below, the value (2) vanishes unless $n \in (R + L) \cap N = Im b$. In the latter case pick $l_1 \in L$ with $b(l_1) = n$. Then

$$\chi_{RL}(n,0) = \psi(-\frac{1}{2} \omega(l_1,b(l_1)))$$

So, we get for $L' = Ker b$

$$(\tilde{F}_{R,N} \ast \tilde{F}_{N,L})(h) = q^{d+1+dim L'} \chi_{RL}(h) \int_{l \in L/L'} \psi(\frac{1}{2} \omega(l,b(l)))dl$$

We are done. □

Sublemma 1. Let $L$ be a $d$-dimensional $k$-vector space, $b \in \text{Sym}^2 L^*$ and $u \in L^*$. View $b$ as a map $b : L \to L^*$, let $L'$ be the kernel of $b$. Then

$$\int_{l \in L} \psi((l,u) + \frac{1}{2} \langle l, b(l) \rangle)dl \quad (3)$$

is supported at $u \in (L/L')^*$ and there equals

$$q^{dim L'} \psi(-\frac{1}{2} b^{-1}u,u) \int_{L/L'} \psi(\frac{1}{2} \langle l, b(l) \rangle)dl$$,
where \( b : L/L' \rightarrow (L/L')^* \), so that \( b^{-1}u \in L/L' \). (Here the scalar product is between \( L \) and \( L^* \), so is symmetric).

**Proof** Let \( L' \subset L \) denote the kernel of \( b : L \rightarrow L^* \). Integrating first along the fibres of the projection \( L \rightarrow L/L' \) we will get zero unless \( u \in (L/L')^* \). For any \( l_0 \in L \) the integral (3) equals

\[
\int_{l \in L} \psi((l+\rho),u)+\frac{1}{2}(l+\rho,b(l)+b(\rho)))dl = \int_{l \in L} \psi((l,u)+\frac{1}{2}(l,b(\rho)))dl
\]

Assuming \( u \in (L/L')^* \) take \( l_0 \) such that \( u = -b(\rho) \). Then (3) becomes

\[
\psi(\frac{1}{2}(\rho),u)) \int_{l \in L} \psi(\frac{1}{2}(l,b(l)))dl
\]

We are done. \( \square \)

**Remark 1.** The expression (3) is the Fourier transform from \( L \) to \( L^* \). In the geometric setting we will use 2) of Lemma 1 only under the additional assumption \( R \cap L = 0 \).

### Geometrization

3.1 Let \( M, H, \mathcal{L}(M) \) and \( \tilde{\mathcal{L}}(M) \) be as in Section 2.1. Remind that \( G = \text{Sp}(M) \). For each \( L \in \mathcal{L}(M) \) we have a rank one local system \( \chi_L \) on \( L = L \times \mathbb{A}^1 \) defined by \( \chi_L = pr^* \mathcal{L}_\psi, \) where \( pr : L \times \mathbb{A}^1 \rightarrow \mathbb{A}^1 \) is the projection. Let \( \mathcal{H}_L \) denote the category of perverse sheaves on \( H \) which are \((\tilde{L}, \chi_L)\)-equivariant under the left multiplication, this is a full subcategory in \( \mathcal{P}(H) \). Write \( D\mathcal{H}_L \subset D(H) \) for the full subcategory of objects whose all perverse cohomologies lie in \( \mathcal{H}_L \).

Denote by \( C \rightarrow \mathcal{L}(M) \) (resp., \( \tilde{C} \rightarrow \mathcal{L}(M) \)) the vector bundle whose fibre over \( L \in \mathcal{L}(M) \) is \( L \) (resp., \( \tilde{L} = L \times \mathbb{A}^1 \)). Its inverse image to \( \tilde{\mathcal{L}}(M) \) is denoted by the same symbol.

Write \( \chi_C \) for the local system \( p^* \mathcal{L}_\psi \) on \( \tilde{C} \), where \( p : \tilde{C} \rightarrow \mathbb{A}^1 \) is the projection on the center sending \( (L \in \mathcal{L}(M), (l,a) \in \tilde{L}) \) to \( a \). Consider the maps

\[
pr, \text{act}_{tr} : \tilde{C} \times \tilde{C} \times H \rightarrow \mathcal{L}(M) \times \mathcal{L}(M) \times H \times H
\]

where \( \text{act}_{tr} \) sends \( (\bar{n}, \bar{l} \in L, h) \) to \( (N, L, \bar{n}h\bar{l}) \), and \( pr \) sends the above point to \( (N, L, h) \). We say that a perverse sheaf \( K \) on \( \mathcal{L}(M) \times \mathcal{L}(M) \times H \) is \( \text{act}_{tr}-\text{equivariant} \) if it admits an isomorphism

\[
\text{act}_{tr}^* K \cong pr^* K \otimes pr^*_C \chi_C \otimes pr^*_C \chi_C
\]

satisfying the usual associativity condition and whose restriction to the unit section is the identity (such isomorphism is unique if it exists). One has a similar definition for \( \tilde{\mathcal{L}}(M) \times \tilde{\mathcal{L}}(M) \times H \).

Let

\[
\text{act}_G : G \times \tilde{\mathcal{L}}(M) \times \tilde{\mathcal{L}}(M) \times H \rightarrow \tilde{\mathcal{L}}(M) \times \tilde{\mathcal{L}}(M) \times H
\]

be the action map sending \( (gN^0, L^0, h) \) to \( (gN^0, gL^0, gh) \).
For this map we have a usual notion of a $G$-equivariant perverse sheaf on $\tilde{\mathcal{L}}(M) \times \tilde{\mathcal{L}}(M) \times H$. As $G$ is connected, a perverse sheaf on $\tilde{\mathcal{L}}(M) \times \tilde{\mathcal{L}}(M) \times H$ admits at most one $G$-equivariant structure.

If $S$ is a stack then for $K, F \in D(S \times H)$ define their convolution $K \ast F \in D(S \times H)$ by

$$K \ast F = \text{mult}_!(\text{pr}_1^* K \otimes \text{pr}_2^* F) \otimes (\overline{Q}_\ell[1](\frac{1}{2}))^{d+1-2 \dim \mathcal{L}(M)},$$

here $\text{pr}_i : S \times H \times H \to S \times H$ is the projection to the $i$-th component in the pair $H \times H$ (and the identity on $S$). The multiplication map $\text{mult} : H \times H \to H$ sends $(h_1, h_2)$ to $h_1 h_2$.

Let

$$(\mathcal{L}(M) \times H)_\Delta \hookrightarrow \mathcal{L}(M) \times H$$

be the closed subscheme of those $(L \in \mathcal{L}(M), h \in H)$ for which $h \in \tilde{L}$. Let

$$\alpha_\Delta : (\mathcal{L}(M) \times H)_\Delta \to \mathbb{A}^1$$

be the map sending $(L, h)$ to $a$, where $h = (l, a), l \in L, a \in \mathbb{A}^1$. Define a perverse sheaf

$$\tilde{F}_\Delta = \alpha_\Delta^* \mathcal{L}_\psi \otimes (\overline{Q}_\ell[1](\frac{1}{2}))^{d+1+\dim \mathcal{L}(M)},$$

which we extend by zero under $[\mathbb{I}]$.

Since $\tilde{\mathcal{L}}(M) \to \mathcal{L}(M)$ is a $\mu_2$-gerb, $\mu_2$ acts on each $K \in D(\tilde{\mathcal{L}}(M))$, and we say that $K$ is genuine if $-1 \in \mu_2$ acts on $K$ as $-1$.

**Theorem 1.** There exists an irreducible perverse sheaf $F$ on $\tilde{\mathcal{L}}(M) \times \tilde{\mathcal{L}}(M) \times H$ (pure of weight zero) with the following properties:

- for the diagonal map $i : \tilde{\mathcal{L}}(M) \times H \to \tilde{\mathcal{L}}(M) \times \tilde{\mathcal{L}}(M) \times H$ the complex $i^* F$ identifies canonically with the inverse image of

$$\tilde{F}_\Delta \otimes (\overline{Q}_\ell[1](\frac{1}{2}))^{\dim \mathcal{L}(M)}$$

under the projection $\tilde{\mathcal{L}}(M) \times H \to \mathcal{L}(M) \times H$.

- $F$ is act$_{\text{act}}$-equivariant;
- $F$ is $G$-equivariant;
- $F$ is genuine in the first and the second variable;
- convolution property for $F$ holds, namely for the $ij$-th projections

$$q_{ij} : \tilde{\mathcal{L}}(M) \times \tilde{\mathcal{L}}(M) \times \tilde{\mathcal{L}}(M) \times H \to \tilde{\mathcal{L}}(M) \times \tilde{\mathcal{L}}(M) \times H$$

inside the triple $\tilde{\mathcal{L}}(M) \times \tilde{\mathcal{L}}(M) \times \tilde{\mathcal{L}}(M)$ we have $(q_{12}^* F) \ast (q_{23}^* F) \simeq q_{13}^* F$ canonically.
The proof of Theorem [1] is given in Sections 3.2-3.4.

Remark 2. In the case $k = \mathbb{F}_q$ define $F^{cl}$ as the trace of the geometric Frobenius on $F$.

3.2 Let $U \subset \mathcal{L}(M) \times \mathcal{L}(M)$ be the open subset of pairs $(N, L) \in \mathcal{L}(M) \times \mathcal{L}(M)$ such that $N \cap L = 0$. Define a perverse sheaf $\tilde{F}_U$ on $U \times H$ as follows. Let

$$\alpha_U : U \times H \to \mathbb{A}^1$$

be the map sending $(N, L, h)$ to $a + \frac{1}{2} \omega(l, n)$, where $l \in L, n \in N, a \in \mathbb{A}^1$ are uniquely defined by $h = (n + l, a)$. Set

$$\tilde{F}_U = \alpha_U^* \mathcal{L}_\psi \otimes (\overline{\mathcal{Q}}_\ell[1](\frac{1}{2}))^{\dim H + 2 \dim \mathcal{L}(M)}$$

(5)

Write $U \times \mathcal{L}(M) U \subset \mathcal{L}(M) \times \mathcal{L}(M) \times \mathcal{L}(M)$ for the open subscheme classifying $(R, N, L)$ with $N \cap L = N \cap R = 0$. Let

$$q_i : U \times \mathcal{L}(M) U \to U$$

be the projection on the $i$-th factor, so $q_1$ (resp., $q_2$) sends $(R, N, L)$ to $(R, N)$ (resp., to $(N, L)$). Let $q : U \times \mathcal{L}(M) U \to \mathcal{L}(M) \times \mathcal{L}(M)$ be the map sending $(R, N, L)$ to $(R, L)$. Write

$$(U \times \mathcal{L}(M) U)_0 = q^{-1}(U)$$

The geometric analog of $\theta(R, N, L)$ is the following (shifted) perverse sheaf $\Theta$ on $U \times \mathcal{L}(M) U$. Let $\pi_C : C_3 \to U \times \mathcal{L}(M) U$ be the vector bundle whose fibre over $(R, N, L)$ is $L$. We have a map $\beta : C_3 \to \mathbb{A}^1$ defined as follows. Given a point $(R, N, L) \in U \times \mathcal{L}(M) U$, there is a unique map $b : L \to N$ such that $R = \{l + b(l) \in L \oplus N = M \mid l \in L\}$. Set $\beta(R, N, L, l) = \frac{1}{2} \omega(l, b(l))$. Set

$$\Theta = (\pi_C)_! \beta^* \mathcal{L}_\psi \otimes (\overline{\mathcal{Q}}_\ell[1](\frac{1}{2}))^{d}$$

Write $Y = \mathcal{L}(M) \times \mathcal{L}(M)$, let $\mathcal{A}_Y$ be the $\mathbb{Z}/2\mathbb{Z}$-graded purely of degree zero) line bundle on $Y$ whose fibre at $(R, L)$ is $\det R \otimes \det L$. Write $\hat{Y}$ for the gerb of square roots of $\mathcal{A}_Y$. Note that $\mathcal{A}_Y$ is $G$-equivariant, so $G$ acts on $\hat{Y}$ naturally.

The following perverse sheaf $S_M$ on $\hat{Y}$ was introduced in (10), Definition 2. Let $Y_i \subset Y$ be the locally closed subscheme given by $\dim(R \cap L) = i$ for $(R, L) \in Y_i$. The restriction of $\mathcal{A}_Y$ to each $Y_i$ admits the following $G$-equivariant square root. For a point $(R, L) \in Y_i$ we have an isomorphism $L/(R \cap L) \cong (R/(R \cap L))^* \cong \det L$. It induces a $\mathbb{Z}/2\mathbb{Z}$-graded isomorphism $\det R \otimes \det L \cong \det(R \cap L)^2$.

So, for the restriction $\hat{Y}_i$ of the gerb $\hat{Y} \to Y$ to $Y_i$ we get a trivialization

$$\hat{Y}_i \cong Y_i \times B(\mu_2)$$

(6)

Write $W$ for the nontrivial local system of rank one on $B(\mu_2)$ corresponding to the covering $\text{Spec} k \to B(\mu_2)$. 

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Definition 1. Let $S_{M,g}$ (resp., $S_{M,s}$) denote the intermediate extension of

$$(\bar{Q}_\ell \boxtimes W) \otimes (\bar{Q}_\ell[1](\frac{1}{2}))^{\dim Y}$$

from $\tilde{Y}_0$ to $\tilde{Y}$ (resp., of $(\bar{Q}_\ell \boxtimes W) \otimes (\bar{Q}_\ell[1](\frac{1}{2}))^{\dim Y-1}$ from $\tilde{Y}_1$ to $\tilde{Y}$). Set $S_M = S_{M,g} \oplus S_{M,s}$.

Let $\pi_Y : U \times_{\mathcal{L}(M)} U \to \tilde{Y}$

be the map sending $(R, N, L)$ to

$$(R, L, B, \epsilon : B^2 \to \text{det } R \otimes \text{det } L),$$

where $B = \text{det } L$ and $\epsilon$ is the isomorphism induced by $\epsilon_0$. Here $\epsilon_0 : L \to R$ is the isomorphism sending $l \in L$ to $l + b(l) \in R$. In other words, $\epsilon_0$ sends $l$ to the unique $r \in R$ such that $r = l \mod N \in M/N$. Write also $\tilde{U} = \tilde{Y}_0$.

Define $E \in D(\text{Spec } k)$ by

$$E = R\Gamma_c(A_1, \beta_0^* L \psi) \otimes \bar{Q}_\ell[1](\frac{1}{2}),$$

where $\beta_0 : A^1 \to A^1$ sends $x$ to $x^2$. Then $E$ is a 1-dimensional vector space placed in cohomological degree zero. The geometric Frobenius $\text{Fr}_q$ acts on $E^2$ by 1 if $-1 \in (\mathbb{F}^*_q)^2$ and by $-1$ otherwise. A choice of $\sqrt{-1} \in k$ yields an isomorphism $E^2 \to \bar{Q}_\ell$, so $E^2 \to \bar{Q}_\ell$ canonically.

As in ([10], Proposition 5), one gets a canonical isomorphism

$$\pi_Y^*(S_{M,g} \otimes E^d \oplus S_{M,s} \otimes E^{d-1}) \to \Theta \otimes (\bar{Q}_\ell[1](\frac{1}{2}))^{2 \dim \mathcal{L}(M)}$$

(7)

Since $d \geq 1$, the restriction $\pi_Y : (U \times_{\mathcal{L}(M)} U)_0 \to \tilde{U}$ is smooth of relative dimension $\dim \mathcal{L}(M)$, with geometrically connected fibres. It is convenient to introduce a rank one local system $\Theta_U$ on $\tilde{U}$ equipped with a canonical isomorphism

$$\Theta \to \pi_Y^* \Theta_U$$

(8)

over $(U \times_{\mathcal{L}(M)} U)_0$. The local system $\Theta_U$ is defined up to a unique isomorphism.

