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MIXED BOUNDARY VALUE PROBLEMS ON CYLINDRICAL DOMAINS

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Abstract. We study second-order divergence-form systems on half-infinite cylindrical domains with a bounded and possibly rough base, subject to homogeneous mixed boundary conditions on the lateral boundary and square integrable Dirichlet, Neumann, or regularity data on the cylinder base. Assuming that the coefficients $A$ are close to coefficients $A_0$ that are independent of the unbounded direction with respect to the modified Carleson norm of Dahlberg, we prove a priori estimates and establish well-posedness if $A_0$ has a special structure. We obtain a complete characterization of weak solutions whose gradient either has an $L^2$-bounded non-tangential maximal function or satisfies a Lusin area bound. Our method relies on the first-order formalism of Axelsson, McIntosh, and the first author and the recent solution of Kato’s conjecture for mixed boundary conditions due to Haller-Dintelmann, Tolksdorf, and the second author.

1. Introduction

We consider elliptic $m \times m$-systems of divergence-form equations

\[(Lu)_l(t, x) := - \sum_{i,j=0}^d \sum_{k=1}^m \partial_i (a^{L,k}_{i,j}(x) \partial_j u_k(t, x)) = 0 \quad (l = 1, \ldots, m)\]

posed on a cylindrical domain $\mathbb{R}^+ \times \Omega$ with a bounded base $\Omega \subseteq \mathbb{R}^d$, $d \geq 2$. Here, and throughout, we write $(t, x) \in \mathbb{R}^{1+d}$, where $t \in \mathbb{R}$ is the distinguished perpendicular coordinate and $x \in \mathbb{R}^d$ is the tangential coordinate. We have set $\partial_0 = \partial_t$ and $\partial_i = \partial_{x_i}$ for $i \geq 1$ and write $\nabla_{t,x}$ for the gradient in all directions and $\nabla_x$ for the tangential gradient. We assume that the coefficient tensor $A(t, x) := (a_{i,j}^{L,k}(x))_{i,j=0}^d_{k=1}^m$ is bounded on $\mathbb{R}^+ \times \Omega$ and strictly accretive on a certain subspace of $L^2(\Omega)^m \times L^2(\Omega)^{dm}$. The equations are complemented with mixed Dirichlet/Neumann conditions

\[(1.2) \quad u = 0 \quad (\text{on } \mathbb{R}^+ \times \partial\Omega) \]
\[\nu \cdot A \nabla_{t,x} u = 0 \quad (\text{on } \mathbb{R}^+ \times (\partial \Omega \setminus \partial \Omega))\]
on the lateral boundary, see Figure 1 below for illustration. Here, $\nu$ denotes the formal outer unit normal vector to the boundary of $\mathbb{R}^+ \times \Omega$. Our focus lies on rough geometric configurations even beyond the Lipschitz class. So, we assume that $\Omega$ is $d$-Ahlfors regular, that $\partial\Omega$ satisfies the Ahlfors-David condition, and only around the Neumann part of the boundary do we require Lipschitz

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The cylinder \( \mathbb{R}^+ \times \Omega \subseteq \mathbb{R}^3 \) is built from a non-Lipschitzian base \( \Omega \subseteq \mathbb{R}^2 \) (the heart) that satisfies the standing geometric assumptions in this article. The lateral boundary splits into a Dirichlet part \( \mathbb{R}^+ \times \mathcal{D} \) (highlighted by bold lines) and its complement carrying homogeneous Neumann boundary conditions. On the bottom of our heart, inhomogeneous boundary conditions for either \( u|_{t=0} \), \( \nu \cdot A \nabla_{t,x} u|_{t=0} \), or \( \nabla_x u|_{t=0} \) are imposed.

Our goal is to classify all weak solutions \( u \) to these equations that satisfy appropriate interior estimates of \( \nabla_{t,x} u \), such as a square-integrable modified non-tangential maximal function or Lusin area bounds. Moreover, we aim for well-posedness, that is, unique solvability in the aforementioned spaces, given either Dirichlet data \( u|_{t=0} \), Neumann data \( \nu \cdot A \nabla_{t,x} u|_{t=0} \), or Dirichlet regularity data \( \nabla_x u|_{t=0} \) in \( L^2(\Omega) \).

Since the coefficients \( A \) may depend on all variables, these boundary value problems are not always solvable in general unless some additional regularity in \( t \)-direction is imposed, see \([5, 14, 19]\) for counterexamples and further background. Following the treatment in \([5, 11]\), we use the modified Carleson norm \( \| \cdot \|_C \) originating from the work of Dahlberg \([22]\) as a fair means to measure the size of perturbations of \( A \) from the class of \( t \)-independent coefficients \( A_0 \).

Assuming finiteness of \( \| A - A_0 \|_C \), we prove \emph{a priori} estimates and representation formulas for all weak solutions with non-tangential maximal function estimates \( \| \tilde{N}_*(\nabla_{t,x} u) \|_{L^2(\Omega)} < \infty \) or Lusin area bounds \( \int_0^\infty \| \nabla_{t,x} u \|_{L^2(\Omega)}^2 t \, dt < \infty \). Here, \( \tilde{N}_* \) is a modified non-tangential maximal function taking \( L^2 \)-averages over truncated cones. Ever since the famous work of Kenig and Pipher \([33]\), the \( L^2 \)-bound for \( \tilde{N}_*(\nabla_{t,x} u) \) is considered a natural interior estimate for the Neumann and regularity problem. Given our method, the Lusin area bound is most natural for the Dirichlet problem but we show that any such solution satisfies \( \| \tilde{N}_*(u) \|_{L^2(\Omega)} < \infty \) as well. Moreover, we prove that any solution with non-tangential maximal bound and Lusin area bound attains a trace \( \nabla_{t,x} u|_{t=0} \) and \( u|_{t=0} \) on \( \{0\} \times \Omega \), respectively, in the sense of almost everywhere convergence of Whitney averages.

Next, assuming smallness of \( \| A - A_0 \|_C \) and that \( A_0 \) is either Hermitean or a block matrix (no mixed derivatives \( \partial_t \partial_{x_i} \) occur), we obtain well-posedness of the inhomogeneous boundary value problems. For a precise formulation of our main results we refer to Section 3. We remark that...
these results match the status quo for elliptic systems with $L^2$ boundary data on the upper half space $\mathbb{R}^d$.

Modern theory for real equations on the upper half space, that is, when $m = 1$, $\Omega = \mathbb{R}^d$, and $A(t, x) \in \mathbb{R}^{(1+d) \times (1+d)}$, dates back to Dahlberg [21], who was first to solve the Dirichlet problem for $\Delta u = 0$ on a Lipschitz domain with boundary data $\varphi \in L^2$. For such equations the picture is rather complete by now, see [30, 32] just to mention a few. All of these results heavily build on real-variable techniques, such as maximum principles and harmonic measures, which for equations with complex coefficients (let alone coupled systems of such) are not available anymore.

In this paper, we follow a completely different approach that has been proposed and developed to full strength in a series of papers by Axelsson, McIntosh, and the first author [5, 8, 9, 11] and which works equally well for real equations and systems, see also [7, 10] for related results. To date, this so-called $DB$-approach as only been followed for systems on the upper half-space or the unit ball [11]. Much more challenging geometric configurations, such as a cylinder with a rough base bear new and interesting challenges arising from the lateral boundary conditions. These have – at least to our knowledge – not been addressed before.

The general idea is to reformulate the second-order system for $u$ as a first-order system for the conormal gradient $f$ of $u$, a vector formed of the conormal derivative and the tangential gradient at each interior point, see Section 5 for definitions. The first-order system for $f$ then has the form of a non-autonomous evolution equation

$$\partial_t f_t + DB_t f_t = 0 \quad (t > 0)$$

for $D$ a first-order self-adjoint operator acting on the tangential variables and $B_t$ a bounded accretive multiplication operator. The lateral boundary conditions are hidden in the domain of $D$. Having rephrased as

$$\partial_t f_t + DB_0 f_t = D(B_0 - B_t)f_t \quad (t > 0),$$

where $B_0$ is independent of $t$ and corresponds to a $t$-independent coefficient tensor $A_0$ just in the same manner as $B$ corresponds to $A$, it is tempting to solve by the semigroup formula $f_t = e^{-DB_0}f_0$ if $B = B_0$ and then use maximal regularity methods to obtain $f$ via a Duhamel formula in the general case. However, since $DB_0$ will have positive and negative spectrum, the underlying evolution for $f$ will be forward on one part of $L^2$ and backward on another part. In order to master the situation, we have to split $L^2$ into spectral subspaces.

In Section 6 we establish boundedness of the spectral projections $E^\pm_0 := \mathbb{1}_{C^\pm}(DB_0)$, which is a highly delicate matter in general and would not have been available before the resolution of the Kato square root problem for elliptic systems with mixed boundary conditions acting on the cylinder base $\Omega$ only [25, 26]. In Section 7, which lies at the heart of this article, we present a careful analysis of the semigroup solutions $f_t = e^{-DB_0}f_0$ to the first-order system for $B = B_0$. In particular, we identify them as elements of the natural solution spaces and prove Whitney average convergence as $t \to 0$ toward the data $f_0$.

As for the extension to $t$-dependent coefficients with modified Carleson control, we can rely on the maximal regularity estimates for elliptic systems on the upper half space due to Rosén and the first author [5], which are mostly formulated on abstract function spaces and therefore hold for our setup as well. Hence, we shall be rather brief here and suggest to keep a copy of [5] handy as duplicated arguments will be omitted. Additionally, we will prove almost everywhere convergence of Whitney averages of solutions, which was left as an open problem in [5] and was partly resolved in [11, 13]. The so-obtained a priori estimates for weak solutions to $t$-dependent systems are presented in Section 8. In the special case of $t$-independent coefficients $A = A_0$, the...
they entail that the semigroup solutions investigated in Section 7 are the only solutions to the first-order system $\partial_t f_t + DB_0 f_t = 0$ satisfying the respective interior estimates on $\mathbb{R}^+ \times \Omega$.

Finally, in Section 9 we prove well-posedness of the three boundary problems for $t$-independent coefficients $A_0$ that are either Hermitean, of block form, or sufficiently close to one of these classes in the $L^\infty$-topology. We also show that this result is stable under $t$-dependent perturbations $A$ satisfying a smallness condition on $\|A - A_0\|_C$.

Of course, weak solutions to the elliptic system with mixed lateral boundary conditions can also be constructed using the Lax-Milgram lemma, provided there is an interior control $E,F$ sets.

## 2. Notation and basic assumptions

### 2.1. General notation

Function spaces in this article are always over the complex number field. For functions $f$ on $\mathbb{R}^{1+d}$ we let $f_t(x) = f(t, x)$ and frequently identify $L^2(\mathbb{R}^{1+d}) \cong L^2(\mathbb{R}; L^2(\mathbb{R}^d))$ in virtue of Fubini’s theorem. We decompose $f \in \mathbb{C}^n$, where $n = (1 + d)m$ and $m$ is the number of equations in our elliptic system (1.1), as

$$f = \begin{bmatrix} f_\perp \\ f_\parallel \end{bmatrix}$$

into its perpendicular part $f_\perp \in \mathbb{C}^m$ and its tangential part $f_\parallel \in \mathbb{C}^{dm}$. We denote inner products by $(\cdot | \cdot)$ and for $f, g \in \mathbb{C}^n$ we write $f \cdot g := \sum_{j=1}^n f_j g_j$. We let $d(E, F)$ be the semi-distance of sets $E, F \subseteq \mathbb{R}^d$ induced by Euclidean distance on $\mathbb{R}^d$ and we abbreviate by $d_E(x)$ if $F = \{x\}$.

Given compatible Banach spaces $\mathcal{X}_0, \mathcal{X}_1$, we write $[\mathcal{X}_0, \mathcal{X}_1]_\theta, 0 < \theta < 1$, for the corresponding scale of complex interpolation spaces. For background on interpolation theory the reader can refer e.g. to [15].

Concerning inequalities we will write $A \lesssim B$ if there exists a constant $c > 0$ not depending on the parameters at stake, such that $A \leq cB$. Similarly, we use the symbols $\gtrsim$ and $\simeq$.

### 2.2. Geometry of the cylinder base.

We require the following geometric quality of the cylinder base $\Omega$ and the Dirichlet part $D \subseteq \partial \Omega$. These are the same assumptions under which the Kato problem for mixed boundary conditions was solved in [26].

**Assumption 2.1.** Let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain and let $D \subseteq \partial \Omega$ be closed.

(i) Assume that $\Omega$ is $d$-Ahlfors regular,

$$|B(x, r) \cap \Omega| \simeq r^d \quad (x \in \Omega, \ 0 < r \leq 1).$$

(ii) Assume that $D$ is either empty or $(d - 1)$-Ahlfors regular,

$$\mathcal{H}_{d-1}(B(x, r) \cap D) \simeq r^{d-1} \quad (x \in D, \ 0 < r \leq 1),$$

where $\mathcal{H}_{d-1}$ denotes the $(d - 1)$-dimensional Hausdorff measure in $\mathbb{R}^d$. 

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**Note:** The notation and basic assumptions are adapted from the original content, ensuring consistency and readability. The document provides a comprehensive overview of the mathematical framework, including key assumptions and notations that are fundamental to the discussion of mixed boundary value problems on cylindrical domains.
(iii) The Lipschitz condition holds around $\partial \Omega \setminus \mathcal{D}$: For every $x \in \partial \Omega \setminus \mathcal{D}$ there is an open neighborhood $U_x$ and a bi-Lipschitz mapping $\Phi_x : U_x \to (-1,1)^d$ such that

$$\Phi_x(U_x \cap \Omega) = (-1,0) \times (-1,1)^{d-1}, \quad \Phi_x(U_x \cap \partial \Omega) = \{0\} \times (-1,1)^{d-1}.$$ 

**Remark 2.2.** Our assumptions entail that $\Omega$ equipped with the restricted Euclidean distance and the restricted Lebesgue measure becomes a doubling metric measure space, see e.g. [16] for this notion. Moreover, given any $r_0 > 0$, comparability $|B(x,r) \cap \Omega| \simeq r^d$ easily extends to all $0 < r \leq r_0$ upon a change of the implicit constants.

2.3. Sobolev spaces. For $\Xi \subseteq \mathbb{R}^d$ an open set and $\partial \subseteq \partial \Xi$ a closed part of its boundary, we define the Sobolev spaces $W^{1,p}(\Xi)$, $1 < p < \infty$, as the closure of the set of test functions

$$C_c^\infty(\Xi) := \{u|_{\Xi} ; u \in C_c^\infty(\mathbb{R}^d), d(\text{supp} u, \partial) > 0\}$$

with respect to the norm $u \mapsto (\int_\Xi |u|^p + |\nabla u|^p \, dx)^{1/p}$. These spaces should be thought of as the subspaces of those functions in the ordinary Sobolev spaces $W^{1,p}(\Xi)$ that vanish on $\partial$ in an appropriate sense. For further information on their structure the reader can refer e.g. to [17,24].

Under Assumption 2.3 there exists a bounded extension operator $E : W^{1,p}_{\mathcal{D}}(\Omega) \to W^{1,p}(\mathbb{R}^d)$ independent of $p$ such that $Eu = u$ a.e. on $\Omega$ for every $u \in W^{1,p}_{\mathcal{D}}(\Omega)$, see e.g. [6, Lem. 3.2]. In particular, this gives the compatibility $W^{1,p}_{\mathcal{D}}(\Omega) = W^{1,p}(\Omega)$ and provides the usual Sobolev embeddings of type $W^{1,p}_{\mathcal{D}}(\Omega) \subseteq L^q(\Omega)$.

2.4. Weak solutions. We write $\mathcal{V} := W^{1,2}_{\mathcal{D}}(\Omega)^m$ for the natural $L^2$-function space allowing to model mixed Dirichlet/Neumann boundary conditions for $m \times m$ elliptic systems and let $\nabla \mathcal{V}$ denote the distributional gradient operator $L^2(\Omega)^m \to L^2(\Omega)^{dm}$ with domain $\mathcal{D}(\nabla \mathcal{V}) := \mathcal{V}$.

**Assumption 2.3.** The coefficient tensor $A(t,x)$ is measurable and essentially bounded,

$$A(t,x) := (a^{\alpha,\beta}(x))_{\alpha,\beta=1,\ldots,m} \in L^\infty(\mathbb{R}^+ \times \Omega; \mathcal{L}(\mathbb{R}^{(1+d)m}))$$

and there exists some $\lambda > 0$ independent of $t > 0$ such that it satisfies the ellipticity/accretivity condition

$$\text{Re} \int_\Omega A(t,x)f(x) \cdot \overline{f(x)} \, dx \geq \lambda \int_\Omega |f(x)|^2 \, dx \quad (f \in L^2(\Omega)^m \times \mathcal{R}(\nabla \mathcal{V})).$$

**Remark 2.4.** Assumption 2.3 is weaker than pointwise uniform accretivity of $A$ and stronger than Gårding’s inequality for $u \in L^2(\mathbb{R}^+; \mathcal{V}) \cap W^{1,2}(\mathbb{R}^+; L^2(\Omega)^m)$. The second statement follows by taking $f = (\nabla_{t,x} u)_t$ for fixed $t > 0$ and integrating over $t$. For further information and related ellipticity concepts the reader can refer to [8, Sec. 2].

A formal integration by parts in (1.1), taking into account the lateral boundary conditions (1.2), leads to our notion of $L^2_{\text{loc}}(L^2)$-weak solutions.

**Definition 2.5.** If $\mathcal{D} \neq \emptyset$, then a weak solution to the elliptic system complemented with mixed lateral boundary conditions is a function $u \in L^2_{\text{loc}}(\mathbb{R}^+; \mathcal{V}) \cap W^{1,2}_{\text{loc}}(\mathbb{R}^+; L^2(\Omega)^m)$ that satisfies

$$\int_0^\infty \int_\Omega A(t,x) \nabla_{t,x} u(t,x) \cdot \overline{\nabla_{t,x} v(t,x)} \, dx \, dt = 0 \quad (v \in C_c^\infty(\mathbb{R}^+; \mathcal{V})).$$

If $\mathcal{D} = \emptyset$, then it is additionally required that $u$ satisfies the no-flux condition

$$\lim_{t \to \infty} \int_\Omega (A(t,x) \nabla_{t,x} u(t,x))_\perp \, dx = 0.$$
The no-flux condition is common to rule out linear growth of solutions at spatial infinity \([1]\). This specialty of the pure lateral Neumann case is a substitute for the Dirichlet boundary condition at \(t = \infty\), which is present in all other cases as the Dirichlet part \(\mathbb{R}^+ \times D\) reaches up to spacial infinity. In fact, the flux \(\int_{\Omega}(A \nabla_t u)\downarrow dx\) is independent of \(t\).

**Lemma 2.6.** Suppose \(\mathcal{D} = \emptyset\). If \(u \in L^2_{\text{loc}}(\mathbb{R}^+; \mathcal{Y}) \cap W^{1,2}_{\text{loc}}(\mathbb{R}^+; L^2(\Omega)^m)\) satisfies (2.1), then there is a constant \(c \in \mathbb{C}^m\) such that \(\int_{\Omega}(A \nabla_t u)\downarrow dx = c\) for all \(t > 0\). In particular, \(c = 0\) if \(u\) is a weak solution.

**Proof.** Let \(y \in \mathbb{C}^m\). For every \(\eta \in C^\infty_c(\mathbb{R}^+; \mathbb{R})\) the choice \(v_t(x) = \eta(t)y, t > 0\), is admissible in (2.1) and

\[
\int_0^\infty \eta'(t) \int_{\Omega} (A(t,x) \nabla_t u(t,x))\downarrow \cdot \mathbf{y} \, dx \, dt = 0
\]

follows. Hence, the integral over \(\Omega\) is independent of \(t\). Letting \(y\) run through the standard orthonormal basis of \(\mathbb{C}^m\) yields the claim. \(\square\)

Finally, we define the conormal gradient of weak solutions, a vector formed from the gradient \(\nabla u\) in such a way that its \(\downarrow\)-component corresponds to Neumann and its \(\|\cdot\|\)-component to regularity boundary conditions.

**Definition 2.7.** The conormal gradient of a function \(u \in L^2_{\text{loc}}(\mathbb{R}^+; W^{1,2}(\Omega)^m) \cap W^{1,2}_{\text{loc}}(\mathbb{R}^+; L^2(\Omega)^m)\) is given by

\[
\nabla_A u := \left[ \begin{array}{c} (A \nabla_t u)\downarrow \\ \nabla_x u \end{array} \right] \in L^2_{\text{loc}}(\mathbb{R}^+; L^2(\Omega)^n).
\]

**2.5. Modified non-tangential maximal function.** Following \([5]\), we define a modified non-tangential maximal function on the cylinder \(\mathbb{R}^+ \times \Omega\) by \(L^2\)-averaging over truncated cones, called Whitney balls below.

**Definition 2.8.** The modified non-tangential maximal function of a function \(f\) on \(\mathbb{R}^+ \times \Omega\) is defined by

\[
\tilde{N}_s f(x) := \sup_{t > 0} \left( \frac{1}{\int_{W(t,x)} f(s,y)^2 \, dy \, ds} \right)^{1/2} (x \in \Omega),
\]

where

\[
W(t,x) = \{(s,y) \in \mathbb{R}^+ \times \Omega; c_0^{-1}t < s < c_0 t, |y-x| < c_1 t\}
\]

is called Whitney ball around \((t,x)\) and \(c_0 > 1\) and \(c_1 > 0\) are fixed constants. The modified Carleson norm of a function \(g\) on \(\mathbb{R}^+ \times \Omega\) is

\[
\|g\|_C := \left( \frac{1}{\sup_B \frac{1}{|B \cap \Omega|} \int_{(0,r(B)) \times (B \cap \Omega)} \left( \sup_{W(t,x)} |g|^2 \right) \frac{dx \, dt}{t} \right)^{1/2},
\]

where the supremum is taken over all balls \(B \subseteq \mathbb{R}^d\) with center in \(\Omega\) and radius \(r(B) > 0\).

The modified Carleson norm will serve as our measure for the deviation of the coefficients \(A\) from the class \(t\)-independent coefficients \(A_0(t,x) = A_0(x)\). The reader should think of \(\|A - A_0\|_C < \infty\) to mean that “\(A(t,x) = A_0(x)\) holds at \(t = 0\) but also that \(A(t,x)\) is close to \(A_0(x)\) at all scales” \([5]\). It turns out that given \(A\), such coefficients \(A_0\) are unique and satisfy Assumption 2.3 with controlled bounds. The proof of this result is deferred until Section 4.
Lemma 2.9. Let $A : \mathbb{R}^+ \times \Omega \to \mathcal{L}(\mathbb{C}^{1+d \cdot m})$ satisfy Assumption \ref{Assumption2.3} with constant of accretivity $\lambda > 0$. Assume that $A_0 : \mathbb{R}^+ \times \Omega \to \mathcal{L}(\mathbb{C}^{1+d \cdot m})$ are $t$-independent measurable coefficients such that $\|A - A_0\|_C < \infty$. Then $A_0$ is uniquely determined by $A$, that is, if $A_0'$ are $t$-independent measurable coefficients such that $\|A - A_0'\|_C < \infty$, then $A_0' = A_0$ almost everywhere. Furthermore, $A_0$ satisfies Assumption \ref{Assumption2.3} with

$$\lambda \leq \lambda_0 \leq \|A_0\|_\infty \leq \|A\|_\infty,$$

where $\lambda_0$ denotes a constant of accretivity for $A_0$.

3. Main results

Our first two results provide a priori estimates for weak solutions to the system

$$Lu = 0 \quad \text{(in } \mathbb{R}^+ \times \Omega)$$

$$u = 0 \quad \text{(on } \mathbb{R}^+ \times \partial D)$$

$$\nu \cdot A \nabla_{t,x} u = 0 \quad \text{(on } \mathbb{R}^+ \times (\partial \Omega \setminus \mathcal{D}))$$

that satisfy appropriate interior estimates of $\nabla_{t,x} u$ and one of the following three classical inhomogeneous boundary conditions on the cylinder bottom:

- The Dirichlet condition $u = \varphi$ on $\{0\} \times \Omega$, given $\varphi \in L^2(\Omega)^m$,
- The Neumann condition $(A \nabla_{t,x} u)_{\perp} = \varphi$ on $\{0\} \times \Omega$, given $\varphi \in L^2(\Omega)^m$,
- The Dirichlet regularity condition $\nabla_{x} u = \varphi$ on $\{0\} \times \Omega$, given $\varphi \in L^2(\Omega)^{dm}$.

Note that $(1,0)$ is the inward pointing normal vector to $\{0\} \times \Omega$ (identified with $\Omega$ for simplicity of exposition), so that $(A \nabla_{t,x} u)_{\perp} = \varphi$ really is a boundary condition of Neumann type.

For the Neumann and regularity problems we impose an $L^2$-bound for the non-tangential maximal function of $u$ and obtain the following.

Theorem 3.1. Let $\Omega$ and $\mathcal{D}$ satisfy Assumption \ref{Assumption2.7} and let the coefficients $A$ be bounded and elliptic as in Assumption \ref{Assumption2.3}.

(i) A priori estimates and traces: Suppose that there exist $t$-independent measurable coefficients $A_0$ such that $\|A - A_0\|_C < \infty$. If $u$ is a weak solution to the elliptic system with estimates $\|\tilde{N}_s(\nabla_{t,x} u)\|_{L^2(\Omega)} < \infty$, then $\nabla_{A} u$ has limits

$$\lim_{t \to 0} \int_{t}^{2t} \|\nabla_{A} u_s - f_0\|_{L^2(\Omega)^n} ds = 0 = \lim_{t \to \infty} \int_{t}^{2t} \|\nabla_{A} u_s\|_{L^2(\Omega)^n} ds$$

for some trace function $f_0 \in L^2(\Omega)^n$ with estimate $\|f_0\|_{L^2(\Omega)^n} \lesssim \|\tilde{N}_s(\nabla_{t,x} u)\|_{L^2(\Omega)}$. Moreover, Whitney averages of $\nabla_{A} u$ converge to $f_0$ almost everywhere,

$$\lim_{t \to 0} \int_{W(t,x)} \nabla_{A} u ds = f_0(x) \quad (a.e. \ x \in \Omega).$$

(ii) Regularity for $t$-independent coefficients: If $A = A_0$ is $t$-independent, then every weak solution $u$ with estimates as in (i) has additional regularity

$$\nabla_{A} u \in C([0, \infty); L^2(\Omega)^n) \cap C^\infty((0, \infty); L^2(\Omega)^n)$$

and converge to $f_0$ and $0$ in the $L^2(\Omega)^n$-sense as $t \to 0$ and $t \to \infty$, respectively.

Proof. Part (i) follows from Theorem \ref{Theorem8.3} and Theorem \ref{Theorem7.20}. Part (ii) is due to Corollary \ref{Corollary8.4}. \hfill $\square$

For the Dirichlet problem a Lusin area bound is more feasible (given our method), though we obtain a priori non-tangential estimates as well.
Theorem 3.2. Let $\Omega$ and $\mathcal{D}$ satisfy Assumption [2.1] and let the coefficients $A$ be bounded and elliptic as in Assumption [2.3].

