Calculation of distances and distance-redshift relationships under the different modes of space expansion

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Abstract

The discovery of the generalized cosmological redshift and of universe expansion is the foundation of modern cosmology, but also a challenge for determining distances. The distance of a celestial body measured in our telescopes is not the distance at the time of light emission, nor at the time of reception because space expands ahead of and behind the light front. In addition, the correspondence between these different distances and the redshift depends on the mode of expansion. Instead of starting from universe models and hypothetical causes of expansion, the inverse approach used here is to systematically examine the relations between redshift and distances expected in all the scenarios of expansion postulated in the course of universe maturation, regardless of their rationale: proportional to cosmic time, power law or exponential, and for different assumptions about the origin of the redshift: kinematic, relativistic and wave stretching effects. None of these combinations gives the same results in terms of redshift, of horizon, of distances and of deviation from the Hubble law. This compendium thereby extends the mathematical toolbox for distinguishing cosmological hypotheses.

Keywords: Universe expansion; cosmological redshift; cosmological distances; informational horizon.

1 Introduction

Distances and redshifts are the basic observational ingredients of cosmology, but their use is delicate. Ambiguities about the Hubble law [1] have long been pointed by Harrison [2], such as the common confusion between the redshift-distance and velocity-distance relationships. It would also be preferable to know what we measure exactly. What is the distance perceived, using parameters such as luminosity and magnitude, through a telescope? It is not the distance of the celestial body when the light was emitted since it was shorter, nor the actual distance because the space separating us from the star continued to expand during the time of travel of light. In addition for distant objects, several modes of expansion could have succeeded during the long journey of light. Other mathematical twists are necessary to establish and interpret the famous distance-redshift Hubble diagram, the black box of the universe in which are inscribed the past modes of expansion. The determination of redshifts is reliable compared to other types of measurements and the distance-redshift diagram is drawn with increasing accuracy thanks to technical progress and to objects which are both distant and bright such as supernovae and gamma ray bursts; but to fully exploit this diagram, the tools for its analysis must be clearly established. The distance-redshift diagram is sometimes used to define the instruments necessary for its own reading or to select convenient parameters (such as Ω and appropriate forms of matter and energy), to adjust the theoretical curves of the mainstream model ΛCDM. Theories are based on observables and conversely certain measurements make use of accepted theories, such as when distances are deduced from redhifts, thereby introducing circular distortions in the reasoning. The present study does not favour a specific pre-designed model, but is aimed at listing basic mathematical relationships preliminary to modeling and curve fitting, such as the dependence of redshifts on the mode of expansion and the true distance of light sources deduced from their measured distance. Instead of starting from Friedmann equations as usual, all cases of expansion are examined using minimalist and original mathematical approaches applied to flat spaces devoid of gravitational influences. Let us start with the basics: the origin of the redshift.

2 The main hypotheses on the origin of the redshift

2.1 Redshift without expansion

General relativity would be sufficient to cause distant objects to appear redshifted as a consequence of an apparent slowing of time. This attractive possibility will not be examined here and space will be approximated as globally flat and not shaped by gravity. Another hypothesis, called "tired light", assumes that the longer is its flight, the more the light loses its energy by interaction with particles encountered along its path (which can come from the quantum vacuum). As the tired light theory is not based on simple optical rules, it will not be treated here, but note that this theory curiously uses the same law as the simple exponential expansion [3], which can of course give it a misleading success. This case will be indirectly treated when studying exponential expansion.

2.2 The redshift caused by expansion

The prevailing view is that the cosmological redshift is related to space expansion, but divergences exist about this relation. Two interpretations of the redshift, Doppler effect and wave stretching, which have been considered equivalent [4], will be compared.

2.2.1 The redshift interpreted as a kinetic Doppler effect

The Hubble Redshift is frequently, perhaps erroneously, interpreted as a Doppler effect. An observer sees a red shift if the light source moved away from him. In case of expansion, the distant stars move away from each other, but passively. The objects can not "feel" acceleration and the mutually receding galaxies can belong to the same inertial reference frame. The question is therefore whether a passive speed can generate a Doppler effect. With uniform motion, the redshift corresponding to the classical Doppler formula is $z = v/c$ and the relativistic redshift based on the Doppler formula of Einstein [5] is

$$z_r = \sqrt{\frac{c+v}{c-v}} - 1 \tag{1}$$

where v is the recession velocity. This equation is little used in astrophysics in which recessional velocities can be superluminal [6]. In addition, this formula has been built in special relativity, whereas recession velocities v_{rec} are generally supposed not uniform.

2.2.2 The redshift interpreted as a stretching of waves during their travel

Before the publication of Hubble [1], Lemaître had shown that wavelengths should follow expansion [7]. For an interval of universe

$$ds^2 = dt^2 - a(t)^2 d\sigma^2 \tag{2}$$

where $d\sigma$ Is the element length of a space of radius equal to 1, the equation of a light beam is

$$\sigma_2 - \sigma_1 = \int_{t_1}^{t_2} \frac{dt}{a} \tag{3}$$

where σ_1 and σ_2 are the coordinates of a source and an observer. A beam emitted later at $t_1+\delta t_1$ and arriving at $t_2 + \delta t_2$ undergoes a shift such that

$$\frac{\delta t_2}{a_2} - \frac{\delta t_1}{a_1} = 0 \tag{4}$$

giving

$$z = \frac{\delta t_2}{\delta t_1} - 1 = \frac{a_2}{a_1} - 1 \tag{5}$$

where δt_1 and δt_2 can be considered as the periods at emission and reception respectively [7]. If a procession of walkers regularly spaced crosses a stretching rubber band, on arrival their spacing will obviously been stretched in the same ratio as the rubber band. The same reasoning applies to a series of wave crests. In his article, Lemaître called this effect a Doppler effect [7]. This term is acceptable if broadly defining the Doppler effect as a wave distortion, but this is not the classical Doppler effect related to the speed of the source. It will be shown here that both mechanisms give different results regardless of the type of expansion. In the wave stretching effect of Lemaître, the ratio between the reception and emission wavelengths simply follows the increase of the distance between the source and the receiver which took place during the light flight between the time point of emission (source and receiver spaced by $D_{\mathtt{E}}$) and that of reception (source and receiver spaced by $D_{\mathtt{R}}$):

$$\frac{\lambda^{\text{app}}}{\lambda} = \frac{D_{\mathtt{R}}}{D_{\mathtt{E}}} \tag{6a}$$