Let $i_U : U \to U \times_{\mathcal{L}(M)} U$ be the map sending $(L, N)$ to $(L, N, L)$. Let $p_1 : U \to \mathcal{L}(M)$ be the projection sending $(L, N)$ to $L$.

Lemma 2. 1) The complex

$$(q_1^* F_U) \ast (q_2^* F_U) \otimes (\bar{Q}_\ell[1](\frac{1}{2}))^{\dim \mathcal{L}(M)}$$

is an irreducible perverse sheaf on $U \times_{\mathcal{L}(M)} U \times H$ pure of weight zero. We have canonically

$$i_U^*((q_1^* F_U) \ast (q_2^* F_U)) \to p_1^* F_\Delta \otimes (\bar{Q}_\ell[1](\frac{1}{2}))^{\dim \mathcal{L}(M)}$$

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over $U \times H$.

2) There is a canonical isomorphism

$$(q_1^* \tilde{F}_U) \ast (q_2^* \tilde{F}_U) \sim q^* \tilde{F}_U \otimes \Theta$$

over $(U \times \mathcal{L}(M))_0 \times H$.

Proof 1) Follows from the properties of the Fourier transform as in Lemma 1, formula (1).

2) The proof of Lemma 1 goes through in the geometric setting. Our additional assumption that $(R,N,L) \in (U \times \mathcal{L}(M))_0$ means that $b : L \to N$ is an isomorphism (it simplifies the argument a little). □

Remark 3. Let $i_\Delta : \mathcal{L}(M) \to \tilde{Y}$ be the map sending $L$ to $(L,L,B = \det L)$ equipped with the isomorphism $\text{id} : B^2 \sim \det L \otimes \det L$. The commutative diagram

$$
\begin{array}{ccc}
U & \xrightarrow{i_U} & U \times \mathcal{L}(M) U \\
\downarrow p_1 & & \downarrow \pi_Y \\
\mathcal{L}(M) & \xrightarrow{i_\Delta} & \tilde{Y}
\end{array}
$$

(9)

Together with (7) yield a canonical isomorphism

$$i_\Delta^* S_M \sim \begin{cases} 
\mathcal{E}^{-d} \otimes (\tilde{Q}_\ell[1](1/2))^{2 \dim \mathcal{L}(M)} - d, & d \text{ is even} \\
\mathcal{E}^{1-d} \otimes (\tilde{Q}_\ell[1](1/2))^{2 \dim \mathcal{L}(M)} - d, & d \text{ is odd}
\end{cases}$$

3.3 Consider the following diagram

$$
\begin{array}{ccc}
\tilde{U} & \xrightarrow{\tilde{q}_1} & (U \times \mathcal{L}(M))_0 \\
\downarrow \tilde{q} & & \downarrow \tilde{q}_2 \\
\tilde{U} & \xrightarrow{\tilde{q}} & \tilde{U}
\end{array}
$$

Here $\tilde{q}$ is the restriction of $\pi_Y$, and the map $\tilde{q}_1$ is the lifting of $q_i$ defined as follows. We set $\tilde{q}_1(R,N,L) = \tilde{q}(R,L,N)$ and $\tilde{q}_2(R,N,L) = \tilde{q}(N,R,L)$.

The following property is a geometric counterpart of the way the Maslov index of $(R,N,L)$ changes under permutations of three lagrangian subspaces.

Lemma 3. 1) For $i = 1, 2$ we have canonically over $(U \times \mathcal{L}(M))_0$

$$\tilde{q}_i^* \Theta_U \otimes \tilde{q}_i^* \Theta_U \sim \tilde{Q}_\ell$$

2) We have $\Theta_U^2 \sim \mathcal{E}^{2d}$ canonically, so $\Theta_U^4 \sim \tilde{Q}_\ell$ canonically.

Proof 1) The two isomorphisms are obtained similarly, we consider only the case $i = 2$. For a point $(R,N,L) \in (U \times \mathcal{L}(M))_0$ we have isomorphisms $b : L \sim N$ and $b_0 : L \sim R$ such that
$R = \{l + b(l) \mid l \in L\}$ and $N = \{l + b_0(l) \mid l \in L\}$. Clearly, $b_0(-l) = l + b(l)$ for $l \in L$. Let $\beta_2 : L \times L \rightarrow k^1$ be the map sending $(l, l_0)$ to $\frac{1}{2}\omega(l, b(l)) + \frac{1}{2}\omega(l, b_0(l))$. We must show that

$$\text{R} \Gamma_c(L \times L, \beta_2^* \mathcal{L}_\psi) \sim \mathbb{Q}_\ell[2d](d)$$

The quadratic form $(l, l_0) \mapsto \omega(l, b(l)) - \omega(l_0, b(l_0))$ is hyperbolic on $L \times L$. Consider the isotropic subspace $Q = \{(l, l) \in L \times L \mid l \in L\}$. Integrating first along the fibres of the projection $L \times L \rightarrow (L \times L)/Q$ and then over $(L \times L)/Q$, one gets the desired isomorphism.

2) This follows from (7).

Define a perverse sheaf $F_U$ on $\tilde{U} \times H$ by

$$F_U = \text{pr}_1^* \Theta_U \otimes \tilde{F}_U,$$

it is understood that we take the inverse image of $\tilde{F}_U$ under the projection $\tilde{U} \times H \rightarrow U \times H$ is the above formula. Let $F$ be the intermediate extension of $F_U$ under the open immersion $\tilde{U} \times H \subset \tilde{Y} \times H$.

Remark 4. In the case $d = 0$ we have $H = k^1$ and $\tilde{Y} = B(\mu_2)$. In this case by definition $F = W \otimes \mathcal{L}_\psi \otimes \mathbb{Q}_\ell[1](\frac{1}{2})$ over $\tilde{Y} \times H = B(\mu_2) \times k^1$.

Combining Lemma 3 and 2) of Lemma 2, we get the following.

**Lemma 4.** We have canonically $(\tilde{q}_1^* F_U) \ast (\tilde{q}_2^* F_U) \sim \tilde{q}^* F_U \otimes \mathcal{E}^{2d}$ over $(U \times \mathcal{L}(M)) U_0 \times H$.

We have a map $\xi : \tilde{L}(M) \times \tilde{L}(M) \rightarrow \tilde{Y}$ sending $(B_1, N, B_1^2 \sim J \otimes \det N; B_2, L, B_2^2 \sim J \otimes \det L)$ to $(N, L, B)$, where $B = B_1 \otimes B_2 \otimes J^{-1}$ is equipped with the natural isomorphism $B^2 \sim \det N \otimes \det L$. The restriction of $F$ under

$$\xi \times \text{id} : \tilde{L}(M) \times \tilde{L}(M) \times H \rightarrow \tilde{Y} \times H$$

is also denoted by $F$. Clearly, $F$ is an irreducible perverse sheaf of weight zero.

Consider the cartesian square

$$
\begin{array}{ccc}
(U \times \mathcal{L}(M)) U_0 \times H & \leftrightarrow & (U \times \mathcal{L}(M)) U \times H \\
\downarrow \pi_Y \times \text{id} & & \downarrow \pi_Y \times \text{id} \\
\tilde{U} \times H & \leftrightarrow & \tilde{Y} \times H
\end{array}
$$

This diagram together with Lemma 2 yield a canonical isomorphism over $(U \times \mathcal{L}(M)) U \times H$

$$(\pi_Y \times \text{id})^* F \sim (q_1^* \tilde{F}_U) \ast (q_2^* \tilde{F}_U) \quad (10)$$

by intermediate extension from $(U \times \mathcal{L}(M)) U_0 \times H$. This gives an explicit formula for $F$.

Consider the diagram

$$
\begin{array}{ccc}
U \times H & \xrightarrow{i_U \times \text{id}} & U \times \mathcal{L}(M) U \times H \\
\downarrow p_1 \times \text{id} & & \downarrow \pi_Y \times \text{id} \\
\mathcal{L}(M) \times H & \xrightarrow{i_\mathcal{L} \times \text{id}} & \tilde{Y} \times H
\end{array}
$$

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obtained from (9) by multiplication with $H$. By Lemma 2 and (10), we get canonically

$$(p_1 \times \text{id})^* (i_\Delta \times \text{id})^* F \cong (p_1 \times \text{id})^* \tilde{F}_\Delta \otimes (\tilde{Q}_t[1](\frac{1}{2}))^{\text{dim} \mathcal{L}(M)}$$

Since $\tilde{F}_\Delta$ is perverse and $p_1$ has connected fibres, this isomorphism descends to a uniquely defined isomorphism

$$(i_\Delta \times \text{id})^* F \cong \tilde{F}_\Delta \otimes (\tilde{Q}_t[1](\frac{1}{2}))^{\text{dim} \mathcal{L}(M)}$$

By construction, $F$ is $act_U$-equivariant and $G$-equivariant (this holds for $F_U$ and this property is preserved by the intermediate extension).

3.4 To finish the proof of Theorem 1, it remains to establish the convolution property of $F$. We actually prove it in the following form.

Write $\tilde{Y} \times_{\mathcal{L}(M)} \tilde{Y}$ for the stack classifying $R,N,L \in \mathcal{L}(M)$, one dimensional $k$-vector spaces $B_1,B_2$ equipped with isomorphisms $B_1^2 \cong \det R \otimes N$ and $B_2^2 \cong \det N \otimes \det L$. We have a diagram

$$\begin{array}{ccc}
\tilde{Y} & \xrightarrow{\tau_1} & \tilde{Y} \times_{\mathcal{L}(M)} \tilde{Y} \\
\downarrow & & \downarrow \\
\tilde{Y} & \xrightarrow{\tau_2} & \tilde{Y}
\end{array}$$

where $\tau_1$ (resp., $\tau_2$) sends the above collection to $(R,N,B_1) \in \tilde{Y}$ (resp., $(N,L,B_2) \in \tilde{Y}$). The map $\tau$ sends the above collection to $(R,L,B)$, where $B = B_1 \otimes B_2 \otimes (\det N)^{-1}$ is equipped with $B^2 \cong \det R \otimes \det L$.

**Proposition 1.** There is a canonical isomorphism over $(\tilde{Y} \times_{\mathcal{L}(M)} \tilde{Y}) \times H$

$$(\tau_1^* F) * (\tau_2^* F) \cong \tau^* F$$

**Proof**

**Step 1.** Consider the diagram

$$\begin{array}{ccc}
(U \times_{\mathcal{L}(M)} U)_0 & \xrightarrow{\tilde{q}_1 \times \tilde{q}_2} & (\tilde{U} \times_{\mathcal{L}(M)} \tilde{U})_0 \\
\downarrow & & \downarrow \\
\tilde{U} & \xrightarrow{\tilde{\tau}} & \tilde{U}
\end{array}$$

It becomes 2-commutative over $\text{Spec} \mathbb{F}_q(\sqrt{-1})$. More precisely, for $K \in \text{D}(\tilde{U})$ we have a canonical isomorphism functorial in $K$

$$\tilde{q}^* K \otimes \mathcal{E}^{2d} \cong (\tilde{q}_1 \times \tilde{q}_2)^* \tau^* K$$

Indeed, let $(R,N,L)$ be a $k$-point of $(U \times_{\mathcal{L}(M)} U)_0$, let $(R,N,L,B_1,B_2)$ be its image under $\tilde{q}_1 \times \tilde{q}_2$. So, $B_1 = \det N$ and $\pi_Y(R,L,N) = (R,N,B_1)$, $B_2 = \det L$ and $\pi_Y(N,R,L) = (N,L,B_2)$. Write

$$\tau(R,N,L,B_1,B_2) = (R,L,B,\delta : B^2 \cong \det R \otimes \det L)$$

Write $\tilde{q}(R,N,L) = (R,L,B,\delta_0 : B^2 \cong \det R \otimes \det L)$. It suffices to show that $\delta_0 = (-1)^d \delta$. 

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Let $\epsilon_1 : N \xrightarrow{\sim} R$ be the isomorphism sending $n \in N$ to $r \in R$ such that $r = n \mod L$. Write $\epsilon_2 : L \xrightarrow{\sim} N$ for the isomorphism sending $l \in L$ to $n \in N$ such that $l = n \mod R$. Let $\epsilon_0 : L \xrightarrow{\sim} R$ be the isomorphism sending $l \in L$ to $r \in R$ such that $r = l \mod N$. We get two isomorphisms

$$id \otimes \det \epsilon_0, \ det \epsilon_1 \otimes \det \epsilon_2 : \ det N \otimes \det L \xrightarrow{\sim} \det R \otimes \det N$$

We must show that $id \otimes \det \epsilon_0 = (-1)^d \det \epsilon_1 \otimes \det \epsilon_2$. Pick a base $\{n_1, \ldots, n_d\}$ in $N$. Define $r_i \in R, l_i \in L$ by $n_i = r_i + l_i$. Then

$$\epsilon_1(n_i) = r_i, \ \epsilon_2(l_i) = n_i, \ \epsilon_0(l_i) = -r_i$$

So, $\epsilon_0(l_1 \wedge \ldots \wedge l_d) = (-1)^d r_1 \wedge \ldots \wedge r_d$. On the other hand, $\det \epsilon_1 \otimes \det \epsilon_2$ sends

$$(n_1 \wedge \ldots \wedge n_d) \otimes (l_1 \wedge \ldots \wedge l_d)$$

to $(r_1 \wedge \ldots \wedge r_d) \otimes (n_1 \wedge \ldots \wedge n_d)$.

**Step 2.** The isomorphism $[\text{6}]$ for $i = 0$ yields $(\tilde{U} \times_{\mathcal{L}(M)} \tilde{U})_0 \xrightarrow{\sim} (U \times_{\mathcal{L}(M)} U)_0 \times B(\mu_2) \times B(\mu_2)$. The corresponding 2-automorphisms $\mu_2 \times \mu_2$ of $(\tilde{Y} \times_{\mathcal{L}(M)} \tilde{Y})$ act in the same way on both sides of $[\text{11}]$. Now from Step 1 it follows that the isomorphism of Lemma $[\text{4}]$ descends under $\tilde{q}_1 \times \tilde{q}_2$ to the desired isomorphism $[\text{11}]$ over $(\tilde{U} \times_{\mathcal{L}(M)} \tilde{U})_0 \times H$.

**Step 3.** To finish the proof it suffices to show that $(\tau_1^* F) \ast (\tau_2^* F)$ is perverse, the intermediate extension under the open immersion

$$(\tilde{U} \times_{\mathcal{L}(M)} \tilde{U})_0 \times H \subset (\tilde{Y} \times_{\mathcal{L}(M)} \tilde{Y}) \times H$$

Let us first explain the idea informally, at the level of functions. In this step for $(N, R, B) \in \tilde{Y}$ we denote by $F_{N,R,B} : H \to \bar{Q}_l$ the function trace of Frobenius of the sheaf $F$.

