Depth, balancing, and limits of the Elo model

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***Abstract*—Much work has been devoted to the computational complexity of games. However, they are not necessarily relevant for estimating the complexity in human terms. Therefore, human-centered measures have been proposed, e.g. the depth. This paper discusses the *depth* of various games, extends it to a continuous measure. We provide new *depth* results and present tool (given-first-move, pie rule, size extension) for increasing it. We also use these measures for analyzing games and opening moves in Y, NoGo, Killall Go, and the effect of pie rules.**

I. Introduction

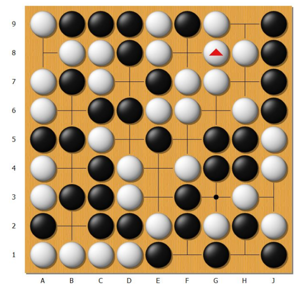
Combinatorial or computational measures of complexity are widely used for games, specific parts of games, or families of games [1], [2], [3], [4]. Nevertheless, they are not always relevant for comparing the complexity of games from a human point of view:

- we cannot compare various board sizes of a same game with complexity classes P, NP, PSPACE, EXP, ... because they are parametrized by the board size; and for some games (e.g. Chess) the principle of considering an arbitrary board size does not make any sense.
- state space complexity is not always clearly defined (for partially observable games), and does not indicate if a game is hard for computers or requires a long learning.

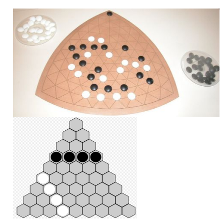
In Section II, we investigate another perspective that aims at better reflecting a human aspect on the *depth* of games. Section II-A defines the *depth* of games and the related Playing-level Complexity (PLC) is described in Section II-B. In Section II-C we review the complexity of various games. In Section II-D we see the impact of board size on complexity. In Section II-E, we compare the *depth* of Killall-Go to the *depth* of Go.

In Section III, we analyze how various rules concerning the first move impact the *depth* and PLC. We focus on the pie rule (PR), which is widely used for making games more challenging, more balanced.

Then, we switch to experimental works in Section IV. We study in Section IV-A how the PR alters the PLC of existing games such as NoGo, Y, and Chinese Dark Chess (Fig. 1). Section IV-B then analyzes the *depth* and PLC of Killall-Go, in particular when using PR. In all the paper, a rational choice means a choice which maximizes the success rate. This is the behavior of a player willing to win and having access to all possible information. In many cases below, we will assume that the opening moves and the pie rule choices (to swap or not to swap) are rational. In all the paper, $\log$ denotes logarithm with basis 10.



(a) NoGo



(b) Y

(c) Chinese Dark Chess

Fig. 1: The NoGo, Y (2 variants) and Chinese Dark Chess games respectively. NoGo has a gameplay similar to Go, but capturing is forbidden which completely changes the strategy (this game is a win by BobNogo against NdhuNogo, July 2012 tournament). Y is a connection game: players play in turn and the first player who connects 3 sides wins (corners are connected to both sides). Chinese Dark Chess is related to Chinese Chess, but with hidden pieces; a special move consists in turning a face-down piece into face-up. Sources: Wikipedia and Kgs.

II. Human-centered complexity measures

In this section, we review some human-centered complexity measures for games: the *depth* and the playing level complexity.

A. *Definition of the* depth *of a game*

Definition 1: Consider a game G, and a set S of players. Then the *depth* Δ of G is the maximal size of a set $p_1, \ldots, p_\Delta$

of players in S such that for each i, p_i wins with probability $\geq 60\%$ against p_{i+1}.

This measure was used in particular in [5]. This definition is less than definitive: the *depth* of a game depends on whether we consider computer players, or just humans, and among them, any possible player, or only players playing in a "reasonable" manner. Also, many stupid games can be very deep for this measure: for example, who was born first, or who has the largest bank account number, or who has the first name in alphabetic order. The *depth* depends on the set of players. Let us consider an example showing that most games are extremely deep for this measure, if we consider the *depth* for all possible players. If, using the gameplay, players can exchange b bits of information, then one can build $1, \dots, 2^b$ players $p_1, \dots, p_d$, with p_i winning almost surely against p_{i+1}, as follows:

- First, p_i starts by writing i on b bits of information (in Go, p_i might encode this on the top of the board if he is black, and on the bottom of the board if he is white);
- Second, if the opponent has written $j < i$, then p_i resigns.

