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November 2, 2015

Abstract

In a proximity region graph $\mathcal{G}$ in $\mathbb{R}^d$, two distinct points x, y of a point process μ are connected when the 'forbidden region' $S(x, y)$ these points determine has empty intersection with μ. The Gabriel graph, where $S(x, y)$ is the open disc with diameter the line segment connecting x and y, is one canonical example. Under broad conditions on the process μ and regions $S(x, y)$, bounds on the Kolmogorov and Wasserstein distances to the normal are produced for functionals of $\mathcal{G}$, including the total number of edges, and total length.

1 Introduction

The family of graphs that we study here, all with vertex sets given by a point process μ in $\mathbb{R}^d$, are motivated by two canonical examples considered in [1], the Gabriel graph and the relative neighborhood graph. Two distinct points x and y of μ are connected by an edge in the Gabriel graph if and only if there does not exist any point z of the process μ lying in the open disk whose diameter is the line segment connecting x and y. The relative neighborhood graph has an edge between x and y if and only if there does not exist a point z of μ such that

$$\max(\|x - z\|, \|z - y\|) < \|x - y\|,$$

that is, if and only if there is no point z of μ that is closer to either x or y than these points are to each other.

These two examples are special cases of 'proximity graphs' as defined in [3], where distinct points x and y of μ are connected if and only if a region $S(x, y)$ determined by x and y contains no points of μ, that is, when $\mu \cap S(x, y) = \emptyset$. As $S(x, y)$ must be free of points of μ in order for x and y to be joined, we call $S(x, y)$ the 'forbidden region' determined by x and y. In particular, with $B(x, r)$ and $B^o(x, r)$ denoting the closed and open ball of radius r centered

[0]The work of the first author was partially supported by NSA-H98230-15-1-0250, and of the second author by NSF grant DMS-1401479

[0]MSC 2010 subject classifications: 60D05, 52A22,60F05

[0]Key words and phrases: Forbidden region graph, stabilization, Poisson functionals

at x, respectively, we see that the Gabriel and relative neighborhood graphs correspond to the forbidden regions

$$S(x, y) = B^o((x + y)/2, \|y - x\|/2) \tag{1}$$

and

$$S(x, y) = B^o(x, \|y - x\|) \cap B^o(y, \|x - y\|) \tag{2}$$

respectively. It is easy to check that the forbidden regions $S(x, y)$ of the Gabriel graph is contained in those of the relative neighbor graph, and hence edges of latter are also edges of former.

We refer to the graphs formed in this manner also as 'forbidden region graphs'. Indeed, when coining the label 'proximity graphs' in [3], one reads that 'this term could be misleading in some cases.' Indeed, forbidden region graphs may depend on 'non-proximate' information, such as the graph considered in Example 5 of [3], whose forbidden region $S(x, y)$ is the infinite strip bounded by the two parallel hyperplanes containing x and y, each perpendicular to $y-x$. Allowing forbidden regions to depend on larger sets of points and to be determined by more complex rules yield well studied graphs with additional structure, including the Minimum Spanning Tree and the Delaunay triangulation, see [1].

For a forbidden region graph $\mathcal{G}$ and μ a Poisson or binomial point process in some bounded observation window we study the distribution of

$$L(\mu) = \frac{1}{2} \sum_{\{x,y\} \subseteq \mu, x \neq y} \mathbf{1}(\mu \cap S(x, y) = \emptyset)\psi(x - y), \tag{3}$$

for some $\psi : \mathbb{R}^d \to \mathbb{R}$. For instance, taking $\psi(x) = \|x\|^\alpha$ for some $\alpha \geq 0$, for $\alpha = 0$ and $\alpha = 1$ the value of $L(\mu)$ is the number of edges and the total length of $\mathcal{G}$, respectively.

Theorem 1, our main result, is a bound on the normal approximation of L, in both the Wasserstein and Kolmogorov metric, that holds under broad conditions on the forbidden regions and underlying point process. First, we will assume that the collection of forbidden regions $\{S(x, y) : \{x, y\} \subseteq \mathbb{R}^d, x \neq y\}$ are nonempty subsets of $\mathbb{R}^d$ that are symmetric in that

$$S(x, y) = S(y, x) \quad \text{for all } \{x, y\} \subseteq \mathbb{R}^d, x \neq y. \tag{4}$$

Nonsymmetric sets $S(x, y)$ would be natural for the construction of directed forbidden region graphs, and though we do not consider them here our methods would apply. We assume also that

$$\{x, y\} \subseteq \overline{S(x, y)} \setminus S(x, y), \tag{5}$$

and that the *normalized diameter* of the collection of forbidden regions is finite, that is,

$$\mathcal{D} < \infty \quad \text{where} \quad \mathcal{D} = \sup\left\{ \frac{\|s - t\|}{\|x - y\|}; \{s, t\} \subseteq S(x, y), \{x, y\} \subseteq \mathbb{R}^d, x \neq y \right\}. \tag{6}$$

Assumption 1 below requires that as x and y become farther apart, the forbidden regions $S(x, y)$ contain increasingly large balls. Note, for instance, that if all forbidden regions have empty interior, then the graph determined by a Poisson input process would be the complete graph almost surely. Also, to ensure that the graph, and functional $L(\mu)$ of (3), are finite, we restrict the graph to some bounded measurable 'viewing window' $\mathbb{X}$.

Assumption 1 (Scaled ball condition)**.** *For some* $\delta > 0$*, it holds for all* $\{x, y\} \subseteq \mathbb{X}$ *that* $S(x, y) \cap \mathbb{X}$ *contains a ball of radius* $\delta\|x - y\|$*.*

With some slight abuse of notation, $|\cdot|$ will be used to denote both the Lebesgue measure of a measurable subset of $\mathbb{R}^d$ and also cardinality of a finite set; use will be clear from context. We assume the following condition on the process η_t.

Assumption 2. *Let* λ *be a probability measure on* $\mathbb{X}$ *satisfying*

$$c_\lambda |B| \leq \lambda(B) \leq b_\lambda |B| \quad \text{for all measurable } B \subseteq \mathbb{X}$$

for some $0 < c_\lambda \leq b_\lambda$*. The point process* η_t *is either Poisson on* $\mathbb{X}$ *with intensity* $\lambda_t = t\lambda, t > 0$*, or a set of i.i.d. variables* $X_1, \ldots, X_t$ *with common distribution* λ*, for* $t \in \mathbb{N}$*.*

For the function ψ in (3) we will assume:

$$\text{There exists } C > 0 \text{ and } \alpha \geq 0 \text{ such that} \quad |\psi(x)| \leq C\|x\|^\alpha \quad \text{for all } x \in \mathbb{R}^d. \tag{7}$$

Lastly, we require the following variance lower bound.

Assumption 3. *For some* $v_\alpha > 0$*,*

$$\operatorname{Var} L(\eta_t) \geq v_\alpha t^{1-2\alpha/d} \quad \text{for all } t \geq 1.$$

Assumption 3 is a serious one, and we separately address the question of when it is satisfied in Section 4.

We write $d_W(X, Y)$ and $d_K(X, Y)$ for the Wasserstein and Kolmogorov distances, respectively, between the laws of the random variables X and Y. We inform the reader that the C that appears in our bounds denotes a positive constant that may not be the same at each occurrence.

Theorem 1. *Let* $\mathbb{X}$ *be a bounded measurable subset of* $\mathbb{R}^d$*, and let* $\{S(x, y) \colon x, y \in \mathbb{X},\, x \neq y\}$ *be a collection of forbidden regions satisfying* (4)–(6) *and Assumption 1. Let* η_t *be a point process on* $\mathbb{X}$ *satisfying Assumption 2, and let*

$$F_t := L(\eta_t), \quad \text{for } t \geq 1,$$

where $L(\cdot)$ *is given in* (3) *with* ψ *satisfying* (7)*.*

If Assumption 3 holds, then there exists a constant C *not depending on* t *such that*

$$\max\left[d_W(\widetilde{F}_t, N), d_K(\widetilde{F}_t, N)\right] \leq Ct^{-1/2} \quad \text{for all } t \geq 1,$$

where $\widetilde{F}_t = (F_t - \mathbb{E}F_t)/\operatorname{Var} F_t$*, and* N *is a standard Gaussian.*

In [9], the authors prove a central limit theorem for statistics of point processes, which is then applied to the Gabriel graph and the nearest neighbor graph on Poisson or binomial input processes. Their result does not give the rate of convergence to normal, however. We note that a variance decay of order $t^{-1/2}$ is typical for stationary Euclidean functionals, meaning the rates obtained in Theorem 1 are likely to be optimal, and could be shown so for discrete valued variables, such as the number of edges, using the methods in [4].

We now address Assumption 3, the lower bound on $\operatorname{Var} L(\eta_t)$. Penrose and Yukich give a general lower bound for the variance of Poisson and binomial statistics in [9]. Their result requires a statistic f to be *strongly stabilized*. (This notion of stabilization is also referred to as stabilization for add-one cost or as external stabilization—see [10] for a general survey.) We cannot apply this result because our statistic L is not strongly stabilized unless we impose additional constraints on the forbidden regions, such as requiring them to be convex. Another possible approach would be to use the results of [7, Section 5]. These are applicable to L, but only for the easier case of Poisson input. We are thus forced to give a new argument to prove that Assumption 3 holds in some generality. We state one additional technical condition required, followed by a sufficient condition for Assumption 3. Let the boundary of a set B be denoted by ∂B.

Assumption 4. *For all $\{w, z\} \subset B(0, 1)$ there exists $y \in \partial S(w, z)$ such that $z \notin \partial S(w, y)$ and $w \notin \partial S(z, y)$.*

Theorem 2. *Suppose the forbidden regions $\{S(x, y), \{x, y\} \subseteq \mathbb{R}^d, x \neq y\}$ form a regular (S, u_0) isotropic family as in Remark 4 below and satisfy Assumption 4. Assume further that the scaled ball condition is satisfied with the role of $\mathbb{X}$ played by $t^{1/d}\mathbb{X} \cap B(x, r)$ for any t and r. Also assume that $\psi(x) = x^\alpha$ in the definition of L. Then there is a constant $v_\alpha > 0$ such that Assumption 3 holds when η_t satisfies Assumption 2.*

An immediate consequence of this theorem is the following result for our two motivating examples, taking, say $\mathbb{X} = B(0, 1)$, for concreteness.

Corollary 3. *Let $\mathbb{X} = B(0, 1)$, and suppose that η_t is either a Poisson process with intensity t on $\mathbb{X}$, or a binomial process of t independent and uniformly distributed points on $\mathbb{X}$. Then the bound of Theorem 1 holds for the Gabriel graph and the relative neighborhood graph on η_t.*

Theorem 1 is based on the methods of [7], in particular on second order Poincaré inequalities, and also the key notion of stabilization. To define stabilization, let $f(\mu)$ be a function of a point process μ in $\mathbb{R}^d$. For $x \in \mathbb{R}^d$ consider the difference (or derivative) at x given by

$$D_x f(\mu) = f(\mu \cup \{x\}) - f(\mu), \tag{8}$$

which is the amount that f changes upon the insertion of the point x into μ. Higher order differences are defined iteratively, for instance $D^2_{x,y} f(\mu) = D_x(D_y f(\mu))$, so

$$D^2_{x,y} f(\mu) = f(\mu \cup \{x, y\}) - f(\mu \cup \{y\}) - f(\mu \cup \{x\}) + f(\mu). \tag{9}$$

There are a number of related notions of a stabilization radius for a functional f. The one we will use is a radius $R(x; \mu)$ such that

$$D^2_{x,y} f(\mu) = 0 \qquad \text{if } \|y - x\| > R(x; \mu). \tag{10}$$

We say in this case that $R(x; \mu)$ is *stabilizing* for f around x.

When μ is a Poisson process with growing intensity λ_t, one key condition from [7] required to obtain bounds to the normal for a properly standardized functional f is that over the observation window $\mathbb{X}$,

$$\sup_{x \in \mathbb{X}, t \geq 1} \int P(D^2_{x,y} f(\eta_t) \neq 0)^a \lambda_t(dy) < \infty, \tag{11}$$

for a some small number, depending on low moments of the derivatives of f. If there exists a stabilization radius for f that is small with sufficiently high probability, then (11) holds. In Section 2, we construct such a radius and prove that it exhibits exponential decay under very weak conditions on the forbidden regions. We end this section by introducing some additional concepts and terminology about forbidden regions, such as the normalized diameter $\mathcal{D}$ and the notion of 'family based' and 'isotropic' forbidden regions.

As already stated, the graph with vertex set a locally finite point configuration μ in $\mathbb{R}^d$ is the $S(x,y)$ forbidden region graph on μ when an edge exists between points x and y of μ if and only if $x \neq y$ and $S(x,y) \cap \mu = \emptyset$. That is, we connect points x and y of μ if and only if they are distinct, and there are no points of μ lying in the forbidden region $S(x,y)$ that these two points generate. Hence, for $x \in \mu$, the set of edges $\mathcal{G}_S(x;\mu)$ incident to x in μ, and the edge set $\mathcal{G}_S(\mu)$ of the forbidden region graph are given, respectively, by

$$\mathcal{G}_S(x;\mu) = \{\{x,y\} : \{x,y\} \subseteq \mu, x \neq y, S(x,y) \cap \mu = \emptyset\} \quad \text{and} \quad \mathcal{G}_S(\mu) = \bigcup_{x \in \mu} \mathcal{G}_S(x;\mu).$$

We may denote, $\mathcal{G}_S(\mu)$ by $\mathcal{G}(\mu)$, say, if S is clear from context.

We say the collection of forbidden regions are *translation invariant* when

$$S(x+z, y+z) = S(x,y) + z \quad \text{for all } \{x,y,z\} \subseteq \mathbb{R}^d, x \neq y.$$

In many natural examples the forbidden region $S(x,y)$ is determined by a 'template' which is shifted and scaled in accordance with the positions of x and y. More precisely, with $\mathcal{S}^{d-1}$ denoting the unit sphere in $\mathbb{R}^d$ we say that the collection of forbidden regions is based on the family of subsets $\{S(u) : u \in \mathcal{S}^{d-1}\}$ of $\mathbb{R}^d$ when

$$S(x,y) = (x+y)/2 + \|x-y\| S\left(\frac{x-y}{\|x-y\|}\right) \quad \text{for all } \{x,y\} \subseteq \mathbb{R}^d, x \neq y.$$

In other words, forbidden regions are based on a family when $S(x,y)$ is a set that depends only on the direction vector from y to x, which is then scaled by the distance between y and x and shifted to have its 'center' at their midpoint.

