Twisted geometric Satake equivalence: reductive case

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Abstract. In this paper we extend the twisted Satake equivalence established in [8] for almost simple groups to the case of split reductive groups.

1. Introduction

Let G be a connected reductive group over an algebraically closed field. Brylinski-Deligne have developed the theory of central extensions of G by K_2. According to Weissman [16], this is a natural framework for the representation theory of metaplectic groups over local and global fields (allowing to formulate a conjectural extension of the Langlands program for metaplectic groups). One may hope the geometric Langlands program could also naturally extend to this setting. As a step in this direction, in this paper we extend the twisted Satake equivalence established in [8] for almost simple groups to the case of reductive groups. Our input data model an extension of G by K_2 (and cover all the isomorphism classes of such extensions).

2. Main result

2.1. **Notations.** Let k be an algebraically closed field. Let G be a split reductive group over k, $T \subset B \subset G$ be a maximal torus and a Borel subgroup. Let Λ (resp., $\check{\Lambda}$) denote the coweights (resp., weights) lattice of T. Let W denote the Weyl group of (T, G). Set $\mathcal{O} = k[[t]] \subset F = k((t))$. As in ([12], Section 3.2), we denote by $\mathcal{E}^s(T)$ the category of pairs: a symmetric bilinear form $\kappa : \Lambda\otimes\Lambda \to \mathbb{Z}$, and a central super extension $1 \to k^* \to \tilde{\Lambda}^s \to \Lambda \to 1$ whose commutator is $(\gamma_1, \gamma_2)_c = (-1)^{\kappa(\gamma_1,\gamma_2)}$.

Let X be a smooth projective connected curve over k. Write Ω for the canonical line bundle on X. Fix once and for all a square root $\Omega^{\frac{1}{2}}$ of Ω.

Let $\mathcal{P}^\theta(X, \Lambda)$ denote the category of theta-data ([3], Section 3.10.3). Recall the functor $\mathcal{E}^s(T) \to \mathcal{P}^\theta(X, \Lambda)$ defined in ([12], Lemma 4.1). Let $(\kappa, \tilde{\Lambda}^s) \in \mathcal{E}^s(T)$, so for $\gamma \in \Lambda$ we are given a super line ϵ^γ and isomorphisms $c^{\gamma_1,\gamma_2} : \epsilon^{\gamma_1} \otimes \epsilon^{\gamma_2} \tilde{\to} \epsilon^{\gamma_1+\gamma_2}$. For $\gamma \in \Lambda$ let $\lambda^\gamma = (\Omega^{\frac{1}{2}})^{\otimes -\kappa(\gamma,\gamma)} \otimes \epsilon^\gamma$. For the evident isomorphisms $'c^{\gamma_1,\gamma_2} : \lambda^{\gamma_1} \otimes \lambda^{\gamma_2} \tilde{\to} \lambda^{\gamma_1+\gamma_2} \otimes \Omega^{\kappa(\gamma_1,\gamma_2)}$ then $(\kappa, \lambda, 'c) \in \mathcal{P}^\theta(X, \Lambda)$. This is the image of $(\kappa, \tilde{\Lambda}^s)$ by the above functor.

Let Sch$/k$ denote the category of k-schemes of finite type with Zarisky topology. The n-th Quillen K-theory group of a scheme form a presheaf on Sch$/k$ as the scheme varies. As in [5], K_n denotes the associated sheaf on Sch$/k$ for the Zariski topology.

Denote by Vect the tensor category of vector spaces. Pick a prime ℓ invertible in k, write $\bar{\mathbb{Q}}_\ell$ for the algebraic closure of $\mathbb{Q}_\ell$. We work with (perverse) $\bar{\mathbb{Q}}_\ell$-sheaves for étale topology.

2.2. **Motivation.** According to Weissman [16], the metaplectic input datum is an integer $n \geq 1$ and an extension $1 \to K_2 \to E \to G \to 1$ as in [5]. It gives rise to a W-invariant quadratic form $Q : \Lambda \to \mathbb{Z}$, for which we get the corresponding even symmetric bilinear form $\kappa : \Lambda \otimes \Lambda \to \mathbb{Z}$ given by $\kappa(x_1, x_2) = Q(x_1 + x_2) - Q(x_1) - Q(x_2)$, $x_i \in \Lambda$.

The extension E yields an extension

$$1 \to K_2(F) \to E(F) \to G(F) \to 1$$

The tame symbol gives a map $(\cdot, \cdot)_{st} : K_2(F) \to k^*$. The push-out by this map is an extension

$$1 \to k^* \to \mathbb{E}(k) \to G(F) \to 1$$

It is the set of k-points of an extension of group ind-schemes over k

$$1 \to \mathbb{G}_m \to \mathbb{E} \to G(F) \to 1 \tag{1}$$

Assume $n \geq 1$ invertible in k. For a character $\zeta : \mu_n(k) \to \bar{\mathbb{Q}}_\ell^*$ denote by $\mathcal{L}_\zeta$ the corresponding Kummer sheaf on $\mathbb{G}_m$.

Pick an injective character $\zeta : \mu_n(k) \to \bar{\mathbb{Q}}_\ell^*$. For a suitable section of (1) over $G(\mathcal{O})$, we are interested in the category $\mathrm{Perv}_{G,\zeta}$ of $G(\mathcal{O})$-equivariant $\bar{\mathbb{Q}}_\ell$-perverse sheaves on $\mathbb{E}/G(\mathcal{O})$ with $\mathbb{G}_m$-monodromy ζ, that is, equipped with $(\mathbb{G}_m, \mathcal{L}_\zeta)$-equivariant structure. One wants to equip it with a structure of a symmetric monoidal category (and actually a structure of a chiral category as in [9]), and prove a version of the Satake equivalence for it.

2.2.1. One has the exact sequence $1 \to T_1 \to T \to G/[G,G] \to 1$, where $T_1 \subset [G,G]$ is a maximal torus. Write Λ_{ab} (resp., $\check{\Lambda}_{ab}$) for the coweights (resp., weights) lattice of $G_{ab} = G/[G,G]$. The kernel of $\Lambda \to \Lambda_{ab}$ is the rational closure in Λ of the coroots lattice. Let J denote the set of connected components of the Dynkin diagram, $\mathcal{J}_j$ denote the set of vertices of the j-th connected component of the Dynkin diagram, $\mathcal{J} = \cup_{j \in J} \mathcal{J}_j$ the set of vertices of the Dynkin diagram. For $i \in \mathcal{J}$ let α_i (resp., $\check{\alpha}_i$) be the corresponding simple coroot (resp., root). One has $G_{ad} = \prod_{j \in J} G_j$, where G_j is a simple group. Let $\mathfrak{g}_j = \mathrm{Lie}\, G_j$.

Write Λ_{ad} for the coweights lattice of G_{ad}. Write R_j (resp., $\check{R}_j$) for the set coroots (resp., roots) of G_j. Let R (resp. $\check{R}$) denote the set of coroots (resp., roots) of G. For $j \in J$ let $\kappa_j : \Lambda_{ad} \otimes \Lambda_{ad} \to \mathbb{Z}$ denote the Killing form for G_j, that is,

$$\kappa_j = \sum_{\check{\alpha} \in \check{R}_j} \check{\alpha} \otimes \check{\alpha}$$

Note that $\frac{\kappa_j}{2} : \Lambda_{ad} \otimes \Lambda_{ad} \to \mathbb{Z}$. We also view κ_j if necessary as a bilinear form on Λ.

There is $m \in \mathbb{N}$ such that $m\kappa$ is of the form

$$\bar{\kappa} = -\beta - \sum_{j \in J} c_j \kappa_j$$

for some $c_j \in \mathbb{Z}$ and some even symmetric bilinear form $\beta : \Lambda_{ab} \otimes \Lambda_{ab} \to \mathbb{Z}$. So, relaxing our assumption on the characteristic, the following setting is sufficient.

2.3. **Input data.** For each $j \in J$ let $\mathcal{L}_j$ be the ($\mathbb{Z}/2\mathbb{Z}$-graded purely of parity zero) line bundle on Gr_G whose fibre at $gG(\mathcal{O})$ is $\det(\mathfrak{g}_j(\mathcal{O}) : \mathfrak{g}_j(\mathcal{O})^g)$. Write E_j^a for the punctured total space of the line bundle $\mathcal{L}_j$ over $G(F)$. This is a central extension

$$1 \to \mathbb{G}_m \to E_j^a \to G(F) \to 1, \tag{2}$$

here a stands for 'adjoint'. It splits canonically over $G(\mathcal{O})$. The commutator of (2) on $T(F)$ is given by

$$(\lambda_1 \otimes f_1, \lambda_2 \otimes f_2)_c = (f_1, f_2)_{st}^{-\kappa_j(\lambda_1,\lambda_2)}$$

for $\lambda_i \in \Lambda_{ab}$, $f_i \in F^*$. Recall that for $f, g \in F^*$ the tame symbol is given by

$$(f, g)_{st} = (-1)^{v(f)v(g)}(g^{v(f)} f^{-v(g)})(0)$$

Assume also given a central extension

$$1 \to \mathbb{G}_m \to E_\beta \to G_{ab}(F) \to 1 \tag{3}$$

in the category of group ind-schemes whose commutator $(\cdot,\cdot)_c : G_{ab}(F) \times G_{ab}(F) \to \mathbb{G}_m$ satisfies

$$(\lambda_1 \otimes f_1, \lambda_2 \otimes f_2)_c = (f_1, f_2)_{st}^{-\beta(\lambda_1,\lambda_2)}$$

for $\lambda_i \in \Lambda_{ab}$, $f_i \in F^*$. Here $\beta : \Lambda_{ab} \otimes \Lambda_{ab} \to \mathbb{Z}$ is an even symmetric bilinear form. This is a Heisenberg β-extension ([3], Definition 10.3.13). Its pull-back under $G(F) \to G_{ab}(F)$ is also denoted E_β by abuse of notations. Assume also given a splitting of E_β over $G_{ab}(\mathcal{O})$.

Let $N \geq 1$, assume N invertible in k. Let $\zeta : \mu_N(k) \to \bar{\mathbb{Q}}_\ell^*$ be an injective character. Assume given $c_j \in \mathbb{Z}$ for $j \in J$.

The sum of the extensions $(E_j^a)^{c_j}$, $j \in J$ and the extension E_β is an extension

$$1 \to \mathbb{G}_m \to \mathbb{E} \to G(F) \to 1 \tag{4}$$

equipped with the induced section over $G(\mathcal{O})$. Set $\mathrm{Gra}_G = \mathbb{E}/G(\mathcal{O})$. Let $\mathrm{Perv}_{G,\zeta}$ denote the category of $G(\mathcal{O})$-equivariant perverse sheaves on Gra_G with $\mathbb{G}_m$-monodromy ζ. This means, by definition, a $(\mathbb{G}_m, \mathcal{L}_\zeta)$-equivariant structure. Set

$$\mathbb{P}\mathrm{erv}_{G,\zeta} = \mathrm{Perv}_{G,\zeta}[-1] \subset \mathrm{D}(\mathrm{Gra}_G)$$

Let $\mathbb{G}_m$ act on $\mathbb{E}$ via the homomorphism $\mathbb{G}_m \to \mathbb{G}_m$, $z \mapsto z^N$. Let $\widetilde{\mathrm{Gr}}_G$ denote the stack quotient of Gra_G by this action of $\mathbb{G}_m$. We view $\mathbb{P}\mathrm{erv}_{G,\zeta}$ as a full category of the category of perverse sheaves on $\widetilde{\mathrm{Gr}}_G$ via the functor $K \mapsto \mathrm{pr}^* K$. Here $\mathrm{pr} : \mathrm{Gra}_G \to \widetilde{\mathrm{Gr}}_G$ is the quotient map. As in [8], the above cohomological shift is a way to avoid some sign problems.