This ensures an exponential *depth* of Go on an $n \times n$ board (and much better than this exponential lower bound is possible), with very weak players only. So, with no restrictions on the set of players, *depth* makes no sense.

An advantage of depth, on the other hand, is that it can compare completely unrelated games (Section II-C) or different board sizes for a same game (Section II-D).

B. Playing-level complexity

We now define a new measure of game complexity, namely the playing level complexity (PLC).

Definition 2: The PLC of a game is the difference between the Elo rating of the strongest player and the Elo rating of a naive player for that game. More precisely, we will here use, as a PLC, the difference between the Elo rating of the best available player (in a given set) and the Elo rating of the worst available player (in the same set).

This implies that the PLC of a game depends on the set of considered players, as for the classical *depth* of games. An additional requirement is that the notion of best player and the notion of worst player make sense - which is the case when the Elo model applies, but not all games verify, even approximately, the Elo model (see a summary on the Elo model in Section V).

When the Elo model holds, the PLC is also equal to the sum of the pairwise differences in Elo rating: if players have ratings $r_1 < r_2 < \dots < r_n$, then the playing level complexity is $plc = r_n - r_1 = \sum_{i=1}^{n-1}(r_{i+1} - r_i)$. Applying the Elo classical formula, this means

$$plc = -\sum_{i=1}^{n-1} 400 \log(1/P_{i+1,i} - 1), \quad (1)$$

where $P_{i+1,i}$ is the probability that player $i+1$ wins against player i. When the Elo model applies exactly, this is exactly the same as

$$plc = -400 \log(1/P(strongest\ wins\ vs\ weakest) - 1). \quad (2)$$

TABLE I: **Left:** lower bounds on the PLC and *depth* for some widely played games. For Leagues of Legends the results are based on the Elo rating at the end of season 2 (data from the website competitive.na.leagueoflegends.com). **Right:** Analysis of the PR for NoGo and Y.

(a) The PLC is obtained by difference between the maximum Elo and the Elo of a beginner. Depths are obtained from these PLC for these games, and/or from [5].

Game	PLC	D.
Go 19 × 19		≥ 40
Chess	2300	≥ 16
Go 9 × 9	2200	≥ 14
Ch. Chess		≥ 14
Shogi		≥ 11
L.o. Legends	1650	≥ 10

Game	PLC	D.
Checkers		≥ 8
Backgammon		≥ 4
Urban Rivals	550	≥ 3
Magic the G.		≥ 3
Poker		≥ 1

(b) Playing-level Complexity for various variants of the games NoGo and Y. RDR, RDR+pr and others refer to rules for managing the first move; see Sections III and IV.

Game, size, # players	RDR	RDR+PR	$\max_i$ $(RDR+$ $+GFM_i)$
NoGo5 (7p)	186.2	185.5	207.6
NoGo6 (7p)	263.2	277.3	357.8
NoGo7 (4p)	305.1	336.6	351.0
Y4 (6p)	885.0	910.2	906.8
Y5 (7p)	946.9	1185.7	1127.9

While both equations are equal in the Elo framework, they differ in our experiments and Eq. 1 is closer to *depth* results.

Unlike the computational complexity and just as the state-space complexity, the PLC allows to compare different board sizes for the same game. We can also use the PLC to compare variants that cannot be compared with the state-space complexity, such as different starting positions or various balancing rules, e.g. imposed first move or pie rules.

When the Elo model applies, the PLC is related to the depth. Let us see why. Let us assume that: (i) the Elo model applies to the game under consideration, (ii) there are infinitely many players, with players available for each level in the Elo range from a player A (weakest) to a player B (strongest). Then, the *depth* Δ of the game, for this set of players, is proportional to the PLC:

$$\Delta = 1 + \left\lfloor \frac{plc}{-400 \log(1/.6 - 1)} \right\rfloor = \left\lfloor \frac{Elo(B) - Elo(A)}{-400 \log(1/.6 - 1)} \right\rfloor .$$

.6 comes from the 60%. The ratio is $-400 \log(1/.6 - 1) \simeq .70.437$. Incidentally, we see that the PLC extends the *depth* in the sense that non-integer "depths" can now be considered.

C. Depth and PLC of various games

According to estimates from Elo scales and Bill Robertie's results [5], Chess would have *depth* 16, Shogi more than 11, checkers 8, Chinese Chess 14 [6], Backgammon a4, Magic The Gathering 3 or 4 (using data at http://www.wizards.com/), Poker 1 or 2 (see posts by Ashley Griffiths Oct. 27th, 2011, on the computer-Go mailing list). The Chinese Chess estimate is based on Chinese Chess player Xiu Innchuan, currently Elo 2683.

Go is the deepest - up to now, as we will see that Killall-Go might outperform it. Go is a fascinating game for the simplicity of its rules, as well as for its tactical and strategic complexity (it is one of the games which can be naturally extended to bigger boards in an EXPTIME-complete manner [7]); it leads naturally to many variants. S. Ravera posted the following evaluation of Go's *depth* on the French Go mailing-list: Beginners have 100 points; 9D players on the EGF database have 2900 points; a difference of 400 points lead to 75% winning probability for players between 20K and 6K; a difference of 300 points lead to 75% winning probability for players between 5K and 3K; a difference of 200 points lead to 75% winning probability for players between 2K and 3D; a difference of 100 points lead to 75% winning probability for players between 4D and 6D. With these statistics, he used the threshold 75% instead of 60%, and concludes to 13 classes of players, namely (20k $<$ 16k $<$ 12k $<$ 8k $<$ 4k $<$ 1k $<$ 2D $<$ 4D $<$ 5D $<$ 6D $<$ 7D $<$ 8D $<$ 9D). Numbers are based on the EGF database. As the Elo difference for 75% is $\frac{\log(1/.75-1)}{\log(1/.6-1)} = 2.7$ times bigger than the Elo difference for 60%, this suggests 35 complexity levels. This is probably underestimated as classes are constrained to be Go classes (1D, 2D, 3D, ...; and not floating point numbers such as 1.4D, 2.3D, ...), so the 40 estimate is probably reasonnable. Using Elo ratings from http://senseis.xmp.net/?EloRating, we got a few more levels, leading to 40, as suggested in Robertie's paper [5]. Let us estimate the *depth* of 9×9-Go for computer players. In 9×9-Go, computers are comparable to the best human players. The Cgos 9×9-Go server gives a Elo range of roughly 2200 between random and the best computer player in the all time ratings, which suggests a *depth* of 14. This is for games with 5 minutes per side, i.e., very fast. We summarize these *depth* results in Table I. Games with relatively high depth are difficult for computers; but games at the bottom are not all that easy, as illustrated by Backgammon [8] or Poker [9].

D. Impact of the board size: NoGo, Y and Chinese Dark Chess

We have seen that Go 19x19 is much deeper than Go 9x9. In this section, we check that this is also the case for NoGo, Y and Chinese Dark Chess. We check this for various rules (RDR, RDR+PR, RDR+GFM), which are introduced later (the details are irrelevant for the comparison here). The results on NoGo (Fig. 1a) and Y (Fig. 1b) in Table Ib are also indicative of the relationship between the PLC and the board size. We see that for all variants, the game of Y appears to have a larger PLC on size 5 than on size 4, with the same set of players (Monte Carlo Tree Search (MCTS) with the same numbers of simulations). Similarly the PLC of NoGo increases from size 5×5 to size 6×6, also with the same set of players (MCTS with the same numbers of simulations). The game of Chinese Dark Chess (CDC) (Fig. 1c) is a stochastic perfect information game usually played on a 4×8 board. It is possible to create a larger 8×8 board by putting two regular boards side by side and using two sets of pieces (Fig. 2). We present in Table II the winning rates of our CDC program (an MCTS program) against an Alpha-beta baseline player, depending on the number of MCTS simulations. We compute the Elo rating of the set of players assuming the baseline has a fix rating of 1000. Using Eq. 2 we derive that the PLC for this set of players is 270.73 (depth 3.84) for 4×8 Chinese Dak Chess and 323.95 (depth 4.60) for 8×8 CDC. We can therefore conclude that the 8×8 version has a bigger PLC, at least for the given set of players.

E. Is Killall-Go deeper than Go?

Finally, we can also use the data to compare Killall-Go (KAG, discussed in details in Section IV-B) to Standard Go. Fig. 3 (right) shows a huge improvement when comparing the PLC of KAG to Go. A detailed analysis shows that there is a regular increase in Elo difference (i.e. player $i+1$ is usually stronger than player i in Fig. 3 (right)), but some specific values lead to a larger gap - in particular player 11 wins very easily against player 10 in KAG. The difference between the PLC of KAG and Go is minor in our other experiments with only 2 players, namely HappyGo 500 simulations per move vs HappyGo 400 simulations per move - but in the case with 15 players there is a huge difference between KAG and Go.

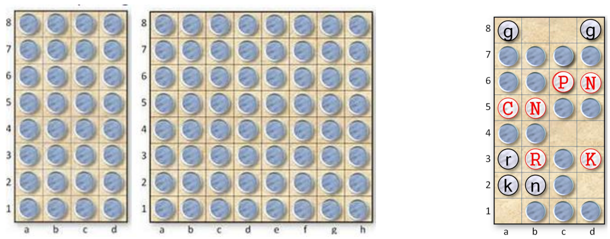
III. Choice rules and depth: theory

There are various systems for choosing, in a game, (i) the initial position (e.g. base placement in Batoo or handicap placement in Go or KAG), (ii) or the first move (e.g. in Twixt, Havannah, Hex or Y, where the first move is crucial), (iii) or some parameter (e.g. komi in Go). These three aspects are very related; it is all about making a fair choice, so that the game is more challenging. (a) It might be fixed by the rules. (b) It might be fixed by the rules, and depend on the context; for example, the relevant komi in Go increases when players' strength increases, and the handicap in KAG decreases when the level increases. (c) It might be chosen by bidding [11] or by pie rule. The pie rule (PR [12]) is a classical tool intended to compensate the advantage of the first player by allowing the second player to exchange (to swap) the roles. It is quite effective for making Hex [13], Havannah [14] or Twixt fairer. We present the PR and show some disappointing properties of it in Section III-A. In particular, the PR cannot systematically increase the winning rate of the strongest player. Various possible effects of PR are illustrated in Section III-B. Then, Section III-C shows that the PR can nonetheless have a positive impact; we will see that this is indeed the usual case.We will refer to Eq. 1 for all evaluations of PLC in this section.

A. Two players only: the PR does not increase the PLC with two players only

Let us first show that the PR cannot systematically increasing the Elo difference between two players, compared to an ad hoc choice of the first move by an all-knowing referee.

Consider the following scenario where Black plays first. There are two players named player 1 and player 2. Without PR, the game starts with player being assigned a color at random, say player i is assigned Black and player $3-i$ is assigned White. The game proceeds with player i playing the first Black move, then player $3-i$ plays White, ...until the



In the opening, both sides revealed a king. In the move sequences, 12. a5(C) b8-a8 13. a3(r) a3-b3 14. a5-a2, Red found Black's king could not move horizontally and tried to capture the king by means of revealing two pieces at squares a5 and a3. However, the pieces flipped at a5 was the red cannon which captured Black's king A2 in the subsequent moves. Estimating probabilities is crucial in CDC.

Fig. 2: **Left:** opening board in 4x8 CDC (left) and 8x8 (right): all pieces are face-down. **Right:** Chinese Dark Chess game won by DarkKnight as the red player, in competition [10].

TABLE II: Estimation of the PLC for two variants of CDC using MCTS programs (4x8 and 8x8).

Thousands of Simulations	4×8					8×8				
	Wins	Losses	Draws	Score	Elo	Wins	Losses	Draws	Score	Elo
0.3	0.24	0.26	0.5	0.49	993.05	0.01	0.46	0.53	0.28	831.60
1	0.45	0.09	0.46	0.68	1130.94	0.06	0.25	0.69	0.41	933.18
2	0.52	0.07	0.41	0.73	1168.40	0.15	0.14	0.72	0.50	1003.44
3	0.61	0.05	0.34	0.78	1219.87	0.19	0.09	0.72	0.55	1034.68
5	0.63	0.05	0.33	0.79	1227.17	0.30	0.06	0.64	0.62	1085.04
25	0.68	0.04	0.28	0.82	1263.42	0.44	0.02	0.54	0.71	1155.54
PLC					270.73					323.95

game is over. With PR, the game starts with one player at random picking a first Black move. The opponent then decides who is assigned Black, say j, and who is playing White, $3-j$ (swap decision). The game proceeds from here with player $3-j$ playing White, then player j playing Black, ...until the game is over. This means that with the PR, the second player has the possibility to change roles. For this reason, the first player should not play a very strong move; he should instead play a move which leads to position as balanced as possible.