Regions based on families enjoy properties (4) and (5) when

$$S(-u) = S(u) \quad \text{and} \quad \{-u,u\} \subseteq 2(\overline{S(u)} \setminus S(u)) \quad \text{for all } u \in \mathcal{S}^{d-1}, \tag{12}$$

are always translation invariant, and have normalized diameter given by

$$\mathcal{D} = \sup\{\|p-q\| : \{p,q\} \subseteq S(u), u \in \mathcal{S}^{d-1}\}.$$

Remark 4. *A particularly simple collection of forbidden regions are the ones we term (S,u_0) isotropic, which are regions based on a family $\{S(u) : u \in \mathcal{S}^{d-1}\}$ determined by a bounded measurable subset $S \subseteq \mathbb{R}^d$ and a unit vector $u_0 \in \mathbb{R}^d$ such that any rotation leaving u_0 invariant also leaves S invariant, that is, u_0 specifies a symmetry axis for S. Assume furthermore that $\{-u_0,u_0\} \subseteq 2(\overline{S} \setminus S)$. Given any unit vector u, let ρ_u be any rotation transforming u_0 into u, and set $S(u) = \rho_u(S)$. The set $S(u)$ is well defined as $\rho_u(S)$ does not depend on the choice of rotation. Indeed, if ρ and ρ' are any two such rotations, the rotation $\rho^{-1}\rho'$ leaves u_0 invariant, and hence also leaves S invariant, whence $\rho'(S) = \rho\rho^{-1}(\rho'(S)) = \rho(S)$. For definiteness, given $u_0 \in \mathbb{S}^{d-1}$, for all $u \in \mathcal{S}^{d-1}$ let ρ_u denote the rotation mapping u_0 to u,* *leaving invariant the orthogonal complement of the space spanned by $\{u_0, u\}$. We say the (S, u_0) isotropic family is regular when S contains an open ball and has negligible boundary.*

We note that in $\mathbb{R}^2$ an isotropic family can be generated by any non-empty subset S and any unit vector u_0, as only the identity rotation leaves u_0 invariant, which necessarily leaves S invariant.

Our two canonical examples, the Gabriel graph and the relative neighborhood graph, are both isotropic families. With $u_0 = (1, 0, \ldots, 0)$, the Gabriel graph is obtained by setting $S = B^o(0, 1/2)$, and the relative neighborhood graph for $S = B^o(u_0/2, 1) \cap B^o(-u_0/2, 1)$. For the Gabriel graph, we then have

$$S(u) = B^o(0, 1/2) \quad \text{for all } u \in \mathcal{S}^{d-1},$$

as we may write (1) as

$$B^o((x+y)/2, \|(y-x)\|/2) = (x+y)/2 + B^o(0, \|y-x\|/2) = (x+y)/2 + \|y-x\| S(u).$$

Rotating the 'base set' S given above for the relative neighborhood graph we obtain

$$S(u) = B^o(u/2, 1) \cap B^o(-u/2, 1),$$

and for given $x \neq y$, with $u = (x-y)/\|x-y\|$, we have (2) expressed as

$$\begin{aligned}
& B^o(x, \|y-x\|) \cap B^o(y, \|x-y\|) \\
&= (x+y)/2 + B^o((x-y)/2, \|y-x\|) \cap B^o((y-x)/2, \|x-y\|) \\
&= (x+y)/2 + \|x-y\| \left(B^o(u/2, 1) \cap B^o(-u/2, 1)\right) \\
&= (x+y)/2 + \|x-y\| S(u).
\end{aligned}$$

We note that (12), and hence both (4) and (5), are satisfied for these two basic examples.

2 Radius of stabilization

We begin this section by constructing a set in (13) that will serve as a stabilizing region about a point $x \in \mathbb{R}^d$, or more generally around a subset $U \subseteq \mathbb{R}^d$. Our radius $R_S(U; \mu)$ is then constructed in (15) in terms of this set, and prove in Lemma 7 that it satisfies (10), that is, that it is stabilizing for $L(\mu)$ around x, is monotone in μ as a consequence of Lemma 5, and show in Proposition 9, under a simple condition on the forbidden regions, that it has exponentially decaying tails with standard Poisson or binomial input.

For $U \subseteq \mathbb{R}^d$, let

$$\begin{aligned}
& \mathcal{R}_S(U; \mu; \mathbb{X}) \\
&\qquad = \bigcup \Big\{ S(w, z) \colon \{w, z\} \subseteq \mathbb{X} \text{ such that } S(w, z) \cap \mu = \emptyset \text{ and } U \cap \overline{S(w, z)} \neq \emptyset \Big\}. \quad (13)
\end{aligned}$$

Intuitively, this set consists of all forbidden regions affected by the addition of a point somewhere in U. The most important case for us is $U = \{x\}$, which we write as $\mathcal{R}_S(x; \mu; \mathbb{X})$.

First, we show $\mathcal{R}_S(U; \mu; \mathbb{X})$ satisfies a monotonicity property in μ.

Lemma 5. *If $\mu \subseteq \nu$, then*

$$\mathcal{R}_S(U;\nu;\mathbb{X}) \subseteq \mathcal{R}_S(U;\mu;\mathbb{X}),$$

with equality if $\nu \setminus \mu$ lies outside of $\mathcal{R}_S(U;\mu;\mathbb{X})$.

Proof. Suppose that $S(w,z)$ satisfies $S(w,z) \cap \nu = \emptyset$ and $U \cap \overline{S(w,z)} \neq \emptyset$. Then this forbidden region also satisfies $S(w,z) \cap \mu = \emptyset$, showing that $\mathcal{R}_S(U;\nu;\mathbb{X}) \subseteq \mathcal{R}_S(U;\mu;\mathbb{X})$.

Now, assume that $\nu \setminus \mu$ lies outside of $\mathcal{R}_S(U;\mu;\mathbb{X})$. Suppose that $S(w,z)$ satisfies $S(w,z) \cap \mu = \emptyset$ and $U \cap \overline{S(w,z)} \neq \emptyset$. Then $S(w,z) \subseteq \mathcal{R}_S(U;\mu;\mathbb{X})$, and hence $\mu = \nu$ on $S(w,z)$. This implies that $S(w,z) \cap \nu = \emptyset$, which means that $S(w,z) \subseteq \mathcal{R}_S(U;\nu;\mathbb{X})$. Therefore $\mathcal{R}_S(U;\mu;\mathbb{X}) \subseteq \mathcal{R}_S(U;\nu;\mathbb{X})$, proving the two sets equal. $\square$

Now we consider the relation between $\mathcal{R}_S(U;\mu;\mathbb{X})$ and the graphs $\mathcal{G}(\mu)$ and $\mathcal{G}(\mu \cup \{x\})$. Let $E_x^+(\mu)$ denote the edges found in $\mathcal{G}(\mu \cup \{x\})$ but not in $\mathcal{G}(\mu)$, and let $E_x^-(\mu)$ denote the edges found in $\mathcal{G}(\mu)$ but not in $\mathcal{G}(\mu \cup \{x\})$, that is

$$E_x^+(\mu) = \mathcal{G}(\mu \cup \{x\}) \setminus \mathcal{G}(\mu) \quad \text{and} \quad E_x^-(\mu) = \mathcal{G}(\mu) \setminus \mathcal{G}(\mu \cup \{x\}). \tag{14}$$

Lemma 6. *Suppose that μ and ν are supported on $\mathbb{X}_1$ and $\mathbb{X}_2$, respectively, and that $U \subseteq \mathbb{X}_1 \cap \mathbb{X}_2$. If $\mathcal{R}_S(U;\mu;\mathbb{X}_1) = \mathcal{R}_S(U;\nu;\mathbb{X}_2)$ and μ and ν agree on the closure of this set, then $E_x^{\pm}(\mu) = E_x^{\pm}(\nu)$ for any $x \in U$.*

Proof. Suppose that $x \in U$ and $\{x,y\} \in E_x^+(\mu)$. Then, by $\{x,y\} \subseteq \mathbb{X}_1$, $S(x,y) \cap \mu = \emptyset$ and (5) we have $S(x,y) \subseteq \mathcal{R}_S(U;\mu;\mathbb{X}_1) = \mathcal{R}_S(U;\nu;\mathbb{X}_2)$. Again by (5) the closure of this set contains y, and μ and ν agree on it. Thus $y \in \nu$ and $S(x,y) \cap \nu = \emptyset$, implying that $\{x,y\} \in E_x^+(\nu)$. Therefore $E_x^+(\mu) \subseteq E_x^+(\nu)$. By symmetry, the opposite inclusion holds as well.

Now suppose that $x \in U$ and $\{w,z\} \in E_x^-(\mu)$. As $\{w,z\} \in \mathcal{G}(\mu)$ we have $\{w,z\} \subseteq \mathbb{X}_1$ and $S(w,z) \cap \mu = \emptyset$. Hence $S(w,z) \subseteq \mathcal{R}_S(U;\mu;\mathbb{X}_1)$, and so is also a subset of $\mathcal{R}_S(U;\nu;\mathbb{X}_2)$. As ν agrees with μ on the closure of this set we have $\{w,z\} \subseteq \mathbb{X}_2$ and $S(w,z) \cap \nu = \emptyset$, so $\{w,z\} \in \mathcal{G}(\nu)$. As $\{w,z\} \notin \mathcal{G}(\mu \cup \{x\})$ we have $x \in S(w,z)$, and therefore $S(w,z) \cap (\nu \cup \{x\}) = \{x\}$. Hence $\{w,z\} \notin \mathcal{G}(\nu \cup \{x\})$, showing $E_x^-(\mu) \subseteq E_x^-(\nu)$. By symmetry, the opposite inclusion also holds. $\square$

Next, for $U \subseteq \mathbb{X}$ and μ supported on $\mathbb{X}$, define

$$R_S(U;\mu) = \sup\left\{\|y-x\| \colon y \in \mathcal{R}_S(U;\mu;\mathbb{X}),\, x \in U\right\}, \tag{15}$$

writing this quantity as $R_S(x;\mu)$ if $U = \{x\}$. The next lemma shows that $R_S(x;\mu)$ is stabilizing.

Lemma 7. *For any $x \in U$ the radius $R_S(U;\mu)$ given in* (15) *is stabilizing for $L(\mu)$, the statistic defined in* (3). *That is,*

$$D^2_{x,y} L(\mu) = 0 \quad \textit{for all } \{x,y\} \subseteq \mathbb{X} \textit{ with } x \in U \textit{ and } \|y-x\| > R_S(U;\mu).$$

Further for any $x \in \mathbb{R}^d$ and $r > 0$ such that $R_S(x;\mu) \leq r$, and $\{x_1,\ldots,x_n\} \subseteq B(x,r)^c$ we have

$$D_x L(\mu) = D_x L(\mu \cup \{x_1, \ldots x_n\}). \tag{16}$$

Proof. Assume that $x \in U$ and $\|y-x\| > R_S(U;\mu)$. We need to show that $D_x L(\mu \cup \{y\}) = D_x L(\mu)$. To do so, we will show that $E_x^{\pm}(\mu \cup \{y\}) = E_x^{\pm}(\mu)$. Since $\|y-x\| > R_S(U;\mu)$, the point y lies outside of $\overline{\mathcal{R}_S(U;\mu;\mathbb{X})}$. By Lemma 5, $\mathcal{R}_S(U;\mu\cup\{y\};\mathbb{X}) = \mathcal{R}_S(U;\mu;\mathbb{X})$. On the closure of this set, μ and $\mu \cup \{y\}$ agree, and so applying Lemma 6 with $\nu = \mu \cup \{y\}$ and $\mathbb{X}_1 = \mathbb{X}_2 = \mathbb{X}$ yields the first conclusion.

For the second claim, the first claim yields that for $x_1 \in B(x,r)^c$ we have

$$D_x L(\mu) = D_x L(\mu \cup \{x_1\}).$$

By Lemma 5, we have $R_S(x;\mu\cup\{x_1\}) \le r$. Thus applying the first claim again yields

$$D_x L(\mu \cup \{x_1\}) = D_x L(\mu \cup \{x_1, x_2\}).$$

Repeating this argument proves (16). $\square$

To prove that our stabilization radius has exponential tails under Poisson or binomial input we now use the scaled ball condition to show that lower and upper bounding $R_S(U;\mu)$ implies that there exists a ball empty of points of μ.

Lemma 8. *Assume the collection of forbidden regions $S(x,y)$ satisfies the scaled ball condition with $\delta > 0$ and let μ be supported on $\mathbb{X}$. If for some $t > 0$ and $0 < r_1 < r_2$ we have $B(u,t) \subseteq \mathbb{X}$ and $0 < r_1 < R_S(B(u,t);\mu) \le r_2$, then with $\mathcal{D}$ the normalized diameter in (6), there exists a ball of radius $(r_1 - 2t)\delta/\mathcal{D}$ lying within $B(u,r_2)\cap\mathbb{X}$ that contains no points of μ.*

Proof. Since $R_S(B(u,t);\mu) > r_1$, there exist $\{w,z\} \subseteq \mathbb{X}$ such that

- $S(w,z)$ contains no points of μ;
- $\overline{S(w,z)}$ contains some point of $B(u,t)$;
- and there exists $y \in S(w,z)$ and $x \in B(u,t)$ with $\|y-x\| > r_1$.

The diameter of $S(w,z)$ is then greater than $r_1 - 2t$ by the triangle inequality, and by the definition of the normalized diameter $\mathcal{D}$, we have $\|z-w\| > (r_1-2t)/\mathcal{D}$. By the scaled ball condition, $S(w,z)\cap\mathbb{X}$ contains a ball of radius $\delta(r_1-2t)/\mathcal{D}$. Since $R_S(B(u,t);\mu) \le r_2$, the set $S(w,z)$ is contained within $B(u,r_2)$ (in fact, it is contained in $B(u,r_2-t)$, but we will not need this fact), and so the ball is also contained within $B(u,r_2)\cap\mathbb{X}$. By virtue of being a subset of $S(w,z)$, the ball contains no points of μ. $\square$

Using Lemma 8 we now show our stabilization radius has exponential tails.