(i) A priori estimates and traces: Suppose that there exists $t$-independent measurable coefficients $A_0$ such that $\|A - A_0\|_C < \infty$. If $u$ is a weak solution to the elliptic system with Lusin area bounds $\int_0^\infty \|\nabla_{t,x} u\|^2_{L^2(\Omega)} \, dt < \infty$, then $u \in C([0,\infty); L^2(\Omega)^m)$ and there are limits

$$\lim_{t \to 0} u_t = u_0 \quad \text{and} \quad \lim_{t \to \infty} u_t = u_\infty$$

in the $L^2(\Omega)^m$-sense for some trace $u_0 \in L^2(\Omega)^m$ and a constant $u_\infty \in \mathbb{C}^m$, which is zero if the lateral Dirichlet part is non-empty. Moreover, there are estimates

$$\|u_0\|_2 \lesssim \|N_s(u)\|_{L^2(\Omega)} + \sup_{t > 0} \|u_t\|_{L^2(\Omega)^m} \lesssim |u_\infty| + \left( \int_0^\infty \|\nabla_{t,x} u\|^2_{L^2(\Omega)} \, dt \right)^{1/2} < \infty$$

and Whitney averages of $u$ converge to $u_0$ almost everywhere,

$$\lim_{t \to 0} \iint_{W(t,x)} u \, ds \, dy = u_0(x) \quad (a.e. \ x \in \Omega).$$

(ii) Regularity for $t$-independent coefficients: If $A = A_0$ is $t$-independent, then every weak solution $u$ with estimates as in (i) has additional regularity

$$u \in C([0,\infty); L^2(\Omega)^m) \cap C^\infty((0,\infty); \mathcal{V}).$$

Proof. Part (i) is due to Theorem 8.10 and Theorem 8.15. Part (ii) follows from Corollary 8.11. \hfill \Box

Our third main result concerns well-posedness of the three boundary value problems. We say that the Dirichlet problem for $A$ is well-posed if for each $\varphi \in L^2(\Omega)^m$ there exists a unique weak solution $u$ to the elliptic system for $A$ with estimate $\|N_s(\nabla_{t,x} u)\|_{L^2(\Omega)} < \infty$ such that Whitney averages of $u$ converge to $\varphi$ a.e. as $t \to 0$.

In the case $\mathcal{D} \neq \emptyset$ we similarly say that the Neumann and regularity problem for $A$ are well-posed if for each $\varphi \in L^2(\Omega)^m$ and $\varphi \in \mathcal{R}(\nabla \mathcal{V})$ there exist a unique weak solution with estimate $\int_0^\infty \|\nabla_{t,x} u\|^2_{L^2(\Omega)} \, dt < \infty$ such that Whitney averages of $(A \nabla_{t,x} u)_\perp$ and $\nabla_x u$ converge to $\varphi$ a.e. as $t \to 0$, respectively. Note that $\varphi \in \mathcal{R}(\nabla \mathcal{V})$ for the regularity problem is a natural compatibility condition for the boundary trace since

$$f_0 = \lim_{t \to 0} \int_t^{2t} \nabla_A u \, ds \in L^2(\Omega)^m \times \mathcal{R}(\nabla \mathcal{V})$$

by Theorem 3.1. If $\mathcal{D} = \emptyset$, then we have to take care of the constant functions. So, well-posedness for the Neumann and regularity problems is defined similarly as before but we require uniqueness of $u$ only modulo constants on $\mathbb{R}^+ \times \Omega$ and for the Neumann problem we include the natural compatibility $\int_0^t \varphi = 0$ stemming from the no-flux condition on $u$.

Theorem 3.3. Let $\Omega$ and $\mathcal{D}$ satisfy Assumption [2.1] let the coefficients $A$ be bounded and elliptic as in Assumption [2.3] and suppose that there exist $t$-independent measurable coefficients $A_0$ such that $\|A - A_0\|_C < \infty$

(i) Well-posedness for $A_0$: Each of the three boundary value problems for $A_0$ is well-posed if $A_0$ is either Hermitian, a block matrix with respect to the block decomposition on $L^2(\mathbb{C}^m \times \mathbb{C}^{dm})$, or sufficiently close in the $L^\infty(\Omega; L(\mathbb{C}^m))$-topology to such coefficients $A'_0$ satisfying Assumption 2.1.
(ii) Well-posedness for $A$: If the Neumann/regularity problem for $A_0$ is well-posed, then there exists $\varepsilon > 0$ such that the Neumann/regularity problem for $A$ is well-posed provided $\|A - A_0\|_C < \varepsilon$. In this case, given appropriate data $\varphi$, the corresponding solution satisfies
\[
\|\tilde{N}_s(\nabla_{t,z} u)\|_{L^2(\Omega)} \simeq \|\varphi\|_{L^2(\Omega)}.
\]
A similar perturbation result holds for the Dirichlet problem with solution estimates
\[
\|\tilde{N}_s(u)\|_{L^2(\Omega)} + |u_\infty| \simeq \sup_{t > 0} \|u_t\|_{L^2(\Omega)}^2 \simeq \int_0^\infty \|\nabla_{t,z} u\|_{L^2(\Omega)}^2 \, dt + |u_\infty| \simeq \|\varphi\|_{L^2(\Omega)}^m,
\]
where $u_\infty = \lim_{t \to \infty} u_t$ and in particular $u_\infty = 0$ as long as the Dirichlet part $\mathcal{D}$ is non-empty.

**Proof.** This result is proved in the final Section 9.3. \qed

4. Natural function spaces

In this short section we introduce the natural function spaces related to boundary value problems with $L^2$-data and review some of their basic properties. For the sake of better reference we adopt notation from [5].

**Definition 4.1.** On $\mathbb{R}^+ \times \Omega$ define the Banach/Hilbert spaces
\[
\mathcal{X} := \{ f : \mathbb{R}^+ \times \Omega \to \mathbb{C}^n; \tilde{N}_s(f) \in L^2(\Omega) \},
\]
\[
\mathcal{Y} := \{ f : \mathbb{R}^+ \times \Omega \to \mathbb{C}^n; \int_0^\infty \|f_t\|_{L^2(\Omega)}^2 \, dt < \infty \}
\]
with their natural norms. Here, $\tilde{N}_s$ is the modified non-tangential maximal function introduced in Definition 2.8. Let $\mathcal{Y}^*$ be the dual of $\mathcal{Y}$ relative to the unweighted space $L^2(\mathbb{R}^+; L^2(\Omega)^n)$,
\[
\mathcal{Y}^* := \left\{ f : \mathbb{R}^+ \times \Omega \to \mathbb{C}^n; \int_0^\infty \|f_t\|_{L^2(\Omega)}^2 \frac{dt}{t} < \infty \right\}.
\]

As outlined in the introduction, $\nabla_{t,z} u \in \mathcal{X}$ is a natural interior control for the Neumann and regularity problems, whereas we shall impose $\nabla_{t,z} u \in \mathcal{Y}$ for the Dirichlet problem and deduce $\tilde{N}_s(u) \in L^2(\Omega)$ *a priori*. The space $\mathcal{X}$ has $\mathcal{Y}^*$ as a subspace and lies locally inside $\mathcal{Y}$.

**Lemma 4.2.** For $f : \mathbb{R}^+ \times \Omega \to \mathbb{C}^n$ it holds
\[
\sup_{t > 0} \frac{1}{t} \int_t^{2t} \|f_s\|_{L^2(\Omega)}^2 \, ds \lesssim \|\tilde{N}_s(f)\|_{L^2(\Omega)} \lesssim \int_0^\infty \|f_s\|_{L^2(\Omega)}^2 \, ds / s.
\]
In particular, $\mathcal{Y}^* \subseteq \mathcal{X}$ with continuous inclusion.

**Proof.** We begin with the lower bound. To this end, we put $t_0 := c_1^{-1} \text{diam}(\Omega)$ and consider the case $t \geq t_0$ first. Then for every $x \in \Omega$,
\[
\frac{1}{t} \int_t^{c_0 t} \|f_s\|_{L^2(\Omega)}^2 \, ds = \frac{1}{t} \int_t^{c_0 t} \int_{B(x,c_1 t) \cap \Omega} |f_s(y)|^2 \, dy \, ds \lesssim (c_0 - c_0^{-1}) |\Omega| \tilde{N}_s(f)(x)^2
\]
and integration over $x$ yields $\frac{1}{t} \int_t^{c_0 t} \|f_s\|_{L^2(\Omega)}^2 \, ds \lesssim \|\tilde{N}_s(f)\|_{L^2(\Omega)}^2$. In order to raise the upper limit for integration to $2t$, we simply have to add the respective estimates for $t = t, c_0 t, \ldots, c_N t$, where $N \in \mathbb{N}$ is minimal subject to $c_N^2 \geq 2$. In the case $0 < t < t_0$ we pull the supremum outside the integral to obtain
\[
\|\tilde{N}_s(f)\|_{L^2(\Omega)}^2 \gtrsim \sup_{0 < t < t_0} \frac{1}{t + d} \int_{c_0^{-1} t}^{c_0 t} \int_{B(x,c_1 t) \cap \Omega} |f(s,y)|^2 \, dy \, ds \, dx.
\]
where we implicitly used d-Ahlfors regularity of $\Omega$. The right-hand side equals
\[
\sup_{0 < t < t_0} \frac{1}{t^{1+\delta}} \int_{c_0 t}^{ct} \int_{\Omega} |f(s, y)|^2 1_{B(y, c_0 t \cap \Omega)}(x) \, dx \, dy \, ds \simeq \sup_{0 < t < t_0} \frac{t^{d}}{t^{1+\delta}} \int_{c_0 t}^{ct} \|f_s\|^2 \, ds
\]
and as before we may raise the upper limit for integration to $2t$ without any difficulty.

For the upper bound we use d-Ahlfors regularity of $\Omega$ to obtain the pointwise estimate
\[
\left(4.1\right) \quad \frac{\int_{W(t, x)} |f_s(y)|^2 \, dy \, ds}{\sup_{0 < t < t_0} \frac{1}{t^{1+\delta}} \int_{c_0 t}^{ct} \int_{\Omega} |f(s, y)|^2 1_{B(y, c_0 t \cap \Omega)}(x) \, dx \, dy \, ds} \lesssim \int_{c_0 t}^{ct} \|f_s\|^2 \, ds
\]
uniformly for $0 < t \leq 1$ and $x \in \Omega$. On the large Whitney balls with $t \geq 1$ we similarly have
\[
\frac{\int_{W(t, x)} |f_s(y)|^2 \, dy \, ds}{\sup_{0 < t < t_0} \frac{1}{t^{1+\delta}} \int_{c_0 t}^{ct} \int_{\Omega} |f(s, y)|^2 1_{B(y, c_0 t \cap \Omega)}(x) \, dx \, dy \, ds} \lesssim \int_{c_0 t}^{ct} \|f_s\|^2 \, ds.
\]
From this the claim follows on taking the supremum over $t \leq 1$ and $t \geq 1$, respectively, and integrating with respect to $x \in \Omega$. 

If $f$ is contained in the subspace $Y^*$ of $X$, then Whitney averages $\int_{W(t, x)} |f|^2 \, ds$ are not only uniformly bounded in $t$ for a.e. $x \in \Omega$, but vanish in the limit $t \to 0$. More precisely, we have the following

**Lemma 4.3.** If $f \in Y^*$, then averages $\int_{W(t, x)} |f_s|^2 \, ds$ vanish as $t \to 0$ and $t \to \infty$, respectively and for almost every $x \in \Omega$ it holds
\[
\lim_{t \to 0} \frac{\int_{W(t, x)} |f(s, y)|^2 \, dy \, ds}{\sup_{0 < t < t_0} \frac{1}{t^{1+\delta}} \int_{c_0 t}^{ct} \int_{\Omega} |f(s, y)|^2 1_{B(y, c_0 t \cap \Omega)}(x) \, dx \, dy \, ds} = 0.
\]

**Proof.** Since $\int_{W(t, x)} |f_s|^2 \, ds \lesssim \int_{W(t, x)} \|t\|_2^2 \, ds$, convergence of the averages follows from integrability of $\|t\|_2^2$ with respect to the measure $\frac{ds}{s}$. For the second claim let $0 < t_0 \leq 1$ be arbitrary. Taking the supremum over $t \leq t_0$ in (4.1) and integrating with respect to $x \in \Omega$ leads to
\[
\int_{\Omega} \sup_{0 < t < t_0} \frac{\int_{W(t, x)} |f_s|^2 \, dy \, ds}{\sup_{0 < t < t_0} \frac{1}{t^{1+\delta}} \int_{c_0 t}^{ct} \int_{\Omega} |f(s, y)|^2 1_{B(y, c_0 t \cap \Omega)}(x) \, dx \, dy \, ds} \lesssim \int_{0}^{c_0 t_0} \|f_s\|^2 \, ds.
\]
Since $f \in Y^*$, the right-hand side vanishes in the limit $t_0 \to 0$ and the conclusion follows. 

The following theorem gives a re-interpretation of the modified Carleson norm from Definition 2.8 as the norm of pointwise multiplication from $X$ into the smaller space $Y^*$. When dealing with $t$-independent coefficients $A(t, x)$, this will be the manner in which we exploit finiteness of $\|A - A_0\|_C$ qualitatively. On $\Omega = \mathbb{R}^d$ the first proof was given by Hytönen and Rosén [31]. Later, Huang gave a different proof ([35, Thm. 3.4]; the required result corresponds to the multiplication $T_2^2 \leftrightarrow T_2^{2, \infty} \cdot T_2^{2, 2}$) which in fact only requires that $\Omega$ is doubling [35, Rem. 6.3f]. In turn, this is guaranteed by our standing assumptions, see Remark 2.2

**Theorem 4.4 ([35, Thm. 3.4]).** For $E : \mathbb{R}^d \times \Omega \to L(\mathbb{C}^n)$ the norm of pointwise multiplication
\[
\|E\|_{\cdot} := \|E\|_{X \to Y^*} = \sup_{\|f\|_X = 1} \|Ef\|_{Y^*}
\]
is equivalent to the modified Carleson norm $\|\cdot\|_C$.

**Remark 4.5.** (i) For $B$ an open ball with center $x_0 \in \Omega$ and radius $r(B)$ let $f_B$ be the characteristic function of the Carleson box $(0, r(B)) \times B$ (times a unit vector field). Splitting the supremum over $t > 0$ in the definition of the non-tangential maximal function at $t = r(B)$, we readily find $\check{N}_s(f_B) \leq 1_{B(x_0, r(B) + c_1 r(B))}$ pointwise on $\Omega$. 


From the estimate \( \| \mathcal{E}_{\mathcal{B}} f \|_{Y'} \leq \| \mathcal{E} \|_C \| f \|_X \) we get that the modified Carleson dominates the standard Carleson norm:

\[
\sup_B \left( \frac{1}{|B \cap \Omega|} \int_{(0, r(B)) \times (B \cap \Omega)} |\mathcal{E}(t, x)|^2 \frac{dtdx}{t} \right)^{1/2} \lesssim \| \mathcal{E} \|_C.
\]

(ii) It holds \( \| \mathcal{E} \|_* \gtrsim \| \mathcal{E} \|_\infty \). In fact, given \( \varepsilon > 0 \) there exist \( t > 0 \) and \( f \in L^2(\mathbb{R}^+; L^2(\Omega)^n) \) with support in \( (t, 2t) \) such that \( \| \mathcal{E}_f \|_2 \| f \|_2 \geq \| \mathcal{E} \|_\infty - \varepsilon \) and therefore Lemma 4.2 implies

\[
\| \mathcal{E} \|_* \geq \| \mathcal{E}_f \|_{Y^*} \gtrsim \frac{t^{-1/2} \| f \|_2}{t^{-1/2} \| f \|_2} \geq \| \mathcal{E} \|_\infty - \varepsilon.
\]

Finally, we can give the proof of Lemma 2.9.

**Proof of Lemma 2.9.** Having at hand the domination of the modified Carleson norm \( \| \cdot \|_C \) by the standard Carleson norm, the proof is essentially the same as the one of Lemma 2.2 in [5]. The only modification is that in our setup \( L^\infty(\Omega) \times V \cap C_0^\infty(\Omega)^m \) plays the role of a dense subset of bounded functions within the space on which \( A \) is accretive (in [5] they use \( C_0^\infty(\mathbb{R}^d)^n \)-functions with curl-free tangential component). \( \Box \)

### 5. Equivalence to a first-order system

In this section we prove that the second-order elliptic system with mixed lateral boundary conditions is equivalent to a non-autonomous evolution equation

\[
(5.1) \quad \partial_t f_t + DB_t f_t = 0 \quad (t > 0),
\]

where \( D \) is a self-adjoint first-order differential operators acting on the tangential variable \( x \) and \( B \) is a bounded multiplication operator related to \( A \) by an algebraic matrix transform.

We begin by defining the relevant operators and function spaces. Recall from Section 2.4 that \( \nabla V : L^2(\Omega)^m \to L^2(\Omega)^{dm} \) denotes the distributional gradient operator with domain \( \mathcal{V} = W^{1,2}_G(\Omega)^m \). This yields a closed operator. The following Hardy and Poincaré inequalities entail that its range is closed and that it is injective if \( \mathcal{D} \) is non-empty and otherwise has an \( m \)-dimensional nullspace containing only the constants.

**Proposition 5.1 ([24, Thm. 3.2/4], [37, Thm. 4.4.2]).** Let \( 1 < p < \infty \) and suppose that \( \Omega \) and \( \mathcal{D} \) satisfy Assumption 2.7.

(i) If \( \mathcal{D} \neq \emptyset \), then Hardy’s inequality

\[
\int_\Omega |u(x)|^p \, dx \lesssim \int_\Omega \left( \frac{|u(x)|}{d_\mathcal{D}(x)} \right)^p \, dx \lesssim \int_\Omega |\nabla u(x)|^p \, dx \quad (u \in W^{1,p}_G(\Omega))
\]

holds and \( W^{1,p}_G(\Omega) \) is the largest subset of \( W^{1,p}(\Omega) \) on which the middle term is finite.

(ii) If \( \mathcal{D} = \emptyset \), then Poincaré’s inequality

\[
\int_\Omega |u(x) - u_\Omega|^p \, dx \lesssim \int_\Omega |\nabla u(x)|^p \, dx \quad (u \in W^{1,p}_G(\Omega))
\]

holds, where \( u_\Omega := \int_\Omega u \) is the average of \( u \) over \( \Omega \).

Integration by parts reveals \( C_0^\infty(\Omega)^{dm} \) as a subset of the domain \( \mathcal{D}(\nabla V)^* \) of the adjoint \( (\nabla V)^* : L^2(\Omega)^{dm} \to L^2(\Omega)^m \), on which this operator acts as the distributional divergence operator. Hence, we shall more suggestively write \( \text{d}V : (\nabla V)^* \). However, note carefully that under our very general geometric assumptions on \( \Omega \) we do not have an explicit description for
$D(\text{div}_V)$ as a space of distributions. The self-adjoint differential operator $D$ in (5.1) will turn out to be

$$D := \begin{bmatrix} 0 & \text{div}_V \\ -\nabla_V & 0 \end{bmatrix}$$

with natural domain in $L^2(\Omega)^n = L^2(\Omega)^m \times L^2(\Omega)^d$. By

$$\mathcal{H} := \overline{R(D)} = \mathcal{N}(\nabla_V)^\perp \times \mathcal{R}(-\nabla_V)$$

we denote the closure of its range, where the orthogonal complement $\mathcal{N}(\nabla_V)^\perp$ in $L^2(\Omega)^m$ coincides with $L^2(\Omega)^m$ provided $\mathcal{D}$ is non-empty and otherwise with the space of $L^2(\Omega)^m$-functions with zero average on $\Omega$.

In order to define the multiplication operator $B$, we consider the decomposition $\mathbb{C}^n = \mathbb{C}^d \times \mathbb{C}^{dm}$, which induces a block decomposition

$$A(t, x) = \begin{bmatrix} A_{\perp\perp}(t, x) & A_{\perp\parallel}(t, x) \\ A_{\parallel\perp}(t, x) & A_{\parallel\parallel}(t, x) \end{bmatrix} \in L^\infty(\mathbb{R}^+ \times \Omega; \mathcal{L}(\mathbb{C}^m \times \mathbb{C}^{dm})).$$

Choosing $f = \begin{bmatrix} 1_E w \\ 0 \end{bmatrix}$ for any measurable $E \subseteq \Omega$ and any $w \in \mathbb{C}^m$ in Assumption 2.3 leads to

$\text{Re}(A_{\perp\perp}(t, x)w \cdot \overline{w}) \geq \lambda \|w\|^2$ for a.e. $(t, x) \in \mathbb{R}^+ \times \Omega$. By separability the exceptional set can be chosen independently of $w$. Hence, $A_{\perp\perp}$ is pointwise strictly accretive and in particular invertible in $L^\infty(\mathbb{R}^+ \times \Omega; \mathcal{L}(\mathbb{C}^m))$. In the space $L^\infty(\mathbb{R}^+ \times \Omega; \mathcal{L}(\mathbb{C}^m))$ we have the matrix-valued functions

$$A := \begin{bmatrix} \text{Id} & 0 \\ A_{\parallel\parallel} & A_{\parallel\perp} \end{bmatrix}, \quad A^{-1} := \begin{bmatrix} A_{\perp\perp} & A_{\perp\parallel} \\ A_{\parallel\perp} & \text{Id} \end{bmatrix}, \quad \overline{A}^{-1} := \begin{bmatrix} A_{\perp\perp}^{-1} & -A_{\perp\parallel}^{-1}A_{\parallel\parallel} \\ A_{\parallel\perp}^{-1} & -A_{\parallel\parallel}^{-1}A_{\perp\parallel} \end{bmatrix}.$$

With this notation the conormal gradient $\nabla_A$ can be written as $\overline{A}\nabla_{t,x}$. Finally, we take $B$ as the bounded multiplication operator on $L^2(\mathbb{R}^+ \times \Omega)^n$ induced by $\overline{A}^{-1}$. Strict accretivity is preserved under the transformation $A \mapsto B$. This follows from the subsequent lemma, whose purely algebraic proof is carried out exactly as in [5, Prop. 4.1].

**Lemma 5.2.** If $\lambda > 0$ is as in Assumption 2.3 then

$$\text{Re}(Bf | f)_{L^2(\Omega)} \geq \lambda \|f\|_{L^\infty(\mathbb{R}^+ \times \Omega; \mathcal{L}(\mathbb{C}^m))} \|f\|_{L^2(\Omega)^n}^2 \quad (f \in L^2(\Omega)^m \times \mathcal{R}(\nabla_V)).$$

By a formal computation we find that

$$\text{div}_{t,x} A \nabla_{t,x} u = 0$$

implies

$$\partial_t \nabla_A u = \left[ -\text{div}_x (A \nabla_{t,x} u) \right] = -\begin{bmatrix} 0 & \text{div}_x \\ -\nabla_x & 0 \end{bmatrix} A \nabla_{t,x} u = DB \nabla_A u,$$

that is $f = \nabla_A u$ satisfies the first-order system (5.1) in a formal sense. This fact is well-known in the case $\Omega = \mathbb{R}^d$, see, e.g., [5] Prop. 4.1, but we stress that due to the lateral boundary conditions the argument for a bounded cylinder base $\Omega$ is more involved and cannot go through on a purely symbolic (i.e. distributional) level. Below, we make this correspondence precise using the following notion of weak solutions to the first-order system.

**Definition 5.3.** A *weak solution* to first-order system (5.1) is a function $f \in L^2_{\text{loc}}(\mathbb{R}^+; \mathcal{H})$ such that

$$\int_0^\infty (f_t | \partial_t g_t)_{L^2(\Omega)^n} \, dt = \int_0^\infty (B_t f_t | D g_t)_{\mathcal{L}(\Omega)^n} \, dt \quad (g \in C_0^\infty(\mathbb{R}^+; D(D))).$$
Remark 5.4. If \( \mathcal{D} = \emptyset \), then the tangential component of \( \mathcal{H} \) is the space of average-free \( L^2(\Omega)^m \) functions and thus captures the no-flux condition.

Proposition 5.5. If \( \mathcal{D} \) is non-empty, then there is a one-to-one correspondence between weak solutions \( u \) to the second-order system with mixed lateral boundary conditions and weak solutions \( f \) to the first-order system given by

\[
f = \nabla A u.
\]

If \( \mathcal{D} \) is empty, then this correspondence becomes one-to-one if \( u \) is considered modulo constants.

Proof. The proof is subdivided into three steps. In order to increase readability, all \( L^2 \)-inner products are abbreviated by \( \langle \cdot | \cdot \rangle \).

Step 1: Weak solutions are mapped to weak solutions. Assume that \( u \) is a weak solution to the second-order system and put \( f := \nabla A u \). Note that \( f \in L^2_{\text{loc}}(\mathbb{R}^+; \mathcal{H}) \) – in the case \( \mathcal{D} = \emptyset \) this is guaranteed by Lemma 2.6. To see that \( f \) satisfies (5.2), fix an arbitrary \( g \in C_\infty(\mathbb{R}^+; \mathcal{D}(D)) \). Then \( g_\perp \) is allowed as test function in Definition 2.5 and (2.1) rewrites as

\[
\int_0^\infty \langle (f_t)_\perp | \partial_t (g_t)_\perp \rangle \; dt = \int_0^\infty \langle (B_t f_t)_\parallel | (D g_t)_\parallel \rangle \; dt.
\]

For the tangential parts note \( g_\parallel \in C_\infty(\mathbb{R}^+; \mathcal{D}(\nabla V)^*) \), so that

\[
\int_0^\infty \langle (f_t)_\parallel | (\partial_t g_t)_\parallel \rangle \; dt = -\int_0^\infty \langle u_t | \partial_t (-\nabla V)^*(g_t)_\parallel \rangle \; dt.
\]

Integration by parts, taking into account that \( g \) has compact support in the \( t \)-direction, leads to

\[
\int_0^\infty \langle (f_t)_\parallel | (\partial_t g_t)_\parallel \rangle \; dt = \int_0^\infty \langle \partial_t u_t | (-\nabla V)^*(g_t)_\parallel \rangle \; dt = \int_0^\infty \langle (B_t f_t)_\perp | (D g_t)_\perp \rangle \; dt.
\]

Adding the identities obtained for the perpendicular and tangential parts yields (5.2).

Step 2: The correspondence is onto. Assume that \( f \in L^2_{\text{loc}}(\mathbb{R}^+; \mathcal{H}) \) is a weak solution to the first-order system. Then, by definition, \( f_\parallel \in L^2_{\text{loc}}(\mathbb{R}^+; \mathcal{R}(\nabla V)) \). We first consider the case that the Dirichlet part \( \mathcal{D} \) is non-empty. In virtue of Poincaré’s inequality, \( \nabla V \) is an isomorphism from \( V \) onto \( \mathcal{R}(\nabla V) \). Hence, there exists a potential \( u \in L^2_{\text{loc}}(\mathbb{R}^+; V) \) such that \( \nabla x u = f_\parallel \). We claim

\[
(5.3) \quad u \in W^{1,2}_{\text{loc}}(\mathbb{R}^+; L^2(\Omega)^m) \quad \text{with} \quad \partial_t u = (B f)_\perp.
\]

Indeed, since \( \mathcal{R}(\nabla V)^* \) is dense in \( L^2(\Omega; C)^m \) by injectivity of \( -\nabla V \), it suffices to prove

\[
\left( \int_0^\infty u_t \partial_t \eta(t) \; dt \right) \neg\neg (-\nabla V)^* y = \left( -\int_0^\infty (B_t f_t)_\perp \eta(t) \; dt \right) \neg\neg (-\nabla V)^* y
\]

for each \( \eta \in C_\infty(\mathbb{R}^+; \mathbb{R}) \) and each \( y \in \mathcal{D}(\nabla V)^* \). Here, the left-hand side equals

\[
\int_0^\infty ( (-\nabla V) u_t | \partial_t \eta(t) y ) \; dt = -\int_0^\infty ( (f_t)_\parallel | \partial_t \eta(t) y ) \; dt,
\]
and we can use \( g(t) := \begin{bmatrix} 0 \\ \eta(t)y \end{bmatrix} \) as test function in (5.2) to continue the chain of equalities by

\[
\begin{align*}
= &- \int_0^\infty (f_t \mid \partial_t g_t) \, dt \\
= &- \int_0^\infty (B_t f_t \mid D g_t) \, dt \\
= &- \int_0^\infty ((B_t f_t)_\perp \mid \eta(t)(-\nabla V)^* y) \, dt,
\end{align*}
\]

which coincides with the right-hand side of the identity in question. Summing up, \( u \) has the required regularity and satisfies

\[
\nabla_A u = \overline{\{B f\}_\perp} = f.
\]

To see that \( u \) is a weak solution to the second-order system, let \( v \in C_c^\infty (\mathbb{R}^+; \mathcal{V}) \). As \( g := \begin{bmatrix} v \\ 0 \end{bmatrix} \) is allowed as test function in (5.2),

\[
0 = \int_0^\infty ((f_t)_\perp \mid \partial_t v_t) + ((B f_t)_\parallel \mid \nabla x v_t) \, dt = \int_0^\infty (A \nabla_{t,x} u \mid \nabla_{t,x} v) \, dt
\]

as required.