Let us make a stronger assumption that we are given a central extension

$$1 \to K_2 \to \mathcal{V}_\beta \to G_{ab} \to 1 \tag{5}$$

as in [5] such that passing to F-points and further taking the push-out by the tame symbol $K_2(F) \to \mathbb{G}_m$ yields the extension (3). Recall that on the level of ind-schemes the tame symbol map

$$(\cdot,\cdot)_{st} : F^* \times F^* \to \mathbb{G}_m \tag{6}$$

is defined in [6], see also ([1], Sections 3.1-3.3). Assume that the splitting of (3) over $G(\mathcal{O})$ is the following one. The composition $K_2(\mathcal{O}) \to K_2(F)$ with the tame symbol map factors through $1 \hookrightarrow \mathbb{G}_m$, hence a canonical section $G_{ab}(\mathcal{O}) \to E_\beta$ of (3). Denote by

$$1 \to \mathbb{G}_m \to V_\beta \to \Lambda_{ab} \to 1 \tag{7}$$

the pull-back of (3) by $\Lambda_{ab} \to G_{ab}(F)$, $\lambda \mapsto t^\lambda$. This is the central extension over k corresponding to (5) by the Brylinski-Deligne classification [5]. The extension (7) is given by a line ϵ^γ (of parity zero as $\mathbb{Z}/2\mathbb{Z}$-graded) for each $\gamma \in \Lambda_{ab}$ together with isomorphisms

$$c^{\gamma_1,\gamma_2} : \epsilon^{\gamma_1} \otimes \epsilon^{\gamma_2} \widetilde{\to} \epsilon^{\gamma_1+\gamma_2}$$

for $\gamma_i \in \Lambda_{ab}$ subject to the conditions in the definition of $\mathcal{E}^s(T)$ ([12], Section 3.2.1). Let

$$1 \to \mathbb{G}_m \to V_{\mathbb{E}} \to \Lambda \to 1 \tag{8}$$

be the pull-back of (4) under $\Lambda \to G(F)$, $\lambda \mapsto t^\lambda$. The commutator in (8) is given by $(\lambda_1, \lambda_2)_c = (-1)^{\bar\kappa(\lambda_1,\lambda_2)}$, where

$$\bar\kappa = -\beta - \sum_{j\in J} c_j \kappa_j$$

Let $\mathbb{G}_m$ act on $V_{\mathbb{E}}$ via the homomoprhism $\mathbb{G}_m \to \mathbb{G}_m$, $z \mapsto z^N$. Let $\bar V_{\mathbb{E}}$ be the stack quotient of $V_{\mathbb{E}}$ by this action of $\mathbb{G}_m$. It fits into an extension of group stacks

$$1 \to B(\mu_N) \to \bar V_{\mathbb{E}} \to \Lambda \to 1 \tag{9}$$

Set

$$\Lambda^\sharp = \{\lambda \in \Lambda \mid \bar\kappa(\lambda) \in N\check\Lambda\}$$

We further assume that (8) is the push-out of the extension

$$1 \to \mu_2 \to V_{\mathbb{E},2} \to \Lambda \to 1 \tag{10}$$

Recall that the exact sequence

$$1 \to \mu_N \to \mu_{2N} \to \mu_2 \to 1 \tag{11}$$

yields a morphism of abelian group stacks $\mu_2 \to B(\mu_N)$, and the push-out of (10) by this map identifies canonically with (9). For N odd the sequence (11) splits canonically, so we get a morphism of group stacks

$$\Lambda \to \bar V_{\mathbb{E}}, \tag{12}$$

which is a section of (9). Our additional input datum is a morphism for any N of group stacks $\mathfrak{t}_{\mathbb{E}} : \Lambda^\sharp \to \bar V_{\mathbb{E}}$ extending $\Lambda^\sharp \hookrightarrow \Lambda$. For N odd $\mathfrak{t}_{\mathbb{E}}$ is required to coincide with the restriction of (12). For N even such $\mathfrak{t}_{\mathbb{E}}$ exists, because the restriction of (8) to $\Lambda^\sharp$ is abelian in that case.

2.4. **Category** $\mathrm{Perv}_{G,\zeta}$.

2.4.1. Let $\mathrm{Aut}(\mathcal{O})$ be the group ind-scheme over k such that , for a k-algebra B, $\mathrm{Aut}(\mathcal{O})(B)$ is the automorphism group of the topological B-algebra $B\hat{\otimes}\mathcal{O}$ (as in [8], Section 2.1). Let $\mathrm{Aut}^0(\mathcal{O})$ be the reduced part of $\mathrm{Aut}(\mathcal{O})$. The group scheme $\mathrm{Aut}^0(\mathcal{O})$ acts naturally on the exact sequence (2) acting trivially on $\mathbb{G}_m$ and preserving $G(\mathcal{O})$. The group scheme $\mathrm{Aut}^0(\mathcal{O})$ acts naturally on F, and the tame symbol (6) is $\mathrm{Aut}^0(\mathcal{O})$-invariant. So, by functoriality, $\mathrm{Aut}^0(\mathcal{O})$ acts on (3) acting trivially on $\mathbb{G}_m$. By functoriality, this gives an action of $\mathrm{Aut}^0(\mathcal{O})$ on (4) such that $\mathrm{Aut}^0(\mathcal{O})$ acts trivially on $\mathbb{G}_m$.

2.4.2. For $\lambda \in \Lambda$ let $t^\lambda \in \mathrm{Gr}_G$ denote the image of t under $\lambda : F^* \to G(F)$. The set of $G(\mathcal{O})$-orbits on Gr_G idetifies with the set Λ^+ of dominant coweights of G. For $\lambda \in \Lambda^+$ write Gr^λ for the $G(\mathcal{O})$-orbit on Gr_G through t^λ. The G-orbit through t^λ identifies with the partial flag variety $\mathcal{B}^\lambda = G/P^\lambda$, where P^λ is a paraboic subgroup whose Levi has the Weyl group $W^\lambda \subset W$ coinciding with the stabilizor of λ in W. For $\lambda \in \Lambda^+$ let Gra^λ be the preimage of Gr^λ in Gra_G.

The action of the loop rotation group $\mathbb{G}_m \subset \mathrm{Aut}^0(\mathcal{O})$ contracts Gr^λ to $\mathcal{B}^\lambda \subset \mathrm{Gr}^\lambda$, we denote by $\tilde{\omega}_\lambda : \mathrm{Gr}^\lambda \to \mathcal{B}^\lambda$ the corresponding map.

For a free $\mathcal{O}$-module $\mathcal{E}$ write $\mathcal{E}_{\bar{c}}$ for its geometric fibre. Let Ω be the completed module of relative differentials of $\mathcal{O}$ over k. For a root $\check{\alpha}$ let $\mathfrak{g}^{\check{\alpha}} \subset \mathfrak{g}$ denote the corresponding root subspace. We fix a pinning Φ of G giving trivializations $\phi_{\check{\alpha}} : \mathfrak{g}^{\check{\alpha}} \widetilde{\to} k$ for all $\check{\alpha} \in \check{R}$.

If $\check{\nu} \in \check{\Lambda}$ is orthogonal to all coroots α of G satisfying $\langle \check{\alpha}, \lambda\rangle = 0$ then we denote by $\mathcal{O}(\check{\nu})$ the G-equivariant line bundle on $\mathcal{B}^\lambda$ corresponding to the character $\check{\nu} : P^\lambda \to \mathbb{G}_m$. The line bundle $\mathcal{O}(\check{\nu})$ is trivialized at $1 \in \mathcal{B}^\lambda$.

Sometimes, we view β as $\beta : \Lambda \to \check{\Lambda}$, similarly for $\kappa_j : \Lambda \to \check{\Lambda}$. The group $\mathrm{Aut}^0(\mathcal{O})$ acts on $\Omega_{\bar{c}}$ by the character denoted $\check{\epsilon}$.

Lemma 2.1. *Let $\lambda \in \Lambda^+$.*
i) For each $j \in J$ the pinning Φ yields a uniquely defined $\mathbb{Z}/2\mathbb{Z}$-graded $\mathrm{Aut}^0(\mathcal{O})$*-equivariant isomorphism*

$$\mathcal{L}_j \mid_{\mathrm{Gr}^\lambda} \widetilde{\to} \Omega_{\bar{c}}^{\frac{\kappa_j(\lambda,\lambda)}{2}} \otimes \tilde{\omega}_\lambda^* \mathcal{O}(\kappa_j(\lambda))$$

ii) The restriction of the line bundle $E_\beta/G(\mathcal{O}) \to \mathrm{Gr}_G$ to Gr^λ is constant with fibre $\epsilon^{\bar{\lambda}}$, where $\bar{\lambda} \in \Lambda_{ab}$ is the image of λ. The group $G(\mathcal{O})$ acts on it by the character $G(\mathcal{O}) \to G \stackrel{\beta(\lambda)}{\to} \mathbb{G}_m$, and $\mathrm{Aut}^0(\mathcal{O})$ *acts on it by $\check{\epsilon}^{\frac{\beta(\lambda,\lambda)}{2}}$.*

Proof We only give the proof of the last part of ii), the rest is left to a reader. Pick a bilinear form $B : \Lambda_{ab} \otimes \Lambda_{ab} \to \mathbb{Z}$ such that $B + {}^tB = \beta$, where ${}^tB(\lambda_1, \lambda_2) = B(\lambda_2, \lambda_1)$ for $\lambda_i \in \Lambda_{ab}$. For this calculation we may assume $E_\beta = \mathbb{G}_m \times G_{ab}(F)$ with the product given by $(z_1, u_1)(z_2, u_2) = (z_1 z_2 \bar{f}(u_1, u_2), u_1 u_2)$ for $u_i \in G_{ab}(F)$, $z_i \in \mathbb{G}_m$. Here $\bar{f} : G_{ab}(F) \times G_{ab}(F) \to \mathbb{G}_m$ is the unique bimultiplicative map such that

$$\bar{f}(\lambda_1 \otimes f_1, \lambda_2 \otimes f_2) = (f_1, f_2)_{st}^{-B(\lambda_1,\lambda_2)}$$

Let $g \in \mathrm{Aut}^0(\mathcal{O})$ and $b = \check{\epsilon}(g)$. Then g sends $(1, t^{\bar{\lambda}})$ to $(1, b^{\bar{\lambda}} t^{\bar{\lambda}}) \in (\bar{f}(t^{\bar{\lambda}}, b^{\bar{\lambda}})^{-1}, 1)(1, t^{\bar{\lambda}}) G_{ab}(\mathcal{O})$. Finally, $\bar{f}(t^{\bar{\lambda}}, b^{\bar{\lambda}}) = b^{-\frac{\beta(\bar{\lambda},\bar{\lambda})}{2}}$. $\square$

Set $\Lambda^{\sharp,+}=\Lambda^\sharp\cap\Lambda^+$. For $\lambda\in\Lambda^+$ the scheme Gra^λ admits a $G(\mathcal{O})$-equivariant local system with $\mathbb{G}_m$-monodromy ζ if and only if $\lambda\in\Lambda^{\sharp,+}$.