Given $(R, N, B_1) \in \tilde{Y}$ and $(N, L, B_2) \in \tilde{Y}$ pick any $S, T \in \mathcal{L}(M)$ such that $(R, S, N) \in U \times_{\mathcal{L}(M)} U$, $(N, T, L) \in U \times_{\mathcal{L}(M)} U$ and $S \cap T = S \cap L = 0$. Assuming

$$(R, N, B_1) = \pi_Y(R, S, N) \quad \text{and} \quad (N, L, B_2) = \pi_Y(N, T, L),$$

by $[\text{11}]$ we get

$$F_{R,N,B_1} \ast F_{N,L,B_2} = (\tilde{F}_{R,S} \ast \tilde{F}_{S,N}) \ast (\tilde{F}_{N,T} \ast \tilde{F}_{T,L}) = q^{d+1} \theta(S, N, T) \tilde{F}_{R,S} \ast \tilde{F}_{S,T} \ast \tilde{F}_{T,L}$$

$$= q^{2d+2} \theta(S, N, T) \theta(S, T, L) \tilde{F}_{R,S} \ast \tilde{F}_{S,L} = q^{2d+2} \theta(S, N, T) \theta(S, T, L) F_{R,L,B},$$

where $(R, L, B) = \pi_Y(R, S, L)$. Now we turn back to the geometric setting.

**Step 4.** Consider the scheme $\mathcal{W}$ classifying $(R, S, N) \in U \times_{\mathcal{L}(M)} U$ and $(N, T, L) \in U \times_{\mathcal{L}(M)} U$ such that $S \cap T = S \cap L = 0$. Let

$$\kappa : \mathcal{W} \to \tilde{Y} \times_{\mathcal{L}(M)} \tilde{Y}$$
be the map sending the above point to \((R, N, L, B_1, B_2)\), where \((R, N, B_1) = \pi_Y(R, S, N)\) and \((N, L, B_2) = \pi_Y(N, T, L)\). The map \(\kappa\) is smooth and surjective. It suffices to show that

\[
\kappa^*((\tau_1^* F) \ast (\tau_2^* F))
\]
is a shifted perverse sheaf, the intermediate extension from \(\kappa^{-1}(\tilde{U} \times_{L(M)} U)\).

Let \(\mu : W \to U \times_{L(M)} U\) be the map sending a point of \(W\) to \((R, S, L)\). Applying (10) several times as in Step 3, we learn that there is a local system of rank one and order two, say \(I\) on \(W\) such that

\[
\kappa^*((\tau_1^* F) \ast (\tau_2^* F)) \cong I \otimes \mu^* \pi_Y^* F
\]

Since \(F\) is an irreducible perverse sheaf, our assertion follows. □

Thus, Theorem 1 is proved.

3.5 Now given \(k\)-points \(N^0, L^0 \in \tilde{L}(M)\), let \(F_{N^0, L^0} \in \mathcal{D}(H)\) be the \(*\)-restriction of \(F\) under \((N^0, L^0) \times \text{id} : H \hookrightarrow \tilde{Y} \times H\). Define the functor \(\mathcal{F}_{N^0, L^0} : \mathcal{D}H \rightarrow \mathcal{D}H\) by

\[
\mathcal{F}_{N^0, L^0}(K) = F_{N^0, L^0} \ast K
\]

To see that it preserves perversity we can pick \(S^0 \in \tilde{L}(M)\) with \(N \cap S = L \cap S = 0\) and use \(\mathcal{F}_{N^0, L^0} = \mathcal{F}_{N^0, S^0} \circ \mathcal{F}_{S^0, L^0}\). This reduces the question to the case \(N \cap L = 0\), in the latter case \(\mathcal{F}_{N^0, L^0}\) is nothing but the Fourier transform.

By Theorem 1 for \(N^0, L^0, R^0 \in \tilde{L}(M)\) the diagram commutes

\[
\begin{array}{ccc}
\mathcal{D}H_L & \xrightarrow{\mathcal{F}_{L^0}} & \mathcal{D}H_R \\
\downarrow \mathcal{F}_{N^0, L^0} & & \downarrow \mathcal{F}_{N^0, R^0} \\
\mathcal{D}H_N & & \\
\end{array}
\]

3.6 Nonramified Weil category

For a \(k\)-point \(L^0 \in \tilde{L}(M)\) let \(i_{L^0} : \tilde{L}(M) \rightarrow \tilde{L}(M) \times \tilde{L}(M) \times H\) be the map sending \(N^0\) to \((N^0, L^0, 0)\). We get a functor \(\mathcal{F}_{L^0} : \mathcal{D}H_L \rightarrow \mathcal{D}H(\tilde{L}(M))\) sending \(K\) to the complex

\[
i_{L^0}\ast(F \ast \text{pr}_3^* K) \otimes (\hat{Q}_\ell[1](\frac{1}{2}))^{\dim L(M) - 2d - 1}
\]

For any \(k\)-points \(L^0, N^0 \in \tilde{L}(M)\) the diagram commutes

\[
\begin{array}{ccc}
\mathcal{D}H_L & \xrightarrow{\mathcal{F}_{L^0}} & \mathcal{D}(\tilde{L}(M)) \\
\downarrow \mathcal{F}_{L^0, N^0} & & \downarrow \mathcal{F}_{N^0} \\
\mathcal{D}H_N & & \\
\end{array}
\]

One checks that \(\mathcal{F}_{L^0}\) is exact for the perverse t-structure.
Definition 2. The non-ramified Weil category $W(\tilde{L}(M))$ is the essential image of $\mathcal{F}_{L_0} : \mathcal{H}_L \to \mathcal{P}(\tilde{L}(M))$. This is a full subcategory in $\mathcal{P}(\tilde{L}(M))$ independent of $L^0$, because (12) is commutative.

The group $G$ acts naturally on $\tilde{L}(M)$, hence also on $\mathcal{P}(\tilde{L}(M))$. This action preserves the full subcategory $W(\tilde{L}(M))$.

At the classical level, for $L \in \mathcal{L}(M)$ the $G$-representation $\mathcal{H}_L \simeq \mathcal{H}_{L,\text{odd}} \oplus \mathcal{H}_{L,\text{even}}$ is a direct sum of two irreducible ones consisting of odd and even functions respectively. The category $W(\tilde{L}(M))$ is a geometric analog of the space $\mathcal{H}_{L,\text{even}}$. The geometric analog of the whole Weil representation $\mathcal{H}_L$ is as follows.

Definition 3. Let $\text{act}_I : \tilde{C} \times H \to \tilde{L}(M) \times H$ be the map sending $(L^0, h, \tilde{l} \in \tilde{L})$ to $(L^0, \tilde{lh})$. A perverse sheaf $K \in \mathcal{P}(\tilde{L}(M) \times H)$ is $\tilde{C}$-equivariant if it is equipped with an isomorphism

$$\text{act}_I^* K \simeq \text{pr}^*_1 K \otimes \text{pr}^*_2 \chi_C$$

satisfying the usual associativity property, and whose restriction to the unit section is the identity.

The complete Weil category $W(M)$ is the category of pairs $(K, \sigma)$, where $K \in \mathcal{P}(\tilde{L}(M) \times H)$ is a $\tilde{C}$-equivariant perverse sheaf, and

$$\sigma : F \circ \text{pr}_{23}^* K \simeq \text{pr}_{13}^* K$$

is an isomorphism for the projections $\text{pr}_{13}, \text{pr}_{23} : \tilde{L}(M) \times \tilde{L}(M) \times H \to \tilde{L}(M) \times H$. The map $\sigma$ must be compatible with the associativity constraint and the unit section constraint of $F$.

The group $G$ acts on $\tilde{L}(M) \times H$ sending $(g \in G, L^0, h)$ to $(gL^0, gh)$. This action extends to an action of $G$ on the category $W(M)$.

4. Compatibility property

4.1 In this section we establish the following additional property of the canonical intertwining operators. Let $V \subset M$ be an isotropic subspace, $V^\perp \subset M$ its orthogonal complement. Let $\mathcal{L}(M)_V \subset \mathcal{L}(M)$ be the open subscheme of $L \in \mathcal{L}(M)$ such that $L \cap V = 0$. Set $M_0 = V^\perp / V$. We have a map $p_V : \mathcal{L}(M)_V \to \mathcal{L}(M_0)$ sending $L$ to $L_V := L \cap V^\perp$.

Write $Y = \mathcal{L}(M) \times \mathcal{L}(M)$ and $Y_V = \mathcal{L}(M)_V \times \mathcal{L}(M)_V$. The gerb $\tilde{Y}$ is defined as in Section 3.2, write $\tilde{Y}_V$ for its restriction to $Y_V$. Set $Y_0 = \mathcal{L}(M_0) \times \mathcal{L}(M_0)$, we have the corresponding gerb $\tilde{Y}_0$ defined as in Section 3.2. We extend the map $p_V : \mathcal{P}(\tilde{L}(M))$ to a map

$$\pi_V : \tilde{Y}_V \to \tilde{Y}_0$$

sending $(L_1, L_2, B, \mathcal{B}^2_0 \simeq \det L_1 \otimes \det L_2)$ to

$$(L_{1,V}, L_{2,V}, B_0, \mathcal{B}^2_0 \simeq \det L_{1,V} \otimes \det L_{2,V})$$

Here $L_{i,V} = L_i \cap V^\perp$ and $B_0 = B \otimes \det V$. We used the exact sequences

$$0 \to L_{i,V} \to L_i \to M / V^\perp \to 0$$

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yielding canonical \( \mathbb{Z}/2\mathbb{Z}\)-graded isomorphisms \( \det L_i \otimes \det V^* \cong \det L_i \).

Write \( H_0 = M_0 \oplus k \) for the Heisenberg group of \( M_0 \). For \( L \in \mathcal{L}(M)V \) we have the categories \( \mathcal{H}_L \) and \( \mathcal{H}_{LV} \) of certain perverse sheaves on \( H \) and \( H_0 \) respectively. To such \( L \) we associate a transition functor \( T^L : \mathcal{H}_{LV} \to \mathcal{H}_L \) which will be fully faithful and exact for the perverse t-structures.

Write for brevity \( H^V = V^\perp \times \mathbb{A}^1 \). First, at the level of functions, given \( f \in \mathcal{H}_{LV} \) consider it as a function on \( H^V \) via the composition \( H^V \xrightarrow{\alpha_V} H_0 \xrightarrow{f} \mathbb{Q}_\ell \), where \( \alpha_V \) sends \((v, a)\) to \((v \mod V, a)\). Then there is a unique \( f_1 \in \mathcal{H}_L \) such that \( f_1(m) = q^{\dim V} f(m) \) for all \( m \in H^V \). We use the property \( V^\perp + L = M \). We set

\[
(T^L)(f) = f_1
\]

(13)

The image of \( T^L \) is

\[
\{ f_1 \in \mathcal{H}_L \mid f(h(v, 0)) = f(h), \ h \in H, v \in V \}
\]

Note that \( H^V \subset H \) is a subgroup, and \( V = \{(v, 0) \in H^V \mid v \in V\} \subset H^V \) is a normal subgroup lying in the center of \( H^V \). The operator \( T^L : \mathcal{H}_{LV} \to \mathcal{H}_L \) commutes with the action of \( H^V \). It is understood that on \( \mathcal{H}_{LV} \) this group acts via its quotient \( H^V \xrightarrow{\alpha_V} H_0 \).

On the geometric level, consider the map \( s : L \times H^V \to H \) sending \((l, (v, a))\) to the product in the Heisenberg group \((l, 0)(v, a)\) \(H \). Note that \( s \) is smooth and surjective, an affine fibration of rank \( \dim L \). Given \( K \in \mathcal{H}_{LV} \) there is a (defined up to a unique isomorphism) perverse sheaf \( T^L K \in \mathcal{H}_L \) equipped with

\[
s^*(T^L K) \otimes (\mathbb{Q}_\ell[1](\frac{1}{2}))^{\dim LV} \cong \mathbb{Q}_\ell \otimes \alpha_V^* K \otimes (\mathbb{Q}_\ell[1](\frac{1}{2}))^{\dim V + \dim L}
\]

The compatibility property of the canonical interwining operators is as follows.

**Proposition 2.** Let \((L, N, \mathcal{B}) \in \tilde{Y}_V \), write \((L_V, N_V, \mathcal{B}_0)\) for the image of \( (L, N, \mathcal{B}) \) under \( \pi_V \). Write \( \mathcal{F}_{NV,L_0} : \mathcal{H}_L \to \mathcal{H}_N \) and \( \mathcal{F}_{NV,L_0}^{LV} : \mathcal{H}_{LV} \to \mathcal{H}_{NV} \) for the corresponding functors defined as in Section 3.5. Then the diagram of categories is canonically 2-commutative

\[
\begin{array}{ccc}
\mathcal{H}_{LV} & \xrightarrow{T^L} & \mathcal{H}_L \\
\downarrow \mathcal{F}_{NV,L_0}^{LV} & & \downarrow \mathcal{F}_{NV,L_0} \\
\mathcal{H}_{NV} & \xrightarrow{T^N} & \mathcal{H}_N \\
\end{array}
\]

One may also replace \( \mathcal{H} \) by \( D\mathcal{H} \) in the above diagram.

4.2 First, we realize the functors \( T^L \) by a universal kernel, namely, we define a perverse sheaf \( T \) on \( \mathcal{L}(M)_V \times H \times H_0 \) as follows.

Remind the vector bundle \( \tilde{C} \to \mathcal{L}(M) \), its fibre over \( L \) is \( \tilde{L} = L \times \mathbb{A}^1 \). Write \( \tilde{C}_V \) for the restriction of \( \tilde{C} \) to the open subscheme \( \mathcal{L}(M)_V \). We have a closed immersion

\[
i_0 : \tilde{C}_V \times H^V \to \mathcal{L}(M)_V \times H \times H_0
\]
Define $T$ for $4.3$ We will prove a geometric version of the equality (up to an explicit power of $s$) satisfying the usual associativity property, and its restriction to the unit section is the identity.

The geometric analog is as follows. Let $0\tilde{C} \rightarrow \mathcal{L}(M)_V$ be the vector bundle, whose fibre over $L \in \mathcal{L}(M)_V$ is $L \times \tilde{L}_V$. Consider the diagram

$$\mathcal{L}(M)_V \times H \times H_0 \overset{pr^V}{\rightarrow} 0\tilde{C} \times H \times H_0 \overset{\text{act}^V_{tr}}{\rightarrow} \mathcal{L}(M)_V \times H \times H_0,$$

where $pr^V$ is the projection, and $\text{act}^V_{tr}$ sends

$$(L \in \mathcal{L}(M)_V, \tilde{l} \in \tilde{L}, \tilde{l}_0 \in \tilde{L}_V, h \in H, h_0 \in H_0)$$

to $(L, \tilde{l}h, \tilde{l}_0h_0)$. Let $0p : 0\tilde{C} \rightarrow \mathbb{A}^1$ be the map sending

$$(L \in \mathcal{L}(M)_V, \tilde{l} \in \tilde{L}, \tilde{l}_0 \in \tilde{L}_V)$$

to $p(\tilde{l}) - p(\tilde{l}_0)$. Here $p : \tilde{L} \rightarrow \mathbb{A}^1$ and $p : \tilde{L}_V \rightarrow \mathbb{A}^1$ are the projections on the center. Set $0\chi = (0p)^*\mathcal{L}_\psi$. Then $T$ is act$^V_{tr}$-equivariant, that is, it admits an isomorphism

$$(\text{act}^V_{tr})^* T \cong (pr^V)^* T \otimes pr^*_1(0\chi),$$

satisfying the usual associativity property, and its restriction to the unit section is the identity.

4.3 We will prove a geometric version of the equality (up to an explicit power of $q$)

$$\int_{u \in H} F_{N^0, L^0}(hu^{-1})T_L(u, h_0)du = \int_{v \in H_0} T_N(h, v)F_{N^0, L^0}(vh_0^{-1})dv$$

for $h \in H, h_0 \in H_0$. Here $(N^0, L^0) \in \tilde{Y}_V$ and

$$(N^0, L^0) = \pi_V(N^0, L^0)$$
Lemma 6. is equivalent to the following.

\[ \tilde{Y}_V \times H \times H_0 \stackrel{\text{pr}_1} \to \tilde{Y}_V \times H_0 \stackrel{\pi_V \times \text{id}} \to \tilde{Y}_0 \times H_0 \]

Proposition 2 is an immediate consequence of the following.