Let us assume that player 1 is stronger than player 2. This means that from any position, if 1 plays against 2, one game as black and one game as white, the expected number of wins for player 1 (over these 2 games) is greater than, or equal to, the expected number of losses.

Let us define p_i the winning probability for player 1 (against player 2) if playing as black for moves 3, 5, 7, 9, ...(player 2 plays moves 2, 4, 6, 8, ...) and first move is $i \in \{1, 2, \ldots, K\}$ (with K the number of possible first moves). Let us define q_i the winning probability for player 1 (against player 2) if playing as white (for moves 2, 4, 6,...; player 2 plays moves 3, 5, 7, ...) and first move is $i \in \{1, 2, \ldots, K\}$. Let us simplify notations by assuming that there is no draw (otherwise, a draw will be half a win). We are interested in finding the method for which the winning rate of player 1 (assumed to be stronger) is the greatest, so that the game has a greater game playing complexity.

The three considered methodologies are:

- RDR: Randomly draw roles, no PR: the winning rate of player 1 depends on the first move as chosen by the player who plays first. Let us assume that player 1 optimizes the first move so that their winning rate is maximum.
- RDR+PR: Randomly draw roles, and possibly change by PR.
- RDR+GFM$_i$: (GFM stands for "given first move") Randomly draw roles, no PR, and the first move is given by the rules - it is the move with index i.

Then the following equations give the success rate w of player 1: (a) RDR+GFM$_i$: $w_{\text{rdr+gfm},i} = (p_i + q_i)/2$ if we decide that the first move is i. (b) RDR: $w_{\text{rdr}} = \frac{1}{2}(\max_i p_i + \min_i q_i)$. (c) RDR+PR: $w_{\text{rdr+pr}} = \frac{1}{2}(\min_i \max(p_i, q_i) + \max_i \min(p_i, q_i))$. Then we claim:

Theorem 1: Consider the PR, when the first move and the PR choice (swap or no swap) are rational.Then,

$$w_{\text{rdr+pr}} \leq \max_i w_{\text{rdr+gfm},i}. \tag{3}$$

Proof: Let us first show that

$$\min_i \left(\frac{p_i + q_i}{2} + \frac{|p_i - q_i|}{2} \right) - \min_i \frac{|p_i - q_i|}{2} \leq \max_i \frac{p_i + q_i}{2} \tag{4}$$

For RDR+PR and RDR+GFM$_i$, exchanging p_i and q_i, for some i, does not change the game. This is because:

- For RDR+GFM$_i$, the role (Black or White) will be randomly chosen, so that exchanging p_i (probability of winning if I am Black) and q_i (probability of winning if I am White) does not change the result.
- For RDR+PR, the role (Black or White) is chosen by the second player, given the choice of i by the first player. Therefore, exchanging p_i and q_i leads to the same choices, just exchanged between Black and White.

So we can assume, without loss of generality, that $p_i \leq q_i$ for any $i \in \{1, \ldots, K\}$. Then, by denoting $q_{i_0} := \min_i q_i =$

$\min_i \left(\frac{p_i+q_i}{2} + \frac{|p_i-q_i|}{2}\right)$ and $\frac{q_{i_1}-p_{i_1}}{2} := \min_i \frac{q_i-p_i}{2} = \min_i \frac{|p_i-q_i|}{2}$, we get:

$$\min_i \left(\frac{p_i+q_i}{2} + \frac{|p_i-q_i|}{2}\right) - \min_i \frac{|p_i-q_i|}{2}$$
$$= q_{i_0} - \frac{q_{i_1}-p_{i_1}}{2} \le \frac{q_{i_1}+p_{i_1}}{2} \le \max_i \frac{p_i+q_i}{2}.$$

We now prove Eq. 3, using the classical equalities $\max(a,b) = \frac{a+b}{2} + \frac{|a-b|}{2}$ and $\min(a,b) = \frac{a+b}{2} - \frac{|a-b|}{2}$ for any $(a,b) \in \mathbb{R}^2$:
$w_{rdr+pr} = \frac{1}{2}(\min_i \max(p_i,q_i) + \max_i \min(p_i,q_i)) = \frac{1}{2}(\min_i[\frac{p_i+q_i}{2} + \frac{|p_i-q_i|}{2}] + \underbrace{\max_i \left[\frac{p_i+q_i}{2} - \frac{|p_i-q_i|}{2}\right]}_{\le \max_i \frac{p_i+q_i}{2} - \min_i \frac{|p_i-q_i|}{2}})$; then, by using Eq. 4, we get $w_{rdr+pr} \le \max_i \frac{p_i+q_i}{2}$, which is the expected result. ■