Now, consider the slightly more involved case that the lateral Dirichlet part is empty. Denote by \( \mathcal{V}_0 \subseteq \mathcal{V} \) the subspace of functions with zero average on \( \Omega \). Poincaré’s inequality on \( \mathcal{V}_0 \) allows to construct a potential \( \tilde{u} \in L^2_{\text{loc}} (\mathbb{R}^+; \mathcal{V}_0) \) such that \( \nabla_x \tilde{u} = f_\parallel \). Repeating the argument succeeding (5.3), at least yields that for every \( \eta \in C_c^\infty (\mathbb{R}^+; \mathbb{R}) \) the \( L^2 \)-valued integral

\[
\int_0^\infty \tilde{u}_t \partial_t \eta(t) \, dt + \int_0^\infty (B_t f_t)_\perp \eta(t) \, dt
\]

is contained in \( \mathcal{R}((\nabla V)^*)_\perp = \mathcal{N}(\nabla V) \) and hence is a constant function on \( \Omega \). Its value is determined as the average integral over \( \Omega \). Since \( \tilde{u}_t \in \mathcal{V}_0 \) for almost every \( t > 0 \), it follows

\[
\int_0^\infty \tilde{u}_t \partial_t \eta(t) \, dt + \int_0^\infty (B_t f_t)_\perp \eta(t) \, dt = \int_0^\infty \eta(t) \left( \int_\Omega (B_t f_t)_\perp \, dx \right) \, dt
\]

for every \( \eta \in C_c^\infty (\mathbb{R}^+; \mathbb{R}) \), that is, \( \partial_t \tilde{u} = (B_t f_t)_\perp - \int_\Omega (B_t f_t)_\perp \) in the sense of \( W^{1,2}_{\text{loc}} (\mathbb{R}^+; L^2(\Omega)^m) \). In order to correct the right-hand side, let \( H \in W^{1,2}_{\text{loc}} (\mathbb{R}^+; \mathbb{C}) \) be an anti-derivative of \( t \mapsto \int_\Omega (B_t f_t)_\perp \, dx \). Note that

\[
\nabla x u = \nabla x \tilde{u} = f_\parallel.
\]

since constant functions on \( \Omega \) are contained in \( \mathcal{V} \) and that by construction \( \partial_t u = (B f)_\perp \) and \( \nabla x u = \nabla x \tilde{u} = f_\parallel \). As in the case of non-empty Dirichlet part this implies that \( u \) is a weak solution to the second-order system satisfying \( \nabla_A u = f \). Note that the no-flux condition automatically holds since \( (\nabla_A u)_\perp = f_\perp \in \mathcal{H}_\perp \) is average-free.

**Step 3:** The correspondence is one-one. If \( u \) is a weak solution with \( \nabla_A u = 0 \), then \( \nabla_{t,x} u = 0 \) by invertibility of \( \overline{A} \). Thus, \( u \) is constant on the domain \( \mathbb{R}^+ \times \Omega \). If in addition \( \mathcal{D} \neq \emptyset \), then \( u = 0 \) by Poincaré’s inequality. \( \square \)
6. Quadratic Value Estimates for $DB_0$ and $B_0D$

We begin our study of the “infinitesimal generator” of the first-order system $\partial_t f_t + DB_0^*f_t = 0$ in case of $t$-independent coefficients $A(t, x) = A_0(x)$ for all $t > 0$. This implies that $B(t, x) = B_0(x)$ is $t$-independent as well and it will be convenient to identify $B_0$ with a bounded accretive multiplication operator on $L^2(\Omega)^n$.

Recall that an operator $T$ in a Hilbert space $K$ is called bisectorial of angle $\omega \in (0, \frac{\pi}{2})$ if its spectrum $\sigma(T)$ is contained in the closure of the double sector

$$S_\omega := \{ z \in \mathbb{C}; \ |\arg z| < \omega \text{ or } |\arg z - \pi| < \omega \}$$

and if the mapping $\lambda \mapsto \lambda(\lambda - T)^{-1}$ is uniformly bounded on $S_\psi$ for every $\psi \in \arg\{\omega, \frac{\pi}{2}\}$. Thanks to Lemma 5.2 the concrete generator $DB_0$ defined in the previous section fits the premise of the following classical result.

**Proposition 6.1** ([8 Prop. 3.3], [23 Prop. 6.2.17]). Let $D$ be a self-adjoint operator in a Hilbert space $K$ and let $B_0 \in \mathcal{L}(K)$. If $B_0$ is accretive on $\mathcal{R}(D)$, that is, there exists $\kappa > 0$ such that $\text{Re}(B_0u | u) \geq \kappa \|u\|^2$ for all $u \in \mathcal{R}(D)$, then the following hold true and implicit constant depend only upon $\kappa$ and an upper bound for the norm of $B_0$.

(i) The operator $DB_0$ has range $\mathcal{R}(DB_0) = \mathcal{R}(D)$ and null space $\mathcal{N}(DB_0) = B_0^{-1}\mathcal{N}(D)$ such that topologically but in general non-orthogonally

$$\mathcal{K} = \mathcal{N}(DB_0) \oplus \mathcal{R}(DB_0).$$

Similarly, $B_0D$ has range $\mathcal{R}(B_0D) = B_0\mathcal{R}(D)$, null space $\mathcal{N}(B_0D) = \mathcal{N}(D)$, and induces a topological splitting

$$\mathcal{K} = \mathcal{N}(B_0D) \oplus \mathcal{R}(B_0D).$$

(ii) The operators $DB_0$ and $B_0D$ are bisectorial of angle $\omega := \arctan\left(\frac{\|B_0\|}{\kappa}\right)$.

Proposition 6.1 holds with $B_0^*$ in place of $B_0$ since this operator satisfies the same accretivity condition. It will also be useful to know the adjoint of the injective part $DB_0|_{\mathcal{R}(D)}$, that is, the maximal restriction of $DB_0$ to an operator on $\mathcal{R}(DB_0)$.

**Corollary 6.2.** In the setup of Proposition 6.1 the Hilbert space adjoint of $DB_0|_{\mathcal{R}(D)}$ is given by $PB_0^*D|_{\mathcal{R}(D)}$, where $P$ is the orthogonal projection in $K$ onto $\mathcal{R}(D)$. Moreover, $\|PB_0^*Du\| \simeq \|Du\|$ for all $u \in D(\mathcal{D}) \cap \mathcal{R}(D)$.

**Proof.** It is straightforward to check that the adjoint of $DB_0|_{\mathcal{R}(D)}$ extends $PB_0^*D|_{\mathcal{R}(D)}$. To obtain equality it suffices to note that these operators share a common resolvent as both are bisectorial: In fact, for the restriction $DB_0|_{\mathcal{R}(D)}$ this is immediate by abstract properties of bisectorial operators [23,29] and $PB_0^*D|_{\mathcal{R}(D)}$ factorizes as $(PB_0^*|_{\mathcal{R}(D)})D|_{\mathcal{R}(D)}$ in the sense of Proposition 6.1. Finally, the required equivalence of norms follows by accretivity of $PB_0^*|_{\mathcal{R}(D)}$. 

As our main result in this section we prove that $DB_0$ and the closely related operator $B_0D$ satisfy quadratic estimates. This will pave the way for everything that follows in this paper.

**Theorem 6.3.** Let Assumption 2.1 be satisfied. Let $B_0$ be a multiplication operator induced by an $L^\infty(\Omega; \mathcal{L}(\mathbb{C}^n))$-function and suppose that $B_0$ is accretive on $\mathcal{R}(D)$. If $T = DB_0$ or $T = B_0D$, then there are quadratic estimates

$$\int_0^\infty \|T(1 + t^2T^2)^{-1}u\|_{L^2(\Omega)}^2 \frac{dt}{t} \simeq \|u\|_{L^2(\Omega)}^2 \quad (u \in \mathcal{R}(T)).$$
Implicit constants can be chosen uniformly for $B_0$ in a bounded subset of $L^\infty(\Omega; \mathcal{L}(\mathbb{C}^n))$ whose members satisfy a uniform lower bound in the accretivity condition.

Before we give the proof of this theorem, let us point out its important consequences. In the following we require basic knowledge on the holomorphic functional calculus for bisectorial operators, allowing to plug in such operators into suitable holomorphic functions defined on a complex bisector enclosing their spectrum. A reader without background in this field can refer to the various comprehensive treatments in the literature, for instance [2,23,29].

(A) The quadratic estimates in Theorem 6.3 remain true for $f(tT)$ in place of $tT(1+t^2T^2)^{-1}$ for every holomorphic $f$ defined on a bisector $S_\psi$ with opening angle $\psi \in (\omega, \frac{\pi}{2})$ that decays polynomially to zero at $0$ and $\infty$ and is non-zero on both connected components of $S_\psi$. Quadratic estimates on $\mathcal{R}(T)$ imply that $T$ has a bounded $H^\infty(S_\psi)$-calculus on $\mathcal{R}(T)$, i.e., for each bounded holomorphic function $f$ defined on $S_\psi$ the operator $f(T)$ in $\mathcal{R}(T)$ satisfies

$$\|f(T)\|_{\mathcal{R}(T) \to \mathcal{R}(T)} \lesssim \|f\|_{L^\infty(S_\psi)}.$$ 

Implicit constants depend only on $\psi$ and the constants in Theorem 6.3. Consequently, the bounds for the $H^\infty(S_\psi)$-calculus enjoy again a uniformity property in $B_0$.

The most important operators defined in the functional calculus for $DB_0$ will be listed below. Proofs of all further statements are carried out in detail e.g. in [23 Sec. 3.3.4].

(B) The characteristic functions $1_{\mathbb{C}_\pm}$ of the right and left complex half planes give the generalized Hardy projections $E_0^\pm := 1_{\mathbb{C}_\pm}(DB_0)$ on $\mathcal{R}(DB_0) = \mathcal{H}$, see Proposition 6.1 for the last equality. Their boundedness yields a topological spectral decomposition $\mathcal{H} = E_0^+ \mathcal{H} \oplus E_0^- \mathcal{H}$.

(C) For $z \in \mathbb{C}$ let $[z] := \sqrt{z^2}$. The exponential functions $z \mapsto e^{-t[z]}$, $t \geq 0$, give the operators $\lambda \mapsto e^{-t|DB_0|}$, $t \geq 0$, on $L^2(\Omega)^n$. They form the bounded holomorphic semigroup generated by $- [DB_0]$. Their restrictions to $E_0^\pm \mathcal{H}$ are the bounded holomorphic semigroups generated by $\pm DB_0|_{E_0^\pm} \mathcal{H}$.

Similar operators can of course be defined in the functional calculus for $B_0 D$. They are related by the following intertwining and duality relations.

(D) For $f$ bounded and holomorphic on $S_\psi$, $\psi \in (\omega, \frac{\pi}{2})$, it holds

$$B_0 f(DB_0)u = f(B_0D)B_0u \quad (u \in \mathcal{H})$$

In fact, this relation is readily checked for resolvents $f(z) = (\lambda - z)^{-1}$, $\lambda \in \mathbb{C} \setminus S_\omega$, and extends to general $f$ by the construction of the functional calculus.

(E) Since $(DB_0)^* = B_0^* D$, for every holomorphic function $f$ on $S_\psi$, $\psi \in (\omega, \frac{\pi}{2})$ with at most polynomial growth at $|z| = 0$ and $|z| = \infty$ it holds

$$f(DB_0)^* = f^*(B_0^* D) \quad \text{where} \quad f^*(z) = \overline{f(z)}.$$ 

Uniformity of the bounds in (A) entails holomorphic dependence of the $H^\infty$-calculus for $DB_0$ with respect to the multiplicative perturbation $B_0$. Most importantly for us, the Hardy projections $E_0^\pm$ depend continuously on $B_0$. For the reader’s convenience, we shortly sketch the standard argument allowing to prove

**Proposition 6.4.** Let $U \subseteq \mathbb{C}$ be open and let $B_0 : U \to \mathcal{L}(L^2(\Omega)^n)$ be a holomorphic function. Assume that each operator $B_0(z)$, $z \in U$, is induced by an $L^\infty(\Omega; \mathcal{L}(\mathbb{C}^n))$-function and that there
exists $K, \kappa > 0$ such that
\[
\text{Re}(B_0(z)u | u)_{L^2(\Omega; \mathbb{C}^n)} \geq \kappa \|u\|_{L^2(\Omega; \mathbb{C}^n)}^2 \quad \text{and} \quad \|B_0(z)\|_{L^\infty(\Omega; \mathbb{C}^n)} \leq K \quad (z \in U, u \in \mathcal{R}(D)).
\]
Then for each $\psi \in (\arctan(\frac{K}{\kappa}), \frac{\pi}{2})$ and each $f \in H^\infty(S_\psi)$ the function $z \mapsto f(DB_0(z)) : U \to \mathcal{L}(\mathcal{H})$ is holomorphic.

**Proof.** We abbreviate $L^2 := L^2(\Omega)^n$. First note that $B_0^* : U \to \mathcal{L}(L^2)$ is holomorphic as well. Given $\lambda \in \mathbb{C} \setminus S_\psi$, holomorphic dependence of $(\lambda - B_0(z)^*D)^{-1} \in \mathcal{L}(L^2)$ on $z$ follows on using the identity
\[
(\lambda - B_0(z_0)^*D)^{-1} - (\lambda - B_0(z_1)^*D)^{-1} = (\lambda - B_0(z_0)^*D)^{-1} (B_0(z_0)^* - B_0(z_1)^*) \ D(\lambda - B_0(z_1)^*D)^{-1} \quad (z_0, z_1 \in U)
\]
on difference quotients. For this we have crucially employed that the domain of $B_0(z)^*D$ is independent of $z$. Taking adjoints, holomorphy of $(\lambda - DB_0(z))^{-1}$ follows. Next, if $f \in H^\infty(S_\psi)$, the subset of functions in $H^\infty(S_\psi)$ it suffices to prove holomorphic dependence of $f(DB_0(z)) \in H^\infty$ on $z$ for each fixed $u \in \mathcal{H}$. Take a bounded sequence $\{f_n\}_n \subseteq H^\infty(S_\psi)$ that converges to $f$ pointwisely on $S_\psi$. Thanks to (B), $\{f_n(DB_0(z))u\}_n$ is a bounded sequence of bounded $\mathcal{H}$-valued holomorphic functions on $U$. The convergence lemma adapted to bisectorial operators ([29] Prop. 5.1.4) or ([23] Prop. 3.3.5) yields pointwise convergence toward $f(DB_0(z))u$. So, holomorphy follows from Vitali’s theorem from complex analysis ([1] Thm. A.5). \hfill \Box

**Proof of Theorem 6.3.** The proof builds upon the tools developed in [25] in order to resolve the Kato problem for mixed boundary conditions under Assumption 2.1. The first ingredient are quadratic estimates for perturbed Dirac type operators acting on $L^2(\Omega)$.

**Proposition 6.5 ([23] Thm. 3.3).** Let Assumption 2.1 be satisfied and let $k \in \mathbb{N}$. On the Hilbert space $L^2 = L^2(\Omega)^{mk}$ consider a triple of operators $\{\Gamma, B_1, B_2\}$ satisfying the following hypotheses.

(H1) $\Gamma$ is nilpotent, i.e. closed, densely defined, and satisfies $\mathcal{R}(\Gamma) \subseteq \mathcal{N}(\Gamma)$.

(H2) $B_1$ and $B_2$ are defined on the whole of $L^2$. There exist $\kappa_1, \kappa_2 > 0$ such that they satisfy the accretivity conditions
\[
\text{Re}(B_1u | u)_{L^2} \geq \kappa_1 \|u\|_{L^2}^2 \quad (u \in \mathcal{R}(\Gamma^*)),
\]
\[
\text{Re}(B_2u | u)_{L^2} \geq \kappa_2 \|u\|_{L^2}^2 \quad (u \in \mathcal{R}(\Gamma))
\]
and there exist $K_1, K_2$ such that they satisfy the boundedness conditions
\[
\|B_1u\|_{L^2} \leq K_1 \|u\|_{L^2} \quad \text{and} \quad \|B_2u\|_{L^2} \leq K_2 \|u\|_{L^2} \quad (u \in L^2).
\]

(H3) $B_2B_1$ maps $\mathcal{R}(\Gamma^*)$ into $\mathcal{N}(\Gamma^*)$ and $B_1B_2$ maps $\mathcal{R}(\Gamma)$ into $\mathcal{N}(\Gamma)$.

(H4) $B_1$ and $B_2$ are multiplication operators induced by $L^\infty(\Omega; \mathcal{L}(\mathbb{C}^{mk}))$-functions.

(H5) For every $\varphi \in C^\infty_c(\mathbb{R}^d; \mathbb{C})$ multiplication by $\varphi$ maps $\mathcal{D}(\Gamma)$ into itself. The commutator $[\Gamma, \varphi]$ is defined on $\mathcal{D}(\Gamma)$ and acts by multiplication with some $c_\varphi \in L^\infty(\Omega; \mathcal{L}(\mathbb{C}^{mk}))$ satisfying pointwise bounds $|c_\varphi(x)| \lesssim |
abla \varphi(x)|$ almost everywhere on $\Omega$.

(H6) Let $Y$ by either $\Gamma$ or $\Gamma^*$. For every open ball $B$ centered in $\Omega$ and for all $u \in \mathcal{D}(\Gamma)$ with compact support in $B \subseteq \Omega$ it holds $\int_{\Omega} |\Gamma u| \lesssim |B|^\frac{1}{2} \|u\|_{L^2}$.

(H7) There exist $\beta_1, \beta_2 \in (0, 1]$ such that the fractional powers of $\Pi := \Gamma + \Gamma^*$ satisfy
\[
\|u\|_{(\mathcal{H}, \mathcal{Y}^k)_{\beta_1}} \lesssim \|((\Pi^2)_{\beta_1/2}u\|_{L^2} \quad \text{and} \quad \|v\|_{(\mathcal{H}, \mathcal{Y}^k)_{\beta_2}} \lesssim \|((\Pi^2)_{\beta_2/2}v\|_{L^2}.
\]
for all \( u \in \mathcal{R}(\Gamma^*) \cap \mathcal{D}(\Pi^2) \) and all \( v \in \mathcal{R}(\Gamma) \cap \mathcal{D}(\Pi^2) \).

Then \( \Pi_B := \Gamma + B_1 \Gamma^* B_2 \) satisfies quadratic estimates

\[
\int_0^\infty \| t \Pi_B (1 + t^2 \Pi_B^2)^{-1} u \|^2 \frac{dt}{t} \approx \| u \|^2 \quad (u \in \overline{\mathcal{R}(\Pi_B)}),
\]

where implicit constants depend on \( B_1 \) and \( B_2 \) only through the constants quantified in (H2).

The second ingredient are extrapolation properties for the weak Laplacian with form domain \( \mathcal{V} \) defined by \( \Delta_V := -\text{div}_V \nabla_V \). For this we need the \( L^2 \)-Bessel potential spaces \( H^\alpha,2(\Omega) \), \( \alpha > 0 \), on \( \Omega \), defined as the restrictions of the ordinary Bessel potential spaces \( H^\alpha(\mathbb{R}^d) \). In [26] the subsequently listed results have been for obtained spaces of scalar-valued functions but they extend to finite Cartesian products in an obvious manner.

**Proposition 6.6** ([26 Thm. 7.1]). Let Assumption 2.1 be satisfied and let \( k \in \mathbb{N} \). Then up to equivalent norms \( [L^2(\Omega)^{mk}, \mathcal{V}^k]^\alpha = H^\alpha,2(\Omega)^{mk} \) for every \( \alpha \in (0, \frac{1}{2}) \).

**Proposition 6.7** ([26 Thm. 4.4]). Let Assumption 2.1 be satisfied. Then there exists \( \alpha \in (0, \frac{1}{2}) \) such that \( \mathcal{D}((\Delta_V)^{\alpha/2}) = H^\alpha,2(\Omega)^m \) with equivalent norms and \( \mathcal{D}((\Delta_V)^{1/2+\alpha/2}) \subseteq H^{1+\alpha,2}(\Omega)^m \) with continuous inclusion.

**Lemma 6.8** (Fractional Poincaré inequality). Let \( \alpha \in (0, 1) \). Under Assumption 2.1 it holds

\[
\| u \|_{L^2(\Omega)^m} \lesssim \| (\Delta_V)^\alpha u \|_{L^2(\Omega)^m} \quad (u \in \mathcal{D}(\Delta_V) \cap \overline{\mathcal{R}(\Delta_V)}).
\]

**Proof.** The restriction \( B := \Delta_V|_{\mathcal{R}(\Delta_V)} \) is an invertible maximal accretivity operator on \( \overline{\mathcal{R}(\Delta_V)} \).

Essentially, this is by Poincaré’s inequality, see also [26 p. 1431]. Invertibility inherits to the fractional powers [29 Prop. 3.1.1], so that \( \| u \|_2 \lesssim \| B^\alpha u \|_2 \) holds for all \( u \in \mathcal{D}(B^\alpha) \). The conclusion follows since \( B^\alpha \) is the restriction of \( (\Delta_V)^\alpha \) to \( \overline{\mathcal{R}(\Delta_V)} \) with domain \( \mathcal{D}((\Delta_V)^\alpha) \cap \overline{\mathcal{R}(\Delta_V)} \), see [29 Prop. 2.6.5]. \( \square \)

**Remark 6.9.** If \( \alpha = \frac{1}{2} \), then Lemma 6.8 is the Poincaré inequality \( \| u \|_{L^2(\Omega)^m} \lesssim \| \nabla u \|_{L^2(\Omega)^{4m}} \). This is due to the Kato estimate \( (\Delta_V)^{1/2} \sim \nabla_V \), see [26 Lem. 4.3] for details.

In order to complete the proof of Theorem 6.3 we apply Proposition 6.5 on \( L^2(\Omega)^n \times L^2(\Omega)^n \) to the operator matrices

\[
\Gamma := \begin{bmatrix} 0 & 0 \\ D & 0 \end{bmatrix}, \quad B_1 := \begin{bmatrix} B_0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{and} \quad B_2 := \begin{bmatrix} 0 & 0 \\ 0 & B_0 \end{bmatrix}.
\]

For these choices

\[
\Pi_B := \begin{bmatrix} 0 & B_0 DB_0 \\ D & 0 \end{bmatrix}, \quad t \Pi_B (1 + t^2 \Pi_B^2)^{-1} = \begin{bmatrix} 0 & tB_0 DB_0 (1 + t^2 (B_0 DB_0)^2)^{-1} \\ 0 & (1 + t^2 (B_0 D)^2)^{-1} \end{bmatrix}.
\]

Since \( \mathcal{R}(B_0 D) = \mathcal{R}(B_0 DB_0) \) and \( \mathcal{R}(DB_0) = \mathcal{R}(D) \) by Proposition 6.1 and as \( B_0 \) is bounded and accretive on \( \mathcal{R}(D) \), we readily see that both quadratic estimates required in the theorem follow from quadratic estimates for \( \Pi_B \).

This being said, it remains to check (H1) - (H7). In fact (H1) - (H4) are met by definition and (H5) and (H6) follow from the product rule and since the integral over the gradient of a compactly supported function vanishes. The only hypothesis that requires a closer inspection is the last one, which due to the symmetry of \( \Pi \) is equivalent to the following:

There exists \( \alpha \in (0, 1] \) such that \( \| u \|_{[L^2(\mathcal{V})]^\alpha} \lesssim \| (D^2)^{\alpha/2} u \|_2 \) for all \( u \in \mathcal{R}(D) \cap \mathcal{D}(D^2) \).
The difficulty lies in that this is a coercivity estimate for a pure first order differential operator. Inevitably, we have to factor out constants if the Dirichlet part of \( \partial \Omega \) is empty: Let \( \alpha \) be as in Proposition 6.7 and fix \( u \in \mathcal{R}(D) \cap \mathcal{D}(D^2) \). Since

\[
D = \begin{bmatrix} 0 & \text{div} v \\ -\nabla v & 0 \end{bmatrix} \quad \text{and} \quad D^2 = \begin{bmatrix} -\Delta v & 0 \\ 0 & (\nabla v) \text{div} v \end{bmatrix}
\]

it follows \( u_\perp \in \mathcal{D}(\Delta v) \cap \mathcal{R}(\text{div} v) \) and \( u_\parallel = -\nabla v v_\perp \) for some \( v_\perp \in \mathcal{D}(\Delta v) \). Note that \( -\nabla v \) and \( (\Delta v)^{1/2} \) share the same nullspace – this is due to the Kato estimate \( (\Delta v)^{1/2} \sim \nabla v \) for the self-adjoint operator \( -\Delta v \). Since the nullspace of fractional powers is independent of their positive exponent [29, Prop. 3.1.1],

\[
\mathcal{R}(\text{div} v) = \mathcal{N}(\nabla v) = \mathcal{N}((-\Delta v)^{1/2}) = \mathcal{N}((-\Delta v)^{1/2}) = \mathcal{R}(\Delta v)
\]

showing \( u_\perp \in \mathcal{R}(\Delta v) \). Due \( \mathcal{N}(\nabla v) = \mathcal{N}(\Delta v) \) and \( L^2(\Omega)^m = \mathcal{N}(\Delta v) \oplus \mathcal{R}(\Delta v) \) we can also assume \( v_\perp \in \mathcal{R}(\Delta v) \). Starting out with the identity

\[
(D^2)^{\alpha/2} u = (D^2)^{\alpha/2} \left( \begin{bmatrix} u_\perp \\ 0 \end{bmatrix} + D \begin{bmatrix} v_\perp \\ 0 \end{bmatrix} \right) = \begin{bmatrix} (\Delta v)^{\alpha/2} u_\perp \\ -\nabla v (-\Delta v)^{\alpha/2} v_\perp \end{bmatrix},
\]

where due to the Kato estimate we may freely replace \( \nabla v \) by \( (-\Delta v)^{1/2} \) as soon as it comes to \( L^2 \)-norms, Lemma 6.8 yields

\[
\| (D^2)^{\alpha/2} u \|_2^2 \lesssim \| u_\perp \|_{\mathcal{R}(\Delta v)^{\alpha/2}}^2 + \| v_\perp \|_{\mathcal{R}(\Delta v)^{1/2 + \alpha/2}}^2.
\]

On the other hand, Proposition 6.6 yields

\[
\| u \|_{L^2(\Omega)^m}^2 \lesssim \| u_\perp \|_{H^{\alpha/2}}^2 + \| \nabla v v_\perp \|_{H^{1/2}}^2 \leq \| u_\perp \|_{H^{\alpha/2}}^2 + \| v_\perp \|_{H^{1 + \alpha/2}}^2,
\]

and invoking Proposition 6.7 the required estimate \( \| u \|_{L^2(\Omega)^m} \lesssim \| (D^2)^{\alpha/2} u \|_2 \) follows. \( \Box \)

7. Analysis of semigroup solutions to \( t \)-independent systems

In this section we restrict ourselves to fixed \( t \)-independent coefficients \( A(t, x) = A_0(x) \). The infinitesimal generator of the corresponding first-order system \( \partial_t f + DB_0 f = 0 \) is bisectorial and hence generates a bounded holomorphic semigroup on the positive Hardy space \( E_0^+ \mathcal{H} \) as we have seen in Section 6. Thus, we can construct semigroup solutions to the first-order system with the following additional limits and regularity.