Lemma 5. There is a canonical isomorphism over \( \tilde{Y}_V \times H \times H_0 \)

\[ (\text{pr}_{12}^* F) \ast_H (p_2 \times \text{id})^* T \simeq (q_0^*(\text{inv} F)) \ast_{H_0} (p_1 \times \text{id})^* T \]

where \( \text{pr}_{12} : \tilde{Y}_V \times H \times H_0 \to \tilde{Y}_V \times H \) and \( p_1 \times \text{id}, p_2 \times \text{id} : \tilde{Y}_V \times H \times H_0 \to \mathcal{L}(M)_V \times H \times H_0 \).

Let \( i_V : H^V \hookrightarrow H \) be the natural closed immersion. It is elementary to check that Lemma 5 is equivalent to the following.

Lemma 6. There is a canonical isomorphism of (shifted) perverse sheaves

\[ (\text{id} \times \alpha_V) ! i_V^* F \simeq (\pi_V \times \text{id})^* F \otimes (\mathcal{Q}_0[1] \langle \frac{1}{2} \rangle)^{\dim \text{rel}(\pi_V) + \dim V} \]  

(15)

for the diagram

\[ \begin{array}{ccc}
\tilde{Y}_V \times H^V & \xrightarrow{i_V} & \tilde{Y}_V \times H \\
\downarrow \text{id} \times \alpha_V & & \downarrow \\
\tilde{Y}_0 \times H_0 & \xleftarrow{\pi_V \times \text{id}} & \tilde{Y}_V \times H_0 
\end{array} \]

Proof Write \( U(M_0) \) for the scheme \( U \) constructed out of the symplectic space \( M_0 \), it classifies pairs of lagrangian subspaces in \( M_0 \) that do not intersect. We have a 2-commutative diagram

\[ \begin{array}{c}
U(M_0) \times \mathcal{L}(M)_0 & \xrightarrow{\pi'_W} & W_V & \xleftarrow{\pi_V} & U \times \mathcal{L}(M) \ U \\
\downarrow \pi_0 & & \downarrow \pi_{Y,V} & & \downarrow \pi_V \\
\tilde{Y}_0 & \xleftarrow{\pi_V} & \tilde{Y}_V & & \tilde{Y}_V 
\end{array} \]

where the square is cartesian thus defining \( W_V, \pi_W \), and \( \pi_{Y,V} \). The map \( i_W \) is a locally closed immersion. Write a point of \( W_V \) as a triple \((N, R, L) \in \mathcal{L}(M)\) such that \( N, L \in \mathcal{L}(M)_V \), \( V \subset R \subset V^\perp \), and \( N \cap R = R \cap L = 0 \). The map \( \pi_W \) sends \((N, R, L)\) to \((N_V, R_V, L_V)\) with \( R_V = R/V \).

Let us establish the isomorphism (15) after restriction under \( \pi_{Y,V} \times \alpha_V : W_V \times H^V \to \tilde{Y}_V \times H_0 \). We first give the argument at the level of functions and then check that it holds through in the geometric setting.

Consider a point of \( W_V \) given by a triple \((N, R, L) \in \mathcal{L}(M)\), so \( N, L \in \mathcal{L}(M)_V \), \( V \subset R \subset V^\perp \), and \( N \cap R = R \cap L = 0 \). We have \( V^\perp = R \oplus L_V \). Let \( h \in H^V \), write \( h = (r, a)(l_1, 0) \) for uniquely
defined $r \in R, l_1 \in L_V, a \in k$. Write $(N^0, L^0) \in \widetilde{Y}_V$ for the image of $(N, R, L)$ under $\pi_{Y;V}$. Using (10), we get

$$
\int_{v \in V} F_{N^0, L^0}(h(v, 0))dv = q^{\dim L(M)} \sum_{l \in L} \tilde{F}_{N,R}(l,0) \tilde{F}_{R,L}(l,0)dv = q^{\dim L(M) + \frac{d+1}{2}} \int_{v \in V, u \in H} \tilde{F}_{N,R}(u) \tilde{F}_{R,L}(u^{-1}(r, a)(v, 0))dv = q^{\dim L(M) + \frac{d+1}{2}} \int_{v \in V, l \in L} \tilde{F}_{N,R}(l,0) \tilde{F}_{R,L}((-l, 0)(r, a)(v, 0))dvd
$$

Since $(-l, 0)(r + v, a) = (r + v, a + \omega(r + v, l))(-l, 0)$, the latter expression equals

$$
q^{-\frac{d}{2}} \int_{v \in V, l \in L} \tilde{F}_{N,R}(l,0) \psi(a + \omega(r + v, l))dvl = q^{\dim L(M) - \frac{d}{2}} \int_{l \in L_V} \tilde{F}_{N,R}(l,0) \psi(a + \omega(r, l))dl
$$

For $l \in L_V$ we get $\tilde{F}_{N,R}(l,0) = q^{\dim L(M) - \dim V} \tilde{F}_{N,R}(l,0)$. Indeed, since $V^\perp = R \oplus N_V$, there are unique $r_1 \in R, n_1 \in N_V$ such that $l = n_1 + r_1$. For $r_1 = r_1 \mod V \in M_0$ we get

$$
\tilde{F}_{N,R}(l,0) = q^{-\dim L(M) - \frac{d}{2}} \chi_{NR}(l,0) = q^{-\dim L(M) - \frac{d}{2}} \chi_{NVR}(-r_1, n_1) = q^{-\dim L(M) - \dim V} \tilde{F}_{N,R}(l,0)
$$

Further, we claim that

$$
\tilde{F}_{R_V,L_V}((-l, 0)\alpha_V(h)) = q^{-\dim L(M) - \dim V} \psi(a + \omega(r, l))
$$

This follows from definition (5) of $\tilde{F}_U$ and the formula $(-l, 0)(r, a) = (r, a + \omega(r, l))(-l, 0)$.

Combining the above we get

$$
\int_{v \in V} F_{N^0, L^0}(h(v, 0))dv = q^c \int_{l \in L_V} \tilde{F}_{N,R}(l,0) \tilde{F}_{R,L}(l,0)((-l, 0)\alpha_V(h))dl = q^{c + \dim V - d} \int_{u \in H_0} \tilde{F}_{N,R}(u) \tilde{F}_{R,L}(u^{-1}\alpha_V(h))du
$$

with $c = \frac{\dim H_0 - d}{2} + 2 \dim L(M_0) - \dim L(M)$. By (10), the latter expression identifies with $F_{N^0, L^0}(h)$ up to an explicit power of $q$.

The argument holds through in the geometric setting yielding the desired isomorphism $\gamma$ over $W_V \times H^V$. For any point $(N_V, L_V B_0) \in \widetilde{Y}_V$ such that $N_V \neq L_V$ the fibre of $\pi_{Y_0}$ over this point is geometrically connected. So, for $\dim V < d$ the isomorphism $\gamma$ descends to a uniquely defined isomorphism (15). The case $\dim V = d$ is easier and is left to the reader. □
Remark 5. Let $i_H : \text{Spec } k \hookrightarrow H$ denote the zero section. Arguing as in Lemma [6] for the map $\text{id} \times i_H : \hat{Y} \rightarrow \hat{Y} \times H$ one gets a canonical isomorphism

$$(\text{id} \times i_H)^* F \cong (S_{M,g} \otimes \mathcal{E} \oplus S_{M,s} \otimes \mathcal{E}^{d-1}) \otimes (\mathbb{Q}_\ell[1](\frac{1}{2}))^{\dim H},$$

it will not be used in this paper.

4.4 The functors $T^L$ satisfy the following transitivity property. Assume that $V_1 \subset V$ is another isotropic subspace in $M$. Let $M_1 = V_1^\perp / V_1$ and $H_1 = M_1 \times \mathbb{A}^1$ be the corresponding Heisenberg group. Then for $L \in \mathcal{L}(M)_V$ we also have $L_{V_1} := L \cap V_1^\perp$ and the category $\mathcal{H}_{L_{V_1}}$ of certain perverse sheaves on $H_1$. Then the diagram is canonically $2$-commutative

$$\begin{array}{c}
\mathcal{H}_{L_{V_1}} \xrightarrow{T^L} \mathcal{H}_{L_{V_1}} \\
\downarrow T^L \quad \downarrow T^L
\end{array}$$

4.5 We will need also one more compatibility property of the canonical interwining operators. Let $V \subset V^\perp \subset M$ be as in 4.1. Write $i_{0,V} : \mathcal{L}(M_0) \rightarrow \mathcal{L}(M)$ for the closed immersion sending $L_0$ to the preimage of $L_0$ under $V^\perp \rightarrow V^\perp / V$.

For $L \in \mathcal{L}(M)$ with $V \subset L$ set $L_V = L / V \in \mathcal{L}(M_0)$. Let $(\mathcal{L}(M_0) \times \mathcal{L}(M)_V)^\wedge$ denote the restriction of the gerb $\hat{Y}$ under

$$\mathcal{L}(M_0) \times \mathcal{L}(M)_V \xrightarrow{i_{0,V} \times \text{id}} \mathcal{L}(M) \times \mathcal{L}(M)_V \subset \hat{Y}$$

Define $\pi_{0,V} : (\mathcal{L}(M_0) \times \mathcal{L}(M)_V)^\wedge \rightarrow \hat{Y}_0$ as the map sending $(L, N, B, B^2 \cong \det L \otimes \det N)$ to

$$(L_V, N_V, B, B^2 \cong \det L_V \otimes \det N_V)$$

Here $L \in \mathcal{L}(M)$ with $V \subset L$. We have used the canonical $\mathbb{Z} / 2\mathbb{Z}$-graded isomorphism $\det L \otimes \det N \cong \det L_V \otimes \det N_V$.

Remind the closed immersion $i_V : H^V \hookrightarrow H$. For $L \in \mathcal{L}(M)$ with $V \subset L$ define the transition functor $T^L : \mathcal{H}_{L^V} \rightarrow \mathcal{H}_L$ by

$$T^L(K) = i_V \alpha_V^* K \otimes (\mathbb{Q}_\ell[1](\frac{1}{2}))^{\dim V}$$

The proof of the following is similar to that of Proposition [2] and is left to the reader.

**Proposition 3.** Let $(L, N, B) \in (\mathcal{L}(M_0) \times \mathcal{L}(M)_V)^\wedge$, let $(L_V, N_V, B)$ denote its image under $\pi_{0,V}$. Write $\mathcal{F}_{N_0, L_0} : \mathcal{H}_L \rightarrow \mathcal{H}_N$ and $\mathcal{F}_{N_0, L_0}^\wedge$ for the corresponding functors defined as in Section 3.5. Then the diagram of categories is canonically $2$-commutative

$$\begin{array}{c}
\mathcal{H}_{L_{V_1}} \xrightarrow{T^L} \mathcal{H}_{L_{V_1}} \\
\downarrow \mathcal{F}_{N_0, L_0} \quad \downarrow \mathcal{F}_{N_0, L_0}^\wedge
\end{array}$$

$$\begin{array}{c}
\mathcal{H}_{N_{V_1}} \xrightarrow{T^N} \mathcal{H}_{N_{V_1}}
\end{array}$$

One may also replace $\mathcal{H}$ by $D\mathcal{H}$ in the above diagram. $\square$
5. DISCRETE LAGRANGIAN LATTICES AND THE METAPLECTIC GROUP

5.1 Set $O = k[[t]] \subset F = k((t))$. Denote by $\Omega$ the completed module of relative differentials of $O$ over $k$. Let $M$ be a free $O$-module of rank $2d$ with symplectic form $\wedge^2 M \to \Omega$. Write $G$ for the group scheme over $\text{Spec} \ O$ of automorphisms of $M$ preserving the symplectic form. Consider the Tate space $M(F)$ (cf. [1], 4.2.13 for the definition), it is equipped with the symplectic form $(m_1, m_2) \mapsto \text{Res} \, \omega \langle m_1, m_2 \rangle$.

For a $k$-subspace $L \subset M(F)$ write

$$L^\perp = \{ m \in M(F) \mid \text{Res} \, \omega \langle m, l \rangle = 0 \text{ for all } l \in L \}$$

For two $k$-subspaces $L_1, L_2 \subset M$ we get $(L_1 + L_2)^\perp = L_1^\perp \cap L_2^\perp$. For a finite-dimensional symplectic $k$-vector space $U$ write $\mathcal{L}(U)$ for the variety of lagrangian subspaces in $U$.

As in loc.cit, we say that an $O$-submodule $R \subset M(F)$ is a $c$-lattice if $M(-N) \subset R \subset M(N)$ for some integer $N$. A lagrangian $d$-lattice in $M(F)$ is a $k$-vector subspace $L \subset M(F)$ such that $L^\perp = L$ and there exists a $c$-lattice $R$ with $R \cap L = 0$. Note that the condition $R \cap L = 0$ implies $R^\perp + L = M(F)$. Let $\mathcal{L}_d(M(F))$ denote the set of lagrangian $d$-lattices in $M(F)$.

For a given $c$-lattice $R \subset M(F)$ write

$$\mathcal{L}_d(M(F))_R = \{ L \in \mathcal{L}_d(M(F)) \mid L \cap R = 0 \}$$

If $R$ is a $c$-lattice in $M(F)$ with $R \subset R^\perp$ then $\mathcal{L}_d(M(F))_R$ is a naturally a $k$-scheme (not of finite type over $k$). Indeed, for each $c$-lattice $R_1 \subset R$ we have the variety

$$\mathcal{L}(R^\perp_1 / R_1)_R := \{ L_1 \in \mathcal{L}(R^\perp_1 / R_1) \mid L_1 \cap R / R_1 = 0 \}$$

For $R_2 \subset R_1 \subset R$ we get a map $p_{R_2, R_1} : \mathcal{L}(R^\perp_1 / R_2)_R \to \mathcal{L}(R^\perp_1 / R_1)_R$ sending $L_2$ to

$$L_1 = (L_2 \cap (R^\perp_1 / R_2)) + R_1$$

The map $p_{R_2, R_1}$ is a composition of two affine fibrations of constant rank. Then $\mathcal{L}_d(M(F))_R$ is the inverse limit of $\mathcal{L}(R^\perp_1 / R_1)_R$ over the partially ordered set of $c$-lattices $R_1 \subset R$.

If $R' \subset R$ is another $c$-lattice then $\mathcal{L}_d(M(F))_R \subset \mathcal{L}_d(M(F))_{R'}$ is an open immersion (as it is an open immersion on each term of the projective system). So, $\mathcal{L}_d(M(F))$ is a $k$-scheme that can be seen as the inductive limit of $\mathcal{L}_d(M(F))_R$.

Let us define the categories $P(\mathcal{L}_d(M(F)))$ and $P_{G(O)}(\mathcal{L}_d(M(F)))$ of perverse sheaves and $G(O)$-equivariant perverse sheaves on $\mathcal{L}_d(M(F))$.

For $r \geq 0$ set

$$r \mathcal{L}_d(M(F)) = \mathcal{L}_d(M(F))_{M(-r)}$$

the group $G(O)$ acts on $r \mathcal{L}_d(M(F))$ naturally. First, define the category $D_{G(O)}(r \mathcal{L}_d(M(F)))$ as follows.

For $N + r \geq 0$ set $N, r \cdot M = t^{-N} M / t^r M$. For $N \geq r \geq 0$ the action of $G(O)$ on $r \mathcal{L}(N, N \cdot M) := \mathcal{L}(N, N \cdot M)_{M(-r)}$ factors through $G(O / t^{2N})$. For $r_1 \geq 2N$ the kernel

$$\text{Ker}(G(O / t^{r_1})) \to G(O / t^{2N})$$
is unipotent, so that we have an equivalence (exact for the perverse t-structures)

\[ D_{G(\mathcal{O}/\mathbb{Z}N)}(r, \mathcal{L}(N,NM)) \simeq D_{G(\mathcal{O}/t^2)}(r, \mathcal{L}(N,NM)) \]

Define \( D_{G(\mathcal{O})}(r, \mathcal{L}(N,NM)) \) as \( D_{G(\mathcal{O}/t^2)}(r, \mathcal{L}(N,NM)) \) for any \( r_1 \geq 2N \). It is equipped with the perverse t-structure.

For \( N_1 \geq N \geq r \geq 0 \) the fibres of the above projection

\[ p : r\mathcal{L}(N_1,N_1M) \to r\mathcal{L}(N,NM) \]

are isomorphic to affine spaces of fixed dimension, and \( p \) is smooth and surjective. Hence, this map yields transition functors (exact for the perverse t-structures and fully faithful embeddings)

\[ D_{G(\mathcal{O})}(r\mathcal{L}(N,NM)) \to D_{G(\mathcal{O})}(r\mathcal{L}(N_1,N_1M)) \]

and

\[ D(r\mathcal{L}(N,NM)) \to D(r\mathcal{L}(N_1,N_1M)) \]

We define \( D_{G(\mathcal{O})}(r\mathcal{L}_d(M(F))) \) as the inductive 2-limit of \( D_{G(\mathcal{O})}(r\mathcal{L}(N,NM)) \) as \( N \) goes to plus infinity. The category \( D(r\mathcal{L}_d(M(F))) \) is defined similarly. Both they are equipped with perverse t-structures.

If \( N_1 \geq N \geq r_1 \geq r \geq 0 \) we have a diagram

\[
\begin{array}{ccc}
r\mathcal{L}(N_1,N_1M) & \xrightarrow{p} & r\mathcal{L}(N,NM) \\
\downarrow j & & \downarrow j \\
r_1\mathcal{L}(N_1,N_1M) & \xrightarrow{p} & r_1\mathcal{L}(N,NM),
\end{array}
\]

where \( j \) are natural open immersions. The restriction functors \( j^* : D_{G(\mathcal{O})}(r_1\mathcal{L}(N,NM)) \to D_{G(\mathcal{O})}(r\mathcal{L}(N,NM)) \) yield (in the limit as \( N \) goes to plus infinity) the functors

\[ j_{r_1,r}^* : D_{G(\mathcal{O})}(r_1\mathcal{L}_d(M(F))) \to D_{G(\mathcal{O})}(r\mathcal{L}_d(M(F))) \]

of restriction with respect to the open immersion \( j_{r_1,r} : r\mathcal{L}_d(M(F)) \hookrightarrow r_1\mathcal{L}_d(M(F)) \). Define \( D_{G(\mathcal{O})}(\mathcal{L}_d(M(F))) \) as the projective 2-limit of

\[ D_{G(\mathcal{O})}(r\mathcal{L}_d(M(F))) \]

as \( r \) goes to plus infinity. Similarly, \( P_{G(\mathcal{O})}(\mathcal{L}_d(M(F))) \) is defined as the projective 2-limit of \( P_{G(\mathcal{O})}(r\mathcal{L}_d(M(F))) \). Along the same lines, one defines the categories \( P(\mathcal{L}_d(M(F))) \) and \( D(\mathcal{L}_d(M(F))) \).

5.2 Relative determinant For a pair of c-lattices \( M_1, M_2 \) in \( M(F) \) define the relative determinant \( \det(M_1 : M_2) \) as the following \( \mathbb{Z}/2\mathbb{Z} \)-graded 1-dimensional \( k \)-vector space. If \( R \) is a c-lattice in \( M(F) \) such that \( R \subset M_1 \cap M_2 \) then

\[ \det(M_1 : M_2) \cong \det(M_1/R) \otimes \det(M_2/R)^{-1}, \]

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it is defined up to a unique isomorphism.

Write $\text{Gr}_G$ for the affine grassmanian $G(F)/G(O)$ of $G$ (cf. [1], Section 4.5). For $R \in \text{Gr}_G, L \in \mathcal{L}_d(M(F))$ define the relative determinant $\text{det}(R : L)$ as the following ($\mathbb{Z}/2\mathbb{Z}$-graded purely of degree zero) 1-dimensional vector space. Pick a c-lattice $R_1 \subset R$ such that $R_1 \cap L = 0$. Then in $R_1^\perp / R_1$ one gets two lagrangian subspaces $R/R_1$ and $L \cap R_1^\perp := L \cap R_1^\perp$. Set

$$\text{det}(R : L) = \text{det}(R/R_1) \otimes \text{det}(L_{R_1})$$

If $R_2 \subset R_1$ is another c-lattice then the exact sequence

$$0 \to L_{R_1} \to L \cap R_2^\perp \to R_2^\perp / R_1^\perp \to 0$$

yields a canonical $\mathbb{Z}/2\mathbb{Z}$-graded isomorphism

$$\text{det}(R/R_2) \otimes \text{det}(L_{R_2}) \cong \text{det}(R_1/R_2) \otimes \text{det}(R/R_1) \otimes \text{det}(L_{R_1}) \otimes \text{det}(R_2^\perp / R_1^\perp) \cong \text{det}(R/R_1) \otimes \text{det}(L_{R_1})$$

So, $\text{det}(R : L)$ is a $\mathbb{Z}/2\mathbb{Z}$-graded line defined up to a unique isomorphism. Another way to say it is as follows. Consider the complex $R \oplus L \xrightarrow{\Delta} M(F)$ placed in cohomological degrees 0 and 1, where $s(r, l) = r + l$. It has finite-dimensional cohomologies and

$$\text{det}(R : L) = \text{det}(R \oplus L \xrightarrow{\Delta} M(F))$$

For $g \in G(F)$ we have canonically

$$\text{det}(gR : gL) \cong \text{det}(R : L)$$

For $R_1, R_2 \in \text{Gr}_G, L \in \mathcal{L}_d(M(F))$ we have canonically

$$\text{det}(R_1 : L) \cong \text{det}(R_1 : R_2) \otimes \text{det}(R_2 : L)$$

5.3 Write $\mathcal{A}_d$ for the line bundle on $\mathcal{L}_d(M(F))$ with fibre $\text{det}(M : L)$ at $L \in \mathcal{L}_d(M(F))$. Clearly, $\mathcal{A}_d$ is $G(O)$-equivariant, so we may see $\mathcal{A}_d$ as the line bundle on the stack quotient $\mathcal{L}_d(M(F))/G(O)$. Let $\tilde{\mathcal{L}}_d(M(F))$ denote the $\mu_2$-gerb of square roots of $\mathcal{A}_d$.

The categories of the corresponding perverse sheaves $P_{G(O)}(\tilde{\mathcal{L}}_d(M(F)))$ and $P(\tilde{\mathcal{L}}_d(M(F)))$ are defined as above. Namely, first for $r \geq 0$ define

$$D_{G(O)}(r, \tilde{\mathcal{L}}_d(M(F)))$$

as follows. For $N \geq r$ take $r_1 \geq 2N$ and consider the stack quotient $r, \mathcal{L}_{(N,NM)}/G(O/t^{r_1})$. We have the line bundle, say $\mathcal{A}_N$ on this stack whose fibre at $L$ is $\text{det}(M/M(-N)) \otimes \text{det} L$. Here $L \subset N,NM$ is a Lagrangian subspace such that $L \cap (M(-r)/M(-N)) = 0$. Write

$$(r, \mathcal{L}_{(N,NM)}/G(O/t^{r_1}))$$

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for the gerb of square roots of this line bundle. Let $D_{G(O)}(\nu \tilde{\mathcal{L}}(N,N,M))$ denote the category

$$D((\nu \mathcal{L}(N,N,M)/G(O/t^r)))$$

for any $r_1 \geq 2N$ (we have canonical equivalences exact for the perverse t-structures between such categories for various $r_1$).

Assume $N_1 \geq N \geq r$ and $r_1 \geq 2N_1$. For the projection

$$p : \nu \mathcal{L}(N_1,N_1,M)/G(O/t^r) \rightarrow \nu \mathcal{L}(N,N,M)/G(O/t^r)$$

we have a canonical $\mathbb{Z}/2\mathbb{Z}$-graded isomorphism $p^* A_N \cong A_{N_1}$. This yields a transition map

$$(\nu \mathcal{L}(N_1,N_1,M)/G(O/t^r)) \rightarrow (\nu \mathcal{L}(N,N,M)/G(O/t^r))$$

The corresponding inverse image yields a transition functor

$$D_{G(O)}(\nu \tilde{\mathcal{L}}(N,N,M)) \rightarrow D_{G(O)}(\nu \tilde{\mathcal{L}}(N_1,N_1,M))$$

exact for the perverse t-structures (and a fully faithful embedding). We define $D_{G(O)}(\nu \tilde{\mathcal{L}}_d(M(F)))$ as the inductive 2-limit of $D_{G(O)}(\nu \tilde{\mathcal{L}}(N,N,M))$ as $N$ goes to plus infinity.

For $N \geq r' \geq r$ and $r_1 \geq 2N$ we have an open immersion

$$\tilde{j} : (\nu \mathcal{L}(N,N,M)/G(O/t^r)) \subset (\nu \mathcal{L}(N,N,M)/G(O/t^r))$$

hence the $*$-restriction functors

$$\tilde{j}^* : D_{G(O)}(\nu \tilde{\mathcal{L}}(N,N,M)) \rightarrow D_{G(O)}(\nu \tilde{\mathcal{L}}(N,N,M))$$

compatible with the transition functors (16). Passing to the limit as $N$ goes to plus infinity, we get the functors

$$\tilde{j}_{r,r'}^* : D_{G(O)}(\nu \tilde{\mathcal{L}}_d(M(F))) \rightarrow D_{G(O)}(\nu \tilde{\mathcal{L}}_d(M(F)))$$

Define $D_{G(O)}(\tilde{\mathcal{L}}_d(M(F)))$ as the projective 2-limit of $D_{G(O)}(\nu \tilde{\mathcal{L}}_d(M(F)))$ as $r$ goes to plus infinity, and similarly for $P_{G(O)}(\tilde{\mathcal{L}}_d(M(F)))$.

Along the same lines one defines the categories $P(\tilde{\mathcal{L}}_d(M(F)))$ and $D(\tilde{\mathcal{L}}_d(M(F)))$.

5.4 Metaplectic group Let $\mathcal{A}_G$ be the line bundle on the ind-scheme $G(F)$ whose fibre at $g$ is $\det(M : gM)$. Write $\tilde{G}(F) \rightarrow G(F)$ for the gerb of square roots of $\mathcal{A}_G$. The stack $\tilde{G}(F)$ has a structure of a group stack. The product map $m : \tilde{G}(F) \times \tilde{G}(F) \rightarrow \tilde{G}(F)$ sends

$$(g_1, B_1, \sigma_1 : B_1^2 \cong \det(M : g_1M), g_2, B_2, \sigma_2 : B_2^2 \cong \det(M : g_2M))$$

to the collection $(g_1g_2, B, \sigma : B^2 \cong \det(M : g_1g_2M))$, where $B = B_1 \otimes B_2$ and $\sigma$ is the composition

$$(B_1 \otimes B_2)^2 \overset{\sigma_1 \otimes \sigma_2}{\longrightarrow} \det(M : g_1M) \otimes \det(M : g_2M) \overset{\text{id} \otimes g_1}{\longrightarrow} \det(M : g_1M) \otimes \det(g_1M : g_1g_2M) \cong \det(M : g_1g_2M)$$

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Informally speaking, one may think of the exact sequence of group stacks

\[ 1 \to B(\mu_2) \to \tilde{G}(F) \to G(F) \to 1 \]

We also have a canonical section \( G(\mathcal{O}) \to \tilde{G}(F) \) sending \( g \) to

\[ (g, \mathcal{B} = k, \text{id} : \mathcal{B}^2 \cong \det(M : M)) \]

The group stack \( \tilde{G}(F) \) acts naturally on \( \mathcal{L}_d(M(F)) \), the action map \( \tilde{G}(F) \times \mathcal{L}_d(M(F)) \to \mathcal{L}_d(M(F)) \) sends

\[ (g, \mathcal{B}_1, \sigma_1 : \mathcal{B}_1^2 \cong \det(M : gM)), (L, \mathcal{B}_2, \sigma_2 : \mathcal{B}_2^2 \cong \det(M : L)) \]

to the collection \((gL, \mathcal{B})\), where \( \mathcal{B} = \mathcal{B}_1 \otimes \mathcal{B}_2 \) is equipped with the isomorphism

\[ (\mathcal{B}_1 \otimes \mathcal{B}_2)^2 \cong \det(M : gM) \otimes \det(M : L) \xrightarrow{\text{id} \otimes g} \det(M : gM) \otimes \det(gM : gL) \cong \det(M : gL) \]

5.5 For \( g \in G(F) \) and a c-lattice \( R \subset R^\perp \) in \( M(F) \) we have an isomorphism of symplectic spaces \( g : R^\perp / R \cong (gR)^\perp / gR \). For each c-lattice \( R_1 \subset R \) we have a diagram

\[
\begin{array}{ccc}
\mathcal{L}(R_1^\perp / R_1)_R & \xrightarrow{g} & \mathcal{L}(gR_1^\perp / gR_1)_{gR} \\
\downarrow p & & \downarrow p \\
\mathcal{L}(R^\perp / R) & \xrightarrow{g} & \mathcal{L}(gR^\perp / gR)
\end{array}
\]

Let \( \mathcal{A}_{R_1} \) be the \((\mathbb{Z}/2\mathbb{Z}\text{-graded purely of degree zero})\) line bundle on \( \mathcal{L}(R_1^\perp / R_1)_R \) whose fibre at \( L \) is \( \det L \otimes \det(M : R_1) \). Assume that \( \tilde{g} = (g, \mathcal{B}, \mathcal{B}_1^2 \cong \det(M : gM)) \) is a \( k \)-point of \( \tilde{G}(F) \) over \( g \). It yields a diagram

\[
\begin{array}{ccc}
\tilde{\mathcal{L}}(R_1^\perp / R_1)_R & \xrightarrow{\tilde{g}} & \tilde{\mathcal{L}}(gR_1^\perp / gR_1)_{gR} \\
\downarrow p & & \downarrow p \\
\tilde{\mathcal{L}}(R^\perp / R) & \xrightarrow{\tilde{g}} & \tilde{\mathcal{L}}(gR^\perp / gR)
\end{array}
\]

Here the top horizontal arrow sends \((gL, \mathcal{B}_1, \mathcal{B}_1^2 \cong \det L \otimes \det(M : R_1)) \) to

\[ (gL, \mathcal{B}_2, \sigma : \mathcal{B}_2^2 \cong \det(gL) \otimes \det(M : gR_1)), \]

where \( \mathcal{B}_2 = \mathcal{B}_1 \otimes \mathcal{B} \) and \( \sigma \) is the composition

\[ (\mathcal{B}_1 \otimes \mathcal{B})^2 \cong \det L \otimes \det(M : R_1) \otimes \det(M : gM) \xrightarrow{g \otimes \text{id}} \det(gL) \otimes \det(gM : gR_1) \otimes \det(M : gM) \cong \det(gL) \otimes \det(M : gR_1) \]

In the limit by \( R_1 \) the corresponding functors \( \tilde{g}^* : \mathcal{P}(\tilde{\mathcal{L}}(gR_1^\perp / gR_1)_{gR}) \cong \mathcal{P}(\tilde{\mathcal{L}}(R_1^\perp / R_1)_R) \) yield an equivalence

\[ \tilde{g}^* : \mathcal{P}(\tilde{\mathcal{L}}_d(M(F))_{gR}) \cong \mathcal{P}(\tilde{\mathcal{L}}_d(M(F))_R) \]

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Taking one more limit by the partially ordered set of $c$-lattices $R$, one gets an equivalence
\[ \tilde{g}^* : P(\tilde{L}_d(M(F))) \cong P(\tilde{L}_d(M(F))) \]
In this sense $\tilde{G}(F)$ acts on $P(\tilde{L}_d(M(F)))$.