Proposition 9. *If the scaled ball condition (Assumption 1) holds for $\delta > 0$, and η_t satisfies Assumption 2 with $c_\lambda > 0$, then for any $0 \le \epsilon < 1/2$ and r such that $B(x,\epsilon r) \subseteq \mathbb{X}$,*

$$\mathbb{P}(R_S(B(x,\epsilon r);\eta_t) \ge r) \le C(1-2\epsilon)^{-d}\exp(-c_\lambda \kappa t r^d) \quad \textit{for all } r > 0 \tag{17}$$

with $\kappa = ((1-2\epsilon)\delta/\mathcal{D}\sqrt{d})^d$, and C a constant that depends only on d, $\mathcal{D}$, and δ. In particular,

$$\mathbb{P}(R_S(x;\eta_t) \ge r) \le C\exp(-c_\lambda \kappa t r^d) \quad \textit{for all } r > 0. \tag{18}$$

Proof. Let π_d be the volume of the d-dimensional ball of radius 1. First, we show that for any $s > 0$ and $0 \le \epsilon < 1/2$,

$$\mathbb{P}[s < R_S(B(x, \epsilon s); \mu) \le 2s] \le \frac{(2\mathcal{D}\sqrt{d})^d \pi_d}{(\delta(1-2\epsilon))^d} \exp\bigl(-c_\lambda \kappa t s^d\bigr). \tag{19}$$

To prove this claim, suppose that $s < R_S(B(x, \epsilon s); \mu) \le 2s$ and apply Lemma 8 to conclude that there exists a ball of radius $(1-2\epsilon)\delta s/\mathcal{D}$ within $B(x, 2s) \cap \mathbb{X}$ containing no points of μ. Now, consider the lattice $((1-2\epsilon)\delta s d^{-1/2}/\mathcal{D})\mathbb{Z}^d$. By a volume argument, $B(x, 2s) \cap \mathbb{X}$ contains at most $|B(0, 2s)|/((1-2\epsilon)\delta s d^{-1/2}/\mathcal{D})^d = (2\mathcal{D}\sqrt{d}/(1-2\epsilon)\delta)^d \pi_d$ lattice cells. Any ball of radius $(1-2\epsilon)\delta s/\mathcal{D}$ contains a cell of this lattice.

In all, we have shown that if $s < R_S(B(x, \epsilon s); \mu) \le 2s$, then at least one of the at most $(2\mathcal{D}\sqrt{d}/(1-2\epsilon)\delta)^d \pi_d$ lattice cells within $B(x, 2s) \cap \mathbb{X}$ contains no point of μ. With binomial input, applying (2), the empty cell probability is bounded by

$$\left[1 - c_\lambda \left(\frac{(1-2\epsilon)\delta}{\mathcal{D}\sqrt{d}}\right)^d s^d\right]^t \le \exp\left[-c_\lambda \left(\frac{(1-2\epsilon)\delta}{\mathcal{D}\sqrt{d}}\right)^d t s^d\right]. \tag{20}$$

With Poisson input, each lattice cell contains no point of μ with probability at most the right hand side of (20). A union bound now proves (19).

Now consider $r > 0$, arbitrary. If $\exp\bigl(-c_\lambda \kappa t r^d\bigr) > 1/2$, then (17) is trivially satisfied with $C = 2$. Otherwise, applying a union bound using (19) with $s = r, 2r, 4r, \ldots$ gives

$$\mathbb{P}[R_S(B(x, \epsilon r); \mu) > r] \le \frac{(2\mathcal{D}\sqrt{d})^d \pi_d}{((1-2\epsilon)\delta)^d} \sum_{i=0}^{\infty} \exp\bigl(-c_\lambda \kappa t (2^i r)^d\bigr).$$

Using $\exp\bigl(-c_\lambda \kappa t r^d\bigr) \le 1/2$, inequality (18) may now be established by bounding the sum in the above inequality by a geometric series summing to $2\exp\bigl(-c_\lambda \kappa t r^d\bigr)$. □

3 Functionals of forbidden regions graphs satisfy a Berry-Esseen bound

In this section we prove Theorem 1, starting with the Poisson case. For $t \ge 1$, let η_t be a Poisson process with intensity $\lambda_t = t\lambda$ for some fixed finite measure λ on $\mathbb{X}$. For a functional F_t on η_t with finite, non-zero variance, let

$$\widetilde{F}_t = (F_t - \mathbb{E}F_t)/\sqrt{\mathrm{Var}(F_t)}.$$

Proposition 10 (Proposition 1.3, Last, Peccati and Schulte [7]). *Let $\mathbb{E}F_t^2 < \infty, t \ge 1$, and assume there are finite positive constants p_1, p_2 and c such that*

$$\mathbb{E}|D_x F_t|^{4+p_1} \le c \quad \lambda\text{-a.e. } x \in \mathbb{X}, t \ge 1 \tag{21}$$

and

$$\mathbb{E}|D^2_{x,y} F_t|^{4+p_2} \le c \quad \lambda^2\text{-a.e. } (x, y) \in \mathbb{X}^2, t \ge 1. \tag{22}$$

Moreover, assume that for some $v > 0$

$$\frac{\mathrm{Var}(F_t)}{t} > v \quad \textit{for all } t \geq 1, \tag{23}$$

and that

$$m := \sup_{x \in \mathbb{X}, t \geq 1} \int \mathbb{P}(D^2_{x,y} F_t \neq 0)^{p_2/(16+4p_2)} \lambda_t(dy) < \infty. \tag{24}$$

Then there exists a finite constant C, depending uniquely on c, p_1, p_2, v, m and $\lambda(\mathbb{X})$ such that for N a standard Gaussian random variable,

$$\max\left\{ d_W(\widetilde{F}_t, N), d_K(\widetilde{F}_t, N) \right\} \leq C t^{-1/2} \quad \textit{for all } t \geq 1.$$

We first prove Lemma 11, a bound on the derivative of the functional L in (3), which is used when considering both Poisson and binomial input processes. In preparation, for any finite point configuration $\mu \subseteq \mathbb{X}$ and $x \in \mathbb{X} \setminus \mu$, we let

$$\begin{aligned} &A(x; \mu) \\ &\quad = \{z \in \mu : \exists w \in \mu, w \neq z, S(w, z) \cap (\mu \cup \{x\}) = \{x\}\} \bigcup \{z \in \mu : S(x, z) \cap \mu = \emptyset\}. \end{aligned}$$

Recalling (13), we see

$$A(x; \mu) \subseteq \bigcup \left\{ \overline{S(w,z)} : \{w, z\} \subseteq \mathbb{X}, S(w,z) \cap \mu = \emptyset, x \in \overline{S(w,z)} \right\} \subseteq \overline{\mathcal{R}_S(x; \mu; \mathbb{X})}. \tag{25}$$

Let $\mathbf{N}_{\mathbb{X}}$ denote the set of all locally finite point processes supported on $\mathbb{X}$.

Lemma 11. *Let $\mu \in \mathbf{N}_{\mathbb{X}}$ and $x \in \mathbb{X}$, and let $F = L(\mu)$ where $L(\cdot)$ is given in (3) with $|\psi(x)| \leq C\|x\|^\alpha$ for some $\alpha \geq 0, C > 0$. Then there is a constant C_α, depending only on α, such that*

$$|D_x F| \leq C_\alpha \sum_{z \in A(x;\mu)} \|z - x\|^\alpha. \tag{26}$$

Proof. Let $\mu \in \mathbf{N}_{\mathbb{X}}$. For $x \in \mu$ we have $D_x F = 0$. Otherwise, we define

$$I(zw;\, \mu) = \mathbf{1}(\mu \cap S(z, w) = \emptyset)\|z - w\|^\alpha.$$

Noting that the insertion of x into μ can only break existing edges and form new edges incident to x, we have

$$\begin{aligned} D_x F &= \frac{1}{2} \sum_{\substack{\{z,w\} \subseteq \mu \\ z \neq w}} \Big[I(zw;\, \mu \cup \{x\}) - I(zw;\, \mu) \Big] + \sum_{z \in \mu} I(zx;\, \mu \cup \{x\}) \\ &= \frac{1}{2} \sum_{\substack{\{z,w\} \subseteq \mu \\ z \neq w, S(z,w) \cap (\mu \cup \{x\}) = \{x\}}} \Big[I(zw;\, \mu \cup \{x\}) - I(zw;\, \mu) \Big] + \sum_{z \in \mu, S(z,x) \cap \mu = \emptyset} I(zx;\, \mu \cup \{x\}) \\ &= -\frac{1}{2} \sum_{\substack{\{z,w\} \subseteq \mu \\ z \neq w, S(z,w) \cap (\mu \cup \{x\}) = \{x\}}} \psi(z - w) + \sum_{z \in \mu, S(z,x) \cap \mu = \emptyset} \psi(z - x). \end{aligned}$$

For the first term we note

$$|\psi(z-w)| \le C\|z-w\|^\alpha \le C_\alpha \left(\|z-x\|^\alpha + \|w-x\|^\alpha\right) \quad \text{where } C_\alpha = C\max(1, 2^{\alpha-1}),$$

so that

$$\begin{aligned} |D_x F| &\le C_\alpha \sum_{\substack{\{z,w\}\subseteq\mu \\ z\ne w, S(z,w)\cap(\mu\cup\{x\})=\{x\}}} \|z-x\|^\alpha + C \sum_{z\in\mu, S(z,x)\cap\mu=\emptyset} \|z-x\|^\alpha \\ &\le C_\alpha \sum_{z\in A(x;\mu)} \|z-x\|^\alpha, \end{aligned}$$

completing the argument. □

Proof of Theorem 1, Poisson input. We apply Proposition 10 to $F_t = t^{\alpha/d}L(\eta_t)$, with L as given in (3). First, the condition $\mathbb{E}F_t^2 < \infty$ is seen to be satisfied in light of the inequality $|F_t| \le t^{\alpha/d} C \left(\sup_{\{x,y\}\subseteq\mathbb{X}} \|y-x\|\right)^\alpha |\eta_t|^2$, where $|\nu|$ denotes the number of points of the point process ν.

As Assumption 3 holds by hypothesis, we have

$$\operatorname{Var}(t^{\alpha/d}L(\eta_t)) \ge v_\alpha t,$$

verifying (23).

Next, choosing p_1 and p_2 both equal to 1 in (21), (22) and (24), we verify, respectively,

$$\mathbb{E}|D_x F_t|^5 \le c, \quad \lambda\text{-a.e., } x \in \mathbb{X}, t \ge 1, \tag{27}$$

$$\mathbb{E}|D^2_{x,y} F_t|^5 \le c, \quad \lambda^2\text{-a.e., } (x,y) \in \mathbb{X}\times\mathbb{X}, t \ge 1, \tag{28}$$

and

$$\sup_{x\in\mathbb{X}, t\ge 1} t \int_{\mathbb{X}} \mathbb{P}(D^2_{x,y}F_t \ne 0)^{1/20} \lambda(dy) < \infty. \tag{29}$$

We first note that by (25),

$$y \in A(x;\mu) \quad \text{implies} \quad R_S(x;\mu) \ge \|y-x\|. \tag{30}$$

Let $\mathcal{A}$ be any finite subset of $\mathbb{X}$. Applying (26) in the first line, the version (2.10) of [8] of Mecke's formula in the second line, Fubini's theorem in the third, (30) in the fourth, Lemma 5 providing the monotonicity of $R(x;\cdot)$ in the fifth, Proposition 9 in the sixth, Assumption 2 in the seventh, and letting σ_d denote the surface measure of the sphere $\mathcal{S}^{d-1}$ in $\mathbb{R}^d$, we obtain

$$
\begin{aligned}
&C_\alpha^{-5}\mathbb{E}|D_xF_t(\eta_t\cup\mathcal{A})|^5 \le t^{5\alpha/d}\mathbb{E}\sum_{y_1,\dots,y_5}\prod_{1\le i\le 5}\|y_i-x\|^\alpha\mathbf{1}(y_i\in A(x;\eta_t\cup\mathcal{A}))\\
&= t^{5\alpha/d}\mathbb{E}\int_{\mathbb{X}^5}\prod_{1\le i\le 5}\|y_i-x\|^\alpha\mathbf{1}(\{y_1,\dots,y_5\}\subseteq A(x;\eta_t\cup\mathcal{A}\cup\{y_1,\dots,y_5\}))t^5\lambda(dy_1)\cdots\lambda(dy_5)\\
&= t^{5(1+\alpha/d)}\int_{\mathbb{X}^5}\prod_{1\le i\le 5}\|y_i-x\|^\alpha\mathbb{P}(\{y_1,\dots,y_5\}\subseteq A(x;\eta_t\cup\mathcal{A}\cup\{y_1,\dots,y_5\}))\lambda(dy_1)\cdots\lambda(dy_5)\\
&\le t^{5(1+\alpha/d)}\int_{\mathbb{X}^5}\prod_{1\le i\le 5}\|y_i-x\|^\alpha\mathbb{P}(R_S(x;\eta_t\cup\mathcal{A}\cup\{y_1,\dots,y_5\})\ge\max_{1\le i\le 5}\|y_i-x\|)\lambda(dy_1)\cdots\lambda(dy_5)\\
&\le t^{5(1+\alpha/d)}\int_{\mathbb{X}^5}\prod_{1\le i\le 5}\|y_i-x\|^\alpha\mathbb{P}(R_S(x;\eta_t)\ge\max_{1\le i\le 5}\|y_i-x\|)\lambda(dy_1)\cdots\lambda(dy_5)\\
&\le Ct^{5(1+\alpha/d)}\int_{\mathbb{X}^5}\prod_{1\le i\le 5}\|y_i-x\|^\alpha\exp(-c_\lambda\kappa t\max_{1\le i\le 5}\|y_i-x\|^d)\lambda(dy_1)\cdots\lambda(dy_5)\\
&\le Cb_\lambda^5t^{5(1+\alpha/d)}\int_{\mathbb{R}^{d\times 5}}\prod_{1\le i\le 5}\|y_i-x\|^\alpha\exp(-c_\lambda\kappa t\max_{1\le i\le 5}\|y_i-x\|^d)dy_1\cdots dy_5\\
&= Cb_\lambda^5t^{5(1+\alpha/d)}\int_{\mathbb{R}^{d\times 5}}\prod_{1\le i\le 5}\|y_i\|^\alpha\exp(-c_\lambda\kappa t\max_{1\le i\le 5}\|y_i\|^d)dy_1\cdots dy_5\\
&\le Cb_\lambda^5t^{5(1+\alpha/d)}\int_{\mathbb{R}^{d\times 5}}\prod_{1\le i\le 5}\|y_i\|^\alpha\exp\left(-c_\lambda\kappa t\sum_{1\le i\le 5}\|y_i\|^d/5\right)dy_1\cdots dy_5\\
&= C\left(b_\lambda t^{1+\alpha/d}\int_{\mathbb{R}^d}\|y\|^\alpha\exp\left(-c_\lambda\kappa t\|y\|^d/5\right)dy\right)^5\\
&= C\left(b_\lambda t^{1+\alpha/d}\sigma_d\int_0^\infty r^{\alpha+d-1}\exp\left(-c_\lambda\kappa tr^d/5\right)dr\right)^5\\
&= C\left(b_\lambda\sigma_d\int_0^\infty r^{\alpha+d-1}\exp(-c_\lambda\kappa r^d/5)dr\right)^5\\
&\le C,
\end{aligned}
$$

where the final constant C depends uniquely on $d,\delta,\mathcal{D},c_\lambda$ and b_λ. Letting $\mathcal{A}=\emptyset$ shows that (27) is satisfied, and letting $\mathcal{A}=\{y\}$ we see that (28) also holds, as (9) yields

$$\mathbb{E}|D_{x,y}^2F_t|^5\le 16\left(\mathbb{E}|D_xF_t(\eta_t\cup\{y\})|^5+\mathbb{E}|D_xF_t(\eta_t)|^5\right).$$