By Lemma 2.1, for $\lambda\in\Lambda^+$ there is a $\mathrm{Aut}^0(\mathcal{O})$-equivariant isomorphism between Gra^λ and the punctured (that is, with zero section removed) total space of the line bundle

$$\Omega_{\bar{c}}^{-\frac{\bar\kappa(\lambda,\lambda)}{2}}\otimes\tilde\omega_\lambda^*\mathcal{O}(-\bar\kappa(\lambda))$$

over Gr^λ. Write $\Omega^{\frac{1}{2}}(\mathcal{O})$ for the groupoid of square roots of Ω. For $\mathcal{E}\in\Omega^{\frac{1}{2}}(\mathcal{O})$ and $\lambda\in\Lambda^{\sharp,+}$ define the line bundle $\mathcal{L}_{\lambda,\mathcal{E}}$ on Gr^λ as

$$\mathcal{L}_{\lambda,\mathcal{E}}=\mathcal{E}_{\bar{c}}^{-\frac{\bar\kappa(\lambda,\lambda)}{N}}\otimes\tilde\omega_\lambda^*\mathcal{O}(-\frac{\bar\kappa(\lambda)}{N})$$

Let $\overset{\circ}{\mathcal{L}}_{\lambda,\mathcal{E}}$ denote the punctured total space of $\mathcal{L}_{\lambda,\mathcal{E}}$. Let $p_\lambda:\overset{\circ}{\mathcal{L}}_{\lambda,\mathcal{E}}\to\mathrm{Gra}^\lambda$ be the map over Gr^λ sending z to z^N. Let $\mathcal{W}^\lambda_\mathcal{E}$ be the $G(\mathcal{O})$-equivariant rank one local system on Gra^λ with $\mathbb{G}_m$-monodromy ζ equipped with an isomorphism $p_\lambda^*\mathcal{W}^\lambda_\mathcal{E}\widetilde{\to}\bar{\mathbb{Q}}_\ell$. Let $\mathcal{A}^\lambda_\mathcal{E}\in\mathbb{P}\mathrm{erv}_{G,\zeta}$ be the intermediate extension of $\mathcal{W}^\lambda_\mathcal{E}[\dim\mathrm{Gr}^\lambda]$ under $\mathrm{Gra}^\lambda\hookrightarrow\mathrm{Gra}_G$. The perverse sheaf $\mathcal{A}^\lambda_\mathcal{E}$ is defined up to a scalar automorphism (for G semi-simple it is defined up to a unique isomorphism).

Let $\widetilde{\mathrm{Gr}}^\lambda$ denote the restriction of the gerb $\widetilde{\mathrm{Gr}}_G$ to Gr^λ. For $\lambda\in\Lambda^{\sharp,+}$ the map p_λ yields a section $s_\lambda:\mathrm{Gr}^\lambda\to\widetilde{\mathrm{Gr}}^\lambda$.

Lemma 2.2. *If $\lambda\in\Lambda^{\sharp,+}$ then $\mathcal{A}^\lambda_\mathcal{E}$ has non-trivial usual cohomology sheaves only in degrees of the same parity.*

Proof Let $\mathcal{F}l_G$ denote the affine flag variety of G, $q:\mathcal{F}l_G\to\mathrm{Gr}_G$ the projection, write $\tilde q:\widetilde{\mathcal{F}l}_G\to\widetilde{\mathrm{Gr}}_G$ for the map obtained from q by the base change $\widetilde{\mathrm{Gr}}_G\to\mathrm{Gr}_G$. It suffices to prove this parity vanishing for $\tilde q^*\mathcal{A}^\lambda_\mathcal{E}$, this is done in [10]. $\square$

Lemma 2.2 implies as in ([2], Proposition 5.3.3) that the category $\mathbb{P}\mathrm{erv}_{G,\zeta}$ is semisimple.

2.5. **Convolution.** Let τ be the automorphism of $\mathbb{E}\times\mathbb{E}$ sending (g,h) to (g,gh). Let $G(\mathcal{O})\times G(\mathcal{O})\times\mathbb{G}_m$ act on $\mathbb{E}\times\mathbb{E}$ so that (α,β,b) sends (g,h) to $(g\beta^{-1}b^{-1},\beta bh\alpha)$. Write $\mathbb{E}\times_{G(\mathcal{O})\times\mathbb{G}_m}\mathrm{Gra}_G$ for the quotient of $\mathbb{E}\times\mathbb{E}$ under this free action. Then τ induces an isomorphism

$$\bar\tau:\mathbb{E}\times_{G(\mathcal{O})\times\mathbb{G}_m}\mathrm{Gra}_G\widetilde{\to}\mathrm{Gr}_G\times\mathrm{Gra}_G$$

sending $(g,hG(\mathcal{O}))$ to $(\bar gG(\mathcal{O}),ghG(\mathcal{O}))$, where $\bar g\in G(F)$ is the image of $g\in\mathbb{E}$. Let m be the composition of $\bar\tau$ with the projection to Gra_G. Let $p_G:\mathbb{E}\to\mathrm{Gra}_G$ be the map $h\mapsto hG(\mathcal{O})$. As in [8], we get a diagram

$$\mathrm{Gra}_G\times\mathrm{Gra}_G\overset{p_G\times\mathrm{id}}{\leftarrow}\mathbb{E}\times\mathrm{Gra}_G\overset{q_G}{\to}\mathbb{E}\times_{G(\mathcal{O})\times\mathbb{G}_m}\mathrm{Gra}_G\overset{m}{\to}\mathrm{Gra}_G,$$

where q_G is the quotient map under the action of $G(\mathcal{O})\times\mathbb{G}_m$.

For $K_i \in \mathbb{P}\mathrm{erv}_{G,\zeta}$ define the convolution $K_1 * K_2 \in \mathrm{D}(\mathrm{Gra}_G)$ by $K_1 * K_2 = m_! K \in \mathrm{D}(\mathrm{Gra}_G)$, where $K[1]$ is a perverse sheaf on $\mathbb{E} \times_{G(\mathcal{O})\times\mathbb{G}_m} \mathrm{Gra}_G$ equipped with an isomorphism

$$q_G^* K \widetilde{\to} p_G^* K_1 \boxtimes K_2$$

Since q_G is a $G(\mathcal{O}) \times \mathbb{G}_m$-torsor, and $p_G^* K_1 \boxtimes K_2$ is naturally equivariant under $G(\mathcal{O}) \times \mathbb{G}_m$-action, K is defined up to a unique isomorphism. As in ([8], Lemma 2.6), one shows that $K_1 * K_2 \in \mathbb{P}\mathrm{erv}_{G,\zeta}$.

For $K_i \in \mathbb{P}\mathrm{erv}_{G,\zeta}$ one similarly defines the convolution $K_1 * K_2 * K_3 \in \mathbb{P}\mathrm{erv}_{G,\zeta}$ and shows that $(K_1 * K_2) * K_3 \widetilde{\to} K_1 * K_2 * K_3 \widetilde{\to} K_1 * (K_2 * K_3)$ canonically. Besides, $\mathcal{A}^0_{\mathcal{E}}$ is a unit object in $\mathbb{P}\mathrm{erv}_{G,\zeta}$.

2.6. **Fusion.** As in [8], we are going to show that the convolution product on $\mathbb{P}\mathrm{erv}_{G,\zeta}$ can be interpreted as a fusion product, thus leading to a commutativity constraint on $\mathbb{P}\mathrm{erv}_{G,\zeta}$.

Fix $\mathcal{E} \in \Omega^{\frac{1}{2}(\mathcal{O})}$. Let $\mathrm{Aut}_2(\mathcal{O}) = \mathrm{Aut}(\mathcal{O}, \mathcal{E})$ be the group scheme defined in ([8], Section 2.3), let $\mathrm{Aut}^0_2(\mathcal{O})$ be the preimage of Aut^0 in $\mathrm{Aut}_2(\mathcal{O})$.

Let $\lambda \in \Lambda^{\sharp,+}$. Since $p_\lambda : \overset{\circ}{\mathcal{L}}_{\lambda,\mathcal{E}} \to \mathrm{Gra}^\lambda$ is $\mathrm{Aut}^0_2(\mathcal{O})$-equivariant, the action of $\mathrm{Aut}^0(\mathcal{O})$ on Gra_G lifts to a $\mathrm{Aut}^0_2(\mathcal{O})$-equivariant structure on $\mathcal{A}^\lambda_{\mathcal{E}}$. As in ([8], Section 2.3) one shows that the corresponding $\mathrm{Aut}^0_2(\mathcal{O})$-equivariant structure on each $\mathcal{A}^\lambda_{\mathcal{E}}$ is unique.

For $x \in X$ let $\mathcal{O}_x$ be the completed local ring at $x \in X$, F_x its fraction field. Write $\mathcal{F}^0_G$ for the trivial G-torsor on a base. Write $\mathrm{Gr}_{G,x} = G(F_x)/G(\mathcal{O}_x)$ for the corresponding affine grassmanian. Recall that $\mathrm{Gr}_{G,x}$ can be seen as the ind-scheme classifying a G-torsor $\mathcal{F}$ on X together with a trivialization $\nu : \mathcal{F} \widetilde{\to} \mathcal{F}^0_G \mid_{X-x}$.

For $m \geq 1$ let Gr_{G,X^m} and G_{X^m} be defined as in ([8], Section 2.3). Recall that Gr_{G,X^m} is the ind-scheme classifying $(x_1, \ldots, x_m) \in X^m$, a G-torsor $\mathcal{F}_G$ on X, and a trivialization $\mathcal{F}_G \widetilde{\to} \mathcal{F}^0_G \mid_{X-\cup x_i}$. Here G_{X^m} is a group scheme over X^m classifying $\{(x_1, \ldots, x_m) \in X^m, \mu\}$, where μ is an automorphism of $\mathcal{F}^0_G$ over the formal neighbourhood of $D = \cup_i x_i$ in X.

For $j \in J$ let $\mathcal{L}_{j,X^m}$ be the ($\mathbb{Z}/2\mathbb{Z}$-graded purely of parity zero) line bundle on Gr_{G,X^m} whose fibre at $(\mathcal{F}_G, x_i)$ is

$$\det \mathrm{R}\Gamma(X, (\mathfrak{g}_j)_{\mathcal{F}^0_G}) \otimes \det \mathrm{R}\Gamma(X, (\mathfrak{g}_j)_{\mathcal{F}_G})^{-1}$$

Here for a G-module V and a G-torsor $\mathcal{F}_G$ on a base S we write $V_{\mathcal{F}_G}$ for the induced vector bundle on S.