**Proposition 7.1.** To each \( h^+ \in E_0^+ \mathcal{H} \) corresponds a weak solution \( f_t = e^{-t|DB_0|} h^+, \ t \geq 0, \) of the first-order system for \( B_0 \). It has additional regularity \( f \in C((0, \infty); E_0^+ \mathcal{H}) \cap C^\infty((0, \infty); E_0^+ \mathcal{H}), \) converges to \( h^+ \) and \( 0 \) in the \( L^2(\Omega) \)-sense as \( t \to 0 \) and \( t \to \infty \), respectively, and there are equivalences

\[
\sup_{t \geq 0} \| f_t \|_{L^2(\Omega)^m} \approx \| h^+ \|_{L^2(\Omega)^m} \approx \| \partial_0 f \|_{\mathcal{Y}}.
\]

**Proof.** The restriction of \( \{ e^{-t|DB_0|} \}_{t \geq 0} \) to \( E_0^+ \mathcal{H} \) is the bounded holomorphic semigroup generated by \( -DB_0 \) on \( E_0^+ \mathcal{H} \), see [C] in Section 6. Hence, \( \partial_0 f_t + DB_0 f_t = 0 \) on \( \mathbb{R}^+ \) in the classical sense and in particular, \( f_t \) is a weak solution in the sense of Definition 5.3. The additional regularity and limits follow from abstract semigroup theory, see, e.g., [29, Sec. 3.4]. The first of the equivalences is by boundedness of the semigroup and the second one is by quadratic estimates for \( DB_0 \) with regularly decaying holomorphic function \( |z| e^{-|z|} \). \( \Box \)
Remark 7.2. If \( u \) is a weak solution to the second-order system that satisfies an \( L^2 \)-Dirichlet condition on the cylinder base, then we expect \( f = \nabla_x u \) to be a weak solution to the first-order system without a trace at \( t = 0 \) in the \( L^2 \)-sense. By the same argument as above, such solutions can be constructed as \( f_t = [DB_0]^a e^{-t[DB_0]}h^+ \), where \( a > 0 \) and \( h^+ \in E_0^+H \).

Below, we present a careful analysis of these semigroup solutions to the first-order system and in particular prove that they are contained in the natural solution space \( X \) for the Neumann and regularity problems.

7.1. Off-diagonal decay. As a technical tool to be utilized in the following, we establish \( L^p \) off-diagonal decay of arbitrary polynomial order for the resolvents of \( DB_0 \) if \( |p - 2| \) is sufficiently small. The case \( p = 2 \) is a standard result for perturbed Dirac-type operators, once the subsequent localization and commutator properties for \( D \) have been verified [9, Prop. 5.1].

Lemma 7.3. Let \( \varphi \in C_c^\infty(\mathbb{R}^d; \mathbb{R}) \) and let \( M_\varphi \) be the associated multiplication operator on \( L^2(\Omega)^n \). Then \( M_\varphi D(D) \subseteq D(D) \) and the commutator \([D, M_\varphi]\) acts on \( D(D) \) as a multiplication operator induced by some \( c_\varphi \in L^\infty(\Omega; \mathcal{L}(C^0(\mathbb{C}^n))) \) with pointwise bound \(|c_\varphi(x)| \lesssim |\nabla \varphi(x)| \) on \( \Omega \).

Proof. Since \( \varphi \mathcal{V} \subseteq \mathcal{V} \) by definition of \( \mathcal{V} \), the claim for \( -\nabla_\mathcal{V} \) in place of \( D \) is immediate by the product rule. By duality, the same holds true for \( \text{div}_{\mathcal{V}} = (\nabla_\mathcal{V})^* \) and thus for \( D \) itself. \( \square \)

Proposition 7.4 (\( L^2 \) off-diagonal estimates). Let \( T = DB_0 \) or \( T = B_0 D \). Then for every \( l \in \mathbb{N}_0 \) there exists a constant \( C_l > 0 \) such that

\[
\parallel 1_F (1 + isT)^{-1}1_Eu \parallel_{L^2(\Omega)^n} \leq C_l \left(1 + \frac{d(E,F)}{s}\right)^{-l} \parallel 1_Eu \parallel_{L^2(\Omega)^n}
\]

holds for all \( u \in L^2(\Omega)^n \), all \( s > 0 \), and all Borel sets \( E, F \subseteq \Omega \).

We will appeal to Šneiberg’s stability result on complex interpolation scales in order to extend the off-diagonal bounds to the \( L^p \)-scale nearby \( L^2 \). In the case \( m = 1 \) the following complex interpolation identities for the scale of Banach spaces \( \{W_{sT}^{1,p}(\Omega)^m\}_{1 \leq p \leq \infty} \) have been established in [6, Cor. 8.3]. As usual, interchangeability of interpolation functors and Cartesian products allows to extend the claim to \( m > 1 \). Scale invariance as stated below can basically be obtained by their method as well. Complete details of this somewhat tedious argument have been worked out in [23, Sec. 2.5].

Proposition 7.5. Let \( 0 < \theta < 1 \), let \( 1 < p_0, p_1 < \infty \), and \( \frac{1}{p_0} = \frac{1-\theta}{p_1} + \frac{\theta}{p_1} \). Then the complex interpolation identity

\[
[W_{sT}^{1,p_0}(\Omega)^m, W_{sT}^{1,p_1}(\Omega)^m]_\theta = W_{sT}^{1,p_0}(\Omega)^m
\]

holds up to equivalent norms. Moreover, implicit constants can be chosen uniformly when replacing \( (\Omega, \mathcal{D}) \) by \( (\frac{1}{s}\Omega, \frac{1}{s}\mathcal{D}) \) for any \( 0 < s \leq 1 \).

In the following \( p' = \frac{p}{p-1} \) denotes the Hölder conjugate of \( p \in [1, \infty] \) and we write \( X^* \) for the space of bounded conjugate-linear functionals on a Banach space \( X \).

Corollary 7.6. Let \( 0 < s \leq 1 \). For \( 1 < p < \infty \) let \( X_s^p(\Omega) \) denote the Banach space \( W_{sT}^{1,p}(\Omega)^m \) with equivalent norm \( u \mapsto (\int_\Omega |u|^p + |s\nabla u|^p \, dx)^{1/p} \). Let \( \theta, p_0, p_1, \) and \( p_0 \) be as in Proposition 7.5. Then

\[
[X_{sT}^{p_0}(\Omega), X_{sT}^{p_1}(\Omega)]_\theta = X_{sT}^{p_0}(\Omega)^* \quad \text{and} \quad [X_{sT}^{p_0}(\Omega)^*, X_{sT}^{p_1}(\Omega)^*]_\theta = X_{sT}^{p_0}(\Omega)^*
\]

up to equivalent norms and the equivalence constants can be chosen independently of \( s \).
Proposition 7.7. There exists $\varepsilon > 0$ such that if $|p - 2| < \varepsilon$, then for every $s_0 > 0$ the resolvents 
\{(1 + i s DB_0)^{-1}\}_0 < s < s_0$ extend/restrict to a uniformly bounded family of bounded operators on $L^p(\Omega)^n$.

Proof. We can assume $s_0 = 1$ since $s_0 B_0$ is an operator in the same class as $B_0$. Given $0 < s \leq 1$ and $f \in L^2(\Omega)^n \cap L^p(\Omega)^n$, we define $g \in L^2(\Omega)^n$ by $g := A^{-1} (1 + i s DB_0)^{-1} f$. On recalling $B_0 = A A^{-1}$, we have $f = Ag + is DB_0 g$, that is,
\[
\begin{bmatrix} f_\perp \\ f_\parallel \end{bmatrix} = \begin{bmatrix} (Ag)_\perp \\ g_\parallel \end{bmatrix} + is \begin{bmatrix} -\nabla_v (Ag)_\parallel \\ -\nabla_v g_\parallel \end{bmatrix}.
\]

We use the second equation to eliminate $g_\parallel$ in the first one and separate the terms containing $g_\parallel$ from those containing $f$. This reveals $g_\parallel \in V$ as a solution of the divergence-form problem
\[
(\ref{7.1}) \quad \int_\Omega A \begin{bmatrix} g_\perp \\ is \nabla x g_\perp \end{bmatrix} \cdot \begin{bmatrix} v \\ is \nabla x v \end{bmatrix} dx = \int_\Omega \begin{bmatrix} f_\perp \\ 0 \end{bmatrix} - A \begin{bmatrix} 0 \\ f_\parallel \end{bmatrix} \cdot \begin{bmatrix} v \\ is \nabla x v \end{bmatrix} dx \quad (v \in V).
\]

Due to their intrinsic scaling with respect to $s$, the natural framework to study such problems in $L^p$ are the spaces $X^p_s(\Omega)$. We write the right-hand of $(\ref{7.1})$ as $T(g_\perp)(v)$ for a bounded operator $T : X^p_s(\Omega) \to X^p_s(\Omega)^*$. Then
\[
\|Tu\|_{X^p_s(\Omega)^*} \geq \lambda \|u\|_{X^p_s(\Omega)} \quad (u \in X^p_s(\Omega))
\]
and
\[
\|T\|_{X^p_s(\Omega) \to X^p_s(\Omega)^*} \leq \|A\|_\infty \quad (1 < p < \infty)
\]

Our main point is that these bounds do not depend on $s$. Moreover, $T : X^p_s(\Omega) \to X^p_s(\Omega)^*$ is an isomorphism by the very Lax-Milgram lemma.

Now, fix $1 < p_- < 2 < p_+ < \infty$. If $p \in (p_-, p_+)$, then Corollary $7.6$ allows to replace $X^p_s(\Omega)$-norms by the norms of the corresponding interpolation space between $X^s_{p_+}(\Omega)$ and $X^s_{p_-}(\Omega)$, each time collecting a constant that depends on the respective value of $p$ but not on $s$. Hence we may apply Šnělberg's stability theorem $[34]$ in its quantitative version as stated, e.g., in $[23]$ Thm. 1.3.25 in order to obtain $\varepsilon > 0$ such that for $|p - 2| < \varepsilon$ the operator $T : X^p_s(\Omega) \to X^p_s(\Omega)^*$ is an isomorphism with lower bound
\[
\|Tu\|_{X^p_s(\Omega)^*} \geq c p_{-5} \|u\|_{X^p_s(\Omega)} \quad (u \in X^p_s(\Omega)).
\]
Here, neither $\varepsilon$ nor $c_p$ depend on $s$. Now, $(\ref{7.1})$ implies $\|Tg_\perp\|_{X^p_s(\Omega)^*} \lesssim \|f\|_p$. Hence, if $|p - 2| < \varepsilon$, then $\|g_\parallel\|_{X^p_s(\Omega)} \lesssim \|f\|_p$. Since $g = A^{-1} (1 + is DB_0)^{-1} f$, we find
\[
\|(1 + is DB_0)^{-1} f\|_p \lesssim \|g_\parallel\|_p = \left( \|g_\parallel\|_p + \|f_\parallel + is \nabla v g_\parallel\|_p \right)^{1/p} \lesssim \|f\|_p
\]
with implicit constants independent of $s$. □
Corollary 7.8 (Lp off-diagonal estimates). Let |p − 2| < ε, where ε > 0 is as in Proposition 7.7 and let s_0 > 0. For every l ∈ \mathbb{N}_0 and there exists a constant C_l > 0 such that

\[ \|1_F(1 + isDB_0)^{-1}1_Eu\|_{L^p(\Omega)^n} \leq C_l \left(1 + \frac{d(E,F)}{s}\right)^{-l} \|1_Eu\|_{L^p(\Omega)^n} \]

holds for all u ∈ L^2(\Omega)^n ∩ L^p(\Omega)^n, all 0 < s ≤ s_0, and all Borel sets E, F ⊆ \Omega.

Proof. The claim follows by complex interpolation of the assertions of Proposition 7.4 and 7.7 using the Riesz-Thorin convexity theorem.

7.2. Reverse Hölder estimates. As a second tool toward proving non-tangential estimates for semigroup solutions to the first-order system, we need weak reverse Hölder-type estimates for solutions of the second-order system. For a later purpose we directly prove them for general coefficients A satisfying Assumption 2.3. For elliptic partial differential equations on the whole space or an upper half-space, the classical estimates are already found in [27]. In the case of mixed boundary value problems such estimates have more recently been studied in [18] but – to the best of our knowledge – none of the existing results comprises our geometric setup beyond Lipschitz domains.

Below, we denote by \( \frac{1}{p_*} = \frac{1}{p} + \frac{1}{d} \) and \( \frac{1}{p^*} = \frac{1}{p} - \frac{1}{d} \) the lower and upper Sobolev conjugate of 1 ≤ p ≤ ∞, respectively. We agree on \( p^* = \infty \) if \( p = \infty \).

We need four preparatory lemmas. The first one is a variant of Caccioppoli’s inequality and is proved exactly as the classical estimate in [27], see also [23, Lem. 6.3.14]. The restriction on \( z \) stems from the fact that \( (u - z) \) multiplied by a cut-off function with support in \( (t - 2r,t + 2r) \times B(x,2r) \) has to be admissible as a test function in Definition 2.5.

Lemma 7.9 (Caccioppoli inequality). Let u be a weak solution to the second-order system for A and let \( t > 0 \), \( x \in \Omega \). Let \( r \in (0, \frac{t}{4}) \) and let \( z \in C^\infty \) be arbitrary if \( B(x,2r) \cap \partial \Omega = \emptyset \) and otherwise let \( z = 0 \). Then the estimate

\[ \int_{t-r}^{t+r} \int_{B(x,r) \cap \Omega} |r \nabla u|^2 \, dy \, ds \lesssim \int_{t-2r}^{t+2r} \int_{B(x,2r) \cap \Omega} |u - z|^2 \, dy \, ds \]

holds for an implicit constant depending on A and d.

The second ingredient is the classical Poincaré inequality found in [28, Lem. 7.12/16].

Lemma 7.10. Let 1 ≤ p ≤ q < p* < ∞ and put \( \delta := \frac{1}{p} - \frac{1}{q} \). Let \( \Xi \subseteq \mathbb{R}^l \), \( l \geq 2 \), be bounded, open, and convex, and let S be a Borel subset of \( \Xi \) with |S| > 0. Then

\[ \|u - u_S\|_{L^q(\Xi)} \leq \frac{(1 - \delta)^{1-\delta}}{d(1/d - \delta)^{1-\delta}} \cdot (\text{diam} \, \Omega)^d |B(0,1)|^{1-1/d} |\Omega|^{1/d} |S| \|\nabla u\|_{L^p(\Xi)^d} \]

for all \( u \in W^{1,p}(\Xi) \), where \( u_S := \frac{1}{|S|} \int_S u \, dx \) denotes the mean value of u on S.

We also require a Poincaré inequality on the Sobolev spaces with partially vanishing trace as it can be deduced from [37, Cor. 4.5.3], see also [23, Cor. 2.3.3] for a self-contained proof. Here we write

\[ \mathcal{H}^l_{l-1}(E) := \inf \left\{ \sum_{j=1}^{\infty} r_j^{l-1}; \; x_j \in \mathbb{R}^l, r_j > 0, E \subseteq \bigcup_{j=1}^{\infty} B(x_j, r_j) \right\} \]

for the (l − 1)-dimensional Hausdorff content of a set \( E \subseteq \mathbb{R}^l \).
Lemma 7.11. Let $\Xi \subseteq \mathbb{R}^l$, $l \geq 2$, be a bounded Lipschitz domain, let $1 < p < l$, and $p \leq q < p^*$. There exists a constant $C > 0$ such that for all compact sets $\mathcal{E} \subseteq \Xi$ and every $u \in W^{1,p}_\mathcal{E}(\Xi)$ it holds

$$\|u\|_{L^q(\mathcal{E})} \leq \frac{C}{H_{l-1}(\mathcal{E})^{1/p}} \|
abla u\|_{L^p(\Xi)}.$$ 

As our fourth and final ingredient we rephrase regularity of weak solutions to the second-order system – which was defined somewhat from the perspective of evolution equations by separating the variables $t \in \mathbb{R}$ and $x \in \mathbb{R}^d$ – using a function space on $\mathbb{R}^{1+d}$. This amounts to finding the pre-image of $L^2(\mathbb{R}; \mathcal{V}) \cap W^{1,2}(\mathbb{R}; L^2(\Omega)^m)$ under the identification

$$L^2(\mathbb{R} \times \Omega) \cong L^2(\mathbb{R}^+; L^2(\Omega)) \quad \text{via} \quad u \mapsto u_\otimes, \quad u_\otimes(t) = u(t, \cdot).$$

Lemma 7.12. Let $1 < p < \infty$. The map $u \mapsto u_\otimes$ extends from $C_\infty^\infty(\mathbb{R} \times \Omega)$ by density to an isometric isomorphism

$$W^{1,p}_{\mathcal{E}}(\mathbb{R} \times \Omega) \cong W^{1,p}(\mathbb{R}; L^p(\Omega)) \cap L^p(\mathbb{R}; W^{1,p}_\mathcal{E}(\Omega)).$$

Proof. We omit the dependence of vector-valued spaces on $\mathbb{R}$ and write for example $W^{1,p}(L^p(\Omega))$. By Fubini’s theorem

$$(u \mapsto u_\otimes) : W^{1,p}_{\mathcal{E}}(\mathbb{R} \times \Omega) \rightarrow W^{1,p}(L^p(\Omega)) \cap L^p(W^{1,p}_\mathcal{E}(\Omega))$$

provides an isometry and it suffices to show that each $f \in W^{1,p}(L^p(\Omega)) \cap L^p(W^{1,p}_\mathcal{E}(\Omega))$ with bounded support in $\mathbb{R}$ is contained in the range. To this end, let us recall from Section 2.2 that $W^{1,p}_\mathcal{E}(\Omega)$ admits a Sobolev extension operator by which means we can construct an extension of $f$ in the space $W^{1,p}(L^p(\mathbb{R}^d)) \cap L^p(W^{1,p}_\mathcal{E}(\mathbb{R}^d))$. Fubini’s theorem allows to identify this extension with a function in $W^{1,p}(\mathbb{R}^{1+d})$. Restricting to $\mathbb{R} \times \Omega$, we can therefore represent $f = h_\otimes$, where $h \in W^{1,p}(\mathbb{R} \times \Omega)$ has bounded support in $\mathbb{R} \times \overline{\Omega}$. So, if $\mathcal{E}$ is empty, then we are done. Otherwise we obtain from Hardy’s inequality as stated in Proposition 5.1 \[C_4\] the estimate

$$\iint_{\mathbb{R} \times \Omega} \left| \frac{h(t, x)}{|x|} \right|^p \, dx \, dt = \int_{-\infty}^\infty \int_{\Omega} \left| \frac{f(t, x)}{|x|} \right|^p \, dx \, dt \leq \int_{-\infty}^\infty \int_{\Omega} |\nabla_x f(t, x)|^p \, dx \, dt < \infty.$$ 

We conclude $h \in W^{1,p}_{\mathcal{E}}(\mathbb{R} \times \Omega)$ by the converse of Hardy’s inequality also stated in Proposition 5.1 applied on a suitably large cylinder $(-T, T) \times \Omega$ outside of which $h$ vanishes. \[Q.E.D.\]

Remark 7.13. If $u$ is a weak solution to the second-order system for $A$, then Lemma 7.12 applies to $\eta u$ for all $\eta \in C_c(\mathbb{R}^+; \mathbb{R})$. This shows $u \in W^{1,2}_{(a,b) \times \mathcal{E}}(I \times \Omega)^m$ for all $0 < a < b < \infty$.

Our central result in this section reads as follows.

Theorem 7.14 (Reverse Hölder inequality). Let $u$ be a weak solution to the second-order system for $A$ and let $2_* < p < 2$. Then for all $t > 0$, all $x \in \Omega$, and all $r \in (0, \frac{1}{2})$,

$$\left( \int_{t-r}^{t+r} \int_{B(x, r)} \|\nabla_t u\|^2 \, dy \, ds \right)^{1/2} \lesssim \left( \int_{t-2r}^{t+2r} \int_{B(x, 2r) \cap \Omega} |\nabla_t x u|^p \, dy \, ds \right)^{1/p}$$

with an implicit constant depending on $p$, $A$, and the geometric parameters.

Proof. We claim that it suffices to prove that there exist $c_0 > 0$ and $C > 1$ such that

$$\left( \int_{t-r}^{t+r} \int_{B(x, r)} \|\nabla_t x u\|^2 \, dy \, ds \right)^{1/2} \lesssim \left( \int_{t-Cr}^{t+Cr} \int_{B(x, Cr) \cap \Omega} |\nabla_t x u|^p \, dy \, ds \right)^{1/p}$$

for all $t > 0$, all $x \in \Omega$, and all radii $r$ that are either small in that $r < \min\{c_0 \frac{1}{2C} \}$ or large in that $\text{diam}(\Omega) < r < \frac{1}{2C}$. In fact, by an easy covering argument our estimate for small radii implies
the one claimed in the theorem for all cylinders with \( r \leq 2C \text{diam}(\Omega) \) and our estimate for large radii implies the one in the theorem for all cylinders with \( r > 2C \text{diam}(\Omega) \).

**Step 1: Strategy for small cylinders.** We fix \( t, x \) and define for \( 0 < r < t \),

\[
V_r := (t-r, t+r) \times B(x, r), \quad W_r := (t-r, t+r) \times (B(x, r) \cap \Omega).
\]

For the time being assume that we can extend \( u \) across the boundary to a function in \( W^{1,2}(V_r) \) in such a way that there is control \( \| \nabla_{t,x} u \|_{L^p(V_r)} \lesssim \| \nabla_{t,x} u \|_{L^p(W_{C,r})} \) for some \( C \geq 4 \) independent of \( t \) and \( x \). (Of course we restrict to \( r < t/2C \) implicitly). Firstly, we apply Caccioppoli’s estimate (C) with some admissible \( z \) to obtain

\[
\left( \iint_{W_r} |r \nabla_{t,x} u|^2 \, dy \, ds \right)^{1/2} \lesssim \left( \iint_{V_r} |u - z|^2 \, dy \, ds \right)^{1/2}.
\]

Secondly, we transform to a reference domain \( \Xi = r^{-1}(-x + V_r) \) which neither depends on \( t \) nor on \( x \), apply a suitable Poincaré inequality (P) thereon, and transform back. This is necessary since constants in the Poincaré inequalities may depend on the underlying domain in an uncontrollable way and also may not scale appropriately with respect to \( r \). The result is

\[
\begin{aligned}
&\left( \iint_{V_{4r}} |r \nabla_{t,x} u|^p \, dy \, ds \right)^{1/p} \\
\lesssim & \left( \iint_{W_{C,r}} |r \nabla_{t,x} u|^p \, dy \, ds \right)^{1/p},
\end{aligned}
\]

the second step following by the control on the extension (E).

**Step 2: Details for small cylinders.** We let \( U_1, \ldots, U_N \) be a covering of the compact set \( \overline{\Omega} \setminus \mathcal{D} \) by open sets provided by the Lipschitz condition around \( \overline{\Omega} \setminus \mathcal{D} \) according to Assumption 2.1.

We denote the corresponding bi-Lipschitz mappings by \( \Phi_\pm \) and let \( L \geq 1 \) be the supremum of the Lipschitz constants of \( \Phi_\pm \). Next, we fix \( \kappa > 0 \) such that \( U_\mathcal{D} := \{ x \in \mathbb{R}^d, d(x, \mathcal{D}) < \kappa < d(x, \overline{\Omega} \setminus \mathcal{D}) \} \) lets \( \Omega, U_\mathcal{D}, U_1, \ldots, U_N \) become an open cover of the compact set \( \overline{\Omega} \). By \( \rho > 0 \) we denote a subordinated Lebesgue number, meaning that every ball in \( \mathbb{R}^d \) with radius less than \( \rho \) and center in \( \overline{\Omega} \) is entirely contained in one of the sets used for the covering.

We shall prove \((7.3)\) for \( t > 0, x \in \Omega, \) and \( r < \min\{c, \frac{\rho}{6}\} \), where \( c := \frac{\rho}{6} \) and \( C := 4L^2 \). By the defining property of the Lebesgue number it suffices to get the estimate sketched in Step 1 started in the following cases.

1. Suppose \( B(x, 2r) \subseteq \Omega \). Then \( W_{2r} = V_{2r} \) are subsets of \( \mathbb{R}^d \times \Omega \) and the extension can be omitted. So, we may apply (C) with \( z = \int_{W_{2r}} u \) and use Lemma 7.10 with \( \Xi = r^{-1}(-x + V_{2r}) \) and \( S = r^{-1}(-x + W_{2r}) \) for the Poincaré estimate (P). This yields the required estimate even with \( C = 2 \).

2. Suppose \( B(x, 6r) \subseteq U_j \) for some \( j \) and in addition that \( B(x, 2r) \) does not intersect \( \mathcal{D} \). Utilizing the bi-Lipschitz coordinate charts, we may extend \( u \) to \( V_{2r} \), by even reflection. Since the changes of coordinates increase distances by a factor of at most \( L \), we have control on the extension even with \( C = L^2 \). Now, we can complete the proof as in the first case.

3. Completing the second case, we assume now that already \( B(x, 2r) \) intersect \( \mathcal{D} \). This forces \( z = 0 \) in (C). The Ahlfors-David condition implies the closely related thickness condition

\[
\mathcal{H}^{d-1}_{d-1}(\mathcal{D} \cap B(x, 4r)) \simeq r^{d-1}
\]
Under an affine transformation in test estimates \( \| C \) for this step it will be sufficient to set \( D = \emptyset \). Thus, so far we have proved \( \Theta = \partial \Omega \setminus (-2,2) \times \emptyset \) to be the Dirichlet part of \( \Omega \), then \( \tilde{v} \in W^{1,2}_{\emptyset} \). Moreover, this geometric setup satisfies Assumption \( 2.1 \) in \( \mathbb{R}^{1+d} \). (Here we make essential use of the fact that bottom and top of \( \Omega \) belong to the Dirichlet part). Hence, we have at hand the Sobolev embedding \( W^{1,p}_{\emptyset}(\Omega) \subseteq L^2(\Omega) \), which under the aforementioned linear transformation corresponds to

\[
\left( \int_{t-r}^{t+r} \int_{\Omega} |r \nabla_t u|^2 \, dy \, ds \right)^{1/2} \leq \left( \int_{t-3r}^{t+3r} \int_{\Omega} |v|^p + |r \partial_t v|^p + |v_x v|^p \, dy \, ds \right)^{1/p},
\]

Thus, so far we have proved

\[
(7.4) \quad \left( \int_{t-r}^{t+r} \int_{\Omega} \nabla_{t,x} u^2 \, dy \, ds \right)^{1/2} \leq \left( \int_{t-3r}^{t+3r} \int_{\Omega} |r^{-1} v|^p + |\partial_t v|^p + |v_x v|^p \, dy \, ds \right)^{1/p},
\]

where \( r^{-1} \) can be bounded by the geometrical parameter \( \text{diam}(\Omega)^{-1} \) if convenient. If \( \emptyset \neq \emptyset \), then the pointwise estimates for \( \tilde{v} \) and Poincaré’s inequality from Proposition \( 5.11 \) applied to each function \( u_t \in W^{1,p}_{\emptyset}(\Omega)^m \) allow to bound the right-hand side of \( (7.4) \) further by

\[
\left( \int_{t-3r}^{t+3r} \int_{\Omega} |u|^p + |v_{t,x} u|^p \, dy \, ds \right)^{1/p} \leq \left( \int_{t-3r}^{t+3r} \int_{\Omega} |v_{t,x} u|^p \, dy \, ds \right)^{1/p},
\]

and the proof is complete. Similarly, if \( \emptyset = \emptyset \), then Poincaré’s inequality on the Lipschitz domain \( (t-3r, t+3r) \times \Omega \) (Thm. 4.4.2) allows to bound the right-hand side of \( (7.4) \) by

\[
\left( \int_{t-3r}^{t+3r} \int_{\Omega} |u - z|^p + |v_{t,x} u|^p \, dy \, ds \right)^{1/p} \leq \left( \int_{t-3r}^{t+3r} \int_{\Omega} |v_{t,x} u|^p \, dy \, ds \right)^{1/p}.
\]

Implicity, we have used an affine change of variables to obtain independence of the constants on \( t \) and \( r \).
Let us consider $X := \mathbb{R} \times \Omega$ as a metric space with distance $d((t, x), (s, y)) := \max\{|t-s|, |x-y|\}$. Since $\Omega$ is $d$-Ahlfors regular, the restricted Lebesgue measure is a doubling measure $\mu$ on $X$ in the usual sense [16]. If we let $Y := \mathbb{R}^+ \times \Omega \subseteq X$, then (7.2) simply amounts to the estimate

$$\left(\int_B |\nabla_{t,x} u|^2 \, d\mu\right)^{1/2} \lesssim \left(\int_{2B} |\nabla_{t,x} u|^p \, d\mu\right)^{1/p}$$

for all balls $B$ in $X$ such that $2B \subseteq Y$. In this context, the self-improving character of reverse Hölder estimates (often referred to as Gehring’s lemma) allows to increase the left-hand integrability index to some $q > 2$. For a proof see either [36 Thm. 3.3] or the textbook [16 Thm. 3.22]. The latter reference gives a very transparent proof for $Y = X$ that literally applies in the general case: In fact, in order to achieve the improved estimate involving $q$ on a ball $B$, the argument makes use of the reverse Hölder estimate only on sub-balls of $B$. Thus, we obtain

**Corollary 7.15.** Let $u$ be a weak solution to the second-order system for $A$ and let $2 < p < 2$. Then there exists $2 < q < \infty$ such that for all $t > 0$, all $x \in \Omega$, and all $r \in (0, \frac{t}{2})$,

$$\left(\int_{t-r}^{t+r} \int_{B(x,r) \cap \Omega} |\nabla_{t,x} u|^q \, dy \, ds\right)^{1/q} \lesssim \left(\int_{t-2r}^{t+2r} \int_{B(x,2r) \cap \Omega} |\nabla_{t,x} u|^p \, dy \, ds\right)^{1/p}.$$

The implicit constant as well as $q$ depend only on $p$, $A$, and the geometric parameters.

In view of Proposition [5.5] we can formulate a similar result for weak solutions to the first-order equation. In this context it will be convenient to work with the Whitney regions, to which we can pass from the cylinders by a straightforward covering argument.

**Corollary 7.16.** Let $f$ be a weak solution to the first-order system for $B$. Then there exists $2 < q < \infty$ such that for all $t > 0$ and all $x \in \Omega$,

$$\left(\iint_{W(t,x)} |f|^2 \, dy \, ds\right)^{1/2} \leq \left(\iint_{W(t,x)} |f|^q \, dy \, ds\right)^{1/q} \lesssim \left(\iint_{2W(t,x)} |f|^p \, dy \, ds\right)^{1/p} \leq \left(\iint_{2W(t,x)} |f|^2 \, dy \, ds\right)^{1/2}.$$

Here, $2W(t, x)$ is an enlarged Whitney region obtain from $W(t, x)$ upon replacing $c_0$ and $c_1$ by $2c_0$ and $2c_1$, respectively. The implicit constant as well as $q$ depend only on $p$, $A$, and the geometric parameters. In particular, this estimate applies to $f(t, x) = e^{-t(Dh_0)}h^+(x)$, where $h^+ \in E^+_0 \mathcal{H}$.

For a later use we also record a side result of the proof of Theorem 7.14.

**Corollary 7.17** (Poincaré inequality on Whitney balls). Let $u$ be a weak solution to the second-order system for $A$. Then for all $0 < t < 1$ and all $x \in \Omega$,

$$\iint_{W(t,x)} |u|^2 \, dy \, ds \lesssim \iint_{2W(t,x)} |t \nabla_{t,x} u|^2 \, dy \, ds \left(\iint_{2W(t,x)} |u| \, dy \, ds\right)^2$$

with an implicit constant depending only on $p$, $A$, and the geometric parameters.

**Proof.** Recall the following Poincaré inequality from Step 2 of the Theorem 7.14 If $t > 0$, $x \in \Omega$, and $r < \min\{c, \frac{t}{2}\}$, where $c, C > 0$ are geometrical constants, then

$$\left(\int_{t-2r}^{t+2r} \int_{B(x,2r) \cap \Omega} |u|^2 \, dy \, ds\right)^{1/2} \lesssim \left(\int_{t-Cr}^{t+Cr} \int_{B(x,Cr) \cap \Omega} |r \nabla_{t,x} u|^p \, dy \, ds\right)^{1/p}.$$
Here, $2 < p < 2$ and either $z = 0$ or $z = \int_{-2r}^{t+2r} \int_{B(x,2r) \cap \Omega} |u|$. In any case,

\[
\left( \int_{-2r}^{t+2r} \int_{B(x,2r) \cap \Omega} |u|^2 \, dy \, ds \right)^{1/2} \lesssim \left( \int_{-2r}^{t+2r} \int_{B(x,2r) \cap \Omega} |r \nabla_{t,x} u|^2 \, dy \, ds \right)^{1/2} + \int_{-2r}^{t+2r} \int_{B(x,2r) \cap \Omega} |u| \, dy \, ds
\]

and the required estimate follow from covering the Whitney boxes by suitable cylinders of comparable size. 

\[\Box\]

7.3. Non-tangential estimates and Whitney average convergence. We are in a position to prove that the semigroup solutions constructed in Proposition 7.1 are contained in the solution space $\mathcal{X}$ for the Neumann and regularity problems.

**Theorem 7.18.** If $h^+ \in E_0^+ \mathcal{H}$ and $f_t = e^{-s|DB_0|}h^+$, $t > 0$, then there is comparability

\[
\|\tilde{N}_*(f)\|_{L^2(\Omega)} \simeq \|h^+\|_{L^2(\Omega)}.
\]

**Proof.** The lower bound follows on letting $t \to 0$ in the estimate $\|\tilde{N}_*(f)\|_2 \gtrsim \frac{1}{\tau} \int_{t \tau}^{2\tau} \|f_s\|_2 \, ds$ provided by Lemma 4.2.

For the upper estimate we fix $p < 2$ sufficiently large in order to have at hand both Corollary 7.16 and Corollary 7.3. We let $\zeta = e^{-|z|} \cdot (1 + i z)^{-1}$, so that $\zeta(t DB_0)h^+ = f_t - (1 + it DB_0)^{-1} h^+$. Splitting the non-tangential maximal function at $t = c_0^{-1}$ and employing Corollary 7.16, we obtain the pointwise bound

\[
\tilde{N}_*(f)(x) \leq \sup_{t \geq c_0^{-1}} \left( \int_{W(t,x)} |f|^2 \right)^{1/2} + \sup_{0 < t < c_0^{-1}} \left( \int_{W(t,x)} |\zeta(s DB_0)h^+(y)|^p \, dy \, ds \right)^{1/p} \]

\[
+ \sup_{0 < t < c_0^{-1}} \left( \int_{W(t,x)} |(1 + is DB_0)^{-1} h^+(y)|^p \, dy \, ds \right)^{1/p}.
\]

We shall estimate the three suprema separately in $L^2(\Omega)$ by a multiple of $\|h^+\|_2$.

(i): Since $\Omega$ is $d$-Ahlfors regular, there is a uniform lower bound for the measure of $B(x, c_1 t) \cap \Omega$, where $x \in \Omega$, $t \geq c_0^{-1}$. Using the uniform bound for the $[DB_0]$-semigroup, we obtain that the first supremum is uniformly bounded on $\Omega$ by

\[
\sup_{t \geq c_0^{-1}} \left( \frac{1}{t} \int_{c_0^{-1} t}^{c_0 t} \int_{\Omega} |f_s(y)|^2 \, dy \, ds \right)^{1/2} = \sup_{t \geq c_0^{-1}} \left( \frac{1}{t} \int_{c_0^{-1} t}^{c_0 t} \|e^{-s|DB_0|}h^+\|_2^2 \, ds \right)^{1/2} \lesssim \|h^+\|_2.
\]

Since $\Omega$ is bounded, the required $L^2$-bound follows.

(ii): As for the second supremum, Jensen’s inequality, Lemma 4.2 and quadratic estimates for $DB_0$ bound its $L^2$-norm by

\[
\|\tilde{N}_*(\zeta(t DB_0)h^+\|_2^2 \lesssim \int_0^\infty \|\zeta(t DB_0)h^+\|_2^2 \, \frac{dt}{t} \simeq \|h^+\|_2^2.
\]

Note that here $\tilde{N}_*$ takes averages over enlarged regions $2W(t, x)$, which simply amounts to replacing the generic constants $c_0$ and $c_1$ by $2c_0$ and $2c_1$, respectively.

(iii): For the third term, we perform a rough estimate as in the proof of Lemma 4.2 to find

\[
\left( \int_{W(t,x)} |\zeta(s DB_0)h(y)|^p \, dy \, ds \right) \lesssim \int_{c_0^{-1} t}^{c_0 t} \int_{\Omega} 1_{B(x, 2c_0 c_1 s)}(y) |\zeta(s DB_0)h(y)|^p \, dy \, ds \quad \frac{ds}{s^{1+d}}
\]
uniformly for all $0 < t \leq c_0^{-1}$ and all $x \in \Omega$. Since $s \leq 1$ in the above domain of integration,

$$
(7.5) \quad \sup_{0 < t \leq c_0^{-1}} \iint_{W(t,x)} |\zeta(sDB_0)h^+(y)|^p \, dy \, ds \lesssim \sup_{0 < s \leq 1} \frac{1}{s^d} \int_{\Omega} |\zeta(sDB_0)h^+(y)|^p \, dy.
$$

For the moment we fix $0 < s < 1$ and $x \in \Omega$. In order to control the integral on the right-hand side of $(7.5)$, we put $B_k := B(x, 2^{k+1}c_0c_1s)$, $k \geq 0$, and split $\mathbb{R}^d$ into annuli $C_0 := B_0$ and $C_k := B_k \setminus B_{k-1}$, $k \geq 1$. Corollary 7.8 on $L^p$ off-diagonal estimates yields

$$
\|1_{B_0}\zeta(sDB_0)h^+\|_{L^p(\Omega)^n} \leq \|1_{B_0}\zeta\|_{L^p(\Omega)^n} + \sum_{k \geq 1} \left(1 + (2^k - 2)c_0c_1\right)^{-l} \|1_{B_k}h^+\|_{L^p(\Omega)^n}
$$

for some natural number $l$ to be specified below. Denoting by $M$ the classical Hardy-Littlewood maximal operator on $L^1_{loc}(\mathbb{R}^d)$, we find

$$
\|1_{B_k}h^+\|_{L^p(\Omega)^n} \lesssim 2^{dk/p} s^{d/p} M(|1_{\Omega}h^+|^p)(x)^{1/p} \quad (k \geq 0).
$$

Specializing to a fixed $l > d/p$, we discover

$$
\|1_{B_0}\zeta(sDB_0)h^+\|_{L^p(\Omega)^n} \lesssim s^{d/p} M(|1_{\Omega}h^+|^p)(x)^{1/p}.
$$

This estimate inserted back on the right-hand side of $(7.5)$ leads us to

$$
\sup_{0 < t \leq c_0^{-1}} \iint_{2W(t,x)} |\zeta(sDB_0)h^+(y)|^p \, dy \, ds \lesssim M(|1_{\Omega}h^+|^p)(x) \quad (x \in \Omega),
$$

from which the appropriate bound for the $L^2$-norm follows on integrating the $\frac{2}{p}$-th power with respect to $x \in \Omega$, taking into account that the maximal operator is bounded on $L^{2/p}(\mathbb{R}^d)$. Note that it is only this final step of the proof where we make use of $p < 2$. \hfill \Box

**Remark 7.19.** The orientation of our half-infinite cylindrical domain does not matter and all results remain true for second and first-order systems on the lower half-infinite cylinder $\mathbb{R}^+ \times \Omega$ (with the obvious modifications of definitions). Since each $h \in E_0^- \mathcal{H}$ corresponds to a classical/weak solution $f_t := e^{-tDB_0}h$ of $\partial_t f_t + DB_0f_t = 0$ for $t < 0$, we similarly obtain $\|N_+(e^{-tDB_0}h^-)\|_2 \simeq \|h\|_2$ for $h^+ \in E_0^- \mathcal{H}$. This implies

$$
\|N_+(e^{-tDB_0}h)\|_2 \lesssim \|h\|_2 \quad (h \in \mathcal{H})
$$

since $\mathcal{H}$ is the topological sum of the two Hardy spaces.

Besides $L^2$-convergence of $e^{-tDB_0}h^+$ toward the boundary data $h^+$ as $t \to 0$, we also obtain pointwise almost everywhere convergence of Whitney averages.

**Theorem 7.20.** Let $T = DB_0$ or $T = B_0D$. For every $h \in L^2(\Omega)^n$ there is convergence

$$
\lim_{t \to 0} \iint_{W(t,x)} |e^{-s[T]h(y) - h(x)}|^2 \, dy \, ds = 0
$$

for almost every $x \in \Omega$ and in particular

$$
\lim_{t \to 0} \iint_{W(t,x)} e^{-s[T]h(y)} \, dy \, ds = h(x).
$$

For the proof we need the following auxiliary estimate.

**Lemma 7.21** (Local coercivity estimate). There exists a constant $C > 0$ such that for every $x \in \Omega$, every $r > 0$ such that $B(x, 2r) \subseteq \Omega$, and every $u \in \mathcal{D}(D)$ it holds

$$
\int_{B(x,r)} |Du|^2 \leq C \left( \int_{B(x,2r)} |B_0Du|^2 + \frac{1}{r^2} \int_{B(x,2r)} |u|^2 \right).
$$
Proof. Let $\eta$ be a smooth function with range in $[0, 1]$, identically 1 on $B(x, r)$, support in $B(x, 2r)$, and $|\nabla \eta| \leq \frac{c_d}{r}$ for a constant $c_d$ depending only on $d$. Using the pointwise control of the commutator $[\eta, D]$ provided by Lemma 7.3, we find

$$\int_{B(x,r)} |Du|^2 \leq \int_{\Omega} |\eta Du|^2 \lesssim \int_{\Omega} |D(\eta u)|^2 + \frac{1}{r^2} \int_{B(x,2r)} |u|^2$$

with implicit constants independent of $r$. As $B_0$ is accretive on $\mathcal{R}(D)$ we have $\int_{\Omega} |D(\eta u)|^2 \lesssim \int_{\Omega} |B_0 D(\eta u)|^2$. Now, the claim follows from boundedness of $B_0$ and once again the pointwise commutator estimate. \hfill \square

Proof of Theorem 7.20. Throughout the proof we fix a representative for $h$. For resolvents of $T$ we use the shorthand notation $R_Z^T = (1 + i s T)^{-1}$, $s > 0$. The argument is subdivided into four consecutive steps.

Step 1: Preliminaries for the case $T = B_0 D$. Given $x \in \Omega$, let $t_x := \frac{1}{2} d(x, \partial \Omega)$ and let $\eta_x$ be a smooth $C^1$-valued function with compact support in $\Omega$ that takes the constant value $h(x)$ everywhere on $B(x, t_x)$. Clearly $\eta_x \in \mathcal{D}(D) = \mathcal{D}(B_0 D)$, see Section 5. If $t \leq t_x c_1^{-1}$, then

$$\iint_{W(t,x)} |e^{-s|T|} h(y) - h(x)|^2 \, dy \, ds$$

is bounded from above by

$$2 \iint_{W(t,x)} |(e^{-s|T|} - R_Z^T) h(y)|^2 + |R_Z^T (h - \eta_x)(y)|^2 + |R_Z^T \eta_x(y) - \eta_x(y)|^2 \, dy \, ds.$$  \(\text{(7.6)}\)

We claim that each of these three terms vanishes as $t \to 0$ for almost every $x \in \Omega$. For the first term this follows from Lemma 4.3 and quadratic estimates for $DB_0$ with holomorphic function $\zeta = e^{-|z|} - (1 + i z)^{-1}$. The other two terms require a closer inspection.

Step 2: Second term estimate. Throughout we may assume $t < 1$. Let $B_k = B(x, 2^k c_1 t)$, $k \geq 0$, and split $\mathbb{R}^d$ into annuli $C_0 := B_0$ and $C_k := B_k \setminus B_{k-1}$, $k \geq 1$. By $L^2$ off-diagonal decay for the resolvents of $T$ we can infer an estimate

$$\|1_{B_k} R_Z^T (h - \eta_x)\|_{L^2(\Omega)} \leq \sum_{k \geq 0} 2^{-dk-k} \|1_{C_k} (h - \eta_x)\|_{L^2(\Omega)}$$

for $s$ in the range $[c_1^{-1} t, c_0 t]$ in which it is comparable to $t$. Integration with respect to $s$ leads to

$$\iint_{W(t,x)} |R_Z^T (h - \eta_x)(y)|^2 \, dy \, ds \lesssim \sum_{k \geq 0} 2^{-k} \int_{B_k} |1_{\Omega} h(y) - \eta_x(y)|^2 \, dy.$$  \(\text{(7.7)}\)

where implicitly we have used $d$-Ahlfors regularity of $\Omega$ on the left-hand side. We break the sum at $k_0$ characterized by $2^{-k_0-1} \leq \sqrt{t} < 2^{-k_0}$ and use the Hardy-Littlewood maximal operator $M$ to control the integrals on the large balls with $k \geq k_0$. In this manner the right-hand side of (7.7) is bounded by

$$\sum_{k=0}^{k_0-1} 2^{-k} \int_{B_k} |1_{\Omega} h(y) - \eta_x(y)|^2 \, dy + \sum_{k=k_0}^{\infty} 2^{-k} M(|1_{\Omega} h - \eta_x|^2)(x).$$

Balls occurring in the first sum are of radius less than $c_1 \sqrt{t}$. Hence, if even $c_1 \sqrt{t} < t_x$, then we have $\eta_x(y) = h(x)$ on each ball. For the second sum we utilize $|\eta_x| \leq |h(x)|$ and $\sum_{k=k_0}^{\infty} 2^{-k} \leq 4 \sqrt{t}.$
Altogether, an upper bound up to multiplicative constants for the right-hand side of (7.7) is provided by

\[
\sup_{s \leq t} \int_{B(x,s)} |\mathbf{1}_\Omega h(y) - \mathbf{1}_\Omega h(x)|^2 \, dy + \sqrt{t} M(|\mathbf{1}_\Omega h|^2)(x) + \sqrt{t} |h(x)|^2
\]

if \( t > 0 \) is sufficiently small. In the limit \( t \to 0 \) the following hold: The first term in (7.8) vanishes for every Lebesgue point of \( \mathbf{1}_\Omega h \in L^2(\mathbb{R}^d)^n \). The middle term vanishes provided \( M(|\mathbf{1}_\Omega h|^2)(x) \) is finite, which by the weak-(1,1) estimate for \( M \) applies again for almost every \( x \in \Omega \). Finally, the third term vanishes for every \( x \in \Omega \).

Note carefully that in the end the exceptional sets for \( x \) did not depend on \( t_x \) and \( \eta_x \) although they had been involved in some of the calculations.

**Step 3: Third term estimate.** As \( \eta_x \in C_c^\infty(\Omega)^n \) is constant on the set \( B(x,t_x) \), we can actually compute in the classical sense

\[
T\eta_x(y) = (B_0D\eta_x)(y) = B_0(y) \left[ \frac{\div_x(\eta_x)\nu(y)}{-\nabla(\eta_x)\nu(y)} \right] = 0 \quad (y \in B(x,t_x)).
\]

We may assume \( t \leq \frac{t_x}{2c_1} \) right from the start, so that we have

\[
\frac{1}{s} d (B(x,c_1t) \cap \Omega, \text{supp}(T\eta_x)) \geq \frac{t_x - c_1t}{s} \geq \frac{t_x}{2c_0t} \quad (c_0^{-1}t \leq s \leq c_0t).
\]

On writing \((R_s^T - 1)\eta_x = -isR_s^T T\eta_x \), the \( L^2 \) off-diagonal estimates for \( R_s^T \) yield

\[
\|1_{B(x,c_1t) \cap \Omega}(R_s^T - 1)\eta_x\|_{L^2(\Omega)}^2 \lesssim s^2 t^{2d-2} \|T\eta_x\|_{L^2(\Omega)} \quad (c_0^{-1}t \leq s \leq c_0t)
\]

with implicit constants depending also on \( t_x \). Integration reveals

\[
\iint_{W(t,x)} |R_s^T \eta_x - \eta_x|^2 \, dy \, ds \lesssim t^{-1-d} t^{2d} \|T\eta_x\|_{L^2(\Omega)}^2,
\]

which in the limit \( t \to 0 \) tends to 0 for every \( x \in \Omega \) anyway.

**Step 4: The case \( T = DB_0 \).** Similar to the case \( T = B_0D \) we bound the average integrals over \( W(t,x) \) by

\[
\iint_{W(t,x)} |(e^{-is[D}\eta_x] - R_s^{DB_0})h|^2 + |R_s^{DB_0}h - h|^2 + |h - h(x)|^2 \, dy \, ds.
\]

Here, the integral over the first term vanishes in the limit \( t \to 0 \) for a.e. \( x \in \Omega \) thanks to Lemma 4.3 and quadratic estimates for \( DB_0 \). The integral over the last term vanishes for every Lebesgue point \( x \) of \( \mathbf{1}_\Omega h \in L^2(\mathbb{R}^d)^n \). It remains to consider the middle term in (7.9). Here, we cannot perform a localization argument as we did for \( B_0D \) since now \( D \) is applied after \( B_0 \). However, by the intertwining relation \( R_s^{DB_0} - 1 = -isR_s^{DB_0}DB_0 \) it suffices to prove

\[
\lim_{t \to 0} \iint_{W(t,x)} |isR_s^{DB_0}DB_0 h|^2 \, dy \, ds = 0 \quad (\text{a.e. } x \in \Omega).
\]

To this end, let \( x \in \Omega \). We abbreviate \( \hat{h} := B_0 h \) and associate with it a function \( \hat{\eta}_x \in C_c^\infty(\Omega)^n \) that takes the constant value \( \hat{h}(x) \) on \( B(x,t_x) \). As in Step 4 we have \( D\hat{\eta}_x = 0 \) almost everywhere on \( B(x,t_x) \). If \( t < \frac{t_x}{2c_1} \), then Lemma 7.21 applies on the ball \( B(x,c_1t) \) with \( u = isR_s^{DB_0}DB_0 \hat{h} - is\hat{\eta}_x \) as follows:

\[
\int_{c_0^{-1}t}^{c_0t} \int_{B(x,c_1t)} |isR_s^{DB_0}DB_0 \hat{h}(y)|^2 \, dy \, ds \lesssim \int_{c_0^{-1}t}^{c_0t} \int_{B(x,2c_1t)} |isB_0DR_s^{DB_0}\hat{h}(y)|^2 + |R_s^{DB_0}\hat{h}(y) - \hat{\eta}_x(y)|^2 \, dy \, ds.
\]
Hence,

\[
\iint_{W(t,x)} |sDR^{B_0}_s \hat{\eta}_x(y)|^2 \, dy \, ds \lesssim \iint_{\Omega} |R^{B_0}_s \hat{h} - \hat{\eta}_x(y)|^2 + |R^{B_0}_s \hat{\eta}_x(y) - \hat{\eta}_x(y)|^2 \, dy \, ds + \int_{B(x,2c_1t)} |1_{\Omega} \hat{h}(y) - 1_{\Omega} \hat{h}(x)|^2 \, dy,
\]

where \( \hat{W}(t,x) := 2W(t,x) \). Upon replacing all ‘hatted’ variables by their ‘unhatted’ counterparts, almost everywhere convergence of the first two terms is precisely the statement of Steps 3 and 4, whereas the third term vanishes at every Lebesgue point of \( 1_{\Omega} \hat{h} \). This completes the proof. \( \square \)

**Remark 7.22.** The organization of the proof of Theorem 7.20 is inspired by [13, Sec. 9.1]. However, our setup bears the significant difficulty that \( D \) is not defined on constant functions on \( \Omega \) – at least when the Dirichlet part \( \mathcal{D} \) is non-empty. Surprisingly, the additional localization argument involving \( \eta_x \) provides a slick way out.

### 8. A priori representation of solutions

In this section we turn to systems with \( t \)-dependent coefficients and prove the a priori estimates claimed in our main results. Throughout this section we fix \( t \)-dependent coefficients \( A_0 \) and \( t \)-independent coefficients \( A_0 \) satisfying Assumption 2.3. As before we let \( B \) and \( B_0 \) correspond to \( A \) and \( A_0 \), respectively. We study the first-order system for the \( t \)-dependent coefficients \( B \), which we formally rewrite as

\[
\partial_t f_t + DB_0 f_t = D\mathcal{E}_t f_t, \quad \text{where } \mathcal{E}_t = B_0 - B_t.
\]

For our results we will impose a Carleson condition on \( \mathcal{E} \). We remark that the modified Carleson norms of \( A_0 - A \) and \( \mathcal{E} \) are comparable: Indeed, the identity

\[
\mathcal{E} = B_0 - B = (A_0 - A)\overline{A}_0^{-1} + \overline{A}A_0^{-1} \overline{(A - A_0)}A^{-1}
\]

along with \( \overline{A}_0^{-1}, \overline{A}, \overline{A} \in L^\infty(\mathbb{R}^+ \times \Omega; \mathcal{L}(\mathbb{C}^n)) \) shows that the norms of \( A_0 - A \) dominate those of \( \mathcal{E} \). The reverse estimate follow since the transformation mapping \( A \mapsto B \) is an involution. Similarly, \( \mathcal{X} \) and \( \mathcal{Y} \) norms of \( \nabla_A u \) and \( \nabla_{t,x} u \) are equivalent.

The starting point is a Duhamel-type formula for weak solutions to the first-order system. This uses the operators

\[
\hat{E}_0^\pm := E_0^\pm B_0^{-1}P_{B_0H},
\]

where \( P_{B_0H} \) is the projection onto \( B_0H \) along the splitting \( L^2(\Omega)^n = \mathcal{N}(D) \oplus B_0H \), see Proposition 6.1, and \( B_0^{-1} \) is the inverse of \( B_0 : \mathcal{H} \to B_0H \), which exists by accretivity of \( B_0 \) on \( \mathcal{H} \), see Lemma 5.2.

**Lemma 8.1.** If \( f \) is a weak solution to the first-order system for \( B \), then

\[
- \int_0^t \partial_s \eta^+(s)e^{-(t-s)|DB_0|}E_0^+ f_s \, ds = \int_0^t \eta^+(s)DB_0 e^{-(t-s)|DB_0|} \hat{E}_0^+ \mathcal{E}_s f_s \, ds,
\]

\[
- \int_t^\infty \partial_s \eta^-(s)e^{-(s-t)|DB_0|}E_0^- f_s \, ds = \int_t^\infty \eta^-(s)DB_0 e^{-(s-t)|DB_0|} \hat{E}_0^- \mathcal{E}_s f_s \, ds
\]

for all \( t > 0 \) and all Lipschitz functions \( \eta^+ : \mathbb{R}^+ \to \mathbb{R} \) such that \( \eta^+ \) is compactly supported in \( (0,t) \) and \( \eta^- \) is compactly supported in \( (t,\infty) \).
Proof. By density it suffices to consider smooth functions $\eta^\pm$ sharing the respective support properties. We concentrate on the identity on $(0,t)$ noting that the $(t,\infty)$-integral formula is established in exactly the same way. Throughout we abbreviate $L^2$ inner products by $(\cdot \mid \cdot)$. Since both integrals in the identity in question are absolutely convergent in $\overline{\mathcal{R}(D)} = \mathcal{H}$, it suffices to prove
\begin{equation}
-\int_0^t (\partial_s \eta^+(s)e^{-(t-s)[DB_0]}E_0^+ f_s \mid h) \, ds = \int_0^t (\eta^+(s)DB_0 e^{-(t-s)[DB_0]}\hat{E}_0^+ \mathcal{E}_s f_s \mid h) \, ds
\end{equation}
for all $h \in \mathcal{H}$. Since $f$ is a weak solution,
\begin{equation}
-\int_0^\infty (f_s \mid \partial_s g_s) \, ds + \int_0^\infty (B_0 f_s \mid D g_s) \, ds = \int_0^\infty (\mathcal{E}_s f_s \mid D g_s) \, ds
\end{equation}
holds for all test functions $g \in C^\infty_c(\mathbb{R}^+:\mathcal{D}(D))$. We shall show that for the special choice $g_s := \eta^+(s)(e^{-(t-s)[DB_0]}E_0^+)^* h$ equation (8.2) transforms into (8.1). Here and throughout, adjoints are taken with respect to $\mathcal{H}$ as ambient Hilbert space. This choice is admissible, since by stability of the functional calculus under restrictions and adjoints [29, Sec. 2.6] the map
\begin{equation*}
(0,\infty) \to \mathcal{H}, \quad s \mapsto (e^{-s[DB_0]}E_0^+)^* h = (\chi^+(z)e^{-s[z]})(DB_0)^* h
\end{equation*}
is an orbit of the holomorphic semigroup generated by $-(DB_0|\mathcal{H})^*|E_0^+\mathcal{H}$ on $E_0^+\mathcal{H}$ and as such, it is holomorphic with values in $\mathcal{D}((DB_0|\mathcal{H})^*)$. Recall from Corollary 6.2 that the latter domain is continuously included in $\mathcal{D}(D)$ and that in fact $(DB_0|\mathcal{H})^* = PB_0^* D|\mathcal{H}$ with $P$ the orthogonal projection in $L^2(\Omega)^n$ onto $\mathcal{H}$. So, for this choice of $g$ the left-hand side of (8.2) becomes
\begin{align*}
&-\int_0^t (f_s \mid \partial_s \eta^+(s)(e^{-(t-s)[DB_0]}E_0^+)^* h) \, ds - \int_0^t (f_s \mid \eta^+(s)PB_0^* D(e^{-(t-s)[DB_0]}E_0^+)^* h) \, ds \\
&+ \int_0^t (B_0 f_s \mid \eta^+(s)(e^{-(t-s)[DB_0]}E_0^+)^* h) \, ds.
\end{align*}
Note that if $u \in \mathcal{H}$ and $v \in \mathcal{D}(D)$, then $(B_0 u \mid D v) = (u \mid PB_0^* D v)$. Since $f$ is $\mathcal{H}$-valued, the last two terms above cancel and the result is the left-hand side of (8.1). The right-hand side of (8.2) can be written as
\begin{equation*}
\int_0^t (PB_0^* \mathcal{H} f_s \mid \eta^+(s)D(e^{-(t-s)[DB_0]}E_0^+)^* h) \, ds
\end{equation*}
since $1-PB_0^* \mathcal{H}$ projects onto $\mathcal{N}(D) = \mathcal{H}^\perp$. Similar as above we have for $u \in L^2(\Omega)^n$ and $v \in \mathcal{D}(D)$ that $(PB_0^* \mathcal{H} u \mid D v) = (B_0^{-1}PB_0^* \mathcal{H} u \mid PB_0^* D v)$ so that altogether the right-hand side of (8.2) equals
\begin{equation*}
\int_0^t (B_0^{-1}PB_0^* \mathcal{H} f_s \mid \eta^+(s)(DB_0 e^{-(t-s)[DB_0]}E_0^+)^* h) \, ds,
\end{equation*}
which by definition of $\hat{E}_0^+$ coincides with the right-hand side of (8.1). \qed

Formally taking limits $\eta^+ \to 1_{(0,t)}$ and $\eta^- \to 1_{(t,\infty)}$, in which the derivatives approach certain differences of Dirac distributions, the Duhamel-type formulas in Lemma 8.1 become
\begin{align*}
E_0^+ f_t - e^{-t[DB_0]}E_0^+ f_0 &= \int_0^t DB_0 e^{-(t-s)[DB_0]} \hat{E}_0^+ \mathcal{E}_s f_s \, ds, \\
0 - E_0^- f_t &= \int_0^\infty DB_0 e^{-(s-t)[DB_0]} \hat{E}_0^- \mathcal{E}_s f_s \, ds,
\end{align*}
that is, since \( f \) is \( \mathcal{H} \)-valued, 
\[
f_t = e^{-t[DB_0]}E_0^+ f_0 + S_A f_t \text{ with the singular integral operator}
\]

\[
S_A f_t = \int_0^t DB_0 e^{-(s-t)[DB_0]} \hat{E}_0^+ \mathcal{E}_s f_s \, ds - \int_t^\infty DB_0 e^{-(s-t)[DB_0]} \hat{E}_0^- \mathcal{E}_s f_s \, ds.
\]

A rigorous argument for this limiting process, as well as a rigorous definition of the maximal regularity operator \( S_A \) has been established by Rosén and the first author in \([5]\) under the additional assumption that either \( f \in \mathcal{X} \) or \( f \in \mathcal{Y} \). Note that \([5]\) deals with elliptic systems on the upper half space but taking limits in Lemma 8.1 has been established in an abstract framework, consisting of a Hilbert space \( \mathcal{H} \), spaces \( \mathcal{Y} := L^2(\mathbb{R}^+, t dt; \mathcal{H}) \), \( \mathcal{Y}^* := L^2(\mathbb{R}^+, \frac{dt}{t}; \mathcal{H}) \), and \( \mathcal{X} \) with continuous embeddings

\[
\mathcal{Y}^* \subseteq \mathcal{X} \subseteq L^2_{\text{loc}}(\mathbb{R}^+, dt; \mathcal{H}),
\]

and a semigroup generator \(-[DB_0]\) on \( \mathcal{H} \) such that \( h \mapsto \{e^{-t[DB_0]}h\}_{t>0} \) is bounded from \( \mathcal{H} \) into \( \mathcal{X} \). As for our setup, the required embeddings have been established in Lemma 4.2 and \( \mathcal{H} \to \mathcal{X} \) boundedness of the semigroup is due to Theorem 7.18 and the subsequent remark. This being said, we may freely use the results from \([5]\) and we suggest to keep a copy of this article handy as we shall only outline the necessary changes for our setup.

### 8.1. The Neumann and regularity problems

We begin with the \( \text{a priori} \) estimates for weak solutions with Neumann data \( (\nabla u)_{\pm} |_{t=0} \) or regularity data \( (\nabla u)_{\pm} |_{t=0} = \nabla_x u |_{t=0} \) and interior control \( \nabla_A u \in \mathcal{X} \). In view of Proposition 5.5 and since all these are boundary conditions for the conormal gradient rather than the potential \( u \) itself, it suffices to prove a priori estimates for weak solutions \( f \in \mathcal{X} \) to the first-order system.

Before we can state and prove the main result, we need to rigorously define the maximal regularity operators \( S_A \) on \( \mathcal{X} \) (and simultaneously do so on \( \mathcal{Y} \) for a later use). This uses a family of pointwise approximations to the characteristic functions of \((0, t)\) and \((t, \infty)\) defined by

\[
\eta^\pm(t, s) = \eta^0(\pm \frac{t-s}{\varepsilon}) \eta_\varepsilon(t) \eta_\varepsilon(s),
\]

where \( \eta^0 \) is the piecewise linear function with support \((1, \infty)\) equal to 1 on \((2, \infty)\) and \( \eta_\varepsilon \) is the piecewise linear function with support \((\varepsilon, \frac{1}{2\varepsilon})\) equal to 1 on \((2\varepsilon, \frac{1}{\varepsilon})\).

**Proposition 8.2** (\([3]\) Prop. 7.1). Suppose \( \|\mathcal{E}\|_C < \infty \). For \( \varepsilon > 0 \) the operators

\[
S_A f_t = \int_0^t \eta_\varepsilon^+(t, s) DB_0 e^{-(s-t)[DB_0]} \hat{E}_0^+ \mathcal{E}_s f_s \, ds - \int_t^\infty \eta_\varepsilon^-(t, s) DB_0 e^{-(s-t)[DB_0]} \hat{E}_0^- \mathcal{E}_s f_s \, ds
\]

are bounded \( \|S_A^+ f\|_{\mathcal{X} \to \mathcal{X}} \lesssim \|\mathcal{E}\|_C \) and \( \|S_A^- f\|_{\mathcal{Y} \to \mathcal{Y}} \lesssim \|\mathcal{E}\|_C \) uniformly in \( \varepsilon > 0 \). In \( \mathcal{X} \) there is a limit operator \( S_A \in \mathcal{L}(\mathcal{X}) \) such that

\[
\lim_{\varepsilon \to 0} \|S_A^+ f - S_A f\|_{\mathbb{L}^2(a,b; L^2(\Omega)^n)} = 0 \quad (f \in \mathcal{X}, \ 0 < a < b < \infty).
\]

In \( \mathcal{Y} \) there is a limit operator \( S_A \in \mathcal{L}(\mathcal{Y}) \) such that \( S_A^+ f \to S_A f \) in \( \mathcal{Y} \) for every \( f \in \mathcal{Y} \). The limit operator for both spaces is given by

\[
S_A f_t = \lim_{\varepsilon \to 0} \left( \int_{-\varepsilon}^t DB_0 e^{-(s-t)[DB_0]} \hat{E}_0^+ \mathcal{E}_s f_s \, ds - \int_{t+\varepsilon}^{\varepsilon-1} DB_0 e^{-(s-t)[DB_0]} \hat{E}_0^- \mathcal{E}_s f_s \, ds \right)
\]

with convergence in \( \mathbb{L}^2(a,b; L^2(\Omega)^n) \) for any \( 0 < a < b < \infty \).

**Theorem 8.3.** Assume that \( \|\mathcal{E}\|_C < \infty \) and let \( f \in \mathcal{X} \). Then \( f \) is a weak solution to the first-order system for \( B \) if and only if for some \( h^+ \in E_0^+ \mathcal{H} \), which then is unique, it satisfies

\[
f_t = e^{-t[DB_0]}h^+ + S_A f_t \quad (a.e. \ t > 0).
\]
In this case let $h^- := \int_0^\infty DB_0 e^{-t[DB_0]} \dot{E}_0^- \mathcal{E}_s f_s \, ds \in E_0^- \mathcal{H}$. Then $f$ converges to $f_0 := h^+ + h^-$ in the sense of Whitney averages

$$\lim_{t \to 0} \iint_{W(t,x)} |f(s,y) - f_0(x)|^2 \, dy \, ds = 0 \quad (a.e. \ x \in \Omega)$$

as well as in the square Dini sense

$$\lim_{t \to 0} \int_t^{2t} \|f_s - f_0\|^2_{L^2(\Omega)^n} \, ds = 0.$$

Moreover, $f$ vanishes at spatial infinity in the square Dini sense

$$\lim_{t \to 0} \int_t^{2t} \|f_s\|^2_{L^2(\Omega)^n} \, ds = 0.$$

Finally, there are estimates

$$\|h^+\|_{L^2(\Omega)^n} + \|h^-\|_{L^2(\Omega)^n} \simeq \|f_0\|_{L^2(\Omega)^n} \lesssim \|f\|_X$$

and if $\|\mathcal{E}\|_C$ is sufficiently small, then all three quantities above are comparable to $\|h^+\|_{L^2(\Omega)^n}$.

**Proof.** Necessity of (8.3) and the estimates are proved in parts (i) and (iv) of [5, Thm. 8.2] by taking limits $\varepsilon \to 0$ in the Duhamel-type formulas from Lemma 4.3 for $\eta^\pm := \eta_{\varepsilon}^\pm$. Moreover, part (iii) of [5, Thm. 8.2] tells

$$f - e^{-t[DB_0]} f_0 = E_0^+ f - e^{-t[DB_0]} h^+ + E_0^- f - e^{-t[DB_0]} h^- \in \mathfrak{Y}^*,$$

where in the first step we have used that $f$ is $\mathcal{H}$-valued. Now, square Dini convergence follows from Lemma 1.3 taking into account that $e^{-t[DB_0]} f_0 \to 0$ in $L^2(\Omega)^n$ as $t \to \infty$, see Proposition 7.1 and a.e. convergence of Whitney averages is a consequence of Lemma 1.3 and Theorem 7.2.

Finally, $h^+$ is uniquely determined by $f$, since by strong continuity of the $DB_0$-semigroup we have $f_t - S_A f_t \to h^+$ as $t \to 0$ in $L^2(\Omega)^n$.

Sufficiency requires a new argument since our notion of weak solutions is different than the purely distributional notion in [5]. Assume $f \in \mathcal{X}$ satisfies (8.3). First of all, $f \in L^2_{loc}(\mathbb{R}^+; \mathcal{H})$: Indeed $e^{-t[DB_0]} h^+$ is continuous and $\mathcal{H}$-valued. By Proposition 8.2 we have $S_A f \in \mathcal{X}$, which implies local $L^2$-integrability by Lemma 1.2 and also $S_A f_t \in \mathcal{H}$ for a.e. $t > 0$ since this is true for the approximants $S_A^\varepsilon f_t$.

From Proposition 7.1 we already know that $e^{-t[DB_0]} h^+$ is a weak solution to the first-order system for $B_0$. So, in order to conclude that $f$ is a weak solution to the system for $B$, it remains to prove

$$\int_0^\infty (S_A f_t \mid \partial_t g_t - B_0^* Dg_t)_2 \, dt = - \int_0^\infty (E_t f_t \mid Dg_t)_2 \quad (g \in C_c^\infty(\mathbb{R}^+; D(D))).$$

Fix $g \in C_c^\infty(\mathbb{R}^+; D(D))$. In view of Proposition 8.2 it suffices to replace $S_A$ by $S_A^\varepsilon$ and prove that the left-hand side converges to the right-hand side as $\varepsilon \to 0$. For the $(0,t)$-integral in the definition of $S_A^\varepsilon$ a short calculation, using Fubini’s theorem and integration by parts in the $t$-variable in the first step, reveals that

$$\int_0^\infty \int_0^t \eta^\pm_{\varepsilon}(t,s) (DB_0 e^{-(t-s)[DB_0]} \dot{E}_0^+ \mathcal{E}_s f_s \mid \partial_t g_t - B_0^* Dg_t)_2 \, ds \, dt$$

$$= - \int_0^\infty \int_s^\infty (\partial_t \eta^\pm_{\varepsilon}(t,s) DB_0 e^{-(t-s)[DB_0]} \dot{E}_0^+ \mathcal{E}_s f_s \mid g_t)_2 \, dt \, ds$$

$$= - \int_0^\infty \left( \dot{E}_0^+ \mathcal{E}_s f_s \mid \int_s^\infty \partial_t \eta^\pm_{\varepsilon}(t,s) e^{-(t-s)[B_0^* D]} B_0^* Dg_t \, dt \right)_2 \, ds.$$
If we let $\varepsilon$ so small that $\text{supp} g \subseteq (2\varepsilon, \frac{1}{\varepsilon})$, then this equals
\[
-\int_{0}^{\infty} \left( \tilde{E}_0^{+} \mathcal{E}_s f_0 \right) \eta_\varepsilon(s) \int_{s+\varepsilon}^{s+2\varepsilon} e^{-(t-s)|B_0 \cdot D|} B_0^* Dg_t \; dt \right)_2 \; ds
\]
by definition of $\eta_\varepsilon^\pm$. Since $e^{-(t-s)|B_0 \cdot D|} B_0^* Dg_t$ is uniformly bounded and continuous in $t$ with respect to the $L^2(\Omega)^n$-topology, the $dt$-integrals are uniformly bounded in $s$ and converge locally uniformly to $B_0^* Dg_t$, as $\varepsilon \to 0$. Note that these integrals are non-zero only when $d(s, \text{supp} g) < 2\varepsilon$, so that for $\varepsilon$ small we are in fact integrating in $s$ over a compact subset of $(0, \infty)$. Due to $\tilde{E}_0^+ \mathcal{E} f \in L^2(0, \infty; \mathcal{H})$, dominated convergence applies as $\varepsilon \to 0$ and yields the limit
\[
-\int_{0}^{\infty} (\tilde{E}_0^+ \mathcal{E}_s f_0 \mid B_0^* Dg_t) \; ds.
\]
A similar calculation applies to the $(t, \infty)$-integral in the definition of $S_A^\varepsilon$, so that altogether
\[
\int_{0}^{\infty} (S_A^\varepsilon f_0 \mid \partial_t g_t - B_0^* Dg_t)_2 \; dt \xrightarrow{\varepsilon \to 0} -\int_{0}^{\infty} ((\tilde{E}_0^+ + \tilde{E}_0^-) \mathcal{E}_t f_0 \mid B_0^* Dg_t)_2 \; dt.
\]
Now, $\tilde{E}_0^+ + \tilde{E}_0^- = B_0^{-1} P_{B_0 \mathcal{H}}$, where $P_{B_0 \mathcal{H}}$ annihilates $\mathcal{N}(D) = \mathcal{H}^\perp$, so that
\[
((\tilde{E}_0^+ + \tilde{E}_0^-) \mathcal{E}_t f_0 \mid B_0^* Dg_t)_2 = (P_{B_0 \mathcal{H}} \mathcal{E}_t f_0 \mid Dg_t)_2 = (\mathcal{E}_t f_0 \mid Dg_t)_2.
\]
Hence, our goal (8.4) follows.

We record an immediate corollary for systems with $t$-independent coefficients.

**Corollary 8.4.** Assume that the coefficients $A = A_0$ are $t$-independent and let $f \in \mathcal{X}$. Then $f$ is a weak solution to the first-order system for $B = B_0$ if and only if for some $h^+ \in E_0^+ \mathcal{H}$, which then is unique, it satisfies
\[
f_t = e^{-t|B_0 D|} h^+ \quad (\text{a.e. } t > 0).
\]
In this case, $f$ has additional regularity as specified in Proposition 7.7.

### 8.2. The Dirichlet problem
Things are a little more involved for the Dirichlet problem since here we cannot work with the first-order system only. In particular, similar to the proof of Proposition 5.5, a dichotomy between the cases $\mathcal{D} \neq \emptyset$ and $\mathcal{D} = \emptyset$ occurs when it comes to recovering the potential $u$ from its conormal gradient.

We begin with a representation theorem for weak solutions $f \in \mathcal{Y}$ to the first-order system. It uses the bounded projections
\[
\tilde{E}_0^\pm := 1_{C^\varepsilon}(B_0 D) P_{B_0 \mathcal{H}}^\varepsilon
\]
on $L^2(\Omega)^n$, where as before $P_{B_0 \mathcal{H}}$ is the projection onto $B_0 \mathcal{H}$ along the splitting $L^2(\Omega)^n = \mathcal{N}(D) \oplus B_0 \mathcal{H}$ and the Hardy projection $1_{C^\varepsilon}(B_0 D)$ is defined on $\mathcal{R}(B_0 D) = B_0 \mathcal{H}$ by means of the $H^\infty$-calculus for $B_0 D$, see Section 6 for details.

**Theorem 8.5.** Assume that $\|\mathcal{E}\|_{C} < \infty$ and let $f \in \mathcal{Y}$. Then $f$ is a weak solution to the first-order system for $B$ if and only if for some $h^+ \in \tilde{E}_0^+ L^2(\Omega)^n$, which then is unique, it satisfies
\[
f_t = D e^{-t|B_0 D|} h^+ + S_A f_t \quad (\text{a.e. } t > 0).
\]

**Proof.** Necessity of (8.3) has been proved in [5, Thm. 9.2] by taking limits $\varepsilon \to 0$ in the Duhamel-type formulas from Lemma 8.1 for $\eta^\pm = \eta_\varepsilon^\pm$. As for uniqueness of $h^+$, we assume $D e^{-t|B_0 D|} h^+ = 0$ for a.e. $t > 0$ and check $h^+ = 0$. In fact, due to $e^{-t|B_0 D|} h^+ \in \mathcal{R}(B_0 D)$ we first conclude $e^{-t|B_0 D|} h^+ = 0$ from Proposition 6.1 and then $h^+ = 0$ follows from strong continuity of the $B_0 D$-semigroup.
For sufficiency we note
\[(8.5) \quad e^{-t[B_0 D]}\tilde{E}_0^\pm = e^{-t[B_0 D]}1_{C^\pm}(B_0 D) B_0 B_0^{-1} P_{B_0 H} = B_0 e^{-t[D B_0]}\tilde{E}_0^\pm \quad (t > 0),\]
due to the intertwining property, see (D) in Section 6. In particular, \(De^{-t[B_0 D]}\tilde{E}_0^+, t > 0\), is a weak solution to the first-order system for \(B_0\) due to Remark 7.2. This being said, the exact same reasoning as in the proof of Theorem 8.3 yields the claim. \(\square\)

In order to recover a potential \(u\) from the representation for \(f = \nabla_A u\) provided by the previous theorem, we introduce integral operators \(\overline{S}_A^\varepsilon\) similar to \(S^\varepsilon_A\). For the definition of \(\eta^\pm\) see Proposition 8.2. Below, we denote spaces of bounded continuous functions by \(C_b\).

**Proposition 8.6** ([3] Prop. 7.2). Suppose \(\|E\|_C < \infty\). For \(\varepsilon > 0\) the operators
\[\overline{S}_A f_t = \int_0^t \eta^+ (t,s) e^{-s(t-s)[B_0 D]} \tilde{E}_0^+ \mathcal{E}_s f_s ds - \int_t^\infty \eta^- (t,s) e^{-(s-t)[B_0 D]} \tilde{E}_0^- \mathcal{E}_s f_s ds\]
are bounded \(\overline{Y} \to C_b(0,\infty); L^2(\Omega)^n\) with \(\sup_{\varepsilon > 0} \|\overline{S}_A f_t\|_{L^2(\Omega)^n} \leq \|E\|_C \|f\|_{\overline{Y}}\) uniformly in \(\varepsilon > 0\). There is a limit operator \(\overline{S}_A \in \mathcal{L}(\overline{Y}; C_b(0,\infty); L^2(\Omega)^n))\) such that \(\lim_{\varepsilon \to 0} \|\overline{S}_A f_t - \overline{S}_A f_t\|_{L^2(\Omega)^n} = 0\) locally uniformly in \(t > 0\) for any \(f \in \overline{Y}\). This operator is given by
\[\overline{S}_A f_t = \int_0^t e^{-(s-t)[B_0 D]} \overline{\tilde{E}}_0^+ \mathcal{E}_s f_s ds - \int_t^\infty e^{-(s-t)[B_0 D]} \overline{\tilde{E}}_0^- \mathcal{E}_s f_s ds\]
where the integrals exist as weak integrals in \(L^2(\Omega)^n\), and it has \(L^2(\Omega)^n\)-limits
\[\lim_{t \to 0} \overline{S}_A f_t = -\int_0^\infty e^{-s[B_0 D]} \overline{\tilde{E}}_0^- \mathcal{E}_s f_s ds \in \overline{\tilde{E}}_0^- L^2(\Omega)^n\quad \text{and} \quad \lim_{t \to \infty} \overline{S}_A f_t = 0.\]

**Corollary 8.7.** Suppose \(\|E\|_C < \infty\) and let \(f \in \overline{Y}\). Then,
(i) \(\overline{S}_A f_t \in \mathcal{D}(D)\) and \(D\overline{S}_A f_t = S_A f_t\) for almost every \(t > 0\),
(ii) \(\overline{S}_A f \in W^{1,2}(\mathbb{R}^+; L^2(\Omega)^n)\) with \(\partial_t \overline{S}_A f = -B_0 S_A f + P_{B_0 H} \mathcal{E} f\).