We now show condition (29) is satisfied. Letting $x\in\mathbb{X}$ be arbitrary, invoking Assumption 2 and Lemma 7, followed by Proposition 9, we obtain

$$
\begin{aligned}
b_\lambda^{-1}t\int_{\mathbb{X}}\mathbb{P}(D_{x,y}^2F_t\ne 0)^{1/20}\lambda(dy) &\le t\int_{\mathbb{X}}\mathbb{P}(D_{x,y}^2F_t\ne 0)^{1/20}dy\le t\int_{\mathbb{X}}\mathbb{P}(R_S(x;\eta_t)\ge\|y-x\|)^{1/20}dy\\
&\le Ct\int_{\mathbb{X}}\exp\left(-c_\lambda\kappa t\|y-x\|^d/20\right)dy = Ct\int_{\mathbb{X}-x}\exp\left(-c_\lambda\kappa t\|y\|^d/20\right)dy\\
&\le Ct\int_{\mathbb{R}^d}\exp\left(-c_\lambda\kappa t\|y\|^d/20\right)dy = Ct\sigma_d\int_0^\infty\exp\left(-c_\lambda\kappa tr^d/20\right)r^{d-1}dr=\frac{20C\sigma_d}{dc_\lambda\kappa}.
\end{aligned}
$$

Hence, the supremum over $x \in \mathbb{X}$ and $t \geq 1$ in (29) is finite, and the proof of the theorem is complete. $\square$

We shall now use the results of [6] to derive similar bounds in the binomial setting. Here $n \in \mathbb{N}$ plays the former role of t and $X = (X_1, \dots, X_n)$ is a vector of independent variables with distribution λ over $\mathbb{X}$, and $\eta_n = \{X_1, \dots, X_n\}$. Let $X', \widetilde{X}$ be independent copies of X. We write $U \overset{a.s.}{=} V$ if two variables U and V satisfy $\mathbb{P}(U = V) = 1$. In the vocabulary of [6], a random vector $Y = (Y_1, \dots, Y_n)$ is a recombination of $\{X, X', \widetilde{X}\}$ if for each $1 \leqslant i \leqslant n$, either $Y_i \overset{a.s.}{=} X_i, Y_i \overset{a.s.}{=} Y_i'$ or $Y_i \overset{a.s.}{=} \widetilde{X}_i$. For a vector $x = (x_1, \dots, x_n)$, and indexes $I = \{i_1, \dots, i_q\}$, define

$$x^{i_1,\dots,i_q} := (x_j, j \notin \{i_1, \dots, i_q\}),$$

For $1 \leq i, j \leq n$, and f a real valued function taking in $n, n-1$ or $n-2$ arguments in $\mathbb{R}^d$, let

$$\begin{aligned} \mathsf{D}_i f(X) &= f(X) - f(X^i) \quad \text{and} \\ \mathsf{D}_{i,j} f(X) &= f(X) - f(X^i) - f(X^j) + f(X^{i,j}), \quad \text{noting that } \mathsf{D}_{i,j} f(X) = \mathsf{D}_{j,i} f(X). \end{aligned} \tag{31}$$

For $X', \widetilde{X}$ independent copies of X let

$$\begin{aligned} B_n(f) &= \sup\{\gamma_{Y,Z}(f);\ (Y,Z) \text{ recombinations of } \{X, X', \widetilde{X}\}\} \quad \text{and} \\ B_n'(f) &= \sup\{\gamma'_{Y,Y',Z}(f);\ (Y,Y',Z) \text{ recombinations of } \{X, X', \widetilde{X}\}\}, \quad \text{where} \\ \gamma_{Y,Z}(f) &= \mathbb{E}\left[\mathbf{1}_{\{\mathsf{D}_{1,2} f(Y) \neq 0\}} \mathsf{D}_2 f(Z)^4\right] \quad \text{and} \\ \gamma'_{Y,Y',Z}(f) &= \mathbb{E}\left[\mathbf{1}_{\{\mathsf{D}_{1,2} f(Y) \neq 0,\ \mathsf{D}_{1,3} f(Y') \neq 0\}} \mathsf{D}_2 f(Z)^4\right] \end{aligned}$$

Assume $\mathbb{E} f(X) = 0$ and that $\sigma^2 := \mathrm{Var}(f(X))$ is nonzero and finite. Then Theorem 6.1 of [6], simplified by [6, Remark 5.2] and [6, Proposition 5.3] yields the following Kolmogorov distance bound for the normal approximation of $f(X)$, properly standardized.

Theorem 12 (Lachièze-Rey and Peccati [6])**.** *Let f be a functional taking in arguments of $n, n-1$, or $n-2$ elements of $\mathbb{X}$. Assume furthermore that f is invariant under permutation of its arguments, and that $\mathbb{E} f(X) = 0$. Define $\sigma^2 = \mathrm{Var}(f(X))$. Let $d(\cdot,\cdot)$ denote either the Kolmogorov or the Wasserstein distance. Then, for some $C > 0$ not depending on n,*

$$\begin{aligned} d(\sigma^{-1} f(X), N) \leqslant C \Bigg[&\frac{4\sqrt{2} n^{1/2}}{\sigma^2} \left(\sqrt{n B_n(f)} + \sqrt{n^2 B_n'(f)} + \sqrt{\mathbb{E} \mathsf{D}_1 f(X)^4} \right) \\ &+ \frac{n}{4\sigma^3} \sqrt{\mathbb{E} |\mathsf{D}_1 f(X)^6|} + \frac{\sqrt{2\pi}}{16\sigma^3} n \mathbb{E} |\mathsf{D}_1 f(X)^3| \Bigg], \end{aligned} \tag{32}$$

where N is a standard normal random variable.

For L as in (3) with $|\psi(x)| \leq C\|x\|^\alpha$ for some $\alpha \geq 0, C > 0$, let $F_n = n^{\alpha/d} L(\eta_n)$, and let the functional f, defined on ordered sets of variables, be given by $f(x_1, \dots, x_q) = F_n(\{x_1, \dots, x_q\})$ for any $q \geqslant 1$ and $\{x_1, \dots, x_q\} \subseteq \mathbb{R}^d$. We note that D defined in (31), and D as in (8), obey the relations

$$\mathsf{D}_i f(X) = D_{X_i} F_n(\eta_n \setminus \{X_i\}), \quad \text{and for } i \neq j \quad \mathsf{D}_{ij} f(X) = D_{X_i, X_j} F_n(\eta_n \setminus \{X_i, X_j\}). \tag{33}$$

We now show how Theorem 12, along with Proposition 13 and Lemma 14 below, proves the Kolmogorov and Wasserstein bounds of Theorem 1 for binomial input.

Proof of Theorem 1 for binomial input. In [6], the authors focus on the Kolmogorov distance, but the same bound is valid for the Wasserstein, even though it is not stated there formally. More precisely, we refer the reader to the inequality in Theorem 2.2 of [2], involving Wasserstein distance. The first term in this inequality, $\sigma^{-2}\sqrt{\mathrm{Var}(\mathbb{E}(T|W))}$, has been shown in [6] to be bounded by the terms of the first line of the right hand member of (32). The second term in the inequality of [2] is equal to $n\sigma^{-3}\mathbb{E}|\mathsf{D}_1 f(X)|^3$, also taken care of in (32). The term $\frac{n}{4\sigma^3}\sqrt{\mathbb{E}|\mathsf{D}_1 f(X)^6|}$ is in fact only necessary for Kolmogorov distance, and can be removed when treating the Wasserstein distance. Hence it suffices to show the bound (32) holds for the Kolmogorov distance.

Assumption 3 yields $\sigma^2 \geqslant Cn$ for some $C > 0$. Using (33) and Proposition 13 we obtain,

$$\sup_n \mathbb{E}[\mathsf{D}_1 f(X)^6] = \sup_n \int_{\mathbb{X}} \mathbb{E}|D_x F_n(\eta_{n-1})|^6 \lambda(dx) < \infty.$$

Applying Hölder's inequality, we find that there exists $C > 0$ such that

$$\frac{4\sqrt{2}n^{1/2}}{\sigma^2}\sqrt{\mathbb{E}\mathsf{D}_1 f(X)^4} + \frac{n}{4\sigma^3}\sqrt{\mathbb{E}|\mathsf{D}_1 f(X)^6|} + \frac{\sqrt{2\pi}}{16\sigma^3} n\mathbb{E}|\mathsf{D}_1 f(X)^3| \leqslant Cn^{-1/2}.$$

Lemma 14 yields C such that

$$B_n(f) \leqslant \frac{C}{n} \quad \text{and} \quad B_n'(f) \leqslant \frac{C}{n^2}. \tag{34}$$

Substituting these bounds into (32) completes the proof. □

We begin by proving the moment bound that was used above.

Proposition 13. *For any non-negative integer k,*

$$M := \sup_{n\geqslant 1, x\in\mathbb{X}, \mathcal{A}\subseteq\mathbb{X} \text{ finite}} \mathbb{E}|D_x F_n(\eta_{n-k} \cup \mathcal{A})|^6 < \infty.$$

Proof. We use a computation similar to the Poisson bound, where in place of Mecke's formula we interchange an integral and a finite sum. Let $[n]^6$ denote the collection of 6-tuples of elements of $\{1, 2, \dots, 6\}$. Let $I = (i_1, \dots, i_6) \in [n]^6$. Call λ^I the law of $(X_{i_1}, \dots, X_{i_q})$. Due to possible repetitions in I, it might be that for $j \neq k, i_j = i_k$, and $X_{i_j} = X_{i_k}$. Let $\tilde{I} = \{i_1, \dots, i_6\}$, the set of indexes that appear in I, and for $i \in \tilde{I}$, let $m(i, I)$ be the multiplicity of i in I. The law λ^I could be written as

$$\lambda^I(dx_1, \dots, dx_6) = \lambda^{|\tilde{I}|}(dx_i, i \in \tilde{I})\mathbf{1}_{\{x_j = x_k, i_j = i_k\}}.$$

Further, let $c' > 0$ satisfy $c_\lambda \kappa(n-6) \geq c'n$ for $n \geq 7$.

Arguing as in the proof of Theorem 1 for the Poisson case, starting with the application of Lemma 11 for the first inequality, we have

$$
\begin{aligned}
C_\alpha^{-6}\mathbb{E}|D_xF(\eta_n\cup\mathcal{A})|^6 &\leqslant \mathbb{E}\sum_{I\in[n]^6}\prod_{1\le k\le 6}\|X_{i_k}-x\|^\alpha\mathbf{1}(X_{i_k}\in A(x;\eta_n\cup\mathcal{A}))\\
&\leqslant \sum_{I\in[n]^6}\int_{\mathbb{X}^6}\mathbb{E}\prod_{1\le k\le 6}\|x_k-x\|^\alpha\mathbf{1}(x_k\in A(x;\eta_{n-|\tilde I|}\cup\{x_k,k\in\tilde I\}\cup\mathcal{A}))\lambda^I(dx_1,\dots,dx_6)\\
&\leqslant \sum_{I\in[n]^6}\int_{\mathbb{X}^{|\tilde I|}}\prod_{k\in\tilde I}\|x_k-x\|^{m(k,I)\alpha}\mathbb{P}(x_k\in A(x;\eta_{n-|\tilde I|}\cup\{x_k,k\in\tilde I\}\cup\mathcal{A}))\lambda(dx_k)\\
&\leqslant \sum_{I\in[n]^6}\int_{\mathbb{X}^{|\tilde I|}}\prod_{k\in\tilde I}\|x_k-x\|^{m(k,I)\alpha}\mathbb{P}(R_S(x;\eta_{n-|\tilde I|}\cup\{x_k,k\in\tilde I\}\cup\mathcal{A})\geqslant\max_{k\in\tilde I}\|x_k-x\|)\lambda(dx_k)\\
&\leqslant \sum_{I\in[n]^6}\int_{\mathbb{X}^{|\tilde I|}}\prod_{k\in\tilde I}\|x_k-x\|^{m(k,I)\alpha}\mathbb{P}(R_S(x;\eta_{n-|\tilde I|})\geqslant\max_{k\in\tilde I}\|x_k-x\|)\lambda(dx_k)\\
&\leqslant C\sum_{I\in[n]^6}\int_{\mathbb{X}^{|\tilde I|}}\prod_{k\in\tilde I}\|x_k-x\|^{m(k,I)\alpha}\exp(-c_\lambda\kappa(n-|\tilde I|)\sum_{k\in\tilde I}\|x_k-x\|^d/6)\lambda(dx_k)\\
&\leqslant C\sum_{I\in[n]^6}\int_{\mathbb{X}^{|\tilde I|}}\prod_{k\in\tilde I}\|x_k-x\|^{m(k,I)\alpha}\exp(-c'n\sum_{k\in\tilde I}\|x-x_k\|^d/6)\lambda(dx_k)\\
&\leqslant C\sum_{I\in[n]^6}\prod_{k\in\tilde I}\int_{\mathbb{X}}\|y-x\|^{m(k,I)\alpha}\exp(-c'n\|y-x\|^d/6)\lambda(dy)\\
&\leqslant C\sum_{I\in[n]^6}\prod_{k\in\tilde I}b_\lambda^{|\tilde I|}\int_{\mathbb{R}^d}\|y-x\|^{m(k,I)\alpha}\exp(-c'n\|y-x\|^d/6)dy\\
&\leqslant C\sum_{I\in[n]^6}\prod_{k\in\tilde I}\int_{\mathbb{R}^d}\|y\|^{m(k,I)\alpha}\exp(-c'n\|y\|^d/6)dy\\
&\leqslant C\sum_{I\in[n]^6}\prod_{k\in\tilde I}\int_{\mathbb{R}^d}n^{-m(k,I)\alpha/d}\|y\|^{m(k,I)\alpha}\exp(-c'\|y\|^d/6)n^{-1}dy\\
&\leqslant C\sum_{I\in[n]^6}n^{-6\alpha/d}\prod_{k\in\tilde I}n^{-1}\int_{\mathbb{R}^d}\|y\|^{m(k,I)\alpha/d}\exp(-c'\|y\|^d/6)\lambda(dy)\\
&\le Cn^{-6\alpha/d}\sum_{I\in[n]^6}n^{-|\tilde I|}
\end{aligned}
$$

where the final constant only depends on α, d, b_λ and the constant of Proposition 9. As there are $O(n^m)$ elements $I\in[n]^6$ where $|\tilde I|=m$ for $1\le m\le 6$, the sum is of order $O(n^{-6\alpha/d})$. Now applying the bound to $n^{\alpha/d}F(\eta_{n-k}\cup\mathcal{A})$ gives the claim. ∎

Lemma 14. *There exists C such that*

$$B_n(f)\leqslant\frac{C}{n}\quad\textit{and}\quad B_n'(f)\leqslant\frac{C}{n^2}.$$

Proof. We begin with the first inequality. Let $Y=(Y_1,\dots,Y_n)$ and $Z=(Z_1,\dots,Z_n)$ be recombinations of $\{X,X',\widetilde X\}$. Note that Y_1 is independent of $\{Y_2,Z_2\}$ because Y_1 is either X_1,X_1' or $\widetilde X_1$ and these three variables are independent of $X_2,X_2',\widetilde X_2$. Also, either

Y_2, Z_2 both equal the same element of $\{X_2, X_2', \tilde{X}_2\}$ (almost surely), or they are assigned to different elements of this set. In the first case, $Y_2 \overset{a.s.}{=} Z_2$, while in the second case Y_2 and Z_2 are independent. Letting λ^{Y_1,Y_2,Z_2} denote the law of (Y_1, Y_2, Z_2), therefore we either have $d\lambda^{Y_1,Y_2,Z_2}(y_1, y_2, z_2) = \mathbf{1}_{\{y_2=z_2\}} d\lambda(y_1) d\lambda(y_2)$ if $Y_2 \overset{a.s.}{=} Z_2$, or $d\lambda^{Y_1,Y_2,Z_2}(y_1, y_2, z_2) = d\lambda(y_1) d\lambda(y_2) d\lambda(z_2)$ if Z_2 and Y_2 are independent.