The corresponding redshift z_s is the relative distance increase

$$z_s = \frac{D_{\mathtt{R}} - D_{\mathtt{E}}}{D_{\mathtt{E}}} \tag{6b}$$

3 Relationships between Doppler effects and distance changes in uniform motion

The simple case of uniform motion is sufficient to perceive the symmetric nature of relative motion and the absence of a static medium. Imagine that a source and a receiver can move relative to one another and in addition, can move relative to an hypothetical static medium supporting light propagation at speed c. The source and the receiver recede form each other at speed v. At time $t_{\mathtt{E}}$, when spaced from the receiver by $D_{\mathtt{E}}$, the source emits a light beam propagating towards the receiver.

3.1 In an immobile medium

Different results are obtained depending on whether this is the source or the receiver which moves relatively to the background medium (Fig.1).

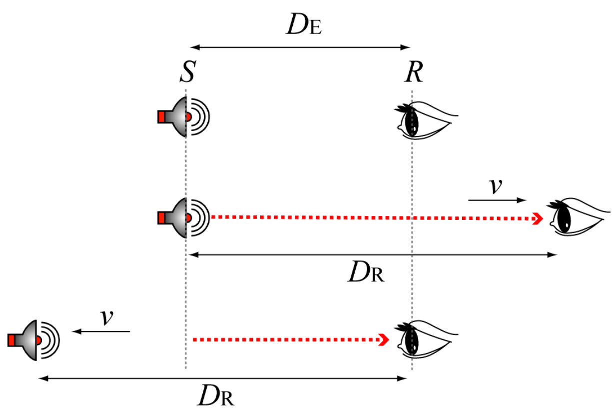

Figure 1. A light pulse is emitted by a source (S) when spaced from a receiver (R) by D_E. Just like a ball thrown between two football players, light travels through a static medium relatively to which either S (middle line) or R (bottom line), is considered immobile.

3.1.1 The receiver is considered immobile relative to the background medium

Light reaches the receiver at time t_R after crossing a distance D_E (bottom scheme of Fig.1). Hence, the duration of the light travel is

$$t_R - t_E = D_E/c \tag{7a}$$

At t_R, the new spacing between the source and the receiver has become

$$D_R = D_E + v(t_R - t_E) \tag{7b}$$

Replacing the duration in Eq.(7b) by its value given by Eq.(7a), yields a distance ratio corresponding to a classical Doppler effect

$$\frac{D_R}{D_E} = 1 + \frac{v}{c} \tag{7c}$$

Distance increases in the same ratio that the classical Doppler effect. In case of collinear approach, the same reasoning gives

$$\frac{D_R}{D_E} = 1 - \frac{v}{c} \tag{7d}$$

3.1.2 The source is considered immobile relative to the background medium

In this case, light is expected to reach the receiver after crossing a distance D_R (middle scheme of Fig.1). The duration of the light travel is

$$t_R - t_E = D_R/c \tag{8a}$$

Replacing the duration in Eq.(7b) by its value given by Eq.(8a), yields a conjectural Doppler formula

$$\frac{D_R}{D_E} = \frac{1}{1 - \frac{v}{c}} \tag{8b}$$

In case of collinear approach, the same reasoning gives

$$\frac{D_R}{D_E} = \frac{1}{1 + \frac{v}{c}} \tag{8c}$$

These contradictory results suggest that the notion of immobility in a static space should be ruled out. Interestingly, the geometric means of the two extreme results obtained by postulating the static medium (Eqs (7c)/(8b) and Eqs (7d)/(8c)), is the relativistic Doppler effect. For the recession:

$$\left\langle \frac{D_R}{D_E} \right\rangle = \sqrt{\frac{c+v}{c-v}} \tag{9a}$$

and for the approach

$$\left\langle \frac{D_R}{D_E} \right\rangle = \sqrt{\frac{c-v}{c+v}} \tag{9b}$$

3.2 Without background medium

In the special relativity theory, uniform motion cannot be attributed specifically to one of the relatively moving frames. The total distance crossed by light is not D_R nor D_E, but

$$D_L = c(t_R - t_E) = \frac{c}{v}(D_R - D_E) \tag{10}$$

The precise relationships between D_R, D_E and D_L will be calculated later.