Remark: The assumption in Theorem 1 means that p_i and q_i are known by both players. We also note that in some cases, the inequality is strict. Importantly, we here compared the result to the best (in terms of PLC) possible choice for the initial move. This is the main limitation; the PR does not perform better than an omniscient referee who would choose the initial move for making the game as deep as possible specifically for these two players. In the following, $\max_i(RDR + GFM_i)$ denotes the best possible choice for the initial move, i.e. the move that give $\max_i w_{\text{rdr+gfm},i}$.

B. General case: examples and counter-examples of impact of the PR on PLC

Two opposite behaviors might happen. (a) Table IIIa presents an example showing that the PR can indeed decrease the success rate of the strongest player, compared to a choice of the 1st move by the 1st player. In this case, $w_{\text{rdr}} > w_{\text{rdr+pr}}$. (b) Table IIIb presents an example showing that the PR can greatly increase the success rate of the strongest player, compared to RDR. In this case, $w_{\text{rdr+pr}} > w_{\text{rdr}}$.

C. An artificial example in which the PLC is increased by the PR, compared to any fixed move

We have seen above (Eq. 3) that, when only two players are considered, $w_{\text{rdr+pr}} \le \max_i w_{\text{rdr+gfm},i}$. Moreover this inequality might be strict. In table IIIc, we present an artificial game (with more than 2 players) in which $w_{\text{rdr+pr}} > \max_i w_{\text{rdr+gfm},i}$. By referring to Table IIIc, if the first move is fixed at A, then $Elo_3 - Elo_2 = 13.905$, $Elo_2 - Elo_1 = 636.426$. If it is fixed at B, then $Elo_3 - Elo_2 = 251.893$, $Elo_2 - Elo_1 = 275.452$. In both cases, the Elo range, i.e. the PLC for those 3 players (Eq. 1), is less than 651. With PR (random choice of initial colors, the second player to play can switch), if each player plays the optimal strategy, then $Elo_3 - Elo_2 = 126.97$ and $Elo_2 - Elo_1 = 530.72$. The PLC for those 3 players is therefore more than 657. Therefore, with two players, PR can be successful in terms of *depth* only against RDR, and not against $\max_i(RDR + GFM_i)$. However, with at least 3 players, depending on the game, PR can be or not successful in terms of *depth* both compared to RDR and compared to $\max_i(RDR + GFM_i)$.

TABLE III: Artificial games to illustrate possible effect of the PR. Each number in the table is the probability of winning for the strongest player, depending on who plays black and what the first move is.

(a) The PR decreases the success rate of the strongest player from 95% to 90%.

	Score of 1 vs. 2	
	as B	as W
move A	100%	90%
move B	50%	90%

(b) B is a very strong move. PR increases the success rate of the strongest player from 50.005% to 75%.

	Score of 1 vs. 2	
	as B	as W
move A	100%	50%
move B	100%	0.01%

(c) PR works better than any fixed move assuming a set of 3 players.

	Score of 1 vs. 2		Score of 2 vs. 3	
	as B	as W	as B	as W
move A	0.96	0.99	0.37	0.67
move B	0.71	0.95	0.68	0.94

IV. Applying the Pie Rule: real games

We study the impact of the PR on NoGo and Y (Section IV-A) and on the handicap choice in KAG (Section IV-B). The short version is that the PR increases the *depth* in particular when a wide range of players is considered.

A. Rules for choosing the first move: NoGo and Y

We now switch to experimental works on the choice of the first move, in particular for the PR. We study the impact of the PR in NoGo and Y. We first make the assumption that each player plays the PR perfectly and chooses the initial move perfectly in RDR (they know the success rates corresponding to a given first move), though later we will check the impact of this assumption by checking its impact on the PLC. We experimented (i) RDR+PR (ii) RDR (iii) RDR+GFM$_i$ with the best choice of i and (iv) RDR+GFM$_i$ with the worst choice of i. The PLC is evaluated by Eq. 1. The pool of players consists of MCTS players with varying number of simulations per move.

The NoGo game (Fig. 1a), designed for the BIRS seminar, has the same gameplay as Go but the goal of the game is different. In NoGo captures and suicides are illegal and the first player with no legal move loses the game. The players are MCTS with 125, 250, 500, 1000, 2000, 4000, 8000 simulations per move in size 5×5 and 6×6, and 125, 250, 500, 1000 simulations per move in size 7×7. We designed 6 opening moves for 5×5, 6 opening moves for 6×6, 10 opening moves for 7×7. The experimental results are displayed in Table Ib. We also evaluated the PLC of 5×5 by Eq. 1 without using the rational first move assumption (i.e. the first move is now chosen by the first player); we got

$$\text{PLC}_{nogo5x5} = 240.32, \quad (5)$$
$$\text{PLC}_{nogo6x6} = 278.58, \quad (6)$$
$$\text{PLC}_{nogo7x7} = 429.71 \quad (7)$$

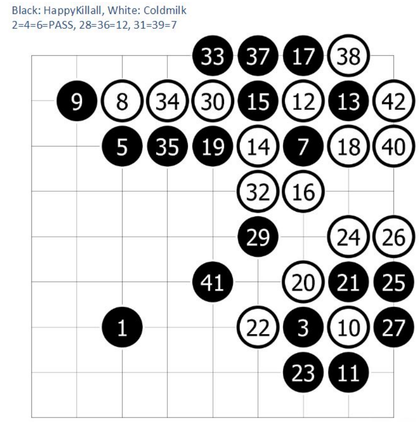
which are all significantly higher than their "rational opening" counterparts and show that opening moves are not rational - and that the ability to decide the first move contributes significantly to the *depth* of the game. We estimated the winrates used in the PLC computations over 500 games for each setting.

The game of Y is a connection board game invented by Shannon in the 50s (Fig. 1b).

We test all the possible opening moves, and the players use 4^0, 4^1,...,4^m simulations per move, with $m = 5$ for board size 4 and $m = 6$ for board size 5. We estimated the winrates used in the PLC computations over 20,000 games for each setting. The experimental results are displayed in Table Ib. In NoGo, we see that, in 6x6 and 7x7, PR improved the PLC compared to RDR, but not always when compared to $\max_i(RDR + GFM_i)$ - though it was successful for both cases with at least 7 players, which is consistent with the fact that with only 2 players PR cannot beat $\max_i(RDR + GFM_i)$ in terms of *depth* (Theorem 1). In the case of the game of Y, the PR is a clear success. Using the Pie Rule in Y extends the *depth* more than any other rule.

B. Rules for choosing the handicap: Killall-Go

In KAG, the gameplay is the same as in Go, and the rules for deciding the winner are the same; but the komi (number of points given to black as a territory bonus for deciding the winner) is such that White wins if and only if he has at least one stone alive at the end. Deciding the handicap placement and the fair handicap is non trivial. For example, most people consider that living is hard for White in 9x9 with handicap 4; but Fig. 3 shows an opening which might be a solution for White to live. The Black player is also known as *Killer* and White is known as *Defender*. In KAG, the handicap is naturally much larger than in a standard Go game, in order to compensate the special komi. For example, 17 handicap stones is common in 19×19 KAG. We use the PR for choosing the number of handicap stones, with, then, free handicap positioning (Black chooses the position). The handicap can be 2, 3, 4 or 5; we consider 9x9 Kill-all Go. We consider N players, which are GnuGo/MCTS with $50, 100, 150, \ldots, 50 \times N$ simulations per move respectively. Besides the best handicap choice (in the sense: the one which maximises the depth) and the pie rule, we consider the 4 stones handicap, which is the most reasonable for strong players. We get results as presented in Fig. 3, right. Except for very weak players the handicap which maximizes the PLC between players is 4. We also see that the pie rule, as soon as the number of players is large enough, increases the PLC compared to fixing once and for all the PLC. The surprising gap between the PLC of KAG and the PLC of Go is discussed in Section II-E - this gap is important but highly depends on the set of considered players.



KAG ; nb of players N	KAG PLC with 4 stones hand.	KAG PLC with best hand.	PLC with pie rule	PLC of Go
5	275	470	434	253
6	554	554	588	268
7	745	745	779	283
8	774	774	1011	292
9	774	999	1290	311
10	774	999	1290	328
11	1191	1191	1570	329
12	1382	1382	1690	329
13	1382	1382	1690	333
14	1382	1382	1690	347
15	1382	1382	1690	349

Fig. 3: Top: KAG with handicap 4 in 9x9. White wins (i.e. White lives), thanks to an elegant opening. Bottom: Comparison between the *depth* of KAG (columns 2 to 4) and Go (column 5) for different numbers N of GnuGo players (first column). Row N corresponds to the *depth* for the set of players $(1, 2, 3, \ldots, N)$, where player i has $50 \times i$ simulations per move.

V. Conclusions

Game complexity measures: beyond computational complexity. Depth and PLC have fundamental shortcomings. They depend on a set of players. Maybe the dependency on a set of players is necessary, in the sense that, as we are looking for a measure relevant for humans, we need humans or human-like AIs somewhere in the definition. One can also easily construct trivial games (e.g. who has the longest right foot) which are very deep. Nonetheless *depth* is interesting as a measure for comparing games. PLC is a nice refinement, and the best formula for measuing it is Eq. 1 as discussed below.

Elo model. A side conclusion is that the Elo model, at least when pushed to the limit for such PLC analysis, is not verified (see also [15]). Under the Elo model, Eq. 1 and Eq. 2 should be equivalent, which is not the case in our experiments. We see that, consistently with [15], the Elo model poorly modelizes large rank differences. For example, IRT models[16] include lower and upper bounds (> 0 and < 1 respectively) on the probability of succeeding, regardless of the ranks; this might be used in Elo scales as well. Further analysis of this point is left for further work.

The pie rule and the PLC. The PR does not necessarily preserve the PLC of a game, compared to a fixed good choice of the initial move. In fact, with 2 players only in the considered set, it cannot increase the PLC compared to the variant of the game which enforces the first move to the deepest opening move (i.e. if it enforces the first move to the one which leads to the deepest game for these two players).

However, it sometimes improves the PLC, with more than 2 players, or compared to a poor choice (in the rules) of an initial move, or compared to the classical case of a choice of the initial move by the first player. PR is very effective when very strong first moves exist. Theorem 1 does not contradict this, as it discusses the case in which the initial move is chosen by the referee. Basically PR discards such first moves from the game, so that the referee does not have to do it. Even with just two players, PR can increase the depth, compared to RDR, which is the most usual case. Our counter-example can be extended to several players, but then it only shows that the pie rule does *not always* increase the depth; whereas for two players we have the stronger result that the pie rule *cannot* increase the depth compared to an ad hoc choice of the first move - i.e. there is at least one first move such that, if it is imposed by the rules, the Elo difference between these two players will be at least the same as with pie rule.

Experimental results on various real-world games: do we improve the *depth* when using the pie rule ? For the game of Y, tested with players with strongly varying strength and moderate board size, the PR is very effective, making the game deeper than any other rule for choosing the first move. Concerning the NoGo game, even if PR made the game deeper compared to RDR, an ad hoc choice of the first move by an omniscient referee leads to the deepest variant of the game.In KAG, the pie-rule is clearly successful for choosing the number of handicap stones in KAG. The wider the range of considered players, the better the PR (Fig. 3, right).

Analyzing games with *depth* analysis. Depth and PLC are also tools for quantifying (for sure not in a perfect manner) the importance and challenging nature of a game. Interestingly, KAG is deeper than Go, by very far for some set of players we have considered, namely variants of GnuGo, in 9x9 - whereas Go is usually considered as the deepest known game. This is however unstable; there are some thresholds at which the winning rate between i and $i+50$ simulations are very large (e.g 550 simulations per move vs 500 simulations per move), and when such gaps are excluded from the set of players, the depth of KAG does not increase that much with pie rule.

Depth as a criterion for choosing exercises. KAG is widely used as an exercise and is particularly deep; this suggests that *depth* might be a good criterion for choosing problems. Generating problems is not that easy, in particular in games such as Go - depth is a criterion for selecting interesting positions.

Chinese Dark Chess. Just doubling the width of CDC makes the game much deeper and far more challenging (Table II). All rules are preserved. This might give birth to a new game for humans.

The first move in NoGo. Last, experiments on NoGo show that the opening move is often not the rational one and that the opening move has a big impact on the game (see Eq. 5-7, showing the depth when using a real first move, compared to Table Ib which uses a rational first move).

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