Using the conditional Hölder inequality with conjugate exponents $3, 3/2$ yields that for every $\{y_1, y_2, z_2\} \subseteq \mathbb{X}$, with the following conditionings valid a.s.,

$$\begin{aligned}\mathbb{E}\left[\mathbf{1}_{\{\mathsf{D}_{1,2}f(Y)\neq 0\}} \mathsf{D}_2 f(Z)^4 | Y_1 = y_1, Y_2 = y_2, Z_2 = z_2\right]&\\ \leqslant \mathbb{P}(\mathsf{D}_{1,2} f(Y) \neq 0 | Y_1 = y_1, Y_2 = y_2, Z_2 = z_2)^{1/3}&\\ \times \mathbb{E}\left[\mathsf{D}_2 f(Z)^6 | Y_1 = y_1, Y_2 = y_2, Z_2 = z_2\right]^{2/3}.& \quad (35)\end{aligned}$$

Either $Z_2 \overset{a.s.}{=} Y_2$, and when conditioning on $Y_2 = y_2, Z_2 = z_2$ we must take $y_2 = z_2$, or Y_2 and Z_2 are independent. In both cases, for λ^{Y_1,Y_2,Z_2}-a.e. (y_1, y_2, z_2), with $\mathcal{L}(U)$ denoting the law of U, and adopting similar notation for the conditional law, by (33) we have

$$\mathcal{L}\left(\mathsf{D}_{1,2} f(Y) | Y_1 = y_1, Y_2 = y_2, Z_2 = z_2\right) = \mathcal{L}(D_{y_1,y_2} F_n(\eta_{n-2})).$$

Similarly, separately studying the cases $Y_1 \overset{a.s.}{=} Z_1$ and (Y_1, Z_1) independent, one has for λ^{Y_1,Y_2,Z_2}-a.e.(y_1, y_2, z_2),

$$\mathcal{L}\left(\mathsf{D}_2 f(Z) | Y_1 = y_1, Y_2 = y_2, Z_2 = z_2\right) = \begin{cases} \mathcal{L}(D_{z_2} F_n(\eta_{n-2} \cup \{y_1\})) & \text{if } Y_1 \overset{a.s.}{=} Z_1 \\ \mathcal{L}(D_{z_2} F_n(\eta_{n-1})) & \text{if } Y_1, Z_1 \text{ are independent.} \end{cases}$$

Applying Proposition 13 with $x = z_2, k = 2$ and $\mathcal{A} = \{y_1\}$ for the first case above, and similarly for the second, shows the final factor in (35) is bounded by M. Now integrating (35) over λ^{Y_1,Y_2,Z_2} and applying Lemma 7 and Proposition 9 yields

$$\begin{aligned}\gamma_{Y,Z}(f) &\leqslant M^{2/3} \int_{\mathbb{X}^2} \mathbb{P}(D_{y_1,y_2} F_n(\eta_{n-2}) \neq 0)^{1/3} dy_1 dy_2 \\ &\leq M^{2/3} \int_{\mathbb{X}^2} \mathbb{P}(R_S(y_1; \eta_{n-2}) \geq \|y_2 - y_1\|)^{1/3} dy_1 dy_2 \\ &\leqslant M^{2/3} \int_{\mathbb{X}^2} C \exp(-c_\lambda \kappa (n-2) \|y_1 - y_2\|^d / 3) dy_1 dy_2 \leqslant \frac{C}{n}\end{aligned}$$

for some final constant $C > 0$, demonstrating the first inequality in (34).

The second inequality in (34) is proved similarly. Let Y, Y', Z be recombinations of $\{X, X', \widetilde{X}\}$. Applying the conditional Hölder inequality for a three way product,

$$\begin{aligned}\gamma'_{Y,Y',Z}(f) \leqslant \int_{\mathbb{X}^5} &\mathbb{P}(\mathsf{D}_{1,2} f(Y) \neq 0 | Y_1 = y_1, Y_2 = y_2, Y_1' = y_1', Y_3' = y_3', Z_2 = z_2)^{1/6} \\ &\mathbb{P}(\mathsf{D}_{1,3} f(Y') \neq 0 | Y_1 = y_1, Y_2 = y_2, Y_1' = y_1', Y_3' = y_3', Z_2 = z_2)^{1/6} \\ &\mathbb{E}\left[\mathsf{D}_2 f(Z)^6 | Y_1 = y_1, Y_2 = y_2, Y_1' = y_1', Y_3' = y_3', Z_2 = z_2\right]^{2/3} \\ &d\lambda^{Y_1,Y_2,Y_1',Y_3',Z_2}(y_1, y_2, y_1', y_3', z_2),\end{aligned}$$

with the conditionings valid $\lambda^{Y_1,Y_2,Y_1',Y_3',Z_2}$-a.e. We have, for some $m \in \{0,1,2\}$ and $\mathcal{A} \subseteq \mathbb{X}$ with $|\mathcal{A}| = m$, depending on how the recombination Z is composed,

$$\mathcal{L}\left(\mathsf{D}_2 f(Z) | Y_1 = y_1, Y_2 = y_2, Y_1' = y_1', Y_3' = y_3', Z_2 = z_2\right) = \mathcal{L}(D_{z_2} F_n(\eta_{n-1-m} \cup \mathcal{A})),$$

whenever $Y_2 \overset{a.s.}{=} Z_2$, necessitating $y_2 = z_2$, or Y_2, Z_2 are independent. Hence, Proposition 13 yields that the last term in the integral is a.e. bounded by $M^{2/3}$.

The values of Y_1', Z_2 are irrelevant to Y once we have conditioned on the values of Y_1, Y_2. Therefore we have

$$\begin{aligned}
&\mathbb{P}(\mathsf{D}_{1,2} f(Y) \neq 0 | Y_1 = y_1, Y_2 = y_2, Y_1' = y_1', Y_3' = y_3', Z_2 = z_2) \\
&\qquad = \begin{cases} \mathbb{P}(D_{y_1,y_2} F_n(\eta_{n-2}) \neq 0) \text{ if } Y_3 \text{ is independent of } Y_3' \\ \mathbb{P}(D_{y_1,y_2} F_n(\eta_{n-3} \cup \{y_3'\}) \neq 0) \text{ if } Y_3 \overset{a.s.}{=} Y_3' \end{cases} \\
&\qquad \leqslant \mathbb{P}(R_S(y_1, \eta_{n-3}) \geqslant \|y_1 - y_2\|) \leqslant C \exp(-c_\lambda \kappa (n-3) \|y_1 - y_2\|^d),
\end{aligned}$$

where we have used that $R_S(x;\mu)$ stabilizes, from Lemma 7,

$$\max(R_S(y_1;\eta_{n-2}), R_S(y_1;\eta_{n-3} \cup \{y_3'\})) \leqslant R_S(y_1;\eta_{n-3}),$$

justifed by the monotonicity property provided by Lemma 5, and Proposition 9. Similarly, as the value of Y_1 is irrelevant to Y' once we condition on Y_1', and Y_2' will either equal one of Y_2 or Z_2 a.s., or be independent of both, for some $m \in \{0,1\}$ and some set $\mathcal{A}$ with m elements,

$$\begin{aligned}
&\mathbb{P}(\mathsf{D}_{1,3} f(Y') \neq 0 | Y_1 = y_1, Y_2 = y_2, Y_1' = y_1', Y_3' = y_3', Z_2 = z_2) \\
&\qquad \leqslant \mathbb{P}(R_S(y_1', \eta_{n-2-m} \cup \mathcal{A}) \geqslant \|y_1' - y_3'\|) \\
&\qquad \leqslant C \exp(-c_\lambda \kappa (n-3) \|y_1' - y_3'\|^d).
\end{aligned}$$

If $Y_1 \overset{a.s.}{=} Y_1'$ and $n \geq 4$ we have

$$\begin{aligned}
\gamma'_{Y,Y',Z}(f) &\leqslant C \int_{\mathbb{X}} \left[\int_{\mathbb{X}} \exp(-c_\lambda \kappa (n-3)\|y_1 - y_2\|^d/6) dy_2\right] \left[\int_{\mathbb{X}} \exp(-c_\lambda \kappa (n-3)\|y_1 - y_3'\|^d/6) dy_3'\right] dy_1 \\
&\leqslant C \int_{\mathbb{X}} \left[\int_{\mathbb{R}^d} \exp(-c_\lambda \kappa (n-3)\|y_1 - y_2\|^d/6) dy_2\right]^2 dy_1 \\
&\leqslant C \int_{\mathbb{X}} \left[(n-3)^{-1} \int_{\mathbb{R}^d} \exp(-c_\lambda \kappa \|y_1 - y_2\|^d/6) dy_2\right]^2 dy_1 \\
&\leqslant \frac{C}{n^2}.
\end{aligned}$$

If Y_1 and Y_1' are independent,

$$\begin{aligned}
\gamma'_{Y,Y',Z}(f) C &\leqslant \int_{\mathbb{X}^2} \exp(-c_\lambda \kappa (n-3)\|y_1 - y_2\|^d/6) dy_1 dy_2 \int_{\mathbb{X}^2} \exp(-c_\lambda \kappa (n-3)\|y_1' - y_3'\|^d/6) dy_1' dy_3' \\
&= \left[\int_{\mathbb{X}} \left[\int_{\mathbb{X}} \exp(-c_\lambda \kappa (n-3)\|y_1 - y_2\|^d/6) dy_1\right] dy_2\right]^2 \\
&\leqslant C \left[\int_{\mathbb{X}} \left[(n-3)^{-1} \int_{\mathbb{R}^d} \exp(-c_\lambda \kappa \|y_1 - y_2\|^d/6) dy_1\right] dy_2\right]^2 \\
&\leqslant \frac{C}{n^2}.
\end{aligned}$$

In both cases, $B_n'(f) \leqslant C/n^2$, which concludes the proof. □

4 Variance lower bounds

In this section, we prove Theorem 2, providing a lower bound on $\operatorname{Var} L(\eta_t)$ under broad conditions on the collection of forbidden regions. One key step of the proof, accomplished in Lemma 27, is to show that if the input process is split into two independent processes then the first process is likely to contain many *influential* point pairs. Intuitively, a point pair $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$ is influential if an additional process point falling in the vicinity of x produces an effect on L that differs from its effect had the point fallen in the vicinity of y. To prove Theorem 2, we show that conditional on the first process containing many influential pairs, the effect of adding the second process contributes at least an amount $\Omega(t)$, a quantity satisfying $\liminf_{t\to\infty} \Omega(t)/t > 0$, to the variance of $L(\eta_t)$.

Throughout this section we assume that the function ψ used to define L in (3) is given by $\psi(x) = \|x\|^\alpha$ for some $\alpha \geq 0$. In addition, we will be working at a different scale from the rest of the paper, considering Poisson and binomial processes of constant intensity on a growing space, rather than of growing intensity on a fixed space. The reason for using this scaling is that we will need to consider the limiting case of a Poisson process on $\mathbb{R}^d$.

For convenience, take the Lebesgue measure of $\mathbb{X}$ to equal one. For any $t \geq 1$ let η_t denote a homogeneous Poisson point process on $t^{1/d}\mathbb{X}$ with intensity 1, and let $\mathcal{U}_t$ be a binomial process of $\lceil t \rceil$ points independently and uniformly placed in $t^{1/d} \cap \mathbb{X}$. We couple all $\eta_t, t \geq 1$ by defining $\eta_t = \eta_\infty \cap t^{1/d}\mathbb{X}$ where η_∞ is a homogeneous Poisson point process on $\mathbb{R}^d$ of intensity 1. We assume throughout that $0 \in \mathbb{X}$ and $\mathbb{X}$ is convex.

For $x \in \mathbb{R}^d$ and $r > 0$ let

$$t_0(x, r) = \inf\{t \colon B(x, r) \subseteq t^{1/d}\mathbb{X}\}.$$

Before stating the following result we recall the definition of $E_x^\pm(\mu)$ from (14), and inform the reader that the constant r_0 may take on different values in the statements below.

Proposition 15. *Assume that the forbidden regions satisfy the scaled ball condition (Assumption 1) for some $\delta > 0$ and all $x \in \mathbb{R}^d$ and positive t, r when the role of $\mathbb{X}$ is played by $t^{1/d}\mathbb{X} \cap B(x, r)$. Then for any $\epsilon > 0$, there exists r_0 such that for all $r > r_0$, all $x \in \mathbb{R}^d$ and all $t > t_0(x, r)$, allowing $t = \infty$,*

$$\mathbb{P}\Big(E_x^\pm(\eta_t) = E_x^\pm(\eta_\infty \cap B(x, r))\Big) \geq 1 - \epsilon, \tag{36}$$

and for all $t \in \mathbb{N}$ satisfying $t > t_0$,

$$\mathbb{P}\Big(E_x^\pm(\mathcal{U}_t) = E_x^\pm(\mathcal{U}_t \cap B(x, r))\Big) \geq 1 - \epsilon. \tag{37}$$

Before proving Proposition 15, first observe that (36) could be equivalently stated with η_t appearing instead of η_∞, since if $B(x, r) \subseteq t^{1/d}\mathbb{X}$, then $\eta_t \cap B(x, r) = \eta_\infty \cap B(x, r)$.