4 Influence of expansion on signal connection

4.1 Validity of the simplistic analogy with the walker on an elastic

The theory of Lemaître recalled above gives results equivalent to a light travel on an elastic medium. In spite of frequent reprobations, this equivalence is perfectly valid because for light following geodesics ($s = 0$), the metric of Friedmann-Robertson is very simple. By rewriting the interval of Eq.(2) in Cartesian coordinates, this metric reads

$$ds^2 = (cdt)^2 - a(t)^2(dx^2 + dy^2 + dz^2) \tag{11}$$

where $a(t)$ is the spatial expansion factor depending on time only [8]. For a light beam oriented along the x axis, this line element reduces to

$$\frac{dx}{dt} = \frac{c}{a(t)} \tag{12}$$

This formula is equivalent to a relative speed introduced below in this study, if adding a unit of distance. Furthermore, a dimensionless equation (time^{-1}), is obtained with the Friedmann-Lemaître-Robertson-Walker approach giving $d\chi/dt = c/a(t)$ where χ is an angle (without dimension) and $a(t)$ is a distance, more appropriately written $R(t)$ for universe radius. It is therefore legitimate to adopt the minimalist treatment used in this article, according to the recommendation of Gibbs: "One of the principal objects of theoretical research in any department of knowledge, is to find the point of view from which the subject appears in its greatest simplicity". Following Gibbs and a proposal of Eddington, light will be compared to a walker on an expandable substrate similar to a rubber band.

Imagine that you want to join your house located at the end of a stretchable road on which you walk at velocity c. The road stretches in front of you so that your house moves away at speed v. You legitimately ask several questions: do you walk fast enough to reach your home? Is this just possible if $v > c$? According to the theory of wave stretching, you realize that your legs themselves are elastic, so that your stride lengthens in the same ratio as the road. So you are first reassured because the number of steps necessary to reach the house remains unchanged. But another rule is imposed on you: you must walk at a constant speed. As your stride is longer, you should slow down your pace. So you will arrive late at home, perhaps much later, if ever; it depends on the initial path length (D_{E}) and especially, as we shall see, on the expansion function. This metaphor is analogous to the expansion of the universe, if comparing the walker to a light beam progressing at a constant speed, a wavelength corresponding to a couple of strides. The speed of light can be expressed in term of wave parameters as follows

$$c = \lambda \text{ meters}/T \text{ seconds} \tag{13}$$

where λ is the length of a stride (wavelength) and T is the stride pace (period). Dilating λ while maintaining the speed c constant, imposes to increase T (decrease ν) in the same ratio, thereby generating a redshift. Before returning to the main two competing interpretations of the redshift, it is useful to determine whether or not you can reach your home. These results will be obtained below for different modes of expansion (uniform, geometric and exponential) and using a classical treatment, without recourse to the usual relativistic approaches.

4.2 Uniform expansion

Several authors have suggested a mode of universe expansion called $a \propto t$, which simply follows the cosmic time [9, 10, 11]. When $a(t) = ct, \dot{a} = c$ and $H = 1/t$, expansion follows the age of the universe according to

$$\frac{D(t)}{D_{\mathrm{E}}} = \frac{t}{t_{\mathrm{E}}} \tag{14}$$

In fact, with a linear cosmic time, this expansion simply corresponds to the uniform expansion at constant speed v. Indeed, Eq.(14) can be re-written

$$\frac{D(t)}{D_{\mathrm{E}}} = 1 + \frac{t - t_{\mathrm{E}}}{t_{\mathrm{E}}} \tag{15a}$$

or

$$D(t) = D_{\mathrm{E}} + \frac{D_{\mathrm{E}}}{t_{\mathrm{E}}}(t - t_{\mathrm{E}}) \tag{15b}$$

that is

$$D(t) = D_{\mathrm{E}} + v(t - t_{\mathrm{E}}) \tag{15c}$$

Uniform expansion means at constant intervals, but not necessarily slow expansion, as the speed can be unlimited. With uniform expansion, even at gigantic superluminal speed, and given sufficient time, the walker will inevitably reach his home, whatever his walking speed (eg walk at $c = 4$ km/h while the road stretches at $v =$ 1,000 km/h). This result is so counterintuitive that it has been popularized as a mathematical game [12]. Speed calculations on an elastic road is an uncomfortable gymnastics due to lack of fixed coordinates. To visualize the phenomenon, let us create a landmark in the form of a television screen on which the entire road always appears complete and fixed. This can be achieved with a movie camera zooming out to exactly compensate the extension of the road. On the screen, we would see the walker move forward more and more slowly but finally reach the house. The relative speed of the walker (time^{-1}) in the fixed frame of the TV screen coordinates (unit size) can be defined in two manners using the correspondence $D(t) = D_{\mathrm{E}}(t/t_{\mathrm{E}}) = D_{\mathrm{E}} + vt$

$$\frac{dx(t)}{dt} = \frac{ct_{\mathrm{E}}}{D_{\mathrm{E}}t} = \frac{c}{D_{\mathrm{E}} + vt} \tag{16}$$

whose integration gives a unitless relative position x, starting from $x(t_{\mathrm{E}}) = 0$, of

$$x(t) = \frac{ct_{\mathrm{E}}}{D_{\mathrm{E}}} \ln \frac{t}{t_{\mathrm{E}}} = \frac{c}{v} \ln \left(1 + \frac{vt}{D_{\mathrm{E}}}\right) \tag{17}$$

These functions without ceiling indicate that the walker continues its merry way and necessarily reaches the other end of the screen (when $x(t) = 1$), even with a very high stretching rate v. Signals cannot be disconnected in the universe. The journey can however be very long as its duration follows

$$\Delta t = \frac{D_{\mathrm{E}}}{v}(\mathrm{e}^{v/c} - 1) \tag{18}$$

If photons are like this walker in an expanding universe, whatever the relative values of v and c, there is no disconnection between the different parts of this universe. To try to stop this persistent walker, one must change the mode of stretching.

4.3 Speculative interlude

Uniform expansion, even at enormous speed, cannot disconnect signals. So how can we disconnect the different regions of the universe? This question can be of practical interest for a clever universe designer. Imagine such a creator who seeded the universe with all the ingredients necessary for the onset of life. She knows that the main danger threatening life, that's life itself, with its lack of regard for life forms seen as inferior. If communication was unrestricted in the universe, a bonus would be given to the rapidly developing forms, but in fact slowly emerging, more complex forms could ultimately be the most effective if sufficient time was given them. The natural solution to this problem is to partition, at least temporarily, the world into regions disconnected from each other by informational borders. If disconnection limits are conveniently adjusted to the probabilities of life emergence, all forms of life could blossom and grow at their own kinetics, without risk of competition with more advanced forms. The recipe for such a fragmentation is the expansion of the space, provided a mode of expansion prevents the signals from reaching other life outbreak sites. As a consequence, if on average, a maximum of one form of life is expected in a connexion sphere, our efforts to not feel alone in the universe may remain in vain. Incidentally, the scale of such fragmentation could also be adjusted to make undetectable any curvature of the universe. Let us examine first a widely used mode of expansion of astrophysics, the geometric progression called "power law".

4.4 Power law expansion

This mode of expansion, predicted by calculations based on the theoretical constituents of the universe, satisfies

$$D(t) = D_{\mathtt{E}} \left(\frac{t}{t_{\mathtt{E}}}\right)^{u} \tag{19}$$

where u can take different values depending on the maturation stage of the universe described in astrophysics courses: $1/2$ for the radiation-dominated era, $2/3$ for the matter-dominated era, $1/3$ for the stiff fluid, $u = 2$ for a minimal condition of inflation satisfying $\ddot{a}/a > 0$, etc. As the strategy of the relative position on our TV screen seems efficient, let us use it again. The relative position of the walker (photon) on the screen is obtained by integration between $t_{\mathtt{E}}$ and t

$$\begin{aligned} x(t) &= \int_{\tau=t_{\mathtt{E}}}^{t} \frac{c}{D_{\mathtt{E}}} \left(\frac{\tau}{t_{\mathtt{E}}}\right)^{-u} d\tau \\ &= \frac{c}{(1-u)} \frac{t_{\mathtt{E}}}{D_{\mathtt{E}}} \left[\left(\frac{t}{t_{\mathtt{E}}}\right)^{1-u} - 1\right] \end{aligned} \tag{20}$$

Things get interesting:

— (i) For $u < 1$, $x(t)$ increases continuously with t and therefore can exceed 1 without problem. As for the previous modes of expansion, the universe is revealed in its entirety.

— (ii) For $u > 1$, we finally succeed in stopping the walker on the television screen. Even if he continues to run at speed c, the walker cannot cross a virtual limit on the screen that can be called a horizon at

$$x(t_\infty) = \frac{c\ t_{\mathtt{E}}}{D_{\mathtt{E}}(u-1)} \tag{21}$$

The success of the connection depends on the initial conditions: the time of departure ($t_{\mathtt{E}}$) and the distance to be covered as it is at this time ($D_{\mathtt{E}}$). Now let us try another accelerated mode of expansion in which time is not raised to a power, but is itself an exponent: the exponential expansion.

4.5 Exponential expansion

With an exponential expansion of rate H, the length of the road stretches according to $D(t) = D_{\mathtt{E}}\ \mathrm{e}^{H\Delta t}$ and the relative speed of the walker is

$$\frac{dx(t)}{dt} = \frac{c}{D_{\mathtt{E}}}\ \mathrm{e}^{-H(t-t_{\mathtt{E}})} \tag{22}$$

whose integration between $t_{\mathtt{E}}$ and t, gives his relative position on the television screen

$$x(t) = \frac{c}{HD_{\mathtt{E}}} \left(1 - \mathrm{e}^{-H\Delta t}\right) \tag{23}$$

This position tends asymptotically to a maximum $c/HD_{\mathtt{E}}$, which means that for given values of H and c, the success of the walker in his crossing depends on $D_{\mathtt{E}}$. To reach the goal, $c/HD_{\mathtt{E}}$ must exceed 1, so $D_{\mathtt{E}}$ must be lower than c/H. The critical distance c/H is a disconnection point in the stretching universe of the walker, that is unreachable because the time of travel

$$\Delta t = \frac{1}{H} \ln \left(\frac{1}{1 - \dfrac{HD_{\mathtt{E}}}{c}}\right) \tag{24}$$

would become infinite, and at t infinite, the relative speed $dx(t)/dt$ is zero. The walker stops on the television screen, even if in reality he goes imperturbably forward at speed c. As there is no particular point in a homogeneous elastic universe, this distance is universal and valid from any starting point.

4.5.1 Consequence of the Hubble law $v_{rec} = HD$

The proportionality between the recession velocity and the distance can be written $dx/dt = Hx$, where H is the slope of a straight line. As the slope of a straight line is a constant, the unique solution of the Hubble law is $x(t) = x(0)e^{Ht}$. This expansion is often called a model (of de Sitter), but in addition to be a model, it is also just the only possible mathematical solution of $v_{rec} = HD$, as long as H is constant. It is recommended in astronomy to write $v_{rec} = H_0 D$ to clearly indicate that $H(t)$ may have been different in the past. Interesting questions are: how much of the universe we can see and how far we have to look for seeing a H different from H_0.

4.5.2 Specificities of the exponential mode of expansion

— (i) It is an infinite mode of expansion without big bang, because $a(t) = a_0 e^{Ht}$ (Hubble law) has no root. Setting $a_0 = 0$ implies that for all t, $a(t) = 0$. So a universe exponentially expanding only has always existed. This property is particularly interesting in the case of an infinite universe, because getting an infinite space by stretching a finite space is a pure mathematical impossibility.

— (ii) This mode of spatial expansion is a direct consequence of the proportionality between velocity and distance $dx/dt = Hx$, which implies $x(t) = x(0)e^{Ht}$. This is the flat version of the universe of de Sitter [13] (for $k = 0$), towards which converge at long times the positively curved version ($x(t) = x(0)\cosh Ht$ for $k = 1$), and the negatively curved version ($x(t) = x(0)\sinh Ht$ for $k = -1$). After a period of success, the model of exponential expansion of de Sitter was disqualified because it has been shown to not satisfy theoretical predictions based on assumptions of pressure and density of the universe. It is however supposed to be the present mode of expansion driven by vaccum energy since we entered the "dark energy-dominated era" in which H_0 is a constant $H_0 = \sqrt{\Lambda/3}$, where Λ is the cosmological constant. Note that H constant no way means that recessional velocities are constant in the sense of uniform motion.

5 Tools and observables

5.1 Nomenclature

The traditional nomenclatures should be adapted to cosmology because astrophysicists use the suffix 0, not for the initial condition as in most usual treatments, but for the final condition (present). The following symbols are therefore used to avoid misunderstanding:

— $t_{\mathtt{E}}$ is the date of light emission (initial, generally written $t_{\mathtt{0}}$ in other scientific contexts). This date can not be measured directly.

— $t_{\mathtt{R}}$ is the date of reception of the light (final). This is the present age of the universe, written t_0 by cosmologists. Of course if running time backward, an acceleration becomes a deceleration and vice versa, which may induce some misunderstandings.

— $D_{\mathtt{E}}$ is the initial distance between the star and the telescope (which did not exist yet!) at $t_{\mathtt{E}}$. $D_{\mathtt{E}}$ is not known *a priori.*

— $D_{\mathtt{R}}$ is the distance between the source and the telescope when light enters the telescope.

— $D_{\mathtt{L}}$ is the distance crossed by light between its emission and its reception. Hence, $D_{\mathtt{L}}/c = t_{\mathtt{R}} - t_{\mathtt{E}}$. $D_{\mathtt{L}}$ is smaller than $D_{\mathtt{R}}$ because the fraction of space already crossed by light continued to stretch until reception.

5.2 Quantities measurable in our telescopes

In the theory of Lemaître, the redshit is given by the ratio $D_{\mathtt{R}}/D_{\mathtt{E}}$ (Eq.(6)), but none of these distances are directly measurable. Only the distance crossed by light ($D_{\mathtt{L}}$) can be estimated through certain parameters of the object, such as the luminosity and magnitude. The same light received at $t_{\mathtt{R}}$ is used to measure both the redshift and the distance.

5.3 Additional recipes

$t_{\mathtt{R}}$, the present age of the universe, is not known but can be replaced by a constant because it is common to all our measurements.
Finally, a useful tool in the present calculations is the fractional distance $x(t)$ crossed by the front of light between the transmitter and the receiver (introduced previously with the TV analogy). This function is defined by

$$x(t) = \int_{\tau=t_{\mathtt{E}}}^{t} \frac{c}{D(\tau)} d\tau$$

where D is the source-receptor distance. Then, the date of arrival $t_{\mathtt{R}}$ is simply obtained by solving

$$x(t_{\mathtt{R}}) = 1$$

6 Redshift-Distance relationships expected for the different modes of expansion

Let us examine successively the different modes of expansion and for each one, the two hypothetical causes of redshift : Doppler effect and wave stretching.

6.1 Power-law expansion

6.1.1 Classical Doppler effect

$$D_{\mathrm{E}} = D_{\mathrm{R}} \left(\frac{t_{\mathrm{E}}}{t_{\mathrm{R}}} \right)^{u} \tag{25a}$$

The speed of the source at emission can be deduced from the distance at reception

$$z_d = \frac{1}{c} \frac{dD_{\mathrm{E}}}{dt_{\mathrm{E}}} \tag{25b}$$

that is, taking D_{R} and t_{R} as constants

$$z_d = \frac{uD_{\mathrm{R}}}{c} \frac{t_{\mathrm{E}}^{u-1}}{t_{\mathrm{R}}^{u}} = \frac{uD_{\mathrm{E}}}{ct_{\mathrm{E}}} \tag{25c}$$

replacing t_{E} by $t_{\mathrm{R}} - D_{\mathrm{L}}/c$ and D_{R} or D_{E} by their values deduced from $x(t_{\mathrm{R}}) = 1$,

$$z_d = \frac{u}{1-u} \left[\left(1 - \frac{D_{\mathrm{L}}}{ct_{\mathrm{R}}} \right)^{u-1} - 1 \right] \tag{25d}$$

where t_{R} can be considered as a constant at our time scale.

6.1.2 Wave stretching

The result is straightforward

$$\frac{D_{\mathrm{R}}}{D_{\mathrm{E}}} = \left(\frac{t_{\mathrm{R}}}{t_{\mathrm{E}}} \right)^{u} = \left(1 - \frac{D_{\mathrm{L}}}{ct_{\mathrm{R}}} \right)^{-u} \tag{26a}$$

and

$$z_s = \left(1 - \frac{D_{\mathrm{L}}}{ct_{\mathrm{R}}} \right)^{-u} - 1 \tag{26b}$$

6.2 The particular case $u = 1$ ($a \propto t$)

As explained previously, this case merely corresponds to uniform expansion.

6.2.1 Doppler effect

Classical Doppler effect

The recession velocity for this mode of expansion is simply D/t, where D and t vary proportionally,

$$\frac{D_{\mathrm{E}}}{t_{\mathrm{E}}} = \frac{D_{\mathrm{R}}}{t_{\mathrm{R}}} = v$$

and

$$z_d = \frac{v}{c} \tag{27}$$

Relativistic Doppler effect

Using special relativity, the redshift is

$$z_r = \sqrt{\frac{c+v}{c-v}} - 1 \tag{28}$$

6.2.2 Wave stretching

By definition for this mode of expansion,

$$\frac{\lambda^{\mathrm{app}}}{\lambda} = \frac{D_{\mathrm{R}}}{D_{\mathrm{E}}} = \frac{t_{\mathrm{R}}}{t_{\mathrm{E}}} \tag{29}$$

and by solving $x(t_{\mathrm{R}}) = 1$,

$$\frac{t_{\mathrm{R}}}{t_{\mathrm{E}}} = \mathrm{e}^{D_{\mathrm{R}}/ct_{\mathrm{R}}} = \mathrm{e}^{v/c} \tag{30}$$

$$z_s = \mathrm{e}^{v/c} - 1 \tag{31}$$

The two postulated causes of redshift give different results, which are similar only near 0 (for $v << c$). Both predict distance-independent redshifts, which clearly disqualifies this mode of expansion.

6.3 Exponential expansion

6.3.1 Doppler effect

Classical Doppler effect

The speed of the source at the time of light emission is

$$\frac{dD_{\mathrm{E}}}{dt} = HD_{\mathrm{E}} \tag{32}$$

whose value is obtained by solving $x(t_{\mathrm{R}}) = 1$,

$$z_d = \frac{HD_{\mathrm{E}}}{c} = 1 - \mathrm{e}^{-HD_{\mathrm{L}}/c} \tag{33}$$

As z_d is always lower than 1, the hypothesis that the redshift results from a classical Doppler effect only can be eliminated for the exponential expansion.

Relativistic Doppler effect

$$z_r = \sqrt{\frac{c + HD_{\mathrm{E}}}{c - HD_{\mathrm{E}}}} - 1 \tag{34}$$

6.3.2 Wave stretching

The wavelengths are stretched according to

$$\frac{\lambda^{\mathrm{app}}}{\lambda} = \frac{D_{\mathrm{R}}}{D_{\mathrm{E}}} = \mathrm{e}^{H\Delta t} = \mathrm{e}^{HD_{\mathrm{L}}/c} \tag{35}$$

so that

$$z_s = \mathrm{e}^{HD_{\mathrm{L}}/c} - 1 \tag{36a}$$

whose reciprocal is

$$D_{\mathrm{L}} = \frac{c}{H} \ln(1 + z_s) \tag{36b}$$

6.4 Mixing Doppler and stretching effects for the exponential expansion

The above results can be combined in a variety of manners. Let us examine only the case of the exponential expansion. The Doppler and wave stretching redshift functions established above display opposite behaviors near 0, clearly visualized by their series expansion. For the classical Doppler effect,

$$z_d \approx \frac{H_0 D_{\rm L}}{c} - \frac{1}{2}\left(\frac{H_0 D_{\rm L}}{c}\right)^2 \tag{37a}$$

and for the wave stretching effect,

$$z_s \approx \frac{H_0 D_{\rm L}}{c} + \frac{1}{2}\left(\frac{H_0 D_{\rm L}}{c}\right)^2 \tag{37b}$$

Hence, by comparison with the traditional redshift formula $z = H_0 D/c$, the Doppler effect seems to include an acceleration parameter, while the wave stretching effect seems to contain a deceleration parameter. For nearby galaxies, the so-called peculiar or ordinary velocities can not be neglected because they can either increase, decrease or cancel the cosmological redshift. This fact poses technical problems for studying expansion-restricted redshifts, but clearly demonstrates that several causes of redshift can really interfere. After all, a recession velocity, though passive, remains a velocity and as such, one wonders why it could not give a kinetic Doppler effect contributing to the redshift. Hence, for completeness, let us cumulate the two effects. In addition to take the two hypotheses on the origin of the redshift into account, two ways to bring them together will be considered.

6.4.1 Combined effects

On the one hand, the dilation of wavelengths caused by kinetic Doppler effect ($\lambda^{\rm dopp}$) is fixed at the emission point and then remains unchanged during light flight. On the other hand, the emission wavelength is expanded during the trip, until received in the form $\lambda^{\rm app}$. So the cumulative effect would logically read

$$\frac{\lambda^{\rm app}}{\lambda} = \frac{\lambda^{\rm app}}{\lambda^{\rm dopp}}\frac{\lambda^{\rm dopp}}{\lambda} \tag{38}$$

using the classical Doppler effect,

$$z_{d+s} = (1+z_d)(1+z_s) - 1 = 2z_s \tag{39}$$

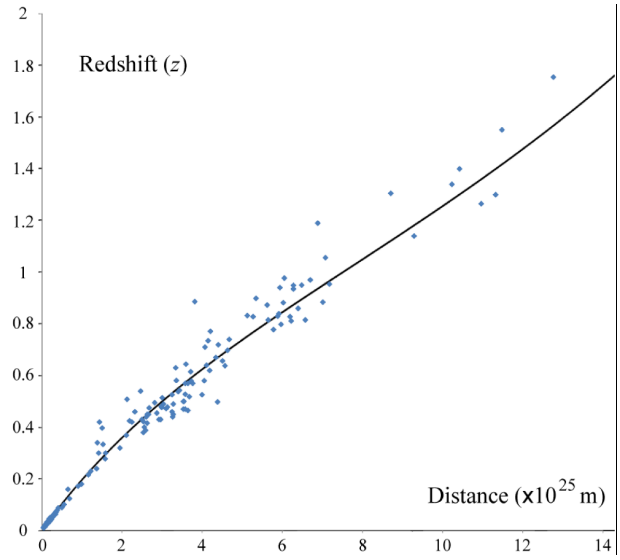

Figure 2. Speculative curve fitting of the distance-redshift supernovae distribution (dots) with an exponential expansion model in which the Doppler effect predominates at short distances while wave stretching predominates for long distances, corresponding to the arithmetic average function $z = 2\sinh(x)/(1+x)$ with $x = HD_{\rm L}/c$ (plain line). The redshift data are from [15] and include 186 supernovae. The fitting parameters are $c = 3\times10^8$m/s and $H = 3.23\times10^{-18}$/s, a value whose precision is poor due to the conversion of Distance modulus (μ) in a non-logarithmic scale using the relationship $10^{1+\mu/5}10^{-6}$ Mpc. The distance in the abscissa should be undersood as the measured distance (written $D_{\rm L}$ here).