As in Section 2.1, our choice of $\Omega^{\frac{1}{2}}$ yields a functor

$$\mathcal{E}^s(G_{ab}) \to \mathcal{P}^\theta(X, \Lambda_{ab}) \tag{13}$$

Let $\theta_0 \in \mathcal{P}^\theta(X, \Lambda_{ab})$ denote the image under this functor of the extension (7) with the bilinear form $-\beta$.

For a reductive group H write Bun_H for the stack of H-torsors on X. Write $\mathcal{P}\mathrm{ic}(\mathrm{Bun}_H)$ for the groupoid of super line bundles on Bun_H. For $\mu \in \pi_1(H)$ write Bun^μ_H for the connected component of Bun_H classifying H-torsors of degree $-\mu$. Similarly, for $\mu \in \pi_1(G)$ we denote by Gr^μ_G the connected component containing $t^\lambda G(\mathcal{O})$ for any $\lambda \in \Lambda$ over μ.

Recall the functor $\mathcal{P}^\theta(X,\Lambda_{ab})\to \mathcal{P}\mathrm{ic}(\mathrm{Bun}_{G_{ab}})$ defined in ([12], Section 4.2.1, formula (18)). Let $\mathcal{L}_\beta\in \mathcal{P}\mathrm{ic}(\mathrm{Bun}_{G_{ab}})$ denote the image of θ_0 under this functor. It is purely of parity zero as $\mathbb{Z}/2\mathbb{Z}$-graded. For $\mu\in\Lambda_{ab}$ we have a map $i_\mu: X\to \mathrm{Bun}_{G_{ab}}$, $x\mapsto \mathcal{O}(\mu x)$. By definition,

$$i_\mu^*\mathcal{L}_\beta \widetilde{\to} (\Omega^{\frac{1}{2}})^{\beta(\mu,\mu)}\otimes \epsilon^\mu$$

For $m\ge 1$ let $\mathcal{L}_{\beta,X^m}$ be the pull-back of $\mathcal{L}_\beta$ under $\mathrm{Gr}_{G,X^m}\to \mathrm{Bun}_{G_{ab}}$. Let Gra_{G,X^m} denote the punctured total space of the line bundle over Gr_G

$$\mathcal{L}_{\beta,X^m}\otimes\big(\underset{j\in J}{\otimes}(\mathcal{L}_{j,X^m})^{c_j}\big)$$

Remark 2.1. The line bundle $\mathcal{L}_{\beta,X^m}$ is G_{X^m}-equivariant. For $(x_1,\dots,x_m)\in X^m$ let

$$\{y_1,\dots,y_s\}=\{x_1,\dots,x_m\}$$

with y_i pairwise different. Let $\mu_i\in\Lambda$ for $1\le i\le s$. Consider a point $\eta\in \mathrm{Gr}_{G,X^m}$ over $\bar\eta\in \mathrm{Gr}_{G_{ab},X^m}$ given by $\mathcal{F}^0_{G_{ab}}(-\sum_{i=1}^s\mu_i y_i)$ with the evident trivialization over $X-\cup_i y_i$. The fibre of G_{X^m} at $(x_1,\dots,x_m)$ is $\prod_{i=1}^s G(\mathcal{O}_{y_i})$, this group acts on the fibre $(\mathcal{L}_{\beta,X^m})_\eta$ by the character

$$\prod_{i=1}^s G(\mathcal{O}_{y_i})\to \prod_{i=1}^s G_{ab}(\mathcal{O}_{y_i})\to\prod_{i=1}^s G_{ab}\stackrel{\prod_i\beta(\mu_i)}{\longrightarrow}\mathbb{G}_m$$

Since the line bundles $\mathcal{L}_{j,X^m}$ are also G_{X^m}-equivariant, the action of G_{X^m} on Gr_{G,X^m} is lifted to an action on Gra_{G,X^m}.

Let $\mathrm{Perv}_{G,\zeta,X^m}$ be the category of G_{X^m}-equivariant perverse sheaves on Gra_{G,X^m} with $\mathbb{G}_m$-monodromy ζ. Set

$$\mathbb{P}\mathrm{erv}_{G,\zeta,X^m}=\mathrm{Perv}_{G,\zeta,X^m}[-m-1]\subset \mathrm{D}(\mathrm{Gra}_{G,X^m})$$

For $x\in X$ let $D_x=\operatorname{Spec}\mathcal{O}_x$, $D_x^*=\operatorname{Spec}F_x$. The analog of the convolution diagram from ([8], Section 2.3) is the following one, where the left and right squares are cartesian:

$$\begin{array}{ccccccc}
\mathrm{Gra}_{G,X}\times\mathrm{Gra}_{G,X} & \stackrel{\tilde p_{G,X}}{\leftarrow} & \tilde C_{G,X} & \stackrel{\tilde q_{G,X}}{\to} & \widetilde{\mathrm{Conv}}_{G,X} & \stackrel{\tilde m_X}{\to} & \mathrm{Gra}_{G,X^2}\\
\downarrow & & \downarrow & & \downarrow & & \downarrow\\
\mathrm{Gr}_{G,X}\times\mathrm{Gr}_{G,X} & \stackrel{p_{G,X}}{\leftarrow} & C_{G,X} & \stackrel{q_{G,X}}{\to} & \mathrm{Conv}_{G,X} & \stackrel{m_X}{\to} & \mathrm{Gra}_{G,X^2}
\end{array}$$

Here the low row is the convolution diagram from [8]. Namely, $C_{G,X}$ is the ind-scheme classifying collections:

(14) $\quad x_1,x_2\in X,\ G\text{-torsors } \mathcal{F}^1_G,\mathcal{F}^2_G \text{ on } X \text{ with } \nu_i:\mathcal{F}^i_G\widetilde{\to}\mathcal{F}^0_G\mid_{X-x_i},\ \mu_1:\mathcal{F}^1_G\widetilde{\to}\mathcal{F}^0_G\mid_{D_{x_2}}$

The map $p_{G,X}$ forgets μ_1. The ind-scheme $\mathrm{Conv}_{G,X}$ classifies collections:

(15) $\quad x_1,x_2\in X,\ G\text{-torsors } \mathcal{F}^1_G,\mathcal{F}^2_G \text{ on } X,$

$$\text{isomorphisms } \nu_1:\mathcal{F}^1_G\widetilde{\to}\mathcal{F}^0_G\mid_{X-x_1}\ \text{ and } \eta:\mathcal{F}^1_G\widetilde{\to}\mathcal{F}_G\mid_{X-x_2}$$

The map m_X sends this collection to $(x_1,x_2,\mathcal{F}_G)$ together with the trivialization $\eta\circ\nu_1^{-1}:\mathcal{F}^0_G\widetilde{\to}\mathcal{F}_G\mid_{X-x_1-x_2}$.

The map $q_{G,X}$ sends (14) to (15), where $\mathcal{F}_G$ is obtained by gluing $\mathcal{F}^1_G$ on $X - x_2$ and $\mathcal{F}^2_G$ on D_{x_2} using their identification over $D^*_{x_2}$ via $\nu_2^{-1} \circ \mu_1$.

For $j \in J$ there is a canonical $\mathbb{Z}/2\mathbb{Z}$-graded isomorphism

$$q^*_{G,X} m^*_X \mathcal{L}_{j,X^2} \widetilde{\to} p^*_{G,X}(\mathcal{L}_{j,X} \boxtimes \mathcal{L}_{j,X}) \tag{16}$$

Lemma 2.3. *There is a canonical $\mathbb{Z}/2\mathbb{Z}$-graded isomorphism*

$$q^*_{G,X} m^*_X \mathcal{L}_{\beta,X^2} \widetilde{\to} p^*_{G,X}(\mathcal{L}_{\beta,X} \boxtimes \mathcal{L}_{\beta,X}) \tag{17}$$

Proof This isomorphism comes from the corresponding isomorphism for G_{ab}, so for this proof we may assume $G = G_{ab}$. For a point (14) of $C_{G,X}$ consider its image under $q_{G,X}$ given by (15). Note that $\mathcal{F}_G = \mathcal{F}^1_G \otimes \mathcal{F}^2_G$ with the trivialization $\nu_1 \otimes \nu_2 : \mathcal{F}^1_G \otimes \mathcal{F}^2_G \widetilde{\to} \mathcal{F}^0_G \mid_{X - x_1 - x_2}$. One gets by ([12], Proposition 4.2)

$$(\mathcal{L}_\beta)_{(\mathcal{F}^1_G, \nu_1)} \otimes (\mathcal{L}_\beta)_{(\mathcal{F}^2_G, \nu_2)} \widetilde{\to} (\mathcal{L}_\beta)_{\mathcal{F}^1_G} \otimes (\mathcal{L}_\beta)_{\mathcal{F}^2_G} \otimes ({}^{-\beta}\mathcal{L}^{univ}_{\mathcal{F}^1_G, \mathcal{F}^2_G}) \widetilde{\to} (\mathcal{L}_\beta)_{\mathcal{F}^1_G \otimes \mathcal{F}^2_G}$$

with the notations of *loc.cit.* Here we used the following trivialization $({}^{-\beta}\mathcal{L}^{univ}_{\mathcal{F}^1_G, \mathcal{F}^2_G}) \widetilde{\to} k$. Forgetting about nilpotents for simplicity, we may assume $\mathcal{F}^2_G \widetilde{\to} \mathcal{F}^0_G(\nu x_2)$ for some $\nu \in \Lambda$ with the evident trivialization over $X - x_2$. Then

$$({}^{-\beta}\mathcal{L}^{univ}_{\mathcal{F}^1_G, \mathcal{F}^2_G}) \widetilde{\to} (\mathcal{L}^{-\beta(\nu)}_{\mathcal{F}^1_G})_{x_2} \widetilde{\to} k,$$

the latter isomorphism is obtained from $\mu_1 : \mathcal{F}^1_G \widetilde{\to} \mathcal{F}^0_G \mid_{D_{x_2}}$. $\square$

The isomorphisms (16) and (17) allow to define the map $\tilde{q}_{G,X}$ exactly as in ([8], Section 2.3), this is the product of the corresponding maps.

Now for $K_i \in \mathbb{P}\mathrm{erv}_{G,\zeta,X}$ there is a (defined up to a unique isomorphism) perverse sheaf $K_{12}[3]$ on $\widetilde{\mathrm{Conv}}_{G,X}$ equipped with $\tilde{q}^*_{G,X} K_{12} \widetilde{\to} \tilde{p}^*_{G,X}(K_1 \boxtimes K_2)$. Moreover K_{12} has $\mathbb{G}_m$-monodromy ζ. We let

$$K_1 *_X K_2 = \tilde{m}_{X!} K_{12}$$

As in ([8], Section 2.3) one shows that $K_1 *_X K_2 \in \mathbb{P}\mathrm{erv}_{G,\zeta,X^2}$.

Let $\mathcal{E} \in \Omega^{\frac{1}{2}}(\mathcal{O})$. As in *loc.cit.*, one has the $\mathrm{Aut}^0_2(\mathcal{O})$-torsor $\hat{X}_2 \to X$ whose fibre over x is the scheme of isomorphisms between $(\Omega^{\frac{1}{2}}_x, \mathcal{O}_x)$ and $(\mathcal{E}, \mathcal{O})$. One has the isomorphisms

$$\mathrm{Gr}_{G,X} \widetilde{\to} \hat{X}_2 \times_{\mathrm{Aut}^0_2(\mathcal{O})} \mathrm{Gr}_G \quad \text{and} \quad \mathrm{Gra}_{G,X} \widetilde{\to} \hat{X}_2 \times_{\mathrm{Aut}^0_2(\mathcal{O})} \mathrm{Gra}_G$$

Since any $K \in \mathrm{Perv}_{G,\zeta}$ is $\mathrm{Aut}^0_2(\mathcal{O})$-equivariant, we get the fully faithful functor

$$\tau^0 : \mathbb{P}\mathrm{erv}_{G,\zeta} \to \mathbb{P}\mathrm{erv}_{G,\zeta,X}$$

sending K to the descent of $\bar{\mathbb{Q}}_\ell \boxtimes K$ under $\hat{X}_2 \times \mathrm{Gra}_G \to \mathrm{Gra}_{G,X}$.

Let $U \subset X^2$ be the complement to the diagonal. Let $j : \mathrm{Gra}_{G,X^2}(U) \hookrightarrow \mathrm{Gra}_{G,X^2}$ be the preimage of U. Let $i : \mathrm{Gra}_{G,X} \to \mathrm{Gra}_{G,X^2}$ be obtained by the base change $X \to X^2$. Recall that $\tilde{m}_X$ is an isomorphism over $\mathrm{Gra}_{G,X^2}(U)$. For $F_i \in \mathbb{P}\mathrm{erv}_{G,\zeta}$ letting $K_i = \tau^0 F_i$ define

$$K_{12} \mid_U := K_{12} \mid_{\mathrm{Gra}_{G,X^2}(U)}$$

as above, it is placed in perverse degree 3. Then $K_1 *_X K_2 \widetilde{\to} j_{!*}(K_{12} \mid_U)$ and $\tau^0(F_1 * F_2) \widetilde{\to} i^*(K_1 *_X K_2)$. So, the involution σ of Gra_{G,X^2} interchanging x_i yields

$$\tau^0(F_1 * F_2) \widetilde{\to} i^* j_{!*}(K_{12} \mid_U) \widetilde{\to} i^* j_{!*}(K_{21} \mid_U) \widetilde{\to} \tau^0(F_2 * F_1),$$

because $\sigma^*(K_{12} \mid_U) \widetilde{\to} K_{21} \mid_U$ canonically. As in [8], the associativity and commutativity constraints are compatible, so $\mathbb{P}\mathrm{erv}_{G,\zeta}$ is a symmetric monoidal category.

Remark 2.2. Let $\mathrm{P}_{G(\mathcal{O})}(\mathrm{Gra}_G)$ denote the category of $G(\mathcal{O})$-equivariant perverse sheaves on Gra_G. One has the covariant self-functor $\star$ on $\mathrm{P}_{G(\mathcal{O})}(\mathrm{Gra}_G)$ induced by the map $\mathbb{E} \to \mathbb{E}$, $z \mapsto z^{-1}$. Then $K \mapsto K^\vee := \mathbb{D}(\star K)[-2]$ is a contravariant functor $\mathbb{P}\mathrm{erv}_{G,\zeta} \to \mathbb{P}\mathrm{erv}_{G,\zeta}$. As in ([8], Remark 2.8), one shows that $\mathrm{RHom}(K_1 * K_2, K_3) \widetilde{\to} \mathrm{RHom}(K_1, K_3 * K_2^\vee)$. So, $K_3 * K_2^\vee$ represents the internal $\mathcal{H}om(K_2, K_3)$ in the sense of the tensor structure on $\mathbb{P}\mathrm{erv}_{G,\zeta}$. Besides, $\star(K_1 * K_2) \widetilde{\to} (\star K_2) * (\star K_1)$ canonically.

2.7. **Main result.** Below we introduce a tensor category $\mathbb{P}\mathrm{erv}^\natural_{G,\zeta}$ obtained from $\mathbb{P}\mathrm{erv}_{G,\zeta}$ by some modification of the commutativity constraint. Let $\check{T}_\zeta = \mathrm{Spec}\, k[\Lambda^\sharp]$ be the torus whose weight lattice is $\Lambda^\sharp$.

For $a \in \mathbb{Q}^*$ written as $a = a_1/a_2$ with $a_i \in \mathbb{Z}$ prime to each other and $a_2 > 0$, say that a_2 is the denominator of a. Recall that we assume N invertible in k.

Theorem 2.1. *There is a connected reductive group $\check{G}_\zeta$ over $\bar{\mathbb{Q}}_\ell$ and a canonical equivalence of tensor categories*

$$\mathbb{P}\mathrm{erv}^\natural_{G,\zeta} \widetilde{\to} \mathrm{Rep}(\check{G}_\zeta).$$

There is a canonical inclusion $\check{T}_\zeta \subset \check{G}_\zeta$ whose image is a maximal torus in $\check{G}_\zeta$. The Weyl groups of G and $\check{G}_\zeta$ viewed as subgroups of $\mathrm{Aut}(\Lambda^\sharp)$ are the same. Our choice of a Borel subgroup $T \subset B \subset G$ yields a Borel subgroup $\check{T}_\zeta \subset \check{B}_\zeta \subset \check{G}_\zeta$. The corresponding simple roots (resp., coroots) of $(\check{G}_\zeta, \check{T}_\zeta)$ are $\delta_i\alpha_i$ (resp., $\check{\alpha}_i/\delta_i$) for $i \in \mathcal{J}$. Here δ_i is the denominator of $\frac{\bar{\kappa}(\alpha_i,\alpha_i)}{2N}$.

Remark 2.3. i) The root datum described in Theorem 2.1 is defined uniquely. The roots are the union of W-orbits of simple roots. For $\alpha \in R$ let δ_α denote the denominator of $\frac{\bar{\kappa}(\alpha,\alpha)}{2N}$. Then $\delta_\alpha\alpha$ is a root of $\check{G}_\zeta$. Any root of $\check{G}_\zeta$ is of this form. Compare with the metaplectic root datum appeared in ([13], [16], [15]).
ii) We hope there could exist an improved construction, which is a functor from the category of central extensions $1 \to K_2 \to E \to G \to 1$ over k to the 2-category of symmetric monoidal categories, $E \mapsto \mathbb{P}\mathrm{erv}_{G,E}$ such that $\mathbb{P}\mathrm{erv}_{G,E}$ is tensor equivalent to the category $\mathrm{Rep}(\check{G}_E)$ of representations of some connected reductive group E.
iii) A similar monoidal category has been studied in [15]. However, only the case when k is of characteristic zero was considered in [15], and it contains some imprecisions, for example, ([15], Proposition II.3.6) is wrong as stated.

3. Proof of Theorem 2.1

3.1. **Functors F'_P.** Let $P \subset G$ be a parabolic subgroup containing B. Let $M \subset P$ be its Levi factor containing T. Let $\mathcal{J}_M \subset \mathcal{J}$ be the subset parametrizing the simple roots of M. Write

$$1 \to \mathbb{G}_m \to \mathbb{E}_M \to M(F) \to 1$$

for the restriction of (4) to $M(F)$. It is equipped with an action of $\mathrm{Aut}^0(\mathcal{O})$ and a section over $M(\mathcal{O})$ coming from the corresponding objects for (4).

Write $\mathrm{Gr}_M, \mathrm{Gr}_P$ for the affine grassmanians for M, P respectively. For $\theta \in \pi_1(M)$ write Gr_M^θ for the connected component of Gr_M containing $t^\lambda M(\mathcal{O})$ for any $\lambda \in \Lambda$ over $\theta \in \pi_1(M)$. The diagram $M \leftarrow P \to G$ yields the following diagram of affine grassmanians

$$\mathrm{Gr}_M \stackrel{\mathfrak{t}_P}{\leftarrow} \mathrm{Gr}_P \stackrel{\mathfrak{s}_P}{\to} \mathrm{Gr}_G \ .$$

Let Gr_P^θ be the connected component of Gr_P such that $\mathfrak{t}_P$ restricts to a map $\mathfrak{t}_P^\theta : \mathrm{Gr}_P^\theta \to \mathrm{Gr}_M^\theta$. Write $\mathfrak{s}_P^\theta : \mathrm{Gr}_P^\theta \to \mathrm{Gr}_G$ for the restriction of $\mathfrak{s}_P$. The restriction of $\mathfrak{s}_P^\theta$ to $(\mathrm{Gr}_P^\theta)_{red}$ is a locally closed immesion.

The section $M \to P$ yields a section $\mathfrak{r}_P : \mathrm{Gr}_M \to \mathrm{Gr}_P$ of $\mathfrak{t}_P$. By abuse of notations, write

$$\mathrm{Gra}_M \stackrel{\mathfrak{r}_P}{\to} \mathrm{Gra}_P \stackrel{\mathfrak{s}_P}{\to} \mathrm{Gra}_G$$

for the diagram obtained from $\mathrm{Gr}_M \stackrel{\mathfrak{r}_P}{\to} \mathrm{Gr}_P \stackrel{\mathfrak{s}_P}{\to} \mathrm{Gr}_G$ by the base change $\mathrm{Gra}_G \to \mathrm{Gr}_G$. Note that $\mathfrak{t}_P$ lifts naturally to a map denoted $\mathfrak{t}_P : \mathrm{Gra}_P \to \mathrm{Gra}_M$ by abuse of notations.

Let $\mathrm{Perv}_{M,G,\zeta}$ denote the category of $M(\mathcal{O})$-equivariant perverse sheaves on Gra_M with $\mathbb{G}_m$-monodromy ζ. Set

$$\mathbb{P}\mathrm{erv}_{M,G,\zeta} = \mathrm{Perv}_{M,G,\zeta}[-1] \subset \mathrm{D}(\mathrm{Gra}_M) \, .$$

Define the functor

$$F'_P : \mathbb{P}\mathrm{erv}_{G,\zeta} \to \mathrm{D}(\mathrm{Gra}_M)$$

by $F'_P(K) = \mathfrak{t}_{P!}\mathfrak{s}_P^* K$. Write Gra_M^θ for the connected component of Gra_M over Gr_M^θ, similarly for Gra_P^θ. Write

$$\mathbb{P}\mathrm{erv}^\theta_{M,G,\zeta} \subset \mathbb{P}\mathrm{erv}_{M,G,\zeta}$$

for the full subcategory of objects that vanish off Gra_M^θ. Set

$$\mathbb{P}\mathrm{erv}'_{M,G,\zeta} = \mathop{\oplus}_{\theta \in \pi_1(M)} \mathbb{P}\mathrm{erv}^\theta_{M,G,\zeta}[\langle \theta, 2\check{\rho}_M - 2\check{\rho}\rangle] \, .$$

As in [8], one shows that F'_P sends $\mathbb{P}\mathrm{erv}_{G,\zeta}$ to $\mathbb{P}\mathrm{erv}'_{M,G,\zeta}$. This is a combination of the hyperbolic localization argument ([14], Theorem 3.5) or ([11], Proposition 12) with the dimension estimates of ([14], Theorem 3.2) or ([4], Proposition 4.3.3).

For the Borel subgroup B the above construction gives $F'_B : \mathbb{P}\mathrm{erv}_{G,\zeta} \to \mathbb{P}\mathrm{erv}'_{T,G,\zeta}$.

Let $B(M) \subset M$ be a Borel subgroup such that the preimage of $B(M)$ under $P \to M$ equals B. The inclusions $T \subset B(M) \subset M$ yield a diagram

$$\mathrm{Gr}_T \stackrel{\mathfrak{r}_{B(M)}}{\to} \mathrm{Gr}_{B(M)} \stackrel{\mathfrak{s}_{B(M)}}{\to} \mathrm{Gr}_M \ . \tag{18}$$

Write

$$\mathrm{Gra}_T \stackrel{\mathfrak{t}_{B(M)}}{\to} \mathrm{Gra}_{B(M)} \stackrel{\mathfrak{s}_{B(M)}}{\to} \mathrm{Gra}_M$$

for the diagram obtained from (18) by the base change $\mathrm{Gra}_M \to \mathrm{Gr}_M$. The projection $B(M) \to T$ yields $\mathfrak{t}_{B(M)} : \mathrm{Gr}_{B(M)} \to \mathrm{Gr}_T$, it lifts naturally to the map denoted $\mathfrak{t}_{B(M)} : \mathrm{Gra}_{B(M)} \to \mathrm{Gra}_T$ by abuse of notations. For $K \in \mathbb{P}\mathrm{erv}'_{M,G,\zeta}$ set

$$F'_{B(M)}(K) = (\mathfrak{t}_{B(M)})_! \mathfrak{s}^*_{B(M)} K\,.$$

As in [8], this defines the functor $F'_{B(M)} : \mathbb{P}\mathrm{erv}'_{M,G,\zeta} \to \mathbb{P}\mathrm{erv}'_{T,G,\zeta}$, and one has canonically

$$F'_{B(M)} \circ F'_P \,\widetilde{\to}\, F'_B\,. \tag{19}$$

3.1.1. For $j \in J$ let $\mathcal{L}_{j,M}$ denote the restriction of $\mathcal{L}_j$ under $\mathfrak{s}_P\mathfrak{r}_P : \mathrm{Gr}_M \to \mathrm{Gr}_G$. Let Λ^+_M denote the coweights dominant for M. For $\lambda \in \Lambda^+_M$ denote by Gr^λ_M the $M(\mathcal{O})$-orbit through $t^\lambda M(\mathcal{O})$. Let Gra^λ_M be the preimage of Gr^λ_M under $\mathrm{Gra}_M \to \mathrm{Gr}_M$. The M-orbit through $t^\lambda M(\mathcal{O})$ is isomorphic to the partial flag variety $\mathcal{B}^\lambda_M = M/P^\lambda_M$, where the Levi subgroup of P^λ_M has the Weyl group coinciding with the stabilizer of λ in W_M. Here W_M is the Weyl group of M. As for G, we have a natural map $\tilde{\omega}_{M,\lambda} : \mathrm{Gr}^\lambda_M \to \mathcal{B}^\lambda_M$.

If $\check{\nu} \in \check{\Lambda}$ is orthogonal to all coroots α of M satisfying $\langle \check{\alpha}, \lambda\rangle = 0$ then we denote by $\mathcal{O}(\check{\nu})$ the M-equivariant line bundle on $\mathcal{B}^\lambda_M$ corresponding to the character $\check{\nu} : P^\lambda_M \to \mathbb{G}_m$. As in Lemma 2.1, for $j \in J$ the pinning Φ yields a uniquely defined $\mathbb{Z}/2\mathbb{Z}$-graded $\mathrm{Aut}^0(\mathcal{O})$-equivariant isomorphism

$$\mathcal{L}_{j,M}\mid_{\mathrm{Gr}^\lambda_M} \,\widetilde{\to}\, \Omega_{\bar{c}}^{\frac{\kappa_j(\lambda,\lambda)}{2}} \otimes \tilde{\omega}^*_{M,\lambda}\mathcal{O}(\kappa_j(\lambda))$$

So, for $\lambda \in \Lambda^+_M$ there is a $\mathrm{Aut}^0(\mathcal{O})$-equivariant isomorphism between Gra^λ_M and the punctured total space of the line bundle

$$\Omega_{\bar{c}}^{-\frac{\bar{\kappa}(\lambda,\lambda)}{2}} \otimes \tilde{\omega}^*_{M,\lambda}\mathcal{O}(-\bar{\kappa}(\lambda))$$

over Gr^λ_M. Set $\Lambda^{\sharp,+}_M = \Lambda^\sharp \cap \Lambda^+_M$. As for G itself, for $\lambda \in \Lambda^+_M$ the scheme Gra^λ_M admits a $M(\mathcal{O})$-equivariant local system with $\mathbb{G}_m$-monodromy ζ if and only if $\lambda \in \Lambda^{\sharp,+}_M$.

As in Section 2.4.2 pick $\mathcal{E} \in \Omega^{\frac{1}{2}}(\mathcal{O})$. For $\lambda \in \Lambda^{\sharp,+}_M$ define the line bundle $\mathcal{L}_{\lambda,M,\mathcal{E}}$ on Gr^λ_M as

$$\mathcal{L}_{\lambda,M,\mathcal{E}} = \mathcal{E}_{\bar{c}}^{-\frac{\bar{\kappa}(\lambda,\lambda)}{N}} \otimes \tilde{\omega}^*_{M,\lambda}\mathcal{O}(-\frac{\bar{\kappa}(\lambda)}{N})$$

Let $\overset{\circ}{\mathcal{L}}_{\lambda,M,\mathcal{E}}$ be the punctured total space of $\mathcal{L}_{\lambda,M,\mathcal{E}}$. Let $p_{\lambda,M} : \overset{\circ}{\mathcal{L}}_{\lambda,M,\mathcal{E}} \to \mathrm{Gra}^\lambda_M$ be the map over Gr^λ_M sending z to z^N. Let $\mathcal{W}^\lambda_{M,\mathcal{E}}$ be the rank one $M(\mathcal{O})$-equivariant local system $\mathcal{W}^\lambda_{M,\mathcal{E}}$ on Gra^λ_M with $\mathbb{G}_m$-monodromy ζ equipped with an isomorphism $p^*_{\lambda,M}\mathcal{W}^\lambda_{M,\mathcal{E}} \,\widetilde{\to}\, \bar{\mathbb{Q}}_\ell$. Let $\mathcal{A}^\lambda_{M,\mathcal{E}} \in \mathbb{P}\mathrm{erv}_{M,G,\zeta}$ be the intermediate extension of $\mathcal{W}^\lambda_{M,\mathcal{E}}[\dim \mathrm{Gr}^\lambda_M]$ to Gra_M, it is defined up to a scalar automorphism.

Set

$$\widetilde{\mathrm{Gr}}_M = \mathrm{Gr}_M \times_{\mathrm{Gr}_G} \widetilde{\mathrm{Gr}}_G$$

For $\lambda \in \Lambda_M^+$ let $\widetilde{\mathrm{Gr}}_M^\lambda$ be the restriction of the gerb $\widetilde{\mathrm{Gr}}_M$ to Gr_M^λ. As for G itself, for $\lambda \in \Lambda_M^{\sharp,+}$ the map $p_{\lambda,M}$ yields a section $s_{\lambda,M} : \mathrm{Gr}_M^\lambda \to \widetilde{\mathrm{Gr}}_M^\lambda$.

The analog of Lemma 2.2 holds for the same reasons. The perverse sheaf $\mathcal{A}_{M,\mathcal{E}}^\lambda$ has non-trivial cohomology sheaves only in degrees of the same parity. It follows that $\mathbb{P}\mathrm{erv}_{M,G,\zeta}$ is semisimple.

3.1.2. *More tensor structures.* One equips $\mathbb{P}\mathrm{erv}_{M,G,\zeta}$ and $\mathbb{P}\mathrm{erv}'_{M,G,\zeta}$ with a convolution product as in Section 2.5. The convolution for these categories can be interpreted as fusion, and this allows to define a commutativity constraint on these categories via fusion.

Each of the line bundles $\mathcal{L}_{j,X^m}, \mathcal{L}_{\beta,X^m}$ on Gr_{G,X^m} admits the factorization structure as in ([8], Section 4.1.2).

As for G, we have the ind-scheme Gr_{M,X^m} for $m \geq 1$ and the group scheme M_{X^m} over X^m defined similarly. Let $\mathrm{Gra}_{M,G,X^m} \to \mathrm{Gra}_{G,X^m}$ be obtained from $\mathrm{Gr}_{M,X^m} \to \mathrm{Gr}_{G,X^m}$ by the base change $\mathrm{Gra}_{G,X^m} \to \mathrm{Gr}_{G,X^m}$. The group scheme M_{X^m} acts naturally on Gra_{M,G,X^m}.

Write $\mathrm{Perv}_{M,G,\zeta,X^m}$ be the category of M_{X^m}-equivariant perverse sheaves on Gra_{M,G,X^m} with $\mathbb{G}_m$-monodromy ζ. Set

$$\mathbb{P}\mathrm{erv}_{M,G,\zeta,X^m} = \mathrm{Perv}_{M,G,\zeta,X^m}[-m-1]\,.$$

Let $\mathrm{Aut}_2^0(\mathcal{O})$ act on Gra_M via its quotient $\mathrm{Aut}^0(\mathcal{O})$. Then every object of $\mathrm{Perv}_{M,G,\zeta}$ admits a unique $\mathrm{Aut}_2^0(\mathcal{O})$-equivariant structure. On has

$$\mathrm{Gra}_{M,G,X} \,\widetilde{\to}\, \hat{X}_2 \times_{\mathrm{Aut}_2^0(\mathcal{O})} \mathrm{Gra}_M,$$

and as above one gets a fully faithful functor

$$\tau^0 : \mathbb{P}\mathrm{erv}_{M,G,\zeta} \to \mathbb{P}\mathrm{erv}_{M,G,\zeta,X}$$

Define the commutativity constraint on $\mathbb{P}\mathrm{erv}_{M,G,\zeta}$ and $\mathbb{P}\mathrm{erv}'_{M,G,\zeta}$ via fusion as in Section 2.6. As in [8], one checks that $\mathbb{P}\mathrm{erv}_{M,G,\zeta}$ and $\mathbb{P}\mathrm{erv}'_{M,G,\zeta}$ are symmetric monoidal categories. Exactly as in ([8], Lemma 4.1), one proves the following.