For $x \in \mathbb{R}^d$, $r > 0$ and $t > t_0(x, r)$, define the event

$$\Phi(x, r, t, \mu) = \big\{\mathcal{R}_S\big(x; \mu; t^{1/d}\mathbb{X}\big) \cap \mathcal{R}_S\big(x; \mu \cap B(x, r); B(x, r)\big)^c \neq \emptyset\big\}.$$

In other words, $\Phi(x, r, t, \mu)$ is the event that "the region that could be affected when we add x to μ" grows when we increase the viewing window from $B(x, r)$ to $t^{1/d}\mathbb{X}$. Similarly,

$$\Psi(x, r, t, \mu) = \big\{\mathcal{R}_S\big(x; \mu \cap B(x, r); B(x, r)\big) \cap \mathcal{R}_S\big(x; \mu; t^{1/d}\mathbb{X}\big)^c \neq \emptyset\big\},$$

is the event that the potentially affected region shrinks when we increase the viewing window from $B(x,r)$ to $t^{1/d}\mathbb{X}$. To prove Proposition 15 we require the following result.

Lemma 16. *Given any $\epsilon > 0$, there exists r_0 such that for all $r > r_0$ and $x \in \mathbb{R}^d$*

$$\mathbb{P}\big(\Phi(x,r,t,\eta_t)\big) < \epsilon/2 \quad \textit{and} \quad \mathbb{P}\big(\Psi(x,r,t,\eta_t)\big) < \epsilon/2 \quad \textit{for all } t_0(x,r) < t \leq \infty,$$

and

$$\mathbb{P}\big(\Phi(x,r,t,\mathcal{U}_t)\big) < \epsilon/2 \quad \textit{and} \quad \mathbb{P}\big(\Psi(x,r,t,\mathcal{U}_t)\big) < \epsilon/2 \quad \textit{for all integers } t > t_0(x,r).$$

Proof. We use the same argument as in Proposition 9. Suppose that $\Phi(x,r,t,\mu)$ holds for $\mu = \eta_t$ or $\mu = \mathcal{U}_t$. Then there exists points $\{w,z\}$ such that

(a) $\{w,z\} \subseteq t^{1/d}\mathbb{X}$;

(b) $S(w,z) \cap \mu = \emptyset$;

(c) $x \in \overline{S(w,z)}$;

(d) $S(w,z) \not\subseteq \mathcal{R}_S\big(x; \mu \cap B(x,r); B(x,r)\big)$.

If $\{w,z\} \subseteq B(x,r)$, then (d) is a contradiction. Thus either w or z is outside this ball. We decompose $\Phi(x,r,t,\mu)$ into subevents. For $u > 0$ let $\widehat{\Phi}(u)$ be the event that there exists $\{w,z\}$ such that (a)–(c) hold and

$$u < \max\big(\|w-x\|, \|z-x\|\big) \leq 2u.$$

In all, we have shown that

$$\Phi(x,r,t,\mu) \subseteq \bigcup_{i=0}^{\infty} \widehat{\Phi}(2^i r). \tag{38}$$

We bound the probability of $\widehat{\Phi}(u)$ and apply a union bound. If $\widehat{\Phi}(u)$ holds, then $\{w,z\} \subseteq t^{1/d}\mathbb{X} \cap B(x,2u)$, and $S(w,z)$ contains no points of μ and has diameter at least u. By the scaled ball condition, with the role of $\mathbb{X}$ played by $t^{1/d}\mathbb{X} \cap B(x,2u)$, the set $S(w,z) \cap t^{1/d}\mathbb{X} \cap B(x,2u)$ contains a ball of radius $\delta u/\mathcal{D}$. Thus, $\widehat{\Phi}(u)$ implies the existence of a ball of radius $\delta u/\mathcal{D}$ within $t^{1/d}\mathbb{X} \cap B(x,2u)$ containing no points of μ. Every ball of radius $\delta u/\mathcal{D}$ contains a cell of the lattice $(\delta u/\mathcal{D}\sqrt{d})\mathbb{Z}^d$, and by considering the volume of $B(x,2u)$, the set $t^{1/d}\mathbb{X} \cap B(x,2u)$ contains at most

$$\frac{\pi_d (2u)^d}{(\delta u/\mathcal{D}\sqrt{d})^d} = \frac{\pi_d (2\mathcal{D}\sqrt{d})^d}{\delta^d}$$

cells of this lattice. Bounding $\widehat{\Phi}(u)$ by the event that all of these cells have no points of μ, in the case $\mu = \eta_t$, recalling that η_t has intensity 1,

$$\mathbb{P}(\widehat{\Phi}(u)) \leq \frac{\pi_d (2\mathcal{D}\sqrt{d})^d}{\delta^d} \exp\left(-\kappa u^d\right),$$

where $\kappa = (\delta/\mathcal{D}\sqrt{d})^d$ If $\mu = \mathcal{U}_t$, a similar statement holds, recalling $|\mathbb{X}| = 1$, as

$$\mathbb{P}(\widehat{\Phi}(u)) \leq \frac{\pi_d(2\mathcal{D}\sqrt{d})^d}{\delta^d}\left(1 - \frac{\kappa u^d}{t}\right)^t \leq \frac{\pi_d(2\mathcal{D}\sqrt{d})^d}{\delta^d}\exp\left(-\kappa u^d\right).$$

Applying the union bound in (38) followed by these two inequalities, and then bounding the resulting sum by a geometric series as in Proposition 9, shows that in either case we have $\mathbb{P}\big(\Phi(x, r, t, \mu)\big) \leq Ce^{-cr^d}$ for constants C and c. Now choose r_0 such that this upper bound is less than $\epsilon/2$ for $r > r_0$.

Bounding $\Psi(x, r, t, \eta_t)$ and $\Psi(x, r, t, \mathcal{U}_t)$ is similar. If $\Psi(x, r, t, \mu)$ holds, then there must exist $\{w, z\} \subseteq B(x, r)$ with $x \in S(w, z)$ such that

$$S(w, z) \cap \mu \cap B(x, r) = \emptyset \quad \text{but} \quad S(w, z) \cap \mu \neq \emptyset.$$

These relations imply that $S(w, z)$ extends outside of $B(x, r)$, which means that $S(w, z)$ has diameter at least r. Hence, by the scaled ball condition, there exists a ball of radius $\delta r/\mathcal{D}$ containing no points of μ, and one may now argue as for $\Phi(x, r, t, \mu)$. $\square$

Proof of Proposition 15. For $\epsilon > 0$ let r_0 be given as in Lemma 16. For $\mu = \eta_t$ or $\mu = \mathcal{U}_t$, for all $r \geq r_0$, $x \in \mathbb{R}^d$, and $t > t_0(x, r)$, it holds except on an event of probability at most ϵ that $\mathcal{R}_S(x; \mu; t^{1/d}\mathbb{X}) = \mathcal{R}_S(x; \mu \cap B(x, r); B(x, r))$. By Lemma 6, and referring to defintion (14), on this event $E^{\pm}(\mu) = E^{\pm}\big(\mu \cap B(x, r)\big)$. $\square$

Since $\mathcal{G}(\eta_\infty)$ is an infinite graph, $L(\eta_\infty)$ does not exist in general. However, when $E_x^{\pm}(\eta_\infty)$ is finite we may define $D_x L(\eta_\infty)$ by the difference

$$D_x L(\eta_\infty) = \sum_{\{x,y\} \in E_x^+(\eta_\infty)} \psi(x - y) - \sum_{\{w,z\} \in E_x^-(\eta_\infty)} \psi(w - z).$$

The following corollary implies that $D_x L(\eta_\infty)$ is also the limit of $D_x L(\eta_\infty \cap B(x, r))$ as $r \to \infty$.

Corollary 17. *For all $x \in \mathbb{R}^d$ the set $E_x^{\pm}(\eta_\infty)$ is finite almost surely, and for any $\epsilon > 0$ there exists r_0 such that for all $r > r_0$*

$$\mathbb{P}\Big(D_x L(\eta_\infty) = D_x L(\eta_\infty \cap B(x, r))\Big) \geq 1 - \epsilon. \tag{39}$$

Proof. Inequality (36) of Proposition 15 with $t = \infty$ yields an r_0 such that $E_x^{\pm}(\eta_\infty) = E_x^{\pm}(\eta_\infty \cap B(x, r))$ for all $x \in \mathbb{R}^d$ and $r > r_0$ with probability at least $1 - \epsilon$, proving that (39) holds. On the event that $D_x L(\eta_\infty) = D_x L(\eta_\infty \cap B(x, r))$, the quantity $E_x^{\pm}(\eta_\infty)$ is finite, and since ϵ is arbitrary, $E_x^{\pm}(\eta_\infty)$ is a.s. finite. $\square$

We will use the next lemma to replace binomial processes with Poisson processes on large regions.

Lemma 18. *For any bounded set $A \subseteq \mathbb{R}^d$, as $t \to \infty$*

$$\mathcal{U}_t \cap A \to \eta_\infty \cap A$$

in total variation.

Proof. Let M and N be the number of points of $\mathcal{U}_t$ and η_t that fall in A, respectively. Once t is large enough that $A \subseteq t^{1/d}\mathbb{X}$, the distribution of M is $\mathrm{Bin}(t, |A|/t)$, and the distribution of N is $\mathrm{Poi}(|A|)$. It is well known that this binomial distribution converges in total variation to this Poisson distribution, and so M and N can be coupled so that they are equal with probability approaching 1 as $t \to \infty$. As $\mathcal{U}_t \cap A$ can be represented as M points uniformly distributed over A and $\eta_\infty \cap A$ as N points uniformly distributed over A, the two point processes can be coupled to be equal with probability tending to 1. ☐

The next piece of the proof is to show that $D_x f(\eta_\infty)$ is nondeterministic. For any concrete collection of forbidden regions, this is typically straightforward, but to show it in more generality we need to present some technical arguments.

Lemma 19. *Suppose $E = \operatorname{int} \overline{E}$. Then for all $x \in \partial E$, every neighborhood of x intersects the interiors of E and E^c.*

Proof. Let $x \in \partial E$ and let U be an open neighborhood of x. By the definition of the boundary, U intersects E and E^c. Since E is open, $E = \operatorname{int} E$. Thus it just remains to show that U intersects $\operatorname{int}(E^c)$.

Since $\overline{E}^c$ is an open set contained in E^c, we have $\overline{E}^c \subseteq \operatorname{int}(E^c)$. Thus $\operatorname{int}(E^c)^c \subseteq \overline{E}$. Now, suppose that U does not intersect $\operatorname{int}(E^c)$. Then $U \subseteq \operatorname{int}(E^c)^c \subseteq \overline{E}$. Since U is open, we have $U \subseteq \operatorname{int}(\overline{E}) = E$. Hence $x \in E$. But this contradicts $x \in \partial E$, since E is open and hence contains none of its boundary. ☐

For a set $E \subseteq \mathbb{R}^d$ and a direction $u \in \mathcal{S}^{d-1}$, let $E_u = \{t \in [0, \infty) \colon tu \in E\}$, the set of all nonnegative t such that the ray of length t in direction u intersects E. Let σ denote uniform measure on $\mathcal{S}^{d-1}$.

Lemma 20. *Suppose that $E \subseteq \mathbb{R}^d$ has Lebesgue measure zero. Then for σ-a.e. $u \in \mathcal{S}^{d-1}$, the set E_u has one-dimensional Lebesgue measure zero.*

Proof. By [5, Theorem 2.49],

$$0 = \int_{\mathbb{R}^d} \mathbf{1}\{x \in E\}\, dx = C \int_{\mathcal{S}^{d-1}} \int_0^\infty \mathbf{1}\{r \in E_u\} r^{d-1}\, dr\, d\sigma(u),$$

where C is the volume of $\mathcal{S}^{d-1}$. This shows that the inner integrand is zero for σ-a.e. u. As the inner integrand is zero if and only if E_u has measure zero, this completes the proof. ☐

Lemma 21. *Suppose that the forbidden regions $S(x, y)$ form an (S, u_0) regular isotropic family as defined in Remark 4. Then for any $w, y \in \mathbb{R}^d$ with $w \neq y$, the set $\{x \in \mathbb{R}^d \colon w \in \partial S(y, x)\}$ has Lebesgue measure zero.*

Proof. First, we note that by translation invariance of the forbidden regions,

$$\begin{aligned} \{x \in \mathbb{R}^d \colon w \in \partial S(y, x)\} &= \{x \in \mathbb{R}^d \colon w - y \in \partial S(0, x - y)\} \\ &= \{x \in \mathbb{R}^d \colon w - y \in \partial S(0, x)\} + y. \end{aligned}$$

Hence it suffices to prove that $\{x \in \mathbb{R}^d \colon w \in \partial S(0, x)\}$ has measure zero for all $w \in \mathbb{R}^d \setminus \{0\}$.

The rest of the argument is easier to follow in $\mathbb{R}^2$, and we present it there first. Let us identify $\mathbb{R}^2$ with $\mathbb{C}$ for convenience. Observe that our isotropic assumption implies that $S(0, re^{i\theta}) = re^{i\theta}S(0,1)$. Thus, with $T = S(0,1)$, for any $w \in \mathbb{C} \setminus \{0\}$,

$$\begin{aligned}\int_{\mathbb{R}^2} \mathbf{1}\{w \in \partial S(0,x)\}\, dx &= \int_0^{2\pi}\int_0^\infty \mathbf{1}\{r^{-1}e^{-i\theta} \in w^{-1}\partial T)\}\, r\, dr\, d\theta \\ &= \int_0^{2\pi}\int_0^\infty \mathbf{1}\{te^{-i\theta} \in w^{-1}\partial T)\}\, t^{-3}\, dt\, d\theta,\end{aligned}$$

making the substitution $t = r^{-1}$. For a given θ, the inner integrand is zero for t off of the ray $(w^{-1}\partial T)_{e^{-i\theta}}$, in the notation of Lemma 20. As $w^{-1}\partial T$ has measure zero, the inner integral is then zero for a.e. θ by Lemma 20, making the entire integral equal to zero.

In higher dimensions, the proof is more complicated because rotation is more complicated, but the idea is the same. First, we record some facts about rotations of $\mathbb{R}^d$ around the origin, which can be represented as elements of $\mathrm{SO}(d)$, the special orthogonal group of order d. The group $\mathrm{SO}(d)$ is isomorphic to $\mathcal{S}^{d-1} \times \mathrm{SO}(d-1)$. The decomposition works by specifying a vector $u \in \mathcal{S}^{d-1}$ that a chosen vector u_0 is mapped to (note that we take this chosen vector to be the same as the axis of symmetry for the isotropic family), and then specifying how the orthogonal complement of the span of u is rotated. As a corollary to this decomposition, if u is chosen uniformly over $\mathcal{S}^{d-1}$, and the rotation of the orthogonal complement of u is chosen from Haar measure on $\mathrm{SO}(d-1)$, then the result is distributed as Haar measure on $\mathrm{SO}(d)$. We let $\rho_u \in \mathrm{SO}(d)$ denote the rotation of $\mathbb{R}^d$ around the origin taking u_0 to u by rotating the plane containing u_0 and u and fixing its orthogonal complement (if $u = u_0$, take ρ_u to be the identity). We use the notation $\mathrm{SO}(u^\perp)$ to denote the subgroup of $\mathrm{SO}(d)$ fixing u, which as discussed above is isomorphic to $\mathrm{SO}(d-1)$.