6. Canonical intertwining operators: local field case

6.1 Keep notations of Section 5. Write $H = M \oplus \Omega$ for the Heisenberg group defined as in Section 2.1, this is a group scheme over $\text{Spec} \mathcal{O}$.

For $L \in \mathcal{L}_d(M(F))$ we have the subgroup $\bar{L} = L \oplus \Omega(F) \subset H(F)$ and the character $\chi_L : \bar{L} \to \bar{Q}_\ell^*$ given by $\chi_L(l,a) = \chi(a)$. Here $\chi : \Omega(F) \to \bar{Q}_\ell^*$ sends $a$ to $\psi(\text{Res} a)$. In the classical setting we let $H_L$ denote the space of functions $f : H(F) \to \bar{Q}_\ell$ satisfying
\[ C1) \ f(lh) = \chi_L(l)f(h), \text{ for } h \in H, l \in \bar{L}; \]
\[ C2) \text{ there exists a c-lattice } R \subset M(F) \text{ such that } f(h(r,0)) = f(h) \text{ for } r \in R, h \in H. \]

Note that such $f$ has automatically compact support modulo $\bar{L}$. The group $H(F)$ acts on $H_L$ by right translations, this is a model of the Weil representation. Let us introduce a geometric analog of $H_L$.

Given a c-lattice $R \subset M(F)$ such that $R \subset \bar{R}$ write $H_R = (R/\bar{R}) \oplus k$ for the Heisenberg group corresponding to the symplectic space $R/\bar{R}$. If $L \in \mathcal{L}_d(M(F))_R$ then $L_R := L \cap R/\bar{R}$ is lagrangian. Set $\bar{L}_R = L_R \oplus k \subset H_R$. Let $\chi_{L,R} : \bar{L}_R \to \bar{Q}_\ell^*$ be the character sending $(l,a)$ to $\psi(a)$. Set
\[ \mathcal{H}_{L_R} = \{ f : H_R \to \bar{Q}_\ell \mid f(lh) = \chi_{L,R}(l)f(h), \ h \in H_R, l \in \bar{L}_R \} \]

**Lemma 7.** There is a canonical embedding $T_{R}^{L} : \mathcal{H}_{L_R} \hookrightarrow \mathcal{H}_{L}$ whose image is the subspace of those $f \in \mathcal{H}_{L}$ which satisfy
\[ f(h(r,0)) = f(h) \text{ for } r \in R, h \in H \tag{17} \]

**Proof** Set
\[ '\mathcal{H}_{L_R} = \{ \phi : R/\bar{R} \to \bar{Q}_\ell \mid \phi(r+l) = \chi(\frac{1}{2} \omega(r,l))\phi(r), \ r \in R/\bar{R}, l \in L_R \} \]

We have an isomorphism $\mathcal{H}_{L_R} \cong '\mathcal{H}_{L_R}$ sending $f$ to $\phi$ given by $\phi(r) = f(r,0)$. Given $f \in \mathcal{H}_{L}$ satisfying (17), we associate to $f$ a function $\phi \in '\mathcal{H}_{L_R}$ given by
\[ \phi(r) = q^{\frac{1}{2} \dim R/\bar{R}} f(r,0) \]
for $r \in R/\bar{R}$. This defines the map $T_{R}^{L}$. □
Assume that $S \subset R \subset M(F)$ are $c$-lattices and $R \cap L = 0$. Remind the operator $\mathcal{H}_{LR}^{T_{LS}} \rightarrow \mathcal{H}_{LS}$ given by $[13]$, it corresponds to the isotropic subspace $R/S \subset S^\perp/S$. The composition $\mathcal{H}_{LR}^{T_{LS}} \rightarrow \mathcal{H}_{LS} \rightarrow \mathcal{H}_L$ equals $T_{R}^L$.

The geometric analog of $\mathcal{H}_L$ is as follows. For a $c$-lattice $R$ such that $R \cap L = 0$ and $R \subset R^\perp$ we have the category $\mathcal{H}_{LR}$ of perverse sheaves on $H_R$ which are $(L_R, \chi_{L,R})$-equivariant, and the corresponding category $D\mathcal{H}_{LR}$. For $S \subset R$ as above we have an (exact for the perverse structure and fully faithful) transition functor $[14]$, which we now denote by

$$T_{S,R}^L : D\mathcal{H}_{LR} \rightarrow D\mathcal{H}_{LS}$$

Define $\mathcal{H}_L$ (resp., $D\mathcal{H}_L$) as the inductive 2-limit of $\mathcal{H}_{LR}$ (resp., of $D\mathcal{H}_{LR}$) over the partially ordered set of $c$-lattices $R$ such that $R \cap L = 0$ and $R \subset R^\perp$. So, $\mathcal{H}_L$ is abelian and $D\mathcal{H}_L$ is a triangulated category.

6.2 Let $R \subset R^\perp$ be a $c$-lattice in $M(F)$. We have a projection

$$\mathcal{L}_d(M(F))_R \rightarrow \mathcal{L}(R^\perp/R)$$

sending $L$ to $L_R$. Let $\mathcal{A}_R$ be the $\mathbb{Z}/2\mathbb{Z}$-graded purely of degree zero line bundle on $\mathcal{L}(R^\perp/R)$ whose fibre at $L_1$ is $\det L_1 \otimes \det(M : R)$. Write $\tilde{\mathcal{L}}(R^\perp/R)$ for the gerb of square roots of $\mathcal{A}_R$. The restriction of $\mathcal{A}_R$ to $\mathcal{L}_d(M(F))_R$ identifies canonically with $A_d$. The above projection lifts naturally to a morphism of gerbs

$$\tilde{\mathcal{L}}_d(M(F))_R \rightarrow \tilde{\mathcal{L}}(R^\perp/R) \quad (18)$$

Given $k$-points $N^0, L^0 \in \tilde{\mathcal{L}}_d(M(F))$ we are going to associate to them in a canonical way a functor

$$\mathcal{F}_{N^0, L^0} : D\mathcal{H}_L \rightarrow D\mathcal{H}_N \quad (19)$$

sending $\mathcal{H}_L$ to $\mathcal{H}_N$. To do so, consider a $c$-lattice $R \subset R^\perp$ in $M(F)$ such that $L, N \in \mathcal{L}_d(M(F))_R$. Write $N^0_R, L^0_R \in \tilde{\mathcal{L}}(R^\perp/R)$ for the images of $N^0$ and $L^0$ under $(18)$. By definition, the enhanced structure on $L_R$ and $N_R$ is given by one-dimensional vector spaces $\mathcal{B}_L, \mathcal{B}_N$ equipped with

$$\mathcal{B}_L \cong \det L_R \otimes \det(M : R), \quad \mathcal{B}_N \cong \det N_R \otimes \det(M : R),$$

hence an isomorphism $\mathcal{B}^2 \cong \det L_R \otimes \det N_R$ for $B := \mathcal{B}_L \otimes \mathcal{B}_N \otimes \det(M : R)^{-1}$. We denote by

$$\mathcal{F}_{N^0_R, L^0_R} : D\mathcal{H}_{LR} \rightarrow D\mathcal{H}_{NR}$$

the canonical intertwining functor defined in Section 3.5 corresponding to $(N_R, L_R, B) \in \tilde{Y}$, here $Y = \mathcal{L}(R^\perp/R) \times \mathcal{L}(R^\perp/R)$. The following is an immediate consequence of Proposition $[2]$

**Proposition 4.** Let $S \subset R \subset R^\perp \subset S^\perp$ be $c$-lattices such that $L^0, N^0 \in \tilde{\mathcal{L}}_d(M(F))_R$. Then the following diagram of categories is canonically 2-commutative

$$
\begin{array}{ccc}
D\mathcal{H}_{LR} & \xrightarrow{T_{S,R}^L} & D\mathcal{H}_{LS} \\
\downarrow \mathcal{F}_{N^0_R, L^0_R} & & \downarrow \mathcal{F}_{N^0_R, L^0_R} \\
D\mathcal{H}_{NR} & \xrightarrow{T_{S,R}^N} & D\mathcal{H}_{NS}
\end{array}
$$

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Define (19) as the limit of functors $\mathcal{F}_{N^0, L^0_R}$ over the partially ordered set of $c$-lattices $R \subset R^\perp$ such that $L, N \in \mathcal{L}_d(M(F))_R$. As in Section 3.5, one shows that for $L^0, N^0, R^0 \in \mathcal{L}_d(M(F))$ the diagram is canonically 2-commutative

$$
\begin{array}{ccc}
\mathcal{D}H_L & \xrightarrow{\mathcal{F}_{L^0, L^0}} & \mathcal{D}H_R \\
\downarrow \mathcal{F}_{N^0, L^0} & & \downarrow \mathcal{F}_{N^0, R^0} \\
\mathcal{D}H_N & & 
\end{array}
$$

Our main result in the local field case is as follows.

**Theorem 2.** For each $k$-point $L^0 \in \mathcal{L}_d(M(F))$ there is a canonical functor

$$
\mathcal{F}_{L^0} : \mathcal{D}H_L \to \mathcal{D}(\mathcal{L}_d(M(F)))
$$

sending $\mathcal{H}_L$ to $\mathcal{P}(\mathcal{L}_d(M(F)))$. For a pair of $k$-points $(L^0, N^0)$ in $\mathcal{L}_d(M(F))$ the diagram

$$
\begin{array}{ccc}
\mathcal{D}H_L & \xrightarrow{\mathcal{F}_{L^0}} & \mathcal{D}(\mathcal{L}_d(M(F))) \\
\downarrow \mathcal{F}_{N^0, L^0} & & \downarrow \mathcal{F}_{N^0} \\
\mathcal{D}H_N & & 
\end{array}
$$

is canonically 2-commutative. Let $W(\mathcal{L}_d(M(F)))$ be the essential image of

$$
\mathcal{F}_{L^0} : \mathcal{H}_L \to \mathcal{P}(\mathcal{L}_d(M(F))),
$$

this is a full subcategory independent of $L^0$. Besides, $W(\mathcal{L}_d(M(F)))$ is preserved under the natural action of $\tilde{G}(F)$ on $\mathcal{P}(\mathcal{L}_d(M(F)))$.

We will refer to $W(\mathcal{L}_d(M(F)))$ as the non-ramified Weil category on $\mathcal{L}_d(M(F))$. Remind that in the classical setting

$$
\mathcal{H}_L = \mathcal{H}_{L, odd} \oplus \mathcal{H}_{L, even}
$$

is a direct sum of two irreducible representations of the metaplectic group (consisting of odd and even functions respectively). The representation $\mathcal{H}_{L, odd}$ is ramified, whence $\mathcal{H}_{L, even}$ is not. The category $W(\mathcal{L}_d(M(F)))$ together with the action of $\tilde{G}(F)$ is a geometric counterpart of the representation $\mathcal{H}_{L, even}$. The proof of Theorem 2 is given in Sections 6.3-6.4.

6.3 Let $L^0$ be a $k$-point of $\mathcal{L}_d(M(F))$. Let $R \subset R^\perp$ be a $c$-lattice with $L \cap R = 0$. Write $L^0_R$ for the image of $L^0$ under (18). Applying the construction of Section 3.6 to the symplectic space $R^\perp/R$ with $L^0_R \in \mathcal{L}(R^\perp/R)$, one gets the functor

$$
\mathcal{F}_{L^0_R} : \mathcal{D}H_{LR} \to \mathcal{D}(\mathcal{L}(R^\perp/R))
$$

If $N^0$ is another $k$-point of $\mathcal{L}_d(M(F))_R$ then writing $N^0_R$ for the image of $N^0$ in $\mathcal{L}(R^\perp/R)$ we also get that the diagram

$$
\begin{array}{ccc}
\mathcal{D}H_{LR} & \xrightarrow{\mathcal{F}_{L^0_R}} & \mathcal{D}(\mathcal{L}(R^\perp/R)) \\
\downarrow \mathcal{F}_{N^0, L^0_R} & & \downarrow \mathcal{F}_{N^0_R} \\
\mathcal{D}H_{NR} & & 
\end{array}
$$

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is canonically 2-commutative.

Let now \( R \mathcal{F}_L^0 : \mathcal{H}_R \to \mathcal{D}(\tilde{\mathcal{L}}_d(M(F))_R) \) denote the composition of \( \mathcal{F}_L^0 \) with the (exact for the perverse t-structures) restriction functor \( \mathcal{D}(\tilde{\mathcal{L}}(R^+ / R)) \to \mathcal{D}(\tilde{\mathcal{L}}_d(M(F))_R) \) for the projection (18).

Let \( S \subset R \) be another c-lattice. As in Section 5.3, for the open immersion \( j_{S,R} : \tilde{\mathcal{L}}_d(M(F))_R \hookrightarrow \tilde{\mathcal{L}}_d(M(F))_S \) we have the restriction functors \( j_{S,R}^* : \mathcal{D}(\tilde{\mathcal{L}}_d(M(F)))_S \to \mathcal{D}(\tilde{\mathcal{L}}_d(M(F)))_R \).

**Lemma 8.** The diagram of functors is canonically 2-commutative

\[
\begin{array}{ccc}
\mathcal{D}(\tilde{\mathcal{L}}_d(M(F)))_R & \xrightarrow{R \mathcal{F}_L^0} & \mathcal{D}(\tilde{\mathcal{L}}_d(M(F)))_R \\
\downarrow T_{S,R}^L & & \uparrow j_{S,R}^* \\
\mathcal{D}(\tilde{\mathcal{L}}_d(M(F)))_S & \xrightarrow{S \mathcal{F}_L^0} & \mathcal{D}(\tilde{\mathcal{L}}_d(M(F)))_S
\end{array}
\]

**Proof** We have an open immersion \( j : \tilde{\mathcal{L}}(S^+ / S)_R \hookrightarrow \tilde{\mathcal{L}}(S^+ / S) \) and a projection \( p_{R/S} : \tilde{\mathcal{L}}(S^+ / S)_R \to \tilde{\mathcal{L}}(R^+ / R) \). Set \( P_{R/S} = p_{R/S}^* \otimes (\mathbb{Q}_{\ell}[1](\frac{1}{2}) \dim_{rel}(p_{R/S}) \). It suffices to show that the diagram is canonically 2-commutative

\[
\begin{array}{ccc}
\mathcal{D}(\tilde{\mathcal{L}}_d(M(F)))_R & \xrightarrow{R \mathcal{F}_L^0} & \mathcal{D}(\tilde{\mathcal{L}}_d(M(F)))_R \\
\downarrow T_{S,R}^L & & \uparrow j_{S,R}^* \\
\mathcal{D}(\tilde{\mathcal{L}}_d(M(F)))_S & \xrightarrow{S \mathcal{F}_L^0} & \mathcal{D}(\tilde{\mathcal{L}}_d(M(F)))_S
\end{array}
\]

This follows from Lemma 5. \( \square \)

Define \( \mathcal{F}_{L^0, R} : \mathcal{H}_R \to \mathcal{D}(\tilde{\mathcal{L}}_d(M(F))) \) as the functor sending \( K_1 \) to the following object \( K_2 \). For a c-lattice \( S \subset R \) we declare the restriction of \( K_2 \) to \( \tilde{\mathcal{L}}_d(M(F))_S \) to be

\[
(s \mathcal{F}_{L^0} \circ T_{S,R}^L)(K_1)
\]

By Lemma 8 the corresponding projective system defines an object \( K_2 \) of \( \mathcal{D}(\tilde{\mathcal{L}}_d(M(F))) \).