Lemma 3.1. *The functors F'_P, $F'_{B(M)}, F'_B$ are tensor functors, and (19) is an isomorphism of tensor functors.* $\square$

3.2. **Fiber functor.** Recall from Section 3.1.1 that for $\lambda \in \Lambda$ the scheme Gra_T^λ admits a $T(\mathcal{O})$-equivariant local system with $\mathbb{G}_m$-monodromy ζ if and only if $\lambda \in \Lambda^\sharp$. View an object of $\mathbb{P}\mathrm{erv}_{T,G,\zeta}$ as a complex on $\widetilde{\mathrm{Gr}}_T$. The map $\mathfrak{t}_{\mathbb{E}}$ from Section 2.3 defines for each $\lambda \in \Lambda^\sharp$ a section $\mathfrak{t}_{\lambda,\mathbb{E}} : \mathrm{Gr}_T^\lambda \to \widetilde{\mathrm{Gr}}_T^\lambda$. For $K \in \mathbb{P}\mathrm{erv}_{T,G,\zeta}$ the complex $\mathfrak{t}_{\lambda,\mathbb{E}}^* K$ is constant and placed in degree zero, so we view as a vector space denoted $F_T^\lambda(K)$. Let

$$F_T = \bigoplus_{\lambda \in \Lambda^\sharp} F_T^\lambda : \mathbb{P}\mathrm{erv}_{T,G,\zeta} \to \mathrm{Vect}$$

This is a fibre functor on $\mathbb{P}\mathrm{erv}_{T,G,\zeta}$. By ([7], Theorem 2.11) we get

$$\mathbb{P}\mathrm{erv}_{T,G,\zeta} \,\widetilde{\to}\, \mathrm{Rep}(\check{T}_\zeta)\,.$$

For $\nu \in \Lambda^\sharp$ write $F'^{\nu}_{B(M)}$ for the functor $F'_{B(M)}$ followed by restriction to Gra^ν_T. Write $F^\nu_M : \mathbb{P}\mathrm{erv}_{M,G,\zeta} \to \mathrm{Vect}$ for the functor

$$F^\nu_T F'^{\nu}_{B(M)}[\langle \nu, 2\check{\rho}_M\rangle]$$

In particular, this definition applies for $M = G$ and gives the functor $F^\nu_G : \mathbb{P}\mathrm{erv}_{G,\zeta} \to \mathrm{Vect}$.

For $\nu \in \Lambda$ as in Section 3.1 one has the map $\mathfrak{t}^\nu_{B(M)} : \mathrm{Gr}^\nu_{B(M)} \to \mathrm{Gr}^\nu_T$. Let $\widetilde{\mathrm{Gr}}^\nu_{B(M)}$ denote the restriction of the gerb $\widetilde{\mathrm{Gr}}_M$ under $\mathrm{Gr}^\nu_{B(M)} \to \mathrm{Gr}_M$. For $\nu \in \Lambda^\sharp$ the section $\mathfrak{t}_{\nu,\mathbb{E}} : \mathrm{Gr}^\nu_T \to \widetilde{\mathrm{Gr}}^\nu_T$ yields by restriction under $\mathfrak{t}^\nu_{B(M)}$ the section that we denote $\mathfrak{t}_{\nu,B(M)} : \mathrm{Gr}^\nu_{B(M)} \to \widetilde{\mathrm{Gr}}^\nu_{B(M)}$.

Lemma 3.2. *If $\nu \in \Lambda^\sharp$, $\lambda \in \Lambda^{\sharp,+}_M$ then $F^\nu_M(\mathcal{A}^\lambda_{M,\mathcal{E}})$ has a canonical base consisting of those connected components of*

$$\mathrm{Gr}^\nu_{B(M)} \cap \mathrm{Gr}^\lambda_M$$

over which the (shifted) local system $\mathfrak{t}^*_{\nu,B(M)}\mathcal{A}^\lambda_{M,\mathcal{E}}$ is constant. Here we view $\mathcal{A}^\lambda_{M,\mathcal{E}}$ as a perverse sheaf on $\widetilde{\mathrm{Gr}}_M$. In particular, for $w \in W_M$ one has

$$F^{w(\lambda)}_M(\mathcal{A}^\lambda_{M,\mathcal{E}}) \widetilde{\to} \bar{\mathbb{Q}}_\ell$$

Proof Exactly as in ([8], Lemma 4.2). $\square$

Consider the following $\mathbb{Z}/2\mathbb{Z}$-grading on $\mathbb{P}\mathrm{erv}'_{M,G,\zeta}$. For $\theta \in \pi_1(M)$ call an object of $\mathbb{P}\mathrm{erv}^\theta_{M,G,\zeta}[\langle \theta, 2\check{\rho}_M - 2\check{\rho}\rangle]$ of parity $\langle \theta, 2\check{\rho}\rangle \mod 2$, the latter expression depends only on the image of θ in $\pi_1(G)$. As in [8], this $\mathbb{Z}/2\mathbb{Z}$-grading on $\mathbb{P}\mathrm{erv}'_{M,G,\zeta}$ is compatible with the tensor structure. In particular, for $M = G$ we get a $\mathbb{Z}/2\mathbb{Z}$-grading on $\mathbb{P}\mathrm{erv}_{G,\zeta}$. The functors F'_P and $F'_{B(M)}$ are compatible with these gradings.

Write Vect^ϵ for the tensor category of $\mathbb{Z}/2\mathbb{Z}$-graded vector spaces. Let $\mathbb{P}\mathrm{erv}^\natural_{M,G,\zeta}$ be the category of even objects in $\mathbb{P}\mathrm{erv}'_{M,G,\zeta} \otimes \mathrm{Vect}^\epsilon$. Let $\mathbb{P}\mathrm{erv}^\natural_{G,\zeta}$ be the category of even objects in $\mathbb{P}\mathrm{erv}_{G,\zeta} \otimes \mathrm{Vect}^\epsilon$. We get a canonical equivalence of tensor categories $sh : \mathbb{P}\mathrm{erv}^\natural_{T,G,\zeta} \widetilde{\to} \mathbb{P}\mathrm{erv}_{T,G,\zeta}$. The functors $F'_{B(M)}, F'_P, F'_B$ yields tensor functors

$$\mathbb{P}\mathrm{erv}^\natural_{G,\zeta} \stackrel{F^\natural_P}{\to} \mathbb{P}\mathrm{erv}^\natural_{M,G,\zeta} \stackrel{F^\natural_{B(M)}}{\to} \mathbb{P}\mathrm{erv}^\natural_{T,G,\zeta} \tag{20}$$

whose composition is $F^\natural_B$. Write $F^\natural : \mathbb{P}\mathrm{erv}^\natural_{G,\zeta} \to \mathrm{Vect}$ for the functor $F_T \circ sh \circ F^\natural_B$. By Lemma 3.2, $F^\natural$ does not annihilate a non-zero object, so it is faithful. By Remark 2.2, $\mathbb{P}\mathrm{erv}^\natural_{G,\zeta}$ is a rigid abelian tensor category. Since $F^\natural$ is exact and faithful, it is a fibre functor. By ([7], Theorem 2.11), $\mathrm{Aut}^\otimes(F^\natural)$ is represented by an affine group scheme $\check{G}_\zeta$ over $\bar{\mathbb{Q}}_\ell$. We get an equivalence of tensor categories

$$\mathbb{P}\mathrm{erv}^\natural_{G,\zeta} \widetilde{\to} \mathrm{Rep}(\check{G}_\zeta) \tag{21}$$

An analog of Remark 2.2 holds also for M, so $F_T \circ sh \circ F^\sharp_{B(M)} : \mathbb{P}\mathrm{erv}^\natural_{M,G,\zeta} \to \mathrm{Vect}$ is a fibre functor that yields an affine group scheme $\check{M}_\zeta$ and an equivalence of tensor categories $\mathbb{P}\mathrm{erv}^\natural_{M,G,\zeta} \widetilde{\to} \mathrm{Rep}(\check{M}_\zeta)$. The diagram (20) yields homomorphisms $\check{T}_\zeta \to \check{M}_\zeta \to \check{G}_\zeta$.

3.3. Structure of $\check{G}_\zeta$.

3.3.1. For $\lambda\in\Lambda^+$ write $\overline{\mathrm{Gr}}^\lambda$ for the closure of Gr^λ in Gr_G. Let $\overline{\mathrm{Gra}}^\lambda_G$ denote the preimage of $\overline{\mathrm{Gr}}^\lambda$ in Gra_G.

Lemma 3.3. *If $\lambda,\mu\in\Lambda^{\sharp,+}_M$ then $\mathcal{A}^{\lambda+\mu}_{M,\mathcal{E}}$ appears in $\mathcal{A}^\lambda_{M,\mathcal{E}}*\mathcal{A}^\mu_{M,\mathcal{E}}$ with multiplicity one.*

Proof We will give a proof only for $M=G$, the generalization to any M being straightforward. Write $\mathbb{E}^\lambda$ (resp., $\overline{\mathbb{E}}^\lambda$) for the preimage of Gra^λ_G (resp., of $\overline{\mathrm{Gra}}^\lambda_G$) in $\mathbb{E}$. As in Section 2.5, we get the convolution map $m^{\lambda,\mu}:\overline{\mathbb{E}}^\lambda\times_{G(\mathcal{O})\times\mathbb{G}_m}\overline{\mathrm{Gra}}^\mu_G\to\overline{\mathrm{Gra}}^{\lambda+\mu}_G$. Let W be the preimage of $\mathrm{Gra}^{\lambda+\mu}_G$ under $m^{\lambda,\mu}$. Then $m^{\lambda,\mu}$ restricts to an isomorphism $W\to\mathrm{Gra}^{\lambda+\mu}_G$ is an isomorphism, and $W\subset\mathbb{E}^\lambda\times_{G(\mathcal{O})\times\mathbb{G}_m}\mathrm{Gra}^\mu_G$ is open. □

Write $\Lambda_0=\{\lambda\in\Lambda\mid\langle\lambda,\check{\alpha}\rangle=0$ for all $\check{\alpha}\in\check{R}\}$. The biggest subgroup in $\Lambda^{\sharp,+}$ is $\Lambda^\sharp\cap\Lambda_0$. If $\lambda_1,\dots,\lambda_r$ generate $\Lambda^{\sharp,+}_M$ as a semi-group then $\oplus_{i=1}^r\mathcal{A}^{\lambda_i}_{M,\mathcal{E}}$ is a tensor generator of $\mathrm{Perv}^\natural_{M,G,\zeta}$ in the sense of ([7], Proposition 2.20), so $\check{M}_\zeta$ is of finite type over $\bar{\mathbb{Q}}_\ell$.

If $\check{M}_\zeta$ acts nontrivially on $Z\in\mathrm{Perv}_{M,G,\zeta}$ then consider the strictly full subcategory of $\mathrm{Perv}^\natural_{M,G,\zeta}$ whose objects are subquotients of $Z^{\oplus m}$, $m\ge 0$. By Lemma 3.3, this subcategory is not stable under the convolution, so $\check{M}_\zeta$ is connected by ([7], Corollary 2.22). Since $\mathrm{Perv}_{M,G,\zeta}$ is semisimple, $\check{M}_\zeta$ is reductive by ([7], Proposition 2.23).

By Lemma 3.2, for $\lambda\in\Lambda^{\sharp,+}_M$, $w\in W_M$ the weight $w(\lambda)$ of $\check{T}_\zeta$ appears in $F^\natural(\mathcal{A}^\lambda_{M,\mathcal{E}})$. So, $\check{T}_\zeta$ is closed in $\check{M}_\zeta$ by ([7], Proposition 2.21).

For $\nu\in\Lambda^{\sharp,+}_M$ write V^ν_M for the irreducible representation of $\check{M}_\zeta$ corresponding to $\mathcal{A}^\nu_{M,\mathcal{E}}$ via the above equivalence $\mathrm{Perv}^\natural_{M,G,\zeta}\;\tilde{\to}\;\mathrm{Rep}(\check{M}_\zeta)$.