Let $\overline{x} \in \mathcal{S}^{d-1}$ denote $x/\|x\|$ for $x \neq 0$. Let $T = S + u_0/2 = S(0, u_0)$. It follows from our isotropic assumption that

$$\partial S(0,x) = \|x\|\rho_{\overline{x}}(\partial T).$$

Thus, the measure of $\{x \in \mathbb{R}^d \colon w \in \partial S(0,x)\}$ can be expressed as

$$\int_{\mathbb{R}^d} \mathbf{1}\{w \in \|x\|\rho_{\overline{x}}(\partial T)\}\, dx = C\int_0^\infty \int_{\mathcal{S}^d} \mathbf{1}\{w \in r\rho_u(\partial T)\} r^{d-1}\, d\sigma(u)\, dr$$

with the (irrelevant) constant determined by the volume of $\mathcal{S}^{d-1}$. By the definition of isotropic family, for any $\tau \in \mathrm{SO}(u^\perp)$, we have $\tau\rho_u(T) = \rho_u(T)$. Letting μ_u denote Haar measure on $\mathrm{SO}(u^\perp)$ normalized to have measure one, we can rewrite the integral as

$$\begin{aligned}&C\int_0^\infty \int_{\mathcal{S}^{d-1}} \int_{\mathrm{SO}(\mu^\perp)} \mathbf{1}\{w \in r\tau\rho_u(\partial T)\} r^{d-1}\, d\mu_u(\tau)\, d\sigma(u)\, dr \\ &\qquad = C\int_0^\infty \int_{\mathcal{S}^{d-1}} \int_{\mathrm{SO}(u^\perp)} \mathbf{1}\{r^{-1}(\tau\rho_u)^{-1}(w) \in \partial T\} r^{d-1}\, d\mu_u(\tau)\, d\sigma(u)\, dr.\end{aligned}$$

As we mentioned before, $\tau\rho_u$ with τ distributed as μ_u and u distributed as σ is Haar-distributed over $\mathrm{SO}(d)$. By the invariance of Haar measure under multiplication, the distribution of $(\tau\rho_u)^{-1}(w)$ under this measure is uniform over $\|w\|\mathcal{S}^{d-1}$. Hence we can rewrite the integral as

$$C\int_0^\infty \int_{\mathcal{S}^{d-1}} \mathbf{1}\{r^{-1}\|w\|u \in \partial T\} r^{d-1}\, d\sigma(u)\, dr = C\int_{\mathcal{S}^{d-1}} \int_0^\infty \mathbf{1}\{tu \in \|w\|^{-1}\partial T\} t^{-(d+1)}\, dt\, d\sigma(u),$$

substituting $t = 1/r$. The inner integral is supported on the ray $(\|w\|^{-1}\partial T)_u$. Since the set $\|w\|^{-1}\partial T$ has measure zero, the inner integral is thus zero for σ-a.e. u by Lemma 20. $\square$

Lemma 22. *Assume the forbidden regions $S(x,y)$ are a (S,u_0) regular isotropic family satisfying Assumption 4, and that $S(x,y) = \operatorname{int}\overline{S(x,y)}$ for all $\{x,y\} \subseteq \mathbb{R}^d$. Let $\{w,z\} \subseteq B(0,1)$ be distinct points. Let μ be a homogeneous Poisson process on $\mathbb{R}^d \setminus B(0,1+2\mathcal{D})$, and let $\mu' = \{w,z\} \cup \mu$. Then a.s.-μ, there exist open sets $A, A' \subseteq \mathbb{R}^d$ such that*

$$D_x L(\mu') \neq D_{x'} L(\mu') \tag{40}$$

for all $x \in A$, $x' \in A'$.

Proof. The main idea of the proof is that adding to μ' any point close to y has the same effect on $\mathcal{G}(\mu')$ except for possibly causing the deletion of the edge wz. Note that wz is always present in $\mathcal{G}(\mu')$, as $S(w,z)$ has at most diameter $2\mathcal{D}$ and hence is contained in $B(0,1+2\mathcal{D})$, while μ has no points in $B(0,1+2\mathcal{D})$ besides w and z.

Step 1. *A.s.-μ, we have $b \notin \partial S(y,a)$ for all $\{a,b\} \subseteq \mu'$ with $a \neq b$.*
By Assumption 4, $w \notin \partial S(y,z)$ and $z \notin \partial S(y,w)$. Since $\partial S(y,z)$ and $\partial S(y,w)$ have measure zero, almost surely no points of μ fall in either of these sets. Now we are left to show that

$$b \notin \partial S(y,a), \quad a \in \mu,\, b \in \mu',\, a \neq b. \tag{41}$$

For a point process configuration χ, let

$$f(\chi,a) = \#\Big(\big((\{w,z\} \cup \chi) \setminus \{a\}\big) \cap \partial S(y,a)\Big).$$

Our goal is then to show that $\sum_{a\in\mu} f(\mu,a) = 0$ a.s. By Mecke's formula,

$$\mathbb{E}\sum_{a\in\mu} f(\mu,a) = \mathbb{E}\int_{\mathbb{R}^d} f(\mu\cup\{a\},a)\,da = \int_{\mathbb{R}^d} \mathbb{E}\Big[\#\big((\{w,z\}\cup\mu)\cap\partial S(y,a)\big)\Big]\,da,$$

with the transposition of the integral and expectation justified by nonnegativity of the integrand. For any $a \in \mathbb{R}^d$, the set $\partial S(y,a)$ has measure zero, and hence no points of μ are in $\partial S(y,a)$ a.s. Thus we can simplify the above expression to

$$\begin{aligned}\mathbb{E}\sum_{a\in\mu} f(\mu,a) &= \int_{\mathbb{R}^d} \#\big(\{w,z\}\cap\partial S(y,a)\big)\,da \\ &= \int_{\mathbb{R}^d} \big(\mathbf{1}\{w\in\partial S(y,a)\} + \mathbf{1}\{z\in\partial S(y,a)\}\big)\,da,\end{aligned}$$

with the expectation removed because there is no longer any randomness in the integrand. Thus it follows from Lemma 21 that the integrand is zero except on a set of measure zero, proving that $\mathbb{E}\sum_{a\in\mu} f(\mu,a) = 0$. This proves (41), completing the proof of this step.

Step 2. *A.s.-μ, we have $y \notin \partial S(a,b)$ for $\{a,b\} \subset \mu'$, $\{a,b\} \neq \{w,z\}$.*
This follows by essentially the same proof as in the previous step.

In the next step, we say that $E_x^+(\mu')$ and $E_y^+(\mu')$ are equivalent if the set of edges $E_x^+(\mu')$ is equal to the set $E_y^+(\mu')$ when all edges of the form $\{y,a\}$ in the latter are replaced by $\{x,a\}$. Note that we do not need a definition like this for $E_x^-(\mu')$ and $E_y^-(\mu')$, since edges with vertices x or y do not appear in these collections.

Step 3. *For some random radius $\rho > 0$, it holds for all $x \in B(y,\rho)$ that $E_x^+(\mu')$ is equivalent to $E_y^+(\mu')$, and it holds for all $x \in B(y,\rho)$ that $E_x^-(\mu')$ is equal to either $E_y^-(\mu')$ or $E_y^-(\mu') \cup \{\{w,z\}\}$.*
Let $R = \mathcal{R}_S(B(y,1);\mu';\mathbb{R}^d)$. The set $\mathcal{R}_S(B(y,1);\eta_\infty;\mathbb{R}^d)$ is bounded a.s.-η_∞ by Proposition 9. Since μ is distributed as η_∞ conditional on an event of positive probability, $\mathcal{R}_S(B(y,1);\mu;\mathbb{R}^d)$ is bounded a.s.-μ. As $\mu \subseteq \mu'$, Lemma 5 shows that the set R is bounded a.s.-μ. Recall that the addition of any point $x \in B(y,1)$ changes the graph $\mathcal{G}(\mu')$ only by the addition of edges xa and deletion of edges ab for $a,b \in R$.

Step 1 shows that for each $a \in \mu' \cap R$, the set $\partial S(y,a)$ does not contain any points of $(\mu' \setminus \{a\}) \cap R$. Since $(\mu' \setminus \{a\}) \cap R$ is almost surely finite, the set $\partial S(y,a)$ has positive distance from $(\mu' \setminus \{a\}) \cap R$, as both sets are compact. By the Hausdorff continuity of the map S, there is a positive distance ρ_a^+ such that for all $x \in B(y,\rho_a^+)$, the set $\partial S(x,a)$ avoids $(\mu' \setminus \{a\}) \cap R$. Set ρ^+ to be the minimum of ρ_a^+ over the almost surely finitely many $a \in \mu' \cap R$. Then for all $x \in B^o(y,\rho^+)$, the collections $E_x^+(\mu')$ and $E_y^+(\mu')$ are equivalent.

Step 2 implies that for all $\{a,b\} \subseteq \mu' \cap R$ except for $\{w,z\}$, the set $\partial S(a,b)$ has a positive distance ρ_{ab}^- from y. Set ρ^- as the minimum of ρ_{ab}^- over this almost surely finite collection of $\{a,b\}$. Then for $x \in B^o(y,\rho^-)$, as $y \in \partial S(w,z)$, and $S(w,z)$ is open, it holds that $E_x^-(\mu')$ is equal to either $E_y^-(\mu')$ or $E_y^-(\mu') \cup \{\{w,z\}\}$. Taking ρ less than ρ^+ and ρ^- completes the step.

Step 4. *Construction of A, A' satisfying* (40).
Let $A = B^o(y,\rho') \cap \operatorname{int} S(w,z)$ and $A' = B^o(y,\rho') \cap \operatorname{int}(S(w,z)^c)$ for ρ' to be specified later. Assume for now that $\rho' < \rho$. By Lemma 19, both sets A and A' are nonempty. By the previous step, $E_x^+(\mu')$ and $E_y^+(\mu')$ are equivalent for $x \in A \cup A'$. For $x' \in A'$, we have $E_{x'}^-(\mu') = E_y^-(\mu')$, and for $x \in A$, we have $E_x^-(\mu') = E_y^-(\mu') \cup \{\{w,x\}\}$. Thus for $x \in A$ and $x' \in A'$,

$$D_{x'}L(\mu') - D_xL(\mu') = \psi(w,z) + \sum_{a\colon \{a,x\} \in E_x^+(\mu')} \big(\psi(a,x') - \psi(a,x)\big).$$

By the continuity of ψ, and that $E_x^+(\mu')$ is finite, the sum can be made arbitrarily small over all $x \in A$, $x' \in A'$ by choosing ρ' small enough. If we choose ρ' to make the sum smaller than $\psi(w,z)$, then (40) holds for $x \in A$, $x' \in A'$. $\square$

Theorem 23. *Assume that the forbidden regions $S(x,y)$ are a (S,u_0) regular isotropic family satisfying Assumption 4. Then for all $x \in \mathbb{R}^d$, the random variable $D_xL(\eta_\infty)$ is nondeterministic.*

Proof. As $\operatorname{int} S(x,y) \subseteq \operatorname{int} \overline{S(x,y)} \subseteq \overline{S(x,y)}$, the sets $S(x,y)$ and $\operatorname{int} \overline{S(x,y)}$ differ only on $\partial S(x,y)$, a set of measure zero. For each $\{a,b\} \subset \eta_\infty$, there are almost surely no points of η_∞ on $\partial S(a,b)$ besides a and b. Thus $\mathcal{G}(\eta_\infty)$ is almost surely unaffected by replacing each forbidden region $S(x,y)$ by $\operatorname{int} \overline{S(x,y)}$. If $B = \operatorname{int} \overline{A}$, then $B = \operatorname{int} \overline{B}$. Hence we can assume that $S(x,y) = \operatorname{int} \overline{S(x,y)}$ for all x,y.

Let w and z be chosen uniformly and independently from $B(0,1)$, and let μ be a homogeneous Poisson process with intensity 1 on $\mathbb{R}^d \setminus B(0,1+2\mathcal{D})$. With positive probability, η_∞ has exactly two points in $B(0,1+2\mathcal{D})$, both of which are contained in $B(0,1)$. Conditional on this event, η_∞ is distributed as $\mu' := \{w,z\} \cup \mu$. By Lemma 22, a.s.-μ there exist open sets $A, A' \subseteq \mathbb{R}^d$ such that $D_xL(\mu') \neq D_{x'}L(\mu')$ for all $x \in A$ and $x' \in A'$. Thus, with positive probability, there exist open sets $A, A' \subseteq \mathbb{R}^d$ such that $D_x L(\eta_\infty) \neq D_{x'} L(\eta_\infty)$ for all $x \in A$ and $x' \in A'$.

Suppose that $D_x L(\eta_\infty) = c$ a.s. for some $x \in \mathbb{R}^d$ and some constant c. By the translation invariance of η_∞, this holds for all $x \in \mathbb{R}^d$. Hence it holds almost surely that $D_x L(\eta_\infty) = c$ for all x in a countable dense set of $\mathbb{R}^d$. But this contradicts the conclusion of the previous paragraph. ◻

We now use Theorem 23 to show that if x and y are far enough apart, then with positive probability adding x or y to the process produces different effects on L.

Lemma 24. *Assume the conditions of Theorem 2. There exist constants $a > b, r_0 \in (0,\infty)$ and $p_0 \in (0,1]$ such that for all $r > r_0$ the following statement holds: for all $x, y \in \mathbb{R}^d$, if the r-balls around x and y are disjoint and $t > t_1(x,y,r) = \max\{t_0(x,r), t_0(y,r), t_2(r)\}$ where t_2 is a function depending only on r, then*

$$\mathbb{P}\big(D_x L(\mu) > a \text{ and } D_y L(\mu) < b\big) \geq p_0$$

for $\mu = \eta_t$ or $\mu = \mathcal{U}_t$.