Finally, for \( S \subset R \) with \( R \cap L = 0 \) the diagram

\[
\begin{array}{ccc}
\mathcal{D}(\tilde{\mathcal{L}}_d(M(F)))_R & \xrightarrow{R \mathcal{F}_L^0} & \mathcal{D}(\tilde{\mathcal{L}}_d(M(F)))_R \\
\downarrow T_{S,R}^L & & \uparrow \mathcal{F}_{L^0,S} \\
\mathcal{D}(\tilde{\mathcal{L}}_d(M(F)))_S & \xrightarrow{S \mathcal{F}_L^0} & \mathcal{D}(\tilde{\mathcal{L}}_d(M(F)))_S
\end{array}
\]

is canonically 2-commutative. We define (20) as the limit of the functors \( \mathcal{F}_{L^0, R} \) over the partially ordered set of c-lattices \( R \subset R^+ \) such that \( L \cap R = 0 \). The commutativity of (21) follows from the commutativity of (22).

**Definition 4.** The non-ramified Weil category \( \mathcal{W}(\tilde{\mathcal{L}}_d(M(F))) \) is the essential image of the functor \( \mathcal{F}_{L^0} : \mathcal{H}_L \to P(\tilde{\mathcal{L}}_d(M(F))) \). It does not depend on a choice of a k-point \( L^0 \) of \( \tilde{\mathcal{L}}_d(M(F)) \).
6.4 Let $R \subset R^\perp$ be a c-lattice in $M(F)$, let $\tilde{g} \in \tilde{G}(F)$ be a $k$-point, write $g$ for its image in $G(F)$. As in Section 5.5, we have an isomorphism $g : H_R \cong H_{gR}$ of algebraic groups over $k$ sending $(x, a) \in (R^\perp/R) \times \mathbb{A}^1$ to $(gx, a) \in (gR^\perp/gR) \times \mathbb{A}^1$. For $L \in \mathcal{L}_d(M(F))_R$ it induces an equivalence
\[ g : \mathcal{H}_{L_R} \cong \mathcal{H}_{gL_{gR}} \]
If $L^0 \in \tilde{\mathcal{L}}_d(M(F))_R$ is a $k$-point then the $G$-equivariance of $F$ implies that the diagram is canonically 2-commutative
\[ \begin{array}{ccc} \mathcal{H}_{L_R} & \overset{\mathcal{F}_{L^0_R}}{\longrightarrow} & P(\mathcal{L}(R^\perp/R)) \\ \downarrow g & & \downarrow \tilde{g} \\ \mathcal{H}_{gL_{gR}} & \overset{\mathcal{F}_{gL^0_{gR}}}{\longrightarrow} & P(\mathcal{L}(gR^\perp/gR)) \end{array} \]
This, in turn, implies that the diagram is 2-commutative
\[ \begin{array}{ccc} \mathcal{H}_{L_R} & \overset{\mathcal{F}_{L^0_R}}{\longrightarrow} & P(\tilde{\mathcal{L}}_d(M(F))) \\ \downarrow g & & \downarrow \tilde{g} \\ \mathcal{H}_{gL_{gR}} & \overset{\mathcal{F}_{gL^0_{gR}}}{\longrightarrow} & P(\tilde{\mathcal{L}}_d(M(F))) \end{array} \]
Thus, Theorem 2 is proved.

6.5 Theta-sheaf Let $L \in \mathcal{L}_d(M(F))_M$, this is equivalent to saying that $L \subset M(F)$ is a lagrangian d-lattice such that $L \oplus M = M(F)$. Then the category $\mathcal{H}_{LM}$ has a distinguished object $L_\psi$ on $\mathbb{A}^1 = H_M$. Write $S_L$ for its image under $\mathcal{H}_{LM} \to \mathcal{H}_L$. The line bundle $A_d$ over $\mathcal{L}_d(M(F))_M$ is canonically trivialized, so $L$ has a distinguished enhanced structure
\[ (L, B) = L^0 \in \tilde{\mathcal{L}}_d(M(F))_M, \]
where $B = k$ is equipped with $\text{id} : B^2 \cong \det(M : L)$. The theta-sheaf $S_{M(F)}$ over $\tilde{\mathcal{L}}_d(M(F))$ is defined as $\mathcal{F}_{L^0}(S_L)$. It does not depend on $L \in \mathcal{L}_d(M(F))_M$ in the sense that for another $N \in \mathcal{L}_d(M(F))_M$ the diagram (21) yields a canonical isomorphism $\mathcal{F}_{L^0}(S_L) \cong \mathcal{F}_{N^0}(S_N)$. The perverse sheaf $S_{M(F)}$ has a natural $G(\mathcal{O})$-equivariant structure.

6.6 Relation with the Schrödinger model
Assume in addition that $M$ is decomposed as $M \cong U \oplus U^* \otimes \Omega$, where $U$ is a free $\mathcal{O}$-module of rank $d$, both $U$ and $U^* \otimes \Omega$ are isotropic, and the form $\omega : \wedge^2 M \to \Omega$ is given by $\omega(u, u^*) = \langle u, u^* \rangle$ for $u \in U, u^* \in U^* \otimes \Omega$, where $\langle \cdot, \cdot \rangle$ is the natural pairing between $U$ and $U^*$. Let $\tilde{U} = U(F) \oplus \Omega(F)$ viewed as a subgroup of $H(F)$, it is equipped with the character $\chi_U : \tilde{U} \to \mathbb{Q}_\ell^*$ given by $\chi_U(u, a) = \psi(\text{Res} a), a \in \Omega(F), u \in U(F)$. Write
\[ \text{ SHR}_U = \{ f : H(F) \to \mathbb{Q}_\ell | f(\bar{u}h) = \chi_U(\bar{u})f(h), \bar{u} \in \tilde{U}, h \in H(F), \ f \text{ is smooth, } \text{ of compact support modulo } \tilde{U} \}, \]
$H(F)$ acts on it by right translations. This is the Schrödinger model of the Weil representation, it identifies naturally with the Schwarz space $\mathcal{S}(U^* \otimes \Omega(F))$.

Remind the definition of the derived category $\mathcal{D}(U^* \otimes \Omega)$ and its subcategory of perverse sheaves $\mathcal{P}(U^* \otimes \Omega)$ given in ([11], Section 4). For $N, r \in \mathbb{Z}$ with $N + r \geq 0$ we write $N, r U = t^N U/t^r U$.

For $N_1 \geq N_2, r_1 \geq r_2$ we have a diagram

$$N_2, r_2 (U^* \otimes \Omega) \xrightarrow{p} N_2, r_1 (U^* \otimes \Omega) \xrightarrow{i} N_1, r_1 (U^* \otimes \Omega),$$

where $p$ is the smooth projection and $i$ is a closed immersion. We have a transition functor

$$\mathcal{D}(N_2, r_2 (U^* \otimes \Omega)) \to \mathcal{D}(N_1, r_1 (U^* \otimes \Omega))$$

sending $K$ to $i p^* K \otimes \mathbb{Q}(l)(\dim \text{rel}(p))$, it is fully faithful and exact for the perverse $t$-structures. Then $\mathcal{D}(U^* \otimes \Omega(F))$ (resp., $\mathcal{P}(U^* \otimes \Omega(F))$) is defined as the inductive 2-limit of $\mathcal{D}(N, r (U^* \otimes \Omega))$ (resp., of $\mathcal{P}(N, r (U^* \otimes \Omega))$) as $r, N$ go to infinity. The category $\mathcal{P}(U^* \otimes \Omega(F))$ is the geometric analog of the space $\text{Shr}_U$.

In this section we prove the following.

**Proposition 5.** For each $k$-point $L^0 \in \tilde{\mathcal{L}}_d(M(F))$ there is a canonical equivalence

$$\mathcal{F}_{U(F), L^0} : \mathcal{D}(U^* \otimes \Omega(F)) \to \mathcal{H}_L$$

which identifies $\mathcal{P}(U^* \otimes \Omega(F))$ with the category $\mathcal{H}_L$. For $L^0, N^0 \in \tilde{\mathcal{L}}_d(M(F))$ the diagram is canonically 2-commutative

$$\begin{array}{ccc}
\mathcal{D}(U^* \otimes \Omega(F)) & \xrightarrow{\mathcal{F}_{U(F), L^0}} & \mathcal{D}_L \\
\downarrow & & \nearrow \mathcal{F}_{L^0, N^0} \\
\mathcal{D}_N & & \mathcal{D}_L
\end{array}$$

For $N \geq 0$ consider the c-lattice $R = t^N M$ in $M(F)$ and the corresponding symplectic space $R^1/R = N, N M$. Set $U_R := N, N U \in \mathcal{L}(N, N M)$. We have the line bundle $\mathcal{A}_N$ on $\mathcal{L}(N, N M)$ whose fibre at $L$ is $\text{det}(0, N M) \otimes \text{det} L$. As above, $\tilde{\mathcal{L}}(N, N M)$ is the gerb of square roots of $\mathcal{A}_N$. Let

$$U^0_R = (U_R, \text{det}(0, N U)) \in \tilde{\mathcal{L}}(N, N M)$$

equipped with a canonical $\mathbb{Z}/2\mathbb{Z}$-graded isomorphism $\text{det}(0, N U)^2 \sim \text{det} U_R \otimes \text{det}(0, N M)$.

Let $H_R = N, N M \times \mathbb{A}_1$ denote the corresponding Heisenberg group, it has the subgroup $U_R = U_R \times \mathbb{A}_1$ equipped with the character $\chi_{U, R} : U_R \to \mathbb{Q}_2^*$ given by $\chi_{U, R}(u, a) = \psi(a), a \in \mathbb{A}_1$. In the classical setting, $\mathcal{H}_{U_R}$ is the space of functions on $H_R$, which are $(U_R, \chi_{U, R})$-equivariant under the left multiplication. Set $\text{Shr}_U^R = \{ f \in \text{Shr}_U \mid f(h(r, 0)) = f(h), r \in R, h \in H \}$.

**Lemma 9.** In the classical setting there is an isomorphism

$$\text{Shr}_U^R \sim \mathcal{H}_{U_R}$$

(25)
Proof Write $\mathcal{H}_{UR}' = \{ \phi' : R^+/R \to \bar{\mathbb{Q}}_{\ell} \mid \phi'(m + u) = \psi(\frac{1}{2}(m, u))\phi'(m), u \in U_R \}$. We identify $\mathcal{H}_{UR}' \simeq \mathcal{H}_{UR}'$ via the map $\phi \mapsto \phi'$, where $\phi'(m) = \phi(m, 0)$. Given $f \in \text{Shr}_U^U$ for $m \in t^{-N}M$ the value $f(m, 0)$ depends only on the image $\tilde{m}$ of $m$ under $t^{-N}M \to N, N M$. The isomorphism (25) sends $f$ to $\phi' \in \mathcal{H}_{UR}'$ given by $\phi(\tilde{m}) = f(m, 0)$. □

In the geometric setting $\mathcal{H}_{UR}$ is the category of $(\bar{U}_R, \chi_{U,R})$-equivariant perverse sheaves on $H_R$. We identify it with $\text{P}(N, N(U^* \otimes \Omega))$ as follows. Let $m_U : \bar{U}_R \times N, N(U^* \otimes \Omega) \to H_R$ be the isomorphism sending $(\tilde{u}, h)$ to their product $\tilde{u}h$ in $H_R$. The functor $D(N, N(U^* \otimes \Omega)) \to D\mathcal{H}_{UR}$ sending $K$ to

$$(m_U)(\chi_{U,R} \otimes K) \otimes (\tilde{\mathbb{Q}}_{\ell}(1)(\frac{1}{2}))^{\dim \bar{U}_R}$$

is an equivalence (exact for the perverse t-structures).

Let $N' \geq N$ and $S = t^{N'}M$. The corresponding transition functor (23) now yields a functor denoted $T^U_{S,R} : D\mathcal{H}_{UR} \to D\mathcal{H}_{US}$.

Let $L^0 \in \bar{\mathcal{L}}_d(M(F))$ be a $k$-point over $L \in \mathcal{L}_d(M(F))$. Assume that $N$ is large enough so that $L \cap R = 0$. Let $L^0_R$ denote the image of $L^0$ under (18). Define $U^0_S, L^0_S \in \bar{\mathcal{L}}(S^+/S)$ similarly.

**Lemma 10.** The diagram is canonically 2-commutative

$$
\begin{array}{ccc}
D\mathcal{H}_{UR} & \xrightarrow{T^U_{S,R}} & D\mathcal{H}_{US} \\
\downarrow F^U_{L^0_{R}, V^0_{R}} & & \downarrow F^U_{L^0_S, V^0_S} \\
D\mathcal{H}_{LR} & \xrightarrow{T^U_{S,R}} & D\mathcal{H}_{LS}
\end{array}
$$

Proof Set $W = t^{N'}U \oplus t^N(U^* \otimes \Omega)$. The subspace $W/S \subseteq S^+/S$ is isotropic, and $U_S \cap (W/S) = L_S \cap (W/S) = 0$. Write $H_W = (W^+/W) \times \mathbb{A}^1$ for the corresponding Heisenberg group. Set $U_W = U_S \cap (W^+/S)$, $L_W = L_S \cap (W^+/S)$. Applying Proposition 2 we get a 2-commutative diagram

$$
\begin{array}{ccc}
D\mathcal{H}_{UW} & \xrightarrow{T^U_{S,W}} & D\mathcal{H}_{US} \\
\downarrow F^U_{L^0_{W}, V^0_{W}} & & \downarrow F^U_{L^0_S, V^0_S} \\
D\mathcal{H}_{LW} & \xrightarrow{T^U_{S,W}} & D\mathcal{H}_{LS}
\end{array}
$$

Now $R/W \subseteq W^+/W$ is an isotropic subspace, and $R/W \subseteq U_W$, $R/W \cap L_W = 0$. Note that $U_R = U_W/(R/W)$. Applying Proposition 3 we get a 2-commutative diagram

$$
\begin{array}{ccc}
D\mathcal{H}_{UR} & \xrightarrow{T^U_{W,R}} & D\mathcal{H}_{UW} \\
\downarrow F^U_{L^0_{R}, V^0_{R}} & & \downarrow F^U_{L^0_{W}, V^0_{W}} \\
D\mathcal{H}_{LR} & \xrightarrow{T^U_{W,R}} & D\mathcal{H}_{LW}
\end{array}
$$

Our assertion easily follows. □

*Proof of Proposition 5*
Passing to the limit as $N$ goes to infinity, the functors $F_{L^0} : D\mathcal{H}_{U^0} \to D\mathcal{H}_{L^0}$ from Lemma 10 yield the desired functor (24). The second assertion follows by construction. □

**Definition 5.** Let $F_{U(F)} : D(U^* \otimes \Omega(F)) \to D(\tilde{L}_d(M(F)))$ denote the composition

$$D(U^* \otimes \Omega(F)) \xrightarrow{F_{U(F), L^0}} D\mathcal{H}_{L^0} \xrightarrow{\sim} D(\tilde{L}_d(M(F)))$$

By Theorem 2 and Proposition 5, it does not depend on the choice of a $k$-point $L^0 \in \tilde{L}_d(M(F))$. By construction, $F_{U(F)}$ is exact for the perverse t-structures.