6.4.2 Additive effects

As wavelenths are lengths, they should perhaps be handled as such. The extra-length caused by Doppler effect is z_d and that caused by stretching is z_s. If the two kinds of redshift exist simultaneously

$$z_{d+s} = z_d + z_s \tag{40}$$

Using the classical Doppler effect,

$$z_{d+s} = \mathrm{e}^{HD_{\rm L}/c} - \mathrm{e}^{-HD_{\rm L}/c} = 2\sinh\frac{HD_{\rm L}}{c} \tag{41}$$

The redshift of dual origin of Eq.(41) gives a straighter redshift-distance diagram compared to its individual components (clearly illustrated by the cancellation of the squared terms in Eq.(37)). But if for some reason the Doppler effect predominates for nearby galaxies, a bend would appear for low redshifts (around $z = 0.5$). A comparison with the observed redshifts of supernovae compiled in [15], is represented in Fig.(2), drawn in linear coordinates because logarithmic scales are not convenient to appreciate the straightness of a linear plot. By cumulating the Doppler and stretching redshifts with constant

H, the bend would have been an universal property of the diagram, constant and always at the same distance from any observer and at any cosmic time. This exercise of redshift additivity is however purely formal because the accepted explanation of this bend is a recent change in the expansion regime [16] driven by dark energy, which began at the level of galaxies of redshift below unity.

7 Distance of a source inferred from its apparent distance

The observed distance of cosmic objects ($D_{\mathtt{L}}$) does not correspond to any real distance. True distances can however be deduced from this measured distance provided a mode of expansion is selected.

— **For the uniform expansion:**

Since $D_{\mathtt{L}} = c\Delta t$, Eq.(18) gives

$$D_{\mathtt{E}} = D_{\mathtt{L}} \frac{v/c}{\mathrm{e}^{v/c} - 1} \tag{42a}$$

and since $D_{\mathtt{R}} = D_{\mathtt{E}} + v\Delta t$,

$$D_{\mathtt{R}} = D_{\mathtt{L}} \frac{v/c}{1 - \mathrm{e}^{-v/c}} \tag{42b}$$

These distances satisfy both the established distances relationship of Eq.(10) and the stretched shift of Eq.(30).

— **For the geometric expansion:**

When $x(t_{\mathtt{R}}) = 1$, Eq.(20) gives

$$D_{\mathtt{E}} = \frac{ct_{\mathtt{R}} - D_{\mathtt{L}}}{1-u} \left[\left(1 - \frac{D_{\mathtt{L}}}{ct_{\mathtt{R}}}\right)^{u-1} - 1 \right] \tag{43a}$$

so that

$$D_{\mathtt{R}} = \frac{ct_{\mathtt{R}}}{1-u} \left[1 - \left(1 - \frac{D_{\mathtt{L}}}{ct_{\mathtt{R}}}\right)^{1-u} \right] \tag{43b}$$

— **For the exponential expansion:**

When $x(t_{\mathtt{R}}) = 1$, Eq.(23) gives

$$HD_{\mathtt{E}} = c\left(1 - \mathrm{e}^{-H\Delta t}\right) \tag{44a}$$

with $D_{\mathtt{L}} = c\Delta t$,

$$D_{\mathtt{E}} = \frac{c}{H}\left(1 - \mathrm{e}^{-HD_{\mathtt{L}}/c}\right) = \frac{cz_d}{H} \tag{44b}$$

and

$$D_{\mathtt{R}} = \frac{c}{H}\left(\mathrm{e}^{HD_{\mathtt{L}}/c} - 1\right) = \frac{cz_s}{H} \tag{44c}$$

8 Conclusions

A compendium of redshift, horizon and distance rules, is presented here for pure modes of expansion. According to current cosmological models, different successive modes of expansion occurred in the past depending on the maturation stage of the universe. The succession and/or combination of pure modes of expansion requires more sophisticated approaches not described here, but the basic tools summarized in Table.1 could help discriminating cosmological hypotheses. Some general lessons can be drawn from these analyses.

8.1 Comparison with the current astrophysical laws

This study has the same goal as that of Harrison [2]: to rationally derive the links between redshift and distance depending on the type of expansion. These relationships are then expected to help interpreting observational data. The traditional redshift formula $z = H_0 D/c$ is close to the exponential expansion found here ($z_s = \mathrm{e}^{H_0 D_{\mathtt{L}}/c} - 1$) only in the vicinity of $D = 0$. Internal contradictions in the establishment of distance-redshift diagrams, arise when their coordinates are obtained using non-independent methods. For example, results are severely distorted when distances are deduced from redshifts (for distant objects) using preconceived redshift-distance relationships whereas the final aim of the diagram is precisely to establish this relationship. The formula $z = H_0 D/c$ is true only for the invisible actual distance but not for the visible distance. Hence, estimating short distances through luminosity and long distances through the redshift using the classical formula would cause another distortion. Generally, the task of astrophysicists is difficult to disentangle the different interfering sources of redshift, including gravity and peculiar velocities, and to determine the specific contribution of expansion.