Lemma 3.4. *The torus $\check{T}_\zeta$ is maximal in $\check{M}_\zeta$. There is a unique Borel subgroup $\check{T}_\zeta\subset\check{B}(M)_\zeta\subset\check{M}_\zeta$ whose set of dominant weights is $\Lambda^{\sharp,+}_M$.*

Proof First, let us show that for $\nu_1,\nu_2\in\Lambda^{\sharp,+}_M$ the $\check{T}_\zeta$-weight $\nu_1+\nu_2$ appears with multiplicity one in $V^{\nu_1}_M\otimes V^{\nu_2}_M$. For $\lambda_1,\lambda_2\in\Lambda$ write $\lambda_1\le\lambda_2$ if $\lambda_2-\lambda_1$ is a sum of some positive coroots for (G,B). By ([14], Theorem 3.2) combined with Lemma 3.2, if $\nu\in\Lambda^\sharp$ appears in V^λ_M then $w\nu\le\lambda$ for any $w\in W$. By Lemma 3.2, the $\check{T}_\zeta$-weight ν appears in V^ν_M with multiplicity one. Our claim follows.

Let $T'\subset\check{M}_\zeta$ be a maximal torus containing $\check{T}_\zeta$. By Lemma 3.2, for each $\nu\in\Lambda^{\sharp,+}_M$ there is a unique character ν' of T' such that the two conditions are verified: the composition $\check{T}_\zeta\to T'\xrightarrow{\nu'}\mathbb{G}_m$ equals ν; the T'-weight ν' appears in V^ν_M. The map $\nu\mapsto\nu'$ is a homomoprhism of semigroups, so we can apply ([8], Lemma 4.4). This gives a unique Borel subgroup $\check{T}_\zeta\subset\check{B}(M)_\zeta\subset\check{M}_\zeta$ whose set of dominant weights is in bijection with $\Lambda^{\sharp,+}_M$. Since $\nu\mapsto\nu'$ is a bijection between $\Lambda^{\sharp,+}_M$ and the dominant weights of $\check{B}(M)_\zeta$, the torus $\check{T}_\zeta$ is maximal in $\check{M}_\zeta$. □

For $M=G$ write $\check{B}_\zeta=\check{B}(G)_\zeta$. So, $\Lambda^{\sharp,+}$ are dominant weights for $(\check{G}_\zeta,\check{B}_\zeta)$. If $\lambda\in\Lambda^{\sharp,+}$ lies in the W-orbit of $\nu\in\Lambda^{\sharp,+}_M$ then as in Lemma 3.2 one shows that $\mathcal{A}^\nu_{M,\mathcal{E}}$ appears in $F^\natural_P(\mathcal{A}^\lambda_{\mathcal{E}})$. By ([7], Proposition 2.21) this implies that $\check{M}_\zeta$ is closed in $\check{G}_\zeta$.

3.3.2. *Rank one.* Let M be the standard subminimal Levi subgroup of G corresponding to the simple root $\check{\alpha}_i$. Let $j\in J$ be such that $i\in\mathcal{I}_j$. Let $\check{\Lambda}^\sharp=\mathrm{Hom}(\Lambda^\sharp,\mathbb{Z})$ denote the coweights lattice of $\check{T}_\zeta$. Note that $\check{\alpha}_i\in\check{\Lambda}^\sharp$. Then

$$\{\check{\nu}\in\check{\Lambda}^\sharp\mid\langle\lambda,\check{\nu}\rangle\ge 0 \text{ for all } \lambda\in\Lambda^{\sharp,+}_M\}$$

is a $\mathbb{Z}_+$-span of a multiple of $\check{\alpha}_i$. So, the group $\check{M}_\zeta$ is of semisimple rank 1, and its unique simple coroot is of the form $\check{\alpha}_i/m_i$ for some $m_i\in\mathbb{Q}$, $m_i>0$.

Take any $\lambda\in\Lambda^{\sharp,+}_M$ with $\langle\lambda,\check{\alpha}_i\rangle>0$. Write $s_i\in W$ for the simple reflection corresponding to $\check{\alpha}_i$. By Lemma 3.2, $F^\lambda_M(\mathcal{A}^\lambda_{M,\mathcal{E}})$ and $F^{s_i(\lambda)}_M(\mathcal{A}^\lambda_{M,\mathcal{E}})$ do not vanish, so $\lambda-s_i(\lambda)$ is a multiple of the positive root of $\check{M}_\zeta$. So, the unique simple root of $\check{M}_\zeta$ is $m_i\alpha_i$. It follows that the simple reflection for $(\check{T}_\zeta,\check{M}_\zeta)$ acts on $\Lambda^\sharp$ as $\lambda\mapsto\lambda-\langle\lambda,\frac{\check{\alpha}_i}{m_i}\rangle(m_i\alpha_i)=s_i(\lambda)$. We must show that $m_i=\delta_i$.

By ([14], Theorem 3.2) the scheme $\mathrm{Gr}^\nu_{B(M)}\cap\mathrm{Gr}^\lambda_M$ is non empty if and only if

$$\nu=\lambda,\lambda-\alpha_i,\lambda-2\alpha_i,\ldots,\lambda-\langle\lambda,\check{\alpha}_i\rangle\alpha_i\,.$$

For $0<k<\langle\lambda,\check{\alpha}_i\rangle$ and $\nu=\lambda-k\alpha_i$ one has

$$\mathrm{Gr}^\nu_{B(M)}\cap\mathrm{Gr}^\lambda_M\,\widetilde{\to}\,\mathbb{G}_m\times\mathbb{A}^{\langle\lambda,\check{\alpha}_i\rangle-k-1}\,.$$

Let M_0 for the simply-connected cover of the derived group of M, Let T_0 be the preimage of $T\cap[M,M]$ in M_0. Let Gr_{M_0} denote the affine grassmanian for M_0. Let $\Lambda_{M_0}=\mathbb{Z}\alpha_i$ denote the coweights lattice of T_0. Write $\mathcal{L}_{M_0}$ for the ample generator of the Picard group of Gr_{M_0}. This is the line bundle with fibre $\det(V_0(\mathcal{O}):V_0(\mathcal{O})^g)$ at $gM_0(\mathcal{O})$, where V_0 is the standard representation of M_0. Let $f_0:\mathrm{Gr}_{M_0}\to\mathrm{Gr}_G$ be the natural map

For $j'\in J$ the line bundle $f_0^*\mathcal{L}_{j'}$ is trivial unless $j'=j$, and

$$f_0^*\mathcal{L}_j\,\widetilde{\to}\,\mathcal{L}_{M_0}^{\frac{\kappa_j(\alpha_i,\alpha_i)}{2}}$$

Besides, the restriction of the line bundle $E_\beta/G_{ad}(\mathcal{O})$ under $\mathrm{Gr}_{M_0}\stackrel{f_0}{\to}\mathrm{Gr}_G\to\mathrm{Gr}_{G_{ab}}$ is trivial.

Assume that $\lambda=a\alpha_i$ with $a>0, a\in\mathbb{Z}$ such that $\lambda\in\Lambda^\sharp$. Let $\nu=b\alpha_i$ with $b\in\mathbb{Z}$ such that $-\lambda<\nu<\lambda$.

Write $U\subset M(F)$ for the one-parameter unipotent subgroup corresponding to the affine root space $t^{-a+b}\mathfrak{g}_{\check{\alpha}_i}$. Let Y be the closure of the U-orbit through $t^\nu M(\mathcal{O})$ in Gr_M. It is a T-stable subscheme $Y\,\widetilde{\to}\,\mathbb{P}^1$. The T-fixed points in Y are $t^\nu M(\mathcal{O})$ and $t^{-\lambda}M(\mathcal{O})$. The natural map $\mathrm{Gr}_{M_0}\to\mathrm{Gr}_M$ induces an isomorphism $\mathrm{Gr}_{M_0}\,\widetilde{\to}\,(\mathrm{Gr}^0_M)_{red}$ at the level of reduced ind-schemes. So, we may consider the restriction of $\mathcal{L}_{M_0}$ to Y, which identifies with $\mathcal{O}_{\mathbb{P}^1}(a+b)$.

The restriction of $\mathcal{L}_j$ to $\mathrm{Gr}^{\nu}_{B(M)}$ is the constant line bundle with fibre $\Omega_{\bar{c}}^{\frac{\kappa_j(\nu,\nu)}{2}}$. Let $a \in \Omega_{\bar{c}}^{\frac{\kappa_j(\nu,\nu)}{2}}$ be a nonzero element. Viewing it as a section of

$$\mathcal{L}_{M_0}^{\frac{\kappa_j(\alpha_i,\alpha_i)}{2}}$$

over Y, it will vanish only at $t^{-\lambda}M(\mathcal{O})$ with multiplicity $(a+b)\kappa_j(\alpha_i,\alpha_i)/2$. It follows that the shifted local system $\mathfrak{t}^*_{\nu,B(M)}\mathcal{A}^{\lambda}_{M,\mathcal{E}}$ will have the $\mathbb{G}_m$-monodromy

$$\zeta^{(a+b)c_j\kappa_j(\alpha_i,\alpha_i)/2}$$

This local system is trivial if and only if $(a+b)\bar{\kappa}(\alpha_i,\alpha_i)/2 \in N\mathbb{Z}$. We may assume $a\bar{\kappa}(\alpha_i,\alpha_i) \in 2N\mathbb{Z}$. Then the above condition is equivalent to $b\bar{\kappa}(\alpha_i,\alpha_i) \in 2N\mathbb{Z}$. The smallest positive integer b satisfying this condition is δ_i. So, $m_i = \delta_i$.

3.3.3. Let now M be a standard Levi corresponding to a subset $\mathcal{J}_M \subset \mathcal{J}$. The semigroup

$$\{\check{\nu} \in \check{\Lambda}^{\sharp} \mid \langle\lambda,\check{\nu}\rangle \geq 0 \text{ for all } \lambda \in \Lambda^{\sharp,+}_M\}$$

is the $\mathbb{Q}_+$-closure in $\check{\Lambda}^{\sharp}$ of the $\mathbb{Z}_+$-span of positive coroots of $\check{M}_{\zeta}$ with respect to the Borel $\check{B}(M)_{\zeta}$. Since the edges of this convex cone are directed by $\check{\alpha}_i$, $i \in \mathcal{J}_M$, the simple coroots of $\check{M}_{\zeta}$ are positive rational multiples of $\check{\alpha}_i$, $i \in \mathcal{J}_M$. Since we know already that $\check{\alpha}_i/\delta_i$, $i \in \mathcal{J}_M$ are coroots of $\check{M}$, we conclude that the simple coroots of $\check{M}_{\zeta}$ are $\check{\alpha}_i/\delta_i$, $i \in \mathcal{J}_M$. In turn, this implies that $\check{M}_{\zeta}$ is a Levi subgroup of $\check{G}_{\zeta}$. Finally, we conclude that the Weyl groups of G and of $\check{G}_{\zeta}$ viewed as subgroups of $\mathrm{Aut}(\Lambda^{\sharp})$ are the same. Theorem 2.1 is proved.

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