**Proof.**
(i) Due to (8.5) and since the integrals defining \(\overline{S}_A f_t\) and \(S_A f_t\) are absolutely convergent Bochner integrals in \(L^2(\Omega)^n\), we have \(D\overline{S}_A f_t = S_A f_t\) for every \(t > 0\). In the limit \(\varepsilon \to 0\) there is convergence \(\overline{S}_A f_t \to S_A f_t\) in \(L^2(\Omega)^n\) for every \(t > 0\), see Proposition 8.6. Moreover, for a subsequence of \(\varepsilon\) there also is convergence \(S^\varepsilon_A f_t \to S_A f_t\) in \(L^2(\Omega)^n\) for almost every \(t > 0\). This follows from Proposition 8.2 using that convergence in \(Y = L^2(\mathbb{R}^+; \mathcal{E}; L^2(\Omega)^n)\) implies pointwise almost everywhere convergence of a subsequence. As \(D\) is a closed operator in \(L^2(\Omega)^n\), the conclusion follows.
(ii) Let \(g \in C^\infty_c(\mathbb{R}^+; L^2(\Omega)^n)\). A calculation identical to the one in the proof of Theorem 8.3 reveals
\[\int_0^\infty \langle \overline{S}_A f_t \ | \ \partial_t g_t \rangle_2 - (B_0 D \overline{S}_A f_t \ | \ g_t \rangle_2 dt = -\int_0^\infty \langle (\tilde{E}^+_0 + \tilde{E}^-_0) \mathcal{E} f_t \ | \ g_t \rangle_2 dt.\]
The right-hand side coincides with \(-\int_0^\infty (P_{B_0 H} \mathcal{E} f_t \ | \ g_t \rangle_2 dt\), whereas the left-hand side tends to \(\int_0^\infty \langle \tilde{S}_A f_t \ | \ \partial_t g_t \rangle_2 - (B_0 S_A f_t \ | \ g_t \rangle_2 dt\) using 1. Hence, \(\partial_t \tilde{S}_A f = -B_0 S_A f + P_{B_0 H} \mathcal{E} f\) in the distributional sense. This derivative is contained in \(L^2_{loc}(\mathbb{R}^+; L^2(\Omega)^n)\) since \(S_A f \in \overline{Y}\) due Proposition 8.2 and as \(E\) is bounded (see Remark 1.5). \(\square\)

For the Dirichlet problem in case of pure lateral Neumann boundary conditions we also need the subsequently introduced integral operator \(T_A\), which will be responsible for a part of \(u\) that is contained in the space of constant functions on \(\Omega\). Note that we obtain its boundedness only on the subspace of \(\overline{Y}\) containing the weak solutions to the first-order system for \(B\). In fact, boundedness on the whole of \(\overline{Y}\) would require a stronger integrability condition on \(E\).
Definition 8.8. On $L^2(\Omega)^n$ define the orthogonal projections $N^\pm$ and the reflection $N$ by

$$N^- h := \begin{bmatrix} h_1 \\ 0 \end{bmatrix}, \quad N^+ h := \begin{bmatrix} 0 \\ h_2 \end{bmatrix}, \quad Nh := N^+ h - N^- h.$$ 

Proposition 8.9. Assume $\|E\|_C < \infty$. For every weak solution $f \in \mathcal{Y}$ to the first-order system for $B$ it holds

$$\sup_{0 < t \leq \infty} \left\| \int_0^t E_s f_s \, ds \right\|_{L^2(\Omega)^n} \lesssim \|E\|_C \|f\|_\mathcal{Y},$$

where the integrals exist as weak integrals in $L^2(\Omega)^n$. In particular, the weak integrals

$$(T_A f)_t := \int_0^t N^-(1 - P_{B_0})E_s f_s \, ds - \int_t^\infty N^+(1 - P_{B_0})E_s f_s \, ds \quad (t > 0)$$

are defined in $L^2(\Omega)^n$ and satisfy $\sup_{t > 0} \|T_A f_t\|_{L^2(\Omega)^n} \lesssim \|E\|_C \|f\|_\mathcal{Y}$. Moreover, if $\mathcal{Y}$ is non-empty, then $(T_A f)_\perp$ is contained in $W^{1,2}_{\text{loc}}(\mathbb{R}^+; \mathbb{C}^m)$, identified with a subset of $W^{1,2}_{\text{loc}}(\mathbb{R}^+; L^2(\Omega)^m)$, has distributional derivative $\partial_t(T_A f)_\perp = ((1 - P_{B_0})E f)_\perp$, and limits

$$\lim_{t \to 0} (T_A f_t)_\perp = 0 \quad \text{and} \quad \lim_{t \to \infty} (T_A f_t)_\perp = \int_0^\infty ((1 - P_{B_0})E_s f_s)_\perp \, ds$$

in $\mathbb{C}^m$, identified with a subset of $L^2(\Omega)^m$.

Proof. For $h \in L^2(\Omega)^n$ we have

$$\int_0^\infty |(E_s f_s \, |h)_{L^2(\Omega)^n}| \, ds \leq \int_0^\infty \int_\Omega |E(s, y)||f(s, y)||h(y)| \, dy \, ds \simeq \int_0^\infty \int_\Omega \left( \iint_{W(t,x)} |E(s, y)||sf(s, y)||h(y)| \, ds \, dy \right) \frac{dx \, dt}{t},$$

where the “averaging trick” in the second step uses Tonelli’s theorem and $|W(t, x)| \simeq |W(s, y)|$ for $(t, x) \in W(s, y)$ with implicit constants depending on $c_0, c_1, s, \Omega$. The latter follows since for $s$ small we have $|W(s, y)| \simeq s^{1+d}$ (d-Ahlfors regularity of $\Omega$) and for $s$ large we have $|W(s, y)| \simeq s$ (boundedness of $\Omega$). We fix $p \in (2, 2)$, let $\frac{1}{p} + \frac{1}{q} = 1$, and introduce

$$F(t, x) := \left( \iint_{2W(t,x)} |sf(s, y)|^2 \, dy \, ds \right)^{1/2}, \quad G(t, x) := \left( \int_{B(x, c_1)/\Omega} |h(y)|^q \right)^{1/q}.$$

By Hölder’s inequality followed by an application of the reverse Hölder estimate for $f$ (Corollary 7.16),

$$\int_0^\infty |(E_s f_s \, |h)_{L^2(\Omega)^n}| \, ds \lesssim \int_0^\infty \int_\Omega \left( \sup_{W(t,x)} |E| \right) F(t, x) G(t, x) \frac{dx \, dt}{t}.$$

This is the only part where we use that $f$ is a weak solution to the first-order system and not just an element of $\mathcal{Y}$. Now, the tent space estimate of Coifman-Meyer-Stein [20] applies (see [3] Prop. 3.15) for a proof on doubling spaces; notation is explained further below:

$$\lesssim \|E\|_C \int_\Omega A(FG)(z) \, dz \lesssim \|E\|_C \int_\Omega A(F)(z) M_{1/h}(1/q)(z) \, dz \lesssim \|E\|_C \|A(F)\|_{L^2(\Omega)} \|M_{1/h}(1/q)(z)\|_{L^{2/q}(\Omega)},$$
where \( \mathcal{A} \) denotes the area function
\[
\mathcal{A}(g)(z) := \left( \iint_{|x-z|<t} |g(t, x)|^2 \frac{dx}{|B(z, t)|} \frac{dt}{t} \right)^{1/2}
\]
and \( M_\Omega \) is the centered Hardy-Littlewood maximal operator in \( \Omega \) defined by
\[
M_\Omega(g)(z) := \sup_{t>0} \int_{B(z,t)\cap \Omega} |g(z)| \, dz.
\]
The same averaging trick as above reveals
\[
\int_\Omega |A(F)(z)|^2 \, dz \simeq \int_0^\infty \int_\Omega |F(t, x)|^2 \, dt \, dx \simeq \int_0^\infty \int_\Omega |sf(s, y)|^2 \, dy \, ds \simeq \|f\|_Y^2.
\]
Moreover, \( M_\Omega \) is bounded on \( L^{q/2}(\Omega) \) since \( q > 2 \) and \( \Omega \) is doubling \cite[Thm. 3.13]{16}. Altogether,
\[
\int_0^\infty |(\mathcal{E} sf_s \mid h)_{L^2(\Omega)^n}| \, ds \lesssim \|\mathcal{E}\|_{C(Y)} \|f\|_Y \|h\|_{L^2(\Omega)^n},
\]
which proves the integral estimate. Boundedness of the operator \( T_A \) is immediate from that. In order to conclude the further properties of \( (T_A)_\perp \), we note that by the first part
\[
(T_A f_\perp) := \int_0^t ((1 - P_{B_0}) \mathcal{E}_s f_s)_\perp \, ds
\]
is defined for \( 0 \leq t \leq \infty \) as a weak integral in \( L^2(\Omega)^m \), but in fact the integrand is valued in the finite-dimensional subspace \( \mathcal{R}(1 - P_{B_0})_\perp = \mathcal{N}(D)_\perp \) containing only the constant functions. Hence, these integrals exist as proper \( C^m \)-valued Bochner integrals and the limits as \( t \to 0 \) and \( t \to \infty \) as well as differentiability in \( t \) follow easily. \( \square \)

Now, we are in a position to prove the \textit{a priori} estimates for the Dirichlet problem claimed in our main result Theorem 3.2. Note carefully that in contrast to Theorem 8.3 the result is formulated at the level of the second-order system and that sufficiency of the representation formulas requires a smallness condition on \( \|\mathcal{E}\|_C \).

**Theorem 8.10.**

(i) Assume \( \|\mathcal{E}\|_C < \infty \) and that the lateral Dirichlet part \( \partial \) is non-empty. If \( u \) is a weak solution to the second-order system with interior estimate \( \nabla_{t,x} u \in \mathcal{Y} \), then there exists \( \tilde{h}^+ \in \tilde{E}_0^+ L^2(\Omega)^n \) such that
\[
u_t = - \left( e^{-t|B_0|} \tilde{h}^+ + \tilde{S}_A(\nabla_A u)_t \right)_\perp \quad (a.e. \ t > 0).
\]
In this case \( u \in C([0,\infty); L^2(\Omega)^m) \). Moreover, let \( \tilde{h}^- := - \int_0^\infty e^{-s|B_0|} \tilde{E}_0^- \mathcal{E} s \nabla_A u_s \, ds \in \tilde{E}_0^- L^2(\Omega)^n \) and \( v_0 := \tilde{h}^+ + \tilde{h}^- \). Then there are \( L^2(\Omega)^m \)-limits
\[
\lim_{t \to 0} u_t = (v_0)_\perp \quad \text{and} \quad \lim_{t \to \infty} u_t = 0
\]
and estimates
\[
\|(v_0)_\perp\|_{L^2(\Omega)^m} \leq \sup_{t>0} \|u_t\|_{L^2(\Omega)^m} \lesssim \|\nabla_{t,x} u\|_\mathcal{Y} < \infty.
\]
If furthermore \( \|\mathcal{E}\|_C \) is sufficiently small, then \( u \) is a weak solution to the second-order system with interior estimate \( \nabla_{t,x} u \in \mathcal{Y} \) if and only if
\[
u = - \left( e^{-t|B_0|} \tilde{h}^+ + \tilde{S}_A f \right)_\perp \quad \text{with} \quad f = (1 - S_A)^{-1} D e^{-t|B_0|} \tilde{h}^+
\]
for some \( \tilde{h}^+ \in \tilde{E}_0^+ L^2(\Omega)^n \) satisfying \( \|\tilde{h}^+\|_{L^2(\Omega)^n} \simeq \|\nabla_{t,x} u\|_\mathcal{Y} \). In this case, \( f = \nabla_A u \).
(ii) Assume \( \|E\|_C < \infty \) and that the lateral Dirichlet \( \mathcal{D} \) part is empty. Then a function \( u \) is a weak solution to the second-order system with interior estimate \( \nabla_{t,x} u \in \mathcal{Y} \) if and only if for some \( \tilde{h}^+ \in E_0^+L^2(\Omega)^n \) and some \( c \in \mathbb{C}^m \) it satisfies
\[
 u_t = -\left(e^{-t[B_0D]}\tilde{h}^+ + \tilde{S}_A(\nabla_A u)_{t} + T_A(\nabla_A u)_t\right) + c \quad (a.e. \ t > 0).
\]
In this case \( u \in C([0, \infty); L^2(\Omega)^m) \). Let \( \tilde{h}^- \) and \( \nu_0 \) be as before. Then there are \( L^2(\Omega)^m \)-limits
\[
\lim_{t \to 0} u_t = c - (v_0)_{\perp} \quad \text{and} \quad \lim_{t \to \infty} u_t = c - \int_0^\infty \left(1 - P_{B_0\mathcal{H}}\right)E_\mathcal{Y}(\nabla_A u)_{s} \, ds =: u_\infty \in \mathbb{C}^m
\]
and estimates
\[
\|c - (v_0)_{\perp}\|_{L^2(\Omega)^m} \leq \sup_{t > 0} \|u_t\|_{L^2(\Omega)^m} \lesssim |c| + \|\nabla_{t,x} u\|_Y \approx \|u_\infty\| + \|\nabla_{t,x} u\|_Y < \infty.
\]
If furthermore \( \|E\|_C \) is sufficiently small, then \( u \) is a weak solution to the second-order system with interior estimate \( \nabla_{t,x} u \in \mathcal{Y} \) if and only if
\[
u = -\left(e^{-t[B_0D]}\tilde{h}^+ + \tilde{S}_A + T_A\right)f_{\perp} + c \quad \text{with} \quad f = (1 - S_A)^{-1}Dc^{-t[B_0D]}\tilde{h}^+
\]
for some \( c \in \mathbb{C}^m \) and some \( \tilde{h}^+ \in E_0^+L^2(\Omega)^n \) satisfying \( \|\tilde{h}^+\|_{L^2(\Omega)^n} \approx \|\nabla_{t,x} u\|_Y \). In this case, \( f = \nabla_A u \).

**Proof.** We begin with the claim for non-empty lateral Dirichlet part. Putting \( f := \nabla_A u \), Theorem 8.5 and Proposition 5.5 yield an \( \tilde{h}^+ \in E_0^+L^2(\Omega)^n \) such that
\[
f_t = Dc^{-t[B_0D]}\tilde{h}^+ + S_A f_{\perp} \quad (a.e. \ t > 0).
\]
We define the potential \( v := e^{-t[B_0D]}\tilde{h}^+ + \tilde{S}_A f \), so that \( Dv = f \) by Corollary 8.7. Invoking Corollary 8.8, we find
\[
\partial_t v_{\perp} = \left(-B_0Dc^{-t[B_0D]}\tilde{h}^+ - B_0S_A f + P_{B_0\mathcal{H}}E_\mathcal{Y} f\right)_{\perp} = \left(-Bf - (1 - P_{B_0\mathcal{H}})E_\mathcal{Y} f\right)_{\perp}.
\]
Since \( 1 - P_{B_0\mathcal{H}} \) projects onto \( \mathcal{N}(D) \) and as \( \mathcal{N}(D)_{\perp} = \{0\} \),
\[
\nabla_{t,x} v_{\perp} = \left[\partial_t v_{\perp} \over \nabla_x v_{\perp}\right] = -\left[Bf\right]_{\perp} = -\mathcal{A}^{-1}f = -\mathcal{A}^{-1}\nabla_A u = \nabla_{t,x} u.
\]
Hence, \( v_{\perp} + u \) is constant on the domain \( \mathbb{R}^+ \times \Omega \) and in fact \( u = -v_{\perp} \) again by Poincaré’s inequality on \( \mathcal{Y} \). Now, continuity and limits for \( u \) follow from the respective properties for \( \tilde{S}_A f \) as provided by Proposition 8.6 and the analog of Proposition 7.1 for the \( [B_0D] \)-semigroup. As for the estimates, Proposition 8.6 implies
\[
\|v_{\perp}\|_2 \leq \sup_{t > 0} \|u_t\|_2 \leq \sup_{t > 0} \|v_{\perp}\|_2 \lesssim \|\tilde{h}^+\|_2 + \|f\|_Y
\]
and, taking into account quadratic estimates for \( B_0D \) (Theorem 6.3), accretivity of \( B_0 \), and Proposition 8.2
\[
\|\tilde{h}^+\|_2 \lesssim \|B_0Dc^{-t[B_0D]}\tilde{h}^+\|_Y \lesssim \|(1 - S_A)f\|_Y \lesssim \|f\|_Y.
\]
If we additionally assume that \( \|E\|_C \) is sufficiently small, then Proposition 8.2 shows that \( 1 - S_A \) is invertible on \( \mathcal{Y} \) and given \( \tilde{h}^+ \in E_0^+L^2(\Omega)^n \) we can solve for \( f \) in (8.6) by \( f = (1 - S_A)^{-1}Dc^{-t[B_0D]}\tilde{h}^+ \). Setting \( v = e^{-t[B_0D]}\tilde{h}^+ + \tilde{S}_A f \), we have already proved that \( u := -v_{\perp} \) is a weak solution to the second-order system with \( \nabla_A u = f \) and that conversely every such weak solution is of that type. Finally, \( \|(1 - S_A)f\|_Y \lesssim \|f\|_Y \approx \|\nabla_{t,x} u\|_Y \) by invertibility of \( S_A \) and therefore \( \|\tilde{h}^+\|_2 \approx \|f\|_Y \) due to (8.8).
Now, we consider the case of empty lateral Dirichlet part. Using the same notation as before, the only critical difference in the argument is that due to $N(D) \perp = \mathbb{C}^m$ we cannot conclude $(\partial_t v) \perp = -(Bf) \perp$ from \[8.7\]. However, we have at hand Proposition \[8.9\] and our substitute for \[8.7\] becomes

$$\partial_t(u + T_A f) \perp = \left( -Bf - (1 - P_{B_0 \mathcal{H}}) \mathcal{E}f \right) \perp + \left( (1 - P_{B_0 \mathcal{H}}) \mathcal{E}f \right) \perp = -(Bf) \perp.$$  

The rest of the argument is identical to the case of non-empty lateral Dirichlet part, except that now $u - (v + T_A f) \perp$ may be a non-zero constant $c \in \mathbb{C}^m$ and $|c| + \|f\|_Y$ is comparable to $|u_\infty| + \|f\|_Y$ due to Proposition \[8.9\].

Again results in the case of $t$-independent coefficients are particularly simple.

**Corollary 8.11.** Suppose that the coefficients $A = A_0$ are $t$-independent. Then $u$ is a weak solution to the second-order system with Lusin area bound $\nabla_{t,x} u \in \mathcal{Y}$ if and only if there exist $\tilde{h}^+ \in \tilde{E}_0^+ L^2(\Omega)^n$ and a constant $c \in \mathbb{C}^m$, which is zero if the lateral Dirichlet part $\mathcal{D}$ is non-empty, such that

$$u_t = -(e^{-t[B_0 D]} \tilde{h}^+) \perp + c \quad (a.e. \ t > 0).$$

In this case, $u$ has additional regularity $u \in C([0, \infty); L^2(\Omega)^m) \cap C^\infty((0, \infty); \mathcal{Y})$.

**Proof.** Necessity and sufficiency of the semigroup representation for $u$ is due to Theorem \[8.10\]. The additional regularity follows since $t \mapsto e^{-t[B_0 D]} \tilde{h}^+$ is holomorphic with values in $\mathcal{D}(B_0 D) = \mathcal{D}(D)$. □

Our final goal in this section is to prove that weak solutions to the Dirichlet problem obtained in Theorem \[8.10\] also have an $L^2$-controlled maximal function and, in particular, that they converge to their trace at $t = 0$ in the sense of Whitney averages. As a technical tool we need the following $L^p$-non-tangential maximal functions.

**Definition 8.12.** For $1 \leq p \leq 2$ define

$$\tilde{N}_t^p(f)(x) := \left( \sup_{0 < \tau < t} \int_{\mathcal{W}(\tau, x)} |f(s, y)|^p \, dy \, ds \right)^{1/p} \quad (f \in L^p_{\text{loc}}(\mathbb{R}^+ \times \Omega)).$$

By Jensen’s inequality $\tilde{N}_t^p \leq \tilde{N}_t^2$ pointwisely almost everywhere on $\Omega$. Moreover, $\tilde{N}_t^2 \leq \tilde{N}_t$ since $\tilde{N}_t$ takes the supremum over all Whitney balls.

The following lemma is proved in \[5\] Lem. 10.2(iii) using the tent space estimate of Coifman, Meyer, and Stein (see again \[3\] for a proof on doubling spaces) and $L^q$ off-diagonal estimates for $DB_0^q$ with $|q - 2|$ sufficiently small. The only necessary change in the argument in \[5\] is that we have to take the supremum in the definition of $\tilde{N}_t^p$ over $0 < t < t_0$ with $t_0 = 1$ instead of $t_0 = \infty$. The reason for this is that off-diagonal estimates are required for all $t < c_0 t_0$ and Corollary \[7.8\] does only provide them on bounded ranges for $t$.

**Lemma 8.13.** Assume $\|\mathcal{E}\|_C < \infty$ and $1 \leq p < 2$. Let $f \in \mathcal{Y}$ be such that $\int_0^{t_0} \mathcal{E}_s f_s \, ds$ is defined as weak integral. Then

$$\left\| \tilde{N}_t^p \left( (1 + itB_0 D)^{-1} \int_0^t \mathcal{E}_s f_s \, ds \right) \right\|_{L^2(\Omega)} \lesssim \|\mathcal{E}\|_C \|f\|_Y$$

**Remark 8.14.** In view of Proposition \[8.9\] this estimate in particular applies to $1_{(0,t_0)} f$, where $f \in \mathcal{Y}$ is a weak solution to the first-order system for $B$ and $t_0 > 0$. 

Theorem 8.15. Assume \( \|E\|_C < \infty \). Let \( u \) be a weak solution with interior estimate \( \nabla_{t,z} u \in \mathcal{Y} \) and let \( u_0 := \lim_{t \to 0} u_t \) and \( u_\infty := \lim_{t \to \infty} u_t \) as in Theorem 8.10. Then

\[
\|u_0\|_{L^2(\Omega)^m} \lesssim \|\tilde{N}_s(u)\|_{L^2(\Omega)} \lesssim \|u_\infty\| + \|\nabla_{t,z} u\|_{\mathcal{Y}}
\]

and \( u \) converges to \( u_0 \) also in the sense of Whitney averages

\[
\lim_{t \to 0} \iint_{W(t,x)} u(s,y) \, ds \, dy = u_0(x) \quad (a.e. \ x \in \Omega).
\]

Proof. The proof of the \( L^2 \)-bounds for \( \tilde{N}_s(u) \) is similar to the one of [5, Thm. 10.1]. In order to carve out the subtle modifications that are necessary in our geometric framework, we decided to reproduce the argument in some detail, though. Also, this sets the stage for the convergence of Whitney averages, which was left as an open question in [5] and was addressed further in [13].

We adopt notation from Theorem 8.10 and put \( f := \nabla_A u \in \mathcal{Y} \). The argument is subdivided into five consecutive steps.

Step 1: Lower bound for \( \tilde{N}_s(u) \). As \( \lim_{t \to 0} u_t = u_0 \) in \( L^2(\Omega)^m \), Lemma 4.2 gives \( \|\tilde{N}_s(u)\|_2^2 \gtrsim \lim_{t \to 0} \int_{0}^{2t} \|u_s\|_2^2 \, ds = \|u_0\|_2^2 \).

Step 2: Non-tangential estimate of \( e^{-t|B_0D|}\tilde{h}^- \). By the intertwining property [1] in Section 6 we have for every \( \tilde{h} \in B_0H \) that \( e^{-t|B_0D|}\tilde{h} = B_0e^{-t|DB_0|}B_0^{-1}\tilde{h} \) and hence \( \|e^{-t|B_0D|}\tilde{h}\|_\mathcal{X} \lesssim \|\tilde{h}\|_2 \) due to Corollary 7.16 and accretivity of \( B_0 \). In particular, the non-tangential maximal function of \( e^{-t|B_0D|}\tilde{h}^- \) is bounded in \( L^2(\Omega) \)-norm by

\[
\|\tilde{h}^-\|_2 = \sup_{\|g\|_2 = 1} \left| \int_{0}^{\infty} (e^{-s|B_0D|}\tilde{E}_0^-E_s f_s \cdot g)_2 \, ds \right| \lesssim \sup_{\|g\|_2 = 1} \|f\|_\mathcal{Y} \cdot \|E^* e^{-s|DB_0^*|} (\tilde{E}_0^-)_* g\|_\mathcal{Y}.
\]

On noting \( \mathcal{R}((\tilde{E}_0^-)_*) = \mathcal{N}(\tilde{E}_0^-) \perp \mathcal{N}(D) \perp \mathcal{H} \) and \( \|E^*\|_C = \|E\|_C \), Theorem 4.4 and Theorem 7.18 for \( B_0^* \) in place of \( B_0 \) yield

\[
\|\tilde{h}^-\|_2 \lesssim \sup_{\|g\|_2 = 1} \|E\|_C \cdot \|\tilde{E}_0^-E_s f_s \|_\mathcal{X} \cdot \|f\|_\mathcal{Y} \lesssim \|E\|_C \|f\|_\mathcal{Y}.
\]

Step 2: Splitting off \( \mathcal{Y}^s \)-terms from the solution formula for \( u \). In this step we split off \( \mathcal{Y}^s \)-terms from the solution formula for \( u \) that are completely harmless when it comes to non-tangential estimates and Whitney average convergence. We consider the case of non-empty lateral Dirichlet part \( \mathcal{D} \) first. Then \( u \) satisfies the representation formula in item 1 of Theorem 8.10. Starting out with

\[
\tilde{S}_{At} - e^{-t|B_0D|}\tilde{h}^- = \int_{0}^{t} e^{-(t-s)|B_0D|}\tilde{E}_0^+E_s f_s \, ds - \int_{t}^{\infty} e^{-(s-t)|B_0D|}\tilde{E}_0^-E_s f_s \, ds + e^{-t|B_0D|} \int_{0}^{\infty} e^{-s|B_0D|}\tilde{E}_0^-E_s f_s \, ds,
\]

we add correction terms in order to obtain regular decaying kernels for the first two terms

\[
= \int_{0}^{t} e^{-(t-s)|B_0D|}(1 - e^{-2s|B_0D|})\tilde{E}_0^+E_s f_s \, ds - \int_{t}^{\infty} e^{-(s-t)|B_0D|}(1 - e^{-2s|B_0D|})\tilde{E}_0^-E_s f_s \, ds
\]

\[
+ e^{-t|B_0D|} \int_{0}^{t} e^{-s|B_0D|}P_{B_0\mathcal{H}}E_s f_s \, ds,
\]
and introduce the holomorphic function $\psi(z) = e^{-|z|} - (1 + iz)^{-1}$ to eventually discover

$$
\begin{align*}
&= \int_0^t e^{-(t-s)[B_0D]}(1 - e^{-2s[B_0D]})E_0 f_s ds - \int_0^\infty e^{-(s-t)[B_0D]}(1 - e^{-2t[B_0D]})E_0 f_s ds \\
&\quad + \psi(tB_0D) \int_0^\infty e^{-s[B_0D]}P_{B_0H}E_s f_s ds - \int_1^\infty \psi(tB_0D)e^{-s[B_0D]}P_{B_0H}E_s f_s ds \\
&\quad + \int_0^t (1 + itB_0D)^{-1}(e^{-s[B_0D]} - 1)P_{B_0H}E_s f_s ds + P_{B_0H}(1 + itB_0D)^{-1} \int_0^t E_s f_s ds \\
&=: I_1 - I_2 + I_3 - I_4 + I_5 + I_6.
\end{align*}
$$

Recall that the integral occurring in $I_6$ is well-defined as a weak integral by Proposition 8.9. Using the bounded $H^\infty$-calculus of $B_0D$, the kernel $e^{-(t-s)[B_0D]}(1 - e^{-2s[B_0D]})E_0^+$ of $I_1$ is controlled in operator norm by $\frac{s}{t}$ (see also [5]). The same is true for the kernel of $I_5$ and similarly the kernels of $I_2$ and $I_4$ are controlled by $\frac{t}{s}$. This implies for instance

$$
\int_0^\infty \|I_1\|_2 \frac{dt}{t} \leq \int_0^\infty \left( \int_0^t \frac{s}{t} ds \right) \left( \int_0^t \|E_s f_s\|_2^2 s ds \right) \frac{dt}{t} \leq \|E\|_2^2 \|f\|_Y^2 \lesssim \|E\|_C \|f\|_Y^2,
$$

see Remark 4.5 for the last estimate, and similarly we control $I_2$, $I_4$, and $I_5$. In particular, these integrals are absolutely convergent in $s$ and contained in $Y^*$ as functions of $t$. For $I_3$ quadratic estimates for $B_0D$ and a duality argument similar to Step 1 give

$$
\|I_3\|_{Y^*} \lesssim \left\| \int_0^\infty e^{-s[B_0D]}P_{B_0H}E_s f_s ds \right\|_2 \lesssim \|E\|_C \|f\|_Y.
$$

This uses $R(P_{B_0H} h) = H$ and we also get that $I_3$ exists as a weak integral.