We have a morphism of group stacks $GL(U)(F) \to \tilde{G}(F)$ sending $g \in GL(U)(F)$ to $(g, B = \det(U : gU))$ equipped with a canonical $\mathbb{Z}/2\mathbb{Z}$-graded isomorphism

$$\det(M : gM) \cong \det(U : gU) \otimes \det(U^* \otimes \Omega : g(U^* \otimes \Omega)) \cong \det(U : gU)^{\otimes 2}$$

Let $GL(U)(F)$ act on $\tilde{L}_d(M(F))$ via this homomorphism, let it also act naturally on $U^* \otimes \Omega(F)$. Then one may show that $F_{U(F)}$ commutes with the action of $GL(U)(F)$.

Note also that over $GL(U)(O)$ the sections $GL(U)(F) \to \tilde{G}(F)$ and $G(O) \to \tilde{G}(F)$ are compatible.

### 7. Global application

7.1 Assume $k$ algebraically closed. Let $X$ be a smooth connected projective curve. Let $\Omega$ be the canonical invertible sheaf on $X$. Let $G$ be the group scheme over $X$ of automorphisms of $\mathcal{O}_X^d \oplus \Omega^d$ perserving the symplectic form $\Lambda^2(\mathcal{O}_X^d \oplus \Omega^d) \to \Omega$.

Write $\text{Bun}_G$ for the stack of $G$-torsors on $X$, it classifies a rank $2d$-vector bundle $\mathcal{M}$ on $X$ together with a symplectic form $\Lambda^2(\mathcal{M}) \to \Omega$. Let $\mathcal{A}$ be the ($\mathbb{Z}/2\mathbb{Z}$-graded purely of degree zero) line bundle on $\text{Bun}_G$ whose fibre at $\mathcal{M}$ is $\det R\Gamma(X, \mathcal{M})$. Write $\tilde{\text{Bun}}_G$ for the gerb of square roots of $\mathcal{A}$ over $\text{Bun}_G$.

Remind the definition of the theta-sheaf $\text{Aut}$ on $\tilde{\text{Bun}}_G$ ([10], Definition 1). Let $\iota_{\text{Bun}_G} : \text{Bun}_G \to \text{Bun}_G$ be the locally closed substack given by $\dim\mathcal{H}^0(X, \mathcal{M}) = i$ for $\mathcal{M} \in \text{Bun}_G$. Write $\iota_{\text{Bun}_G}$ for the restriction of $\tilde{\text{Bun}}_G$ to $\iota_{\text{Bun}_G}$.

Let $\mathcal{B}$ be the line bundle on $\iota_{\text{Bun}_G}$ whose fibre at $\mathcal{M} \in \iota_{\text{Bun}_G}$ is $\det\mathcal{H}^0(X, \mathcal{M})$, we view it as $\mathbb{Z}/2\mathbb{Z}$-graded of degree $i \mod 2$. For each $i$ we have a canonical $\mathbb{Z}/2\mathbb{Z}$-graded isomorphism $\mathcal{B}^2 \cong \mathcal{A}$, it yields a trivialization $\iota_{\text{Bun}_G} \cong \iota_{\text{Bun}_G} \times B(\mu_2)$.

Define $\text{Aut}_g \in P(\tilde{\text{Bun}}_G)$ (resp., $\text{Aut}_s \in P(\tilde{\text{Bun}}_G)$) as the intermediate extension of

$$(\mathbb{Q}_\ell \otimes W) \otimes (\mathbb{Q}_\ell[1](\frac{1}{2})^{\dim\text{Bun}_G})$$

(resp., of $(\mathbb{Q}_\ell \otimes W) \otimes (\mathbb{Q}_\ell[1](\frac{1}{2})^{\dim\text{Bun}_G - 1})$ under $\iota_{\text{Bun}_G} \to \tilde{\text{Bun}}_G$. Set $\text{Aut} = \text{Aut}_g \oplus \text{Aut}_s$. 

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7.2 Fix a closed point \( x \in X \). Write \( O_X \) for the completed local ring of \( X \) at \( x \), \( F_x \) for its fraction field. Fix a \( G \)-torsor over \( \text{Spec} \, O_X \), we think of it as a free \( O_X \)-module \( M \) of rank \( 2d \) with symplectic form \( \wedge^2 M \rightarrow \Omega(O_X) \) and an action of \( G(O_X) \). We have a map

\[
\xi_x : \text{Bun}_d \rightarrow \mathcal{L}_d(M(F_x))/G(O_X),
\]

where \( \mathcal{L}_d(M(F_x))/G(O_X) \) is the stack quotient. It sends \( M \in \text{Bun}_d \) to the Tate space \( M(F_x) \) with lagrangian c-lattice \( \mathcal{M}(O_x) \) and lagrangian d-lattice \( H^0(X - x, \mathcal{M}) \).

The line bundle \( A_d \) on \( \mathcal{L}_d(M(F_x))/G(O_X) \) is that of Section 5.3. Write \( \mathcal{L}_d(M(F_x))/G(O_X) \) for the gerb of square roots of \( O_X \) for the gerb of square roots of \( A_d \).

We have canonically \( \xi_x^* A_d \rightarrow A_d \), so \( \xi \) lifts naturally to a map of gerbs

\[
\tilde{\xi}_x : \tilde{\text{Bun}}_d \rightarrow \tilde{\mathcal{L}}_d(M(F_x))/G(O_X)
\]

For \( r \geq 0 \) let \( r_x \text{Bun}_d \subset \text{Bun}_d \) be the open substack given by \( H^0(X, \mathcal{M}(-r x)) = 0 \). Write \( r_x \tilde{\text{Bun}}_d \) for the restriction of the gerb \( \tilde{\text{Bun}}_d \) to \( r_x \text{Bun}_d \). If \( r' \geq r \) then \( r_x \text{Bun}_d \subset r'_x \text{Bun}_d \) is an open substack, so we consider the projective 2-limit

\[
2\text{-}\lim_{r \rightarrow \infty} D(r_x \tilde{\text{Bun}}_d)
\]

Note that \( 2\text{-}\lim_{r \rightarrow \infty} P(r_x \tilde{\text{Bun}}_d) \cong P(\tilde{\text{Bun}}_d) \) is a full subcategory in the above limit. Let us define the restriction functor

\[
\tilde{\xi}_x^* : D_{G(O)}(\tilde{\mathcal{L}}_d(M(F))) \rightarrow 2\text{-}\lim_{r \rightarrow \infty} D(r_x \tilde{\text{Bun}}_d)
\]

(26)

To do so, for \( N \geq r \geq 0 \) and \( r_1 \geq 2N \) let

\[
\xi_N : r_x \text{Bun}_d \rightarrow r \mathcal{L}(N,N,M)/G(O/t^r)
\]

(27)

be the map sending \( M \) to the lagrangian subspace \( H^0(X, \mathcal{M}(N_x)) \subset N,N,M \). If \( N_1 \geq N \geq r \) and \( r_1 \geq 2N_1 \) then the diagram commutes

\[
r_x \text{Bun}_d \xrightarrow{\xi_N} r \mathcal{L}(N,N,M)/G(O/t^r) \\
\downarrow \xi_{N_1} \quad \uparrow p \quad \uparrow
\]

\[
\tilde{\mathcal{L}}(N,N,M)/G(O/t^r) \xrightarrow{r} \tilde{\mathcal{L}}(N,N,M)/G(O/t^r)
\]

It induces a similar diagram between the gerbs (cf. Section 5.3 for their definition)

\[
r_x \tilde{\text{Bun}}_d \xrightarrow{\tilde{\xi}_N} (r \mathcal{L}(N,N,M)/G(O/t^r)) \quad \uparrow \quad \uparrow
\]

\[
\tilde{\mathcal{L}}(N,N,M)/G(O/t^r) \xrightarrow{r} \tilde{\mathcal{L}}(N,N,M)/G(O/t^r)
\]

The functors \( K \mapsto \tilde{\xi}_N^* K \otimes (\mathbb{Q}_t[1](\frac{1}{2}))^{\text{dim.rel}(\xi_N)} \) from \( D_{G(O)}(r \tilde{\mathcal{L}}(N,N,M)) \) to \( D(r_x \tilde{\text{Bun}}_d) \) are compatible with the transition functors, so yield a functor

\[
r_x \xi_x^* : D_{G(O)}(r \tilde{\mathcal{L}}(M(F))) \rightarrow D(r_x \tilde{\text{Bun}}_d)
\]

Passing to the limit by \( r \), one gets the desired functor \( 26 \).
Theorem 3. The object $\hat{\xi}^*_xS_M(F_x)$ lies in $P(\widetilde{\text{Bun}}_G)$, and there is an isomorphism of perverse sheaves

$$\hat{\xi}^*_xS_M(F_x) \simeq \text{Aut}$$

Proof For $r \geq 0$ consider the map

$$\hat{\xi}_r : rx \widetilde{\text{Bun}}_G \to (\mathcal{L}(r,rM)/G(\mathcal{O}/t^{2r}))^\natural$$

Set $Y = \mathcal{L}(r,rM) \times \mathcal{L}(r,rM)$. Write $\mathcal{Y}$ for the stack quotient of $Y$ by the diagonal action of $\text{Sp}(r,rM)$. Let $A_Y$ be the $\mathbb{Z}/2\mathbb{Z}$-graded purely of degree zero line bundle on $\mathcal{Y}$ with fibre $\text{det} L_1 \otimes \text{det} L_2$ at $(L_1, L_2)$. Write $\mathcal{Y}$ for the gerb of square roots of $A_Y$ over $\mathcal{Y}$. The map $\mathcal{L}(r,rM) \to Y$ sending $L_1$ to $(0, rM, L_1) \in Y$ yields a morphism of stacks

$$\rho : (\mathcal{L}(r,rM)/G(\mathcal{O}/t^{2r}))^\natural \to \mathcal{Y}$$

Write $S_{r,rM}$ for the perverse sheaf on $\mathcal{Y}$ introduced in (Section 3.2, Definition 1). Set $\tau = \rho \circ \hat{\xi}_r$. It suffices to establish for any $r \geq 0$ a canonical isomorphism

$$\tau^* S_{r,rM} \otimes (\overline{\mathbb{Q}}_\ell [1](\frac{1}{2}))^{\dim \text{rel}(\tau)} \simeq \text{Aut} \quad (28)$$

over $rx \widetilde{\text{Bun}}_G$.

Remind that $Y_i \subset Y$ is the locally closed subscheme given by $\dim(L_1 \cap L_2) = i$ for $(L_1, L_2) \in Y$. Let $Y_i$ be the stack quotient of $Y_i$ by the diagonal action of $\text{Sp}(r,rM)$, set $\mathcal{Y}_i = Y_i \times_\mathcal{Y} \mathcal{Y}$. Set

$$rx,i \widetilde{\text{Bun}}_G = rx \widetilde{\text{Bun}}_G \cap i \widetilde{\text{Bun}}_G \quad \text{and} \quad rx,i \text{Bun}_G = rx \text{Bun}_G \cap i \text{Bun}_G$$

For each $i$ the map $\tau$ fits into a cartesian square

$$\begin{array}{ccc}
rx,i \widetilde{\text{Bun}}_G & \to & \mathcal{Y}_i \\
\downarrow & & \downarrow \\
rx \widetilde{\text{Bun}}_G & \to & \mathcal{Y}
\end{array}$$

Indeed, for $\mathcal{M} \in_{rx} \text{Bun}_G$ the space $H^0(X, \mathcal{M})$ equals the intersection of $\mathcal{M}/\mathcal{M}(−rx)$ and $H^0(X, \mathcal{M}(rx))$ inside $\mathcal{M}(rx)/\mathcal{M}(−rx)$. By (10), Theorem 1), the $*$-restriction of $\text{Aut}$ to $i \text{Bun}_G \cong i \text{Bun}_G \times B(\mu_2)$ identifies with

$$(\overline{\mathbb{Q}}_\ell \boxtimes W) \otimes (\overline{\mathbb{Q}}_\ell [1](\frac{1}{2}))^{\dim \text{Bun}_G − i}$$

Similarly, by (10), Proposition 1 and 5), the $*$-restriction of $S_M$ to $\mathcal{Y}_i \cong \mathcal{Y}_i \times B(\mu_2)$ identifies with

$$(\overline{\mathbb{Q}}_\ell \boxtimes W) \otimes (\overline{\mathbb{Q}}_\ell [1](\frac{1}{2}))^{\dim \mathcal{Y} − i}$$

Since the map $\tau_i$ is compatible with our trivializations of the corresponding gerbs, we get the isomorphism (28) over $rx,i \text{Bun}_G$ for each $i$. Since $\text{Aut}$ is perverse, this also shows that the LHS
of (28) is placed in perverse degrees \( \leq 0 \), and its \(*\)-restriction to \( \leq 2 \widetilde{\text{Bun}}_G \) is placed in perverse degrees \( < 0 \).

The map \( \tau \) is not smooth, we overcome this difficulty as follows. Let us show that the LHS of (28) is placed in perverse degrees \( \geq 0 \). Consider the stack \( \mathcal{X} \) classifying \((\mathcal{M}, \mathcal{B}) \in r_x \widetilde{\text{Bun}}_G\) and a trivialization

\[
\mathcal{M} \mid \text{Spec } \mathcal{O}_{x/t_x^{2}} \xrightarrow{\sim} \mathcal{M} \mid \text{Spec } \mathcal{O}_{x/t_x^{2}}
\]

of the corresponding \( G \)-torsor. Let \( \nu : \mathcal{X} \to \tilde{Y} \) be the map sending a point of \( \mathcal{X} \) to the triple \((\mathcal{M}/r_x \mathcal{B}, H^0(X, \mathcal{M}(r_x)), \mathcal{B})\). Define \( \mathcal{X}_1 \) and \( \mathcal{X}_3 \) by the cartesian squares

\[
\begin{array}{ccc}
\mathcal{X}_3 & \to & C_3 \\
\downarrow \pi_{X_3} & & \downarrow \pi_C \\
\mathcal{X}_1 & \to & U \times_{\mathcal{L}(r_x \mathcal{M})} U \\
\downarrow & & \downarrow \pi_Y \\
\mathcal{X} & \to & \tilde{Y},
\end{array}
\]

Using (7), we get an isomorphism

\[
\mu^* \tau^* S_{r,r_x} \otimes (\mathbb{Q}_\ell[1](1/2))^{\dim \text{rel}(\mathcal{M})+\dim \text{rel}(\tau)} \xrightarrow{\sim} (\pi_{X_3})^! \mathcal{E} \otimes (\mathbb{Q}_\ell[1](1/2))^{\dim \mathcal{X}_3}
\]

for some rank one local system \( \mathcal{E} \) on \( \mathcal{X}_3 \). Here \( \mu : \mathcal{X}_1 \to r_x \widetilde{\text{Bun}}_G \) is the projection, it is smooth. Since \( \pi_{X_3} \) is affine and \( \mathcal{X}_3 \) is smooth, the LHS of (28) is placed in perverse degrees \( \geq 0 \).

Thus, there exists an exact sequence of perverse sheaves \( 0 \to K \to K_1 \to \text{Aut} \to 0 \) on \( r_x \widetilde{\text{Bun}}_G \), where \( K_1 = \tau^* S_{r,r_x} \otimes (\mathbb{Q}_\ell[1](1/2))^{\dim \text{rel}(\tau)} \), and \( K \) is the extension by zero from \( \leq 2 \widetilde{\text{Bun}}_G \). But we know already that \( K_1 \) and \( \text{Aut} \) are isomorphic in the Grothendieck group of \( r_x \widetilde{\text{Bun}}_G \). So, \( K \) vanishes in this Grothendieck group, hence \( K = 0 \). We are done. \( \square \)

References


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