Proof. Let first $\mu = \mathcal{U}_t$. By Theorem 23, there exist $a > b$ and $p > 0$ such that for all $z \in \mathbb{R}^d$,

$$\mathbb{P}\big(D_z L(\eta_\infty) > a\big) \geq p \quad \text{and} \quad \mathbb{P}\big(D_z L(\eta_\infty) < b\big) \geq p.$$

Let $p_0 = (p-\epsilon)^2 - 3\epsilon$, choosing $\epsilon > 0$ small enough that $p_0 > 0$. By Corollary 17, for all sufficiently large r and for all $z \in \mathbb{R}^d$ the random variables $D_z L(\eta_\infty)$ and $D_z L(\eta_\infty \cap B(z,r))$ are within ϵ in total variation distance, and hence

$$\mathbb{P}\big(D_z L(\eta_\infty \cap B(z,r)) > a\big) \geq p - \epsilon \quad \text{and} \quad \mathbb{P}\big(D_z L(\eta_\infty \cap B(z,r)) < b\big) \geq p - \epsilon. \tag{42}$$

Next, from the total variation convergence given by invoking Lemma 18 with $A = B(x,r) \cup B(y,r)$, for all r large enough that (42) holds, and $t > t_2(r)$ depending on r, for any $\{x,y\} \subseteq \mathbb{R}^d$ satisfying $\|x - y\| > 2r$,

$$\begin{aligned} \mathbb{P}\big(D_x L(\mathcal{U}_t \cap B(x,r)) > a \text{ and } D_y L(\mathcal{U}_t \cap B(y,r)) < b\big) \\ \geq \mathbb{P}\big(D_x L(\eta_\infty \cap B(x,r)) > a \text{ and } D_y L(\eta_\infty \cap B(y,r)) < b\big) - \epsilon \\ \geq (p-\epsilon)^2 - \epsilon, \end{aligned} \tag{43}$$

with the last line following from (42) and the independence of $\eta_\infty \cap B(x,r)$ and $\eta_\infty \cap B(y,r)$. By Proposition 15, for all sufficiently large r and all $t > \max\{t_0(x,r), t_0(y,r)\}$, it holds that

$$\mathbb{P}\big(D_x L(\mathcal{U}_t \cap B(x,r)) = D_x L(\mathcal{U}_t)\big) \geq 1 - \epsilon \quad \text{and} \quad \mathbb{P}\big(D_y L(\mathcal{U}_t \cap B(y,r)) = D_y L(\mathcal{U}_t)\big) \geq 1 - \epsilon.$$

Hence, by a union bound,

$$\mathbb{P}\Big(D_x L(\mathcal{U}_t \cap B(x,r)) = D_x L(\mathcal{U}_t) \text{ and } D_y L(\mathcal{U}_t \cap B(y,r)) = D_y L(\mathcal{U}_t)\Big) \geq 1 - 2\epsilon. \tag{44}$$

Now, for all r_0 so that (42), (43) and (44) hold for all $r > r_0$ and all $t > t_1(x,y,r)$, by (43) and (44),

$$\mathbb{P}\big(D_x L(\mathcal{U}_t) > a \text{ and } D_y L(\mathcal{U}_t) < b\big) \geq (p-\epsilon)^2 - \epsilon - 2\epsilon = p_0.$$

The proof for the Poisson case is the same, except that the step involving Lemma 18 is unnecessary. ◻

We will need the following elementary lemma, which is essentially just Markov's inequality applied to a bounded random variable.

Lemma 25. *Suppose that X is a random variable supported on $[0,n]$, and $\mathbb{E}X \geq np$. Then*

$$\mathbb{P}\Big(X > \frac{np}{2}\Big) \geq \frac{p}{2-p}. \tag{45}$$

Proof. Let $Y = n - X$. Then $\mathbb{E}Y \leq n(1-p)$, and applying Markov's inequality to Y yields

$$\mathbb{P}\Big(X \leq \frac{np}{2}\Big) = \mathbb{P}\Big(Y \geq n\Big(1 - \frac{p}{2}\Big)\Big) \leq \frac{1-p}{1-p/2},$$

yielding (45). □

In the remainder of this section let a, b, r_0, and p_0 be the constants given by Lemma 24. For some $m > 0$ and $1 < r < \infty$, we say that a pair of points x and y with $\|x - y\| > 2r$ is (m,r)-*influential* for μ if

> Influential$_1(\mu)$: There exist sets $A \subseteq B(x,1)$ and $B \subseteq B(y,1)$ each of Lebesgue measure m such that $D_zL(\mu) > a$ for $z \in A$ and $D_zL(\mu) < b$ for $z \in B$, and
>
> Influential$_2(\mu)$: $R_S(B(x,1);\mu) \leq r$ and $R_S(B(y,1);\mu) \leq r$.

Note that a pair of influential points for μ are not required to be, and in fact will in general not be, points *of* μ. We have made the radii of the balls containing x and y equal to 1 in these definitions, but the value is unimportant.

Lemma 26. *Assume the conditions of Theorem 2. There exist constants $m \in (0,\infty), p \in (0,1]$ and $r \in (1,\infty)$ such that if x and y are any two points such that the $(r+1)$–balls centered around each are disjoint, then for all sufficiently large t*

$$\mathbb{P}\big((x,y) \text{ is } (m,r)\text{-influential for } \mu\big) \geq p$$

for $\mu = \eta_t$ and $\mu = \mathcal{U}_t$.

Proof. By Proposition 9, for all $\{x,y\} \subseteq \mathbb{R}^d$ and $t > \max\{t_0(x,r), t_0(y,r)\}$, as $r \to \infty$ the probability of Influential$_2(\mu)$ is lower bounded by a quantity tending to one, not depending on $\{x,y\}$. With r_0 and p_0 the constants given by Lemma 24, let $p_0' = p_0/(2-p_0)$, and choose $r > r_0$ large enough that Influential$_2(\mu)$ holds with probability at least $1 - p_0'/2$. Let X and Y be independent and distributed uniformly over $B(x,1)$ and $B(y,1)$, respectively. Let

$$\begin{aligned} P(\mu) &:= \mathbb{P}\big(D_XL(\mu) > a \text{ and } D_YL(\mu) < b \mid \mu\big) \\ &= \mathbb{P}\big(D_XL(\mu) > a \mid \mu\big)\mathbb{P}\big(D_YL(\mu) < b \mid \mu\big). \end{aligned} \tag{46}$$

Note that

$$\mathbb{P}\big(D_XL(\mu) > a \mid \mu\big) = \frac{|\{z \in B(x,1) \colon D_zL(\mu) > a\}|}{|B(x,1)|},$$

with an analogous statement holding for the second factor in (46). By Lemma 24, using that the r-balls around points in $B(x,1)$ and $B(y,1)$ do not intersect, by averaging X and Y over their supports we see that for $t > \sup_{u\in B(x,1), v\in B(y,1)} t_1(u,v,r)$ we have $\mathbb{E}P(\mu) \geq p_0$. Since $P(\mu)$ is supported on $[0,1]$, we apply Lemma 25 with $n=1$ and $p=p_0$ to conclude that $\mathbb{P}(P(\mu) > p_0/2) \geq p_0/(2-p_0) = p_0'$. If $P(\mu) \geq p_0/2$, then both factors in (46) are larger than $p_0/2$. Therefore, with probability at least p_0', the pair (x,y) satisfies $\textsc{Influential}_1(\mu)$ with $m = p_0|B(x,1)|/2$.

Since $\textsc{Influential}_1(\mu)$ holds with probability at least p_0' and $\textsc{Influential}_2(\mu)$ holds with probability at least $1-p_0'/2$, by a union bound both hold simultaneously with probability at least $p_0'/2$. □

From now on, we take m, r, and p to be constants provided by Lemma 26.

Lemma 27. *Assume the conditions of Theorem 2. Let* $\textsc{Influential}(\mu,t,\beta)$ *be the event that there are at least* βt *pairs of* (m,r)*-influential points for* μ*, all of whose* $(r+1)$*-neighborhoods are disjoint and contained in* $t^{1/d}\mathbb{X}$*. For some* $\beta, q > 0$ *independent of* t*, for either* $\mu = \eta_t$ *or* $\mu = \mathcal{U}_t$*, it holds for all sufficiently large* t *that*

$$\mathbb{P}(\textsc{Influential}(\mu,t,\beta)) \geq q.$$

Proof. For some $\beta' > 0$, for all sufficiently large t one can place at least $2\lceil \beta' t\rceil$ points in $t^{1/d}\mathbb{X}$ so that all points have disjoint $(r+1)$-neighborhoods contained in $t^{1/d}\mathbb{X}$. Let $n = \lceil \beta' t\rceil$, and arbitrarily form these $2n$ points into n disjoint pairs. For large enough t, by Lemma 26, each pair has probability at least p of being (m,r)-influential, so the expected number of such (m,r)-influential pairs is at least np. By Lemma 25, there are at least $np/2$ pairs with probability at least $p/(2-p)$. Now we can take $q = p/(2-p)$ and $\beta = p\beta'/3$, say. □

Proof of Theorem 2. It suffices to show that there exists v such that $\operatorname{Var} L(\mu) \geq vt$ where μ is either Poisson on $t^{1/d}\mathbb{X}$ with intensity 1 or binomial with t points. Indeed, as $\psi(x) = \|x\|^\alpha$, for any $a > 0$ we have $L(a\mu) = a^\alpha L(\mu)$, where $a\mu = \{ax, x\in\mu\}$. Hence, when $\operatorname{Var} L(\mu) \geq vt$, scaling a process μ on $t^{1/d}\mathbb{X}$ to one on $\mathbb{X}$, we have

$$\operatorname{Var}(L(t^{-1/d}\mu)) = \operatorname{Var}(t^{-\alpha/d}L(\mu)) = t^{-2\alpha/d}\operatorname{Var}(L(\mu)) \geq vt^{1-2\alpha/d}.$$

The argument will go by splitting μ into a sum of independent point processes μ_1 and μ_2. Initially, take μ_1 to be a deterministic set of points such that $\textsc{Influential}(\mu_1,t,\beta)$ holds for some $\beta > 0$, and define μ_2 as a point process on $t^{1/d}\mathbb{X}$ that is either Poisson with intensity $1/2$ or binomial with $\lfloor t/2\rfloor$ points. We start by arguing that $\operatorname{Var} L(\mu_1\cup\mu_2) > Ct$ for some C.

Since $\textsc{Influential}(\mu_1,t,\beta)$ holds, there exist point pairs $(x_1,y_1),\ldots,(x_n,y_n)$ with $n \geq \beta t$ with sets $A_i \subseteq B(x_i,1)$ and $B_i \subseteq B(y_i,1)$ of measure m such that $\textsc{Influential}_1(\mu_1)$ and $\textsc{Influential}_2(\mu_1)$ hold for each pair. For some $\gamma > 0$ to be specified, consider the event

$$F = \Bigl\{\Bigl|\Bigl\{1\leq i\leq n\colon \bigl|\mu_2\cap(B(x_i,r+1)\cup B(y_i,r+1))\bigr| = \bigl|\mu_2\cap(A_i\cup B_i)\bigr| = 1\Bigr\}\Bigr| \geq \gamma n\Bigr\}, \tag{47}$$

that is, that for at least γn of the pairs (x_i,y_i), exactly one point of μ_2 lands in the $(r+1)$-neighborhoods of x_i and y_i, and it lands in either A_i or B_i. We claim that F occurs with positive probability not depending on t. Indeed, for any fixed i, the process μ_2 will satisfy

$$\bigl|\mu_2\cap(B(x_i,r+1)\cup B(y_i,r+1))\bigr| = \bigl|\mu_2\cap(A_i\cup B_i)\bigr| = 1 \tag{48}$$

with at least with some fixed, positive probability for large enough t. Choosing γ small enough, the event F then holds with some positive probability independent of t by Lemma 25.

Partition μ_2 into $\{X_1, \dots, X_l\}$ and $\{Y_1, \dots, Y_{l'}\}$, where the first set consists of the points of μ_2 that are contained in $A_i \cup B_i$ for some i satisfying (48). Thus $l \geq \gamma n$ when F holds. Now, let $\widetilde{\mu} = \mu_1 \cup \{Y_1, \dots, Y_{l'}\}$, and express $L(\mu_1 \cup \mu_2)$ as the telescoping sum

$$L(\mu_1 \cup \mu_2) = L(\widetilde{\mu}) + D_{X_1} L(\widetilde{\mu}) + D_{X_2} L(\widetilde{\mu} \cup \{X_1\}) + \cdots + D_{X_l} L(\widetilde{\mu} \cup \{X_1, \dots, X_{l-1}\}).$$

By $\textsc{Influential}_2(\mu_1)$, $R_S(X_i; \mu_1) \leq r$. All points of μ_2 except for X_i lie outside of $B(X_i, r)$. By (16) of Lemma 7,

$$D_{X_i} L(\widetilde{\mu} \cup \{X_1, \dots, X_{i-1}\}) = D_{X_i} L(\mu_1).$$

Thus we can rewrite $L(\mu_1 \cup \mu_2)$ as

$$L(\mu_1 \cup \mu_2) = L(\widetilde{\mu}) + D_{X_1} L(\mu_1) + D_{X_2} L(\mu_1) + \cdots + D_{X_l} L(\mu_1). \tag{49}$$

By construction, X_i falls into exactly one of the sets $A_1 \cup B_1, \dots, A_n \cup B_n$; let us say it falls in $A_{\pi(i)} \cup B_{\pi(i)}$. Conditional on F, the point X_i is equally likely to be in $A_{\pi(i)}$ or $B_{\pi(i)}$. Furthermore, which of these it lands in is independent for $1 \leq i \leq l$ conditional on F. If X_i lands in $A_{\pi(i)}$, then $D_{X_i} L(\mu_1) > a$, and if X_i lands in $B_{\pi(i)}$, then $D_{X_i} L(\mu_1) < b$, by the definition of $\textsc{Influential}_1(\mu_1)$. Thus, conditional on F and on $\widetilde{\mu}$, from the expression of $L(\mu_1 \cup \mu_2)$ in (49) as a sum of independent terms, the conditional variance of $L(\mu_1 \cup \mu_2)$ grows at least as a constant times $l \geq \gamma n \geq \gamma \beta t$. Decomposing the unconditional variance of $L(\mu_1 \cup \mu_2)$ as a sum of expections of conditional variances, since F occurs with probability bounded from below, the unconditional variance of $L(\mu_1 \cup \mu_2)$ grows at least as a constant times t as well.

To complete the proof, we now let μ_1 be a point process on $t^{1/d}\mathbb{X}$, independent of μ_2, and either Poisson with intensity $1/2$ or binomial with $\lceil t/2 \rceil$ points. Thus μ can be expressed as $\mu_1 \cup \mu_2$. By Lemma 27, for all t sufficiently large, the event $\textsc{Influential}(\mu_1, t, \beta)$ holds with probability at least q for some $\beta, q > 0$ not depending on t. By the previous argument, the variance of $L(\mu)$ conditional on $\textsc{Influential}(\mu_1, t, \beta)$ for sufficiently large t is at least Ct for a constant $C > 0$ not depending on t, from which the theorem follows. $\square$

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