Recalling the representation formula for $u$ from Theorem 8.10 and $(P_{B_0H} h)_{\perp} = h_{\perp}$ for all $h \in L^2(\Omega)^n$ due to $\mathcal{N}(D)_{\perp} = \mathcal{N}(\nabla \gamma) = \{0\}$, the upshot of all this is

$$
\begin{align*}
(8.9) \quad \left\| u + \left( e^{-t[B_0D]} \tilde{h}_{\perp} + \tilde{h}_{\perp} \right) + (1 + itB_0D)^{-1} \int_0^t E_s f_s ds \right\|_{Y^*} \lesssim \|E\|_C \|f\|_Y.
\end{align*}
$$

A similar result holds in the case of empty lateral Dirichlet part $\mathcal{D}$. In fact, employing $e^{-s[B_0D]} = \text{Id}$ on $R(1 - P_{B_0H}) = \mathcal{N}(B_0D)$, we can write

$$
\begin{align*}
\bar{S}_A f_t + T_A f_t - e^{-t[B_0D]}\tilde{h}_{\perp} - e^{-t[B_0D]} \int_0^\infty N^+(1 - P_{B_0H})E_s f_s ds \\
= \int_0^t e^{-(t-s)[B_0D]}(\tilde{E}_0^+ + N^-(1 - P_{B_0H}))E_s f_s ds - \int_t^\infty e^{-(s-t)[B_0D]}(\tilde{E}_0^- + N^+(1 - P_{B_0H}))E_s f_s ds \\
+ e^{-t[B_0D]} \int_0^\infty e^{-s[B_0D]}(\tilde{E}_0^- + N^+(1 - P_{B_0H}))E_s f_s ds,
\end{align*}
$$

where the integrals on the right-hand side can be split as before. The only difference is that due to $(\tilde{E}_0^+ + N^-(1 - P_{B_0H})) + (\tilde{E}_0^- + N^+(1 - P_{B_0H})) = \text{Id}$ on $L^2(\Omega)^n$ the projection $P_{B_0H}$ does not occur in $I_5$ and $I_6$. Note that $I_3$ and $I_4$ stay the same since $\psi(tB_0D) = 0$ on $\mathcal{N}(B_0D)$. Altogether,

$$
\begin{align*}
(8.10) \quad \left\| u - c + \left( e^{-t[B_0D]} \tilde{h}_{\perp} + \tilde{h}_{\perp} - \int_0^\infty N^+(1 - P_{B_0H})E_s f_s ds \right) + (1 + itB_0D)^{-1} \int_0^t E_s f_s ds \right\|_{Y^*} \\
\lesssim \|E\|_C \|f\|_Y.
\end{align*}
$$
Step 3: The non-tangential estimate for $u$. We only have to consider Whitney balls of size $t < 1$ and prove the estimate $\|\tilde{N}_2^1(u)\|_2 \lesssim \|f\|_Y$. In fact, for Whitney balls of size $t \geq 1$ the estimate $\sup_{s > 0} \|u_s\|_{L^2(\Omega)} \lesssim \|f\|_Y + |u_\infty|$ provided by Theorem 8.10 directly gives

$$\iint_{W(t,x)} |u|^2 \lesssim \int_{c_0 t}^{c_0 t} \|u_s\|_{L^2(\Omega)}^2 \, ds \lesssim \|f\|_Y^2 + |u_\infty|^2,$$

uniformly in $t > 1$ and $x \in \Omega$. For this we have implicitly used $d$-Ahlfors regularity of $\Omega$ and we have $u_\infty = 0$ if the lateral Dirichlet part is non-empty. Now, for $x \in \Omega$ and $t < 1$ we recall from Corollary 7.17 that

$$\iint_{W(t,x)} |u|^2 \lesssim \left( \iint_{2W(t,x)} |t \nabla_{t,z} u|^2 \right) + \left( \iint_{2W(t,x)} |u|^2 \right).$$

Taking the supremum over $t$ and integrating with respect to $x$ leads us to

$$(8.11) \quad \|\tilde{N}_2^1(u)\|_2 \lesssim \|t \nabla_{t,z} u\|_X + \|\tilde{N}_1^1(u)\|_2 \lesssim \|f\|_Y + \|\tilde{N}_1^1(u)\|_2,$$

where for the second step we have utilized the embedding $\mathcal{Y}^* \subseteq \mathcal{X}$ from Lemma 4.2. To be precise, we are using maximal functions that take averages over enlarged Whitney regions $2W(t,x)$ here, but of course this is just a matter of choosing the generic constants $C_0$ and $C_1$. Concerning the estimate of $\tilde{N}_1^1(u)$, we only consider the more difficult case $\mathcal{D} = \emptyset$. The simplifications in the other case are obvious. We subtract from $u$ all terms we have control on by $8.10$. By the triangle inequality along with the embedding $\mathcal{Y}^* \subseteq \mathcal{X}$ and the pointwise estimate $\tilde{N}_1^1 \lesssim \tilde{N}_0$, we obtain

$$\|\tilde{N}_1^1(u)\|_2 \lesssim |c| + \left\| e^{-t[B_0 D]} \left( \tilde{h}_- + \tilde{h}_+ \right) \right\|_X + \left\| e^{-t[B_0 D]} \int_0^\infty N^+(1 - P_{B_0 \mathcal{H}}) E \mathcal{S}_f \, ds \right\|_X$$

$$+ \left\| \tilde{N}_1^1 \left( 1 + itB_0 D \right)^{-1} \int_0^t E \mathcal{S}_f \, ds \right\|_2 + \|\mathcal{E}\|_C \|f\|_Y.$$

For the second term on the right-hand side we intertwine as in (8.5) and then use the $\tilde{N}_*-$bound for the $[DB_0]$-semigroup from Remark 7.19. For the third term uniform boundedness of the semigroup and Proposition 8.9 applies and the fourth term can be controlled by means of Lemma 8.13. Thus,

$$\|\tilde{N}_1^1(u)\|_2 \lesssim |c| + \|\tilde{h}_-\|_2 + \|\tilde{h}_+\|_2 + \|f\|_Y.$$ 

Now, $\|\tilde{h}_+\|_2 \lesssim \|f\|_Y$ by Theorem 8.10 and $\|\tilde{h}_-\|_2 \lesssim \|f\|_Y$ by Step 1. Reinserting these estimates back on the right-hand side of (8.11) shows $\|\tilde{N}_2^1(u)\|_2 \lesssim |c| + \|f\|_Y \lesssim |u_\infty| + \|f\|_Y$, where the last equivalence follows again from Theorem 8.10.

Step 4: Almost everywhere convergence of Whitney averages. Finally we prove that Whitney averages of $u$ converge to the trace $c - \langle v_0 \rangle$ for a.e. $x \in \Omega$ as $t \to 0$. Here, as usual, $c = 0$ if the lateral Dirichlet part is non-empty. Since the right-hand side of (8.10) and (8.9), respectively, are contained in $\mathcal{Y}^*$, Whitney averages converge to 0 thanks to Lemma 4.3. Concerning the resolvent term we fix $t_0 \in (0, c_0)$ and note for $t < t_0$ and $x \in \Omega$ that

$$R(f)(t,x) := \left( (1 + itB_0 D)^{-1} \int_0^t \mathcal{E} s \, ds \right)(x) = R(1_{(0,t_0)} f)(t,x).$$
By Lemma 8.13 we have

\[
\int_{\Omega} \sup_{s < c_{0}^{-1}t_{0}} \left( \int_{W(s,y)} |R(f)(t,x)| \, dx \, dt \right)^{2} \, dy \leq \int_{\Omega} \sup_{s < c_{0}^{-1}t_{0}} \left( \int_{W(s,y)} |R(1_{\{t<t_{0}\}} f)(t,x)| \, dx \, dt \right)^{2} \, dy \\
\lesssim \|E\|^{2} \int_{0}^{t_{0}} \int_{\Omega} |f(s,y)|^{2} \, dy \, ds,
\]

where by dominated convergence the final term converges to 0 in the limit \(t_{0} \to \infty\). This implies that Whitney averages of \(Rf\) converge to zero almost everywhere.

We conclude that in the sense of Whitney averages the limit of \(-(u - c)\) as \(t \to 0\) is the same as the perpendicular part of the limit of the semigroup term in (8.10) and (8.9), respectively. It follows from Theorem 7.20 and since \((N^{+}h)_{\perp} = 0\) for every \(h \in L^{2}(\Omega)^{n}\), that this latter limit is precisely \(\tilde{h}^{-} + \tilde{h}^{+} = v_{0}\). This completes the proof. \(\square\)

9. WELL-POSEDNESS

We are finally ready to study well-posedness of the three boundary value problems in the sense of Section 3. Eventually, we will prove our third main result, Theorem 3.3. Throughout this section we fix \(t\)-dependent coefficients \(A\) and \(t\)-independent coefficients \(A_{0}\) satisfying Assumption 2.3 and as before let \(B\) and \(B_{0}\) correspond to \(A\) and \(A_{0}\), respectively.

We begin by rephrasing well-posedness of the boundary value problems for \(A_{0}\) in terms of Hardy projections. Recall the operators \(N^{\pm}\) and \(N\) from Definition 8.8.

**Lemma 9.1.**

(i) The Neumann and regularity problem for \(A_{0}\) are well-posed if and only if \(N^{-} : E^{+}_{0}\mathcal{H} \to N^{-}\mathcal{H}\) and \(N^{+} : E^{+}_{0}\mathcal{H} \to N^{+}\mathcal{H}\) are isomorphisms, respectively.

(ii) If \(\mathcal{D} \neq \emptyset\), then the Dirichlet problem for \(A_{0}\) is well-posed if and only if \(N^{-} : E^{+}_{0}L^{2}(\Omega)^{n} \to N^{-}\mathcal{H}\) is an isomorphism.

(iii) If \(\mathcal{D} = \emptyset\), then the Dirichlet problem for \(A_{0}\) is well-posed if and only if

\[
N^{-} : E^{+}_{0}L^{2}(\Omega)^{n} \oplus \left\{ \begin{bmatrix} c \\ 0 \end{bmatrix} : c \in \mathbb{C}^{m} \right\} \to L^{2}(\Omega)^{m}
\]

is an isomorphism.

**Proof.** Part \(i\) is a direct consequences of Proposition 5.5 and Corollary 8.4. The map under consideration in part \(ii\) is well-defined since \(N^{-}\mathcal{H} = L^{2}(\Omega)^{m}\) and if it is an isomorphism, then in view of Corollary 8.11 the Dirichlet problem is well-posed. Conversely assume the Dirichlet problem is well-posed. By Corollary 8.11 the map \(N^{-} : E^{+}_{0}L^{2}(\Omega)^{n} \to N^{-}\mathcal{H}\) is onto. Now, suppose \(N^{-}\tilde{h}^{-} = 0\) for some \(\tilde{h}^{+} \in E^{+}_{0}L^{2}(\Omega)^{n}\) and define \(v_{t} = -e^{-t|B_{0}D|}\tilde{h}^{+}, t \geq 0\). Corollary 8.11 reveals \(v_{t} \perp v_{t}\) as a solution of the Dirichlet problem with Lusin area bound and data \(N^{-}\tilde{h}^{+} = 0\). By well-posedness, \(v_{1} = 0\). As in the proof of Theorem 8.10, 0 = \(\nabla A v_{1} = Dv\). This means \(v_{t} \in \mathcal{N}(D)\) for all \(t > 0\) and the topological decomposition \(\mathcal{N}(D) \oplus B_{0}\mathcal{H}\) forces \(v_{t} = 0\) for all \(t > 0\). By strong continuity of the semigroup \(\tilde{h}^{+} = 0\) follows. Part \(iii\) is proved analogously, taking into account \(\mathbb{C}^{m} \times \{0\} \subseteq \mathcal{N}(D)\) and that \(\mathcal{N}(D) \oplus B_{0}\mathcal{H}\) is a topological decomposition. \(\square\)
9.1. Small perturbations. In this section we establish stability of well-posedness for $t$-independent coefficients under small perturbations of the coefficients with respect to the $L^\infty$-topology. For the Neumann and regularity problem this is almost immediate from holomorphic dependence of the Hardy projections on $B_0$ and the following lemma.

Lemma 9.2 (§ Lem. 4.2). Let $\delta > 0$. Let $P_t$, $-\delta \leq t \leq \delta$ be bounded projections on a Hilbert space $K$ depending continuously on $t$ in the $\mathcal{L}(K)$-topology. Let $S : K \to J$ be a bounded operator into a Hilbert space $J$. If $S : P_0 K \to J$ is an isomorphism, then there exists $0 < \varepsilon < \delta$, such that $S : P_t K \to J$ is an isomorphism when $|t| < \varepsilon$.

Proposition 9.3. Consider the set of those $t$-independent coefficients satisfying Assumption 2.3 for which the Neumann problem for $A_0$ is well-posed. This is an open subset of $L^\infty(\Omega; \mathcal{L}(\mathbb{C}^n))$. A similar result holds for the regularity problem.

Proof. If $A_0$ satisfies Assumption 2.3 with respective constant $\lambda > 0$ and $C \in L^\infty(\Omega; \mathcal{L}(\mathbb{C}^n))$ is any matrix, then for $z \in \mathbb{C}$ sufficiently close to 1 the matrices $A_0(z) := (1 - z)C + zA_0$ satisfy Assumption 2.3 with respective constant $\frac{\lambda}{2}$. Let $B_0(z)$ correspond to $A_0(z)$ as usual. Lemma 5.2 and Proposition 6.4 yield holomorphy of $z \mapsto 1_{C^+}(DB_0(z))$. The claim follows from Lemma 9.2 and the characterization for well-posedness given in Lemma 9.1.

The inhomogeneity of considering $N^-$ on $\tilde{E}_0^+ L^2(\Omega)^n$ for the Dirichlet problem can be circumvented by the Dirichlet-regularity duality.

Proposition 9.4 (Dirichlet-regularity duality). The Dirichlet problem for $A_0$ is well-posed if and only if the regularity problem for $A_0^\star$ is well-posed.

This principle is known in the setting $\Omega = \mathbb{R}^d$. As the adaption to our framework bears some subtle difficulties, we include an elementary and completely abstract proof building on the following two lemmas.

Lemma 9.5. Let $P$ be the orthogonal projection in $L^2(\Omega)^n$ onto $\mathcal{H}$. There are similarities of operators

$$DB_0|_{\mathcal{R}(DB_0)} = R^{-1}(B_0 D_{\mathcal{R}(B_0 D)})R \quad \text{and} \quad B_0 D_{\mathcal{R}(B_0 D)} = S^{-1}(P B_0 D|_{\mathcal{R}(DB_0)})S.$$

The isomorphisms $R, S^{-1} : \mathcal{R}(DB_0) \to \mathcal{R}(B_0 D)$ are given by $R = B_0|_{\mathcal{R}(DB_0)}$ and $S = P|_{\mathcal{R}(B_0 D)}$. Moreover, $S^{-1}$ is the restriction to $\mathcal{R}(DB_0)$ of the projection $Q$ onto $\mathcal{R}(B_0 D)$ along the splitting $L^2(\Omega)^n = \mathcal{N}(D) \oplus \mathcal{R}(B_0 D)$.

Proof. The first similarity of operators is proved in [12 Prop. 2.1(2)] and the second one follows as in [13 Prop. 2.2].

Lemma 9.6 ([7 p. 37], [23 Lem. 6.5.9]). Assume that $N^\pm$ and $E^\pm$ are two pairs of complementary bounded projections on a Hilbert space $K$, i.e., $(N^\pm)^2 = N^\pm$ and $N^+ + N^- = \text{Id}$, and similarly for $E^\pm$. Then the adjoint operators $(N^\pm)^* : (E^\pm)^*$ are also two pairs of complementary projections on $K$ and the restricted projection $N^+ : E^+ K \to N^+ K$ is an isomorphism if, and only if the restricted adjoint projection $(N^-)^* : (E^-)^* K \to (N^-)^* K$ is an isomorphism.

Proof of Proposition 9.4. Note that $A_0^\star$ satisfies the same accretivity condition as $A_0$ and that replacing $A_0$ with $A_0^\star$ amounts to replacing $B_0$ with $B_0^\star = N B_0^\star N$ and $D B_0$ with $D B_0^\star = -N D B_0^\star N$. We abbreviate $E_\star^\pm := 1_{C^+}(DB_0^\star)$. 
Step 1: Rephrasing well-posedness of the Dirichlet problem. We begin with establishing a representation for the space $E_0^+L^2(\Omega)^n$ that better suits our circumstance. First, $E_0^+L^2(\Omega)^n = B_0E_0^+\mathcal{H}$ by the intertwining property (\ref{D}) in Section 6. The similarities from Lemma \ref{9.5} are inherited to the functional calculus. So, $E_0^+ = R^{-1}S^{-1}1_{C}(PB_0D|_{\mathcal{H}})SR$. As $SR$ is an automorphism of $\mathcal{H}$ and since $B_0R^{-1} = \text{Id}$ on $B_0\mathcal{H}$, it follows $B_0E_0^+\mathcal{H} = S^{-1}1_{C}(PB_0D|_{\mathcal{H}})\mathcal{H}$. Corollary \ref{6.2} with the roles of $B_0$ and $B_0^*$ interchanged yields

$$PB_0D|_{\mathcal{H}} = (DB_0^+|_{\mathcal{H}})^* = (-NDB_0^+N|_{\mathcal{H}})^* = -N(DB_0^+|_{\mathcal{H}})^*N|_{\mathcal{H}},$$

where all adjoints are taken within $\mathcal{H}$. Taking into account $1_{C^+}(z) = 1_{C^-}(z')$, $z \in \mathbb{C}$, and $N^{-1} = N$, this carries over to $1_{C^+}(PB_0D|_{\mathcal{H}}) = N(E_{\star}^*)^*N|_{\mathcal{H}}$ as before. Altogether,

$$E_0^+L^2(\Omega)^n = B_0E_0^+\mathcal{H} = S^{-1}1_{C^+}(PB_0D|_{\mathcal{H}})\mathcal{H}.$$

Step 2: The claim for non-empty lateral Dirichlet part. Assume $\mathcal{D} \neq \emptyset$. By Lemma \ref{9.5} and Step 1, well-posedness of the Dirichlet problem for $A_0$ is equivalent to $N^{-1} : S^{-1}1_{C^+}(PB_0D|_{\mathcal{H}})\mathcal{H} \rightarrow N^{-1}\mathcal{H}$ being an isomorphism. From Lemma \ref{9.5} we recall that $S^{-1}$ agrees with the projection $Q$ onto $B\mathcal{H}$ which annihilates $\mathcal{N}(D)$. Since the first map in the chain

$$N^{-1} : S^{-1}1_{C^+}(PB_0D|_{\mathcal{H}})\mathcal{H} \rightarrow N^{-1}\mathcal{H}$$

is an isomorphism, well-posedness of the Dirichlet problem is equivalent to the composite map being an isomorphism. From the identity

$$N^{-1}S^{-1}Nh = N^{-1}Nh - N^{-1}(1 - Q)Nh = -N^{-1}h - N^{-1}(1 - Q)Nh \quad (h \in \mathcal{H})$$

and the fact that $N^{-1}\mathcal{N}(D) = \{0\}$ by injectivity of $\nabla_V$, we see that the composite map in \ref{9.2} acts as $N^{-1} : (E_{\star}^*)^*\mathcal{H} \rightarrow N^{-1}\mathcal{H}$. Lemmas \ref{9.6} and \ref{9.1} yield the claim.

Step 3: The claim for empty lateral Dirichlet part. Finally, we consider the case $\mathcal{D} = \emptyset$. First assume that the regularity problem for $A_0^*$ is well-posed. In view of \ref{9.1} and Lemmas \ref{9.6} and \ref{9.1} we have that $N^{-1} : (E_{\star}^*)^*\mathcal{H} \rightarrow N^{-1}\mathcal{H}$ is an isomorphism and have to show that so is

$$N^{-1} : S^{-1}1_{C^+}(PB_0D|_{\mathcal{H}})\mathcal{H} \oplus \left\{ \begin{bmatrix} c \\ 0 \end{bmatrix} : c \in \mathbb{C}^m \right\} \rightarrow L^2(\Omega)^m.$$

Suppose $h \in (E_{\star}^*)^*\mathcal{H}$ and $c \in \mathbb{C}^m$ satisfy $N^{-1}(S^{-1}Nh) + c = 0$. By \ref{9.3},

$$-N^{-1}h - N^{-1}(1 - Q)Nh + c = 0,$$

where the first term has zero average on $\Omega$ and the second and third terms are constant on $\Omega$. This forces $N^{-1}h = 0$ and $N^{-1}(1 - Q)Nh = c$. By assumption $h = 0$ and therefore $c = 0$, proving that the map in \ref{9.4} is one-to-one. As for onto, let $g \in L^2(\Omega)^m$ be given and define $g_0 := \frac{1}{|\Omega|}g$. By assumption, there exists $h \in (E_{\star}^*)^*\mathcal{H}$ such that $-N^{-1}h = g - g_0$. Putting $c = g_0 + N^{-1}(1 - Q)Nh$, it follows once again from \ref{9.3} that

$$N^{-1}(S^{-1}Nh) + c = -N^{-1}h + N^{-1}(1 - Q)Nh + c = g - g_0 + g_0 = g.$$

Conversely, assume that \ref{9.4} provides an isomorphism. In order to prove that $N^{-1} : (E_{\star}^*)^*\mathcal{H} \rightarrow N^{-1}\mathcal{H}$ is an isomorphism as well, first let $h \in (E_{\star}^*)^*\mathcal{H}$ satisfy $N^{-1}h = 0$. With $c := -N^{-1}(1 - Q)Nh$ we obtain from \ref{9.3} that $N^{-1}S^{-1}Nh = c$, whence $S^{-1}Nh = \begin{bmatrix} c \\ 0 \end{bmatrix}$. The topological decomposition $\mathcal{N}(D) \oplus B_0\mathcal{H}$ yields $S^{-1}Nh = 0$ and thus $h = 0$. Also, given $g \in N^{-1}\mathcal{H}$, by assumption there exist $h \in (E_{\star}^*)^*\mathcal{H}$ and $c \in \mathbb{C}^m$ such that

$$g = N^{-1}(S^{-1}Nh) + c = -N^{-1}h - N^{-1}(1 - Q)Nh + c.$$
Since $g$ and $-N^-h$ have zero average on $\Omega$ and as the other two terms are constant, $g = N^-(-h)$ follows. \hfill \square

**Corollary 9.7.** Consider the set of those $t$-independent coefficients satisfying Assumption 2.3 for which the Dirichlet problem for $A_0$ is well-posed. This is an open subset of $L^\infty(\Omega; L^\infty(\mathbb{C}^n))$.

9.2. **Well-posedness for block and Hermitean matrices.** For the Neumann and regularity problem there are at least two classes of $t$-independent coefficients for which invertibility of the projections in Lemma 9.1 is known nowadays: The class of matrices $A_0$ in block-form

$$A_0 = \begin{bmatrix} (A_0)_{\perp\perp} & 0 \\ 0 & (A_0)_{\parallel\parallel} \end{bmatrix},$$

and the Hermitean matrices satisfying $A_0^* = A_0$. (There are also some results for block-triangular matrices [10]). For elliptic systems on the upper halfspace these results have first been obtained in [8]. More precisely, the following was shown [8, Sec. 4.1/2]:

- If $A_0$ is of block form, then the projections in Lemma 9.1 can be inverted by a purely algebraic formula relying on the identity $N^{-1}B_0N = B_0$ valid since $B_0$ is of block form as well. In fact, well-posedness in this case is equivalent to the solution of the Kato problem for elliptic systems on $\Omega$ with Dirichlet condition on $D$, recently solved in [26].
- Well-posedness of the Neumann and regularity problem for Hermitean $A_0$ follows from well-posedness of these problems with $A_0 = Id$ the identity matrix (which is of block form), the method of continuity (which we have at our disposal thanks to holomorphic dependence of the Hardy projections on $A_0$ as in the proof of Proposition 9.3), and the so-called Rellich identity. The proof of the latter is again an abstract argument that literally applies in our situation as well.

Finally, well-posedness of the Dirichlet problem follows on using Proposition 9.4. Summing up, we can record the following result.

**Proposition 9.8.** Each of the three boundary value problems for $A_0$ is well-posed if $A_0$ is of block-form

$$A_0 = \begin{bmatrix} (A_0)_{\perp\perp} & 0 \\ 0 & (A_0)_{\parallel\parallel} \end{bmatrix},$$

or Hermitean, that is, $A_0^* = A_0$.

9.3. **The proof of Theorem 3.3** The claim for the $t$-independent coefficients follows from Propositions 9.3, Corollary 9.7, and Proposition 9.8

Next, we inquire well-posedness of the Neumann and regularity problems for $A$. Throughout, we assume that the Neumann and regularity problems for $A_0$ are well-posed and that $\|A - A_0\|_C$ is small enough so that $1 - S_A$ is invertible on $X$ thanks to Proposition 8.2. In view of Proposition 5.5 and Theorem 8.3 the conormal gradients $f$ of weak solutions $u$ to the second-order system are precisely the functions $f = (1 - S_A)^{-1}e^{-[tDB_0]}h^+$, $h^+ \in E_0^+ \mathcal{H}$, which converge in the sense of Whitney averages as well as in square Dini sense to

$$\Gamma_A h^+ := h^+ + \int_0^\infty DB_0 e^{-s[DB_0]} \hat{E}_0 \mathcal{E}_s f_s \, ds$$

as $t \to 0$. Note that $\Gamma_A$ really is a linear operator acting on $h^+$ since $f$ is determined by $h^+$. It follows that the Neumann problem and regularity problem for $A$ is well-posed, if $N^- \Gamma_A :
Theorem 8.10 and Theorem 8.15 we already know that a solution with \( \tilde{\varphi} \) for \( \tilde{\varphi} \) in Theorem 8.15) and \( N \)
we use \( B \).
Employing Theorems 4.4 and 7.18 we can infer a bound by \( \| \mathcal{E} f \|_{\mathcal{Y}} \lesssim \| \mathcal{E} c \|_{\mathcal{Y}} \| h^+ \|_2 \)
for the first term. For the second term, we note that \( B_0 \)
attain there boundary trace on \( \Omega \) in the sense
of \( \gamma \mathcal{Y} \) thanks to Proposition 8.2. We 
and know from Theorem 8.3.
Neumann and regularity problems, we only have to consider the operator \( \mathcal{E} \) and compute
on \( \gamma \mathcal{Y} \) and \( \| \mathcal{E} \|_{\mathcal{Y}} \| \mathcal{E} c \|_{\mathcal{Y}} \| h^+ \|_2 \)
To this end, we use \( \tilde{\varphi} \).
Finally, we prove the required estimates for the Dirichlet problem. To this end let \( u \)
be a weak solution with \( \nabla_{t,z} u \in \gamma \mathcal{Y} \) and let \( u_0 \) and \( u_\infty \) be its limits at \( t = 0 \) and \( t = \infty \), respectively. From
Theorem 8.10 and Theorem 8.15 we already know that
\[
\| u_0 \|_2 \lesssim \| \tilde{N}_c(u) \|_2 + \sup_{t > 0} \| u_t \|_2 \lesssim (|u_\infty| + \| \nabla_{t,z} u \|_{\gamma \mathcal{Y}}).
\]
To see that all four quantities are equivalent if \( \|A - A_0\|_C \) is small enough, we note firstly that \( \|\nabla_{t,x} u\|_Y \sim \|\tilde{h}\|_2 \) by Theorem 8.10 and secondly, since \( \Gamma_A \) is an isomorphism as we have seen above, that \( \|u_0\|_2 \sim \|\tilde{h}^+\|_2 \) if \( \mathcal{D} \) is non-empty. If \( \mathcal{D} \) is empty, then similarly

\[
\|u_0\|_2 \sim \left\| \tilde{h}^+ + \frac{c}{0} \right\|_2 \leq |c| + \|\tilde{h}^+\|_2 \sim |c| + \|\nabla_{t,x} u\|_Y \sim |u_\infty| + \|\nabla_{t,x} u\|_Y,
\]

the final step following again from Theorem 8.10.

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### References