8.2 The different interpretations of the redshift are not equivalent

Whatever the expansion modes, the results listed here show that the two postulated origins of the redshift: Doppler effect and wave stretching, always give different results, as stated by Harrison [2]. If the universal redshift is entirely due to wave stretching, the recession velocity of galaxies mutually inert ("comoving") is not really a velocity. The mere existence of redshifts higher than 1 is sufficient to rule out the Doppler effect as the sole cause of redshift in the case of the exponential expansion.

Table 1 – Summarized consequences of the modes of expansion on the redshift, the horizon and the source-receiver distances. V_{rec} is the apparent source velocity at emission. z_d and z_s are the redshifts calculated as Doppler or wave-stretching effects respectively. $D_{\mathtt{E}}$ is the source-receiver distance at emission, $D_{\mathtt{R}}$ is the source-receiver distance at reception and $D_{\mathtt{L}}$ is the apparent distance actually measured. $t_{\mathtt{R}}$ is the present time, which can be approximated as a constant as it is identical for all our measurements.

Mode	$a \propto t$ $(u = 1)$	Power law $(u \neq 1)$	Exponential
$D(t) =$	$D_{\mathtt{E}}\left(\frac{t}{t_{\mathtt{E}}}\right)$	$D_{\mathtt{E}}\left(\frac{t}{t_{\mathtt{E}}}\right)^u$	$D_{\mathtt{E}}\ \mathrm{e}^{H(t-t_{\mathtt{E}})}$
$\frac{\dot{a}}{a}$	$\frac{1}{t}$	$\frac{u}{t}$	H
Horizon	No	$u < 1$: No $\quad u > 1$: for $D_{\mathtt{E}} \geq \frac{c\ t_{\mathtt{E}}}{u-1}$	for $D_{\mathtt{E}} \geq \frac{c}{H}$
V_{rec}	$\frac{D}{t} = v$	$\frac{uD_{\mathtt{E}}}{t_{\mathtt{E}}}$	$HD_{\mathtt{E}}$
z_d	$\frac{v}{c}$	$\frac{u}{1-u}\left[\left(1-\frac{D_{\mathtt{L}}}{ct_{\mathtt{R}}}\right)^{u-1}-1\right]$	$1-\mathrm{e}^{-HD_{\mathtt{L}}/c}$
z_s	$\mathrm{e}^{v/c}-1$	$\left(1-\frac{D_{\mathtt{L}}}{ct_{\mathtt{R}}}\right)^{-u}-1$	$\mathrm{e}^{HD_{\mathtt{L}}/c}-1$
$D_{\mathtt{E}}$	$D_{\mathtt{L}}\frac{v/c}{\mathrm{e}^{v/c}-1}$	$\frac{ct_{\mathtt{R}}}{1-u}\left(1-\frac{D_{\mathtt{L}}}{ct_{\mathtt{R}}}\right)\left[\left(1-\frac{D_{\mathtt{L}}}{ct_{\mathtt{R}}}\right)^{u-1}-1\right]$	$\frac{c}{H}\left(1-\mathrm{e}^{-HD_{\mathtt{L}}/c}\right)$
$D_{\mathtt{R}}$	$D_{\mathtt{L}}\frac{v/c}{1-\mathrm{e}^{-v/c}}$	$\frac{ct_{\mathtt{R}}}{1-u}\left[1-\left(1-\frac{D_{\mathtt{L}}}{ct_{\mathtt{R}}}\right)^{1-u}\right]$	$\frac{c}{H}\left(\mathrm{e}^{HD_{\mathtt{L}}/c}-1\right)$

8.3 Rejection of the uniform mode of expansion

Another clear conclusion of this systematic survey is that uniform expansion can be definitely ruled out. In fact, whatever the origin of the redshift (Doppler or wave stretching), it would give redshifts independent of distance, which is contradicted by observations since 1929 [1].

8.4 Expansion does not imply big bang

The exponential mode of expansion (synonymous to $V = HD$) is elegant in that it does not imply an original singularity. Rewinding the film of expansion intuitively led to the idea of big bang in the scientific community. But a less intuitive consequence of this hypothesis is that running backward the linear cosmic time t (and disconnecting it from our terrestrial calendar), would allow determining the date of the big bang, for example on a Monday afternoon of a given year. To get rid of this hardly acceptable incongruity, one could recourse either to a non-constant flow of cosmic time, or to an exponential mode of expansion compatible with an eternal universe.

8.5 Deviation from a linear Hubble plot

All the modes of expansion examined here give apparent straight $z = f(D_{\mathtt{L}})$ plots only in the vicinity of $D_{\mathtt{L}} = 0$ in a much wider universe. The redshifts at very long distances will perhaps allow discriminating the different behaviours predicted here. If the universe is much more gigantic than it appears, most combinations of redshift-distances listed in this article could be approximated as straight lines. In this case, the most precise Hubble diagram would remain of little utility. But this apprehension could be, hopefully, too pessimistic, based on the confident feeling of Werner Heisenberg that "Nature is made such that she can be understood".

8.6 Horizon

The present calculations show that the only two modes of expansion listed here which are capable of generating disconnection barriers, are the exponential expansion (Horizon at $D_{\mathtt{E}} = c/H$) and the geometric expansion if $u > 1$ (Horizon at $D_{\mathtt{E}} = ct_{\mathtt{E}}/(u-1)$). The horizon is speculatively suggested to be the condition for breaking causality in the universe and giving their chance to all forms of life.

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