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Abstract. We study the ω-limit set of solutions of a nonlocal ordinary differential equation, where the nonlocal term is such that the space integral of the solution is conserved in time. Using the monotone rearrangement theory, we show that the rearranged equation in one space dimension is the same as the original equation in higher space dimensions. In many cases, this property allows us to characterize the ω-limit set for the nonlocal differential equation. More precisely, we prove that the ω-limit set only contains one element.

1   Introduction

The aim of this present paper is to study the ω-limit set of solutions of the initial value problem

$$(P) \quad \begin{cases} u_t = g(u)p(u) - g(u)\dfrac{\displaystyle\int_\Omega g(u)p(u)}{\displaystyle\int_\Omega g(u)} & x \in \Omega,\ t \geq 0, \\[2em] u(x,0) = u_0(x) & x \in \Omega. \end{cases}$$

Here $\Omega \subset \mathrm{I\!R}^N (N \geq 1)$ is an open bounded set, $g, p : \mathrm{I\!R} \to \mathrm{I\!R}$ are continuously differentiable and u_0 is a bounded function. More precise conditions on g, p and u_0 will be given later. A typical example is given by the functions $g(u) = u(1-u)$ and $p(u) = u$. In this case, the equation becomes

$$u_t = u^2(1-u) - u(1-u)\frac{\displaystyle\int_\Omega u^2(1-u)}{\displaystyle\int_\Omega u(1-u)}.$$

Problem (P) has been originally proposed by M. Nagayama [6] to express bubble motion with chemical reaction where the volume constraint is considered. By integrating the equation in (P), it is easy see that solutions of (P)

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satisfy the mass conservation property:

$$\int_\Omega u(x,t)\,dx = \int_\Omega u_0(x)\,dx \quad \text{for all} \ \ t \geq 0.$$

We refer to Proposition 2.2 for a rigorous proof of this property.

We will consider Problem (P) under some different hypotheses on the initial function u_0. Problem (P) possesses a Lyapunov functional whose form depends on the hypothesis satisfied by u_0 (see section 4 for more details).

In this paper, we always consider the following hypotheses on the functions g and p:

$$\begin{cases} p \in C^1(\mathbb{R}) \ \text{ is strictly increasing on } \mathbb{R}, \\ g \in C^1(\mathbb{R}), g(0) = g(1) = 0, g > 0 \text{ on } (0,1) \text{ and } g < 0 \text{ on } (-\infty,0) \cup (1,\infty). \end{cases}$$

We suppose that the initial function satisfies one of the following hypotheses:
$(\mathbf{H_1})$ $u_0 \in L^\infty(\Omega)$, $u_0(x) \geq 1$ for a.e. $x \in \Omega$, and $u_0 \not\equiv 1$.
$(\mathbf{H_2})$ $u_0 \in L^\infty(\Omega)$, $0 \leq u_0(x) \leq 1$ for a.e. $x \in \Omega$, and $\int_\Omega g(u_0(x))\,dx \neq 0$.
$(\mathbf{H_3})$ $u_0 \in L^\infty(\Omega)$, $u_0(x) \leq 0$ for a.e. $x \in \Omega$, and $u_0 \not\equiv 0$.
Note that Hypothesis $(\mathbf{H_1})$ (and also $(\mathbf{H_3})$) implies that $\int_\Omega g(u_0) \neq 0$.

Before defining a solution of Problem (P), we introduce the notation

$$F(u) := g(u)p(u) - g(u)\frac{\displaystyle\int_\Omega g(u)p(u)}{\displaystyle\int_\Omega g(u)}. \tag{1}$$

Definition 1.1. *Let $0 < T \leq \infty$. The function $u \in C^1([0,T); L^\infty(\Omega))$ is called a solution of Problem (P) on $[0,T)$ if the three following properties hold*

(i) $u(0) = u_0$,

(ii) $\displaystyle\int_\Omega g(u(t)) \neq 0$ *for all* $t \in [0,T)$,

(iii) $\dfrac{du}{dt} = F(u)$ *for all* $t \in [0,T)$.

The ω-limit sets are important and interesting objects in the theory of dynamical systems. Understanding their structure allows us to apprehend the long time behavior of solutions of dynamical systems. In this paper, we characterize the ω-limit set of solutions of Problem (P), which is defined as follows:

Definition 1.2. *We define the ω-limit set of u_0 by*

$$\omega(u_0) := \{\varphi \in L^1(\Omega) : \exists t_n \to \infty, u(t_n) \to \varphi \ \text{ in } \ L^1(\Omega) \text{ as } n \to \infty\}.$$

In the above definition, we do not use the L^∞-topology to define $\omega(u_0)$ because the solution often develops sharp transition layers which cannot be captured by the L^∞-topology. Note also that as we will see in Theorem 2.5, solutions of (P) are uniformly bounded so that the topology of L^1 is equivalent to that of L^p with $p \in [1, \infty)$. For convenience, we refer to the books [7, 8] for studies about dynamical systems as well as the structure of ω-limit sets.

An essential step to study $\omega(u_0)$ is to show the relative compactness of the solution orbits in $L^1(\Omega)$. In local problems, the standard comparison principle can be applied to obtain the uniform boundedness of solutions. Furthermore, in local problems with a diffusion term, such as local parabolic problems, the uniform boundedness of solutions implies the relative compactness of solution orbits in some suitable spaces by using Sobolev imbedding theorems. However, the above scheme cannot be applied to Problem (P), due to the presence of the nonlocal term as well as to the lack of a diffusion term.

By careful observation of the dynamics of pathwise trajectories (i.e. the sets $\{u(x,t) : t \geq 0\}$ for $x \in \Omega$), we show the existence of invariant sets and hence the uniform boundedness of solutions. The difficulties connected with the lack of diffusion term will be overcome by using ideas presented in [2]. More precisely, applying the rearrangement theory, we introduce the equi-measurable rearrangement $u^\sharp$ and show that it is the solution of a one-dimensional problem $(P^\sharp)$ (see section 3). Since the orbit $\{u^\sharp(t) : t \geq 0\}$ is bounded in $BV(\Omega^\sharp)$, where $\Omega^\sharp := (0, |\Omega|) \subset \mathbb{R}$, it is relatively compact in $L^1(\Omega^\sharp)$. We then deduce the relative compactness of solution orbits of Problem (P), by using the fact that

$$\|u(t) - u(\tau)\|_{L^1(\Omega)} = \|u^\sharp(t) - u^\sharp(\tau)\|_{L^1(\Omega^\sharp)}. \tag{2}$$

Note that the inequality $\|u(t) - u(\tau)\|_{L^1(\Omega)} \geq \|u^\sharp(t) - u^\sharp(\tau)\|_{L^1(\Omega^\sharp)}$ follows from a general property of the rearrangement theory. The important point is that (2) involves an equality.

An other advantage of considering Problem $(P^\sharp)$ is that the differential equations in $(P^\sharp)$ and (P) have the same form. Therefore we will study the ω-limit set for Problem $(P^\sharp)$ rather than for Problem (P). Although $(P^\sharp)$ possesses many stationary solutions, the one-dimensional structure of Problem $(P^\sharp)$ allows us to characterize its ω-limit set, and then deduce results for that of (P).

The organization of this article is as follows: In section 2, we prove the global existence and uniqueness of the solution as well as its uniform boundedness. Next in section 3, we recall and apply results from the arrangement theory presented in [2] to obtain the relative compactness of the solution in $L^1(\Omega)$. In section 4, we prove that Problem (P) possesses Lyapunov functionals and use them together with the relative compactness of the solution to show that $\omega(u_0)$ is nonempty and consists of stationary solutions. Moreover, these stationary solutions are step functions. More precise properties of these functions are given in Theorems 4.4 and 4.5. In section 5, we suppose that

one of the hypotheses ($\mathbf{H_1}$) or ($\mathbf{H_3}$) holds and prove that $\omega(u_0)$ only contains one element.

If Hypothesis ($\mathbf{H_2}$) is satisfied, the properties obtained in Theorems 4.4 and 4.5 are not sufficient to show that $\omega(u_0)$ contains one element even in one space dimension. This is an open problem and we conjecture that if ($\mathbf{H_2}$) holds, then $\omega(u_0)$ only contains one element.

2 Existence and uniqueness of solutions of (P)

2.1 Local existence

First we proves the local Lipschitz property of the nonlocal nonlinear term F, given by (1), in the space $L^\infty(\Omega)$.

Lemma 2.1 (Local Lipschitz continuity of F). *Let $v \in L^\infty(\Omega)$ be such that $\int_\Omega g(v(x))\,dx \neq 0$. Then there exist a $L^\infty(\Omega)$-neigbourhood $\mathcal{V}$ of v and a constant $L > 0$ such that $F(\widetilde{v})$ is well-defined for all $\widetilde{v} \in \mathcal{V}$ and that*

$$\|F(v_1) - F(v_2)\|_{L^\infty(\Omega)} \leq L\|v_1 - v_2\|_{L^\infty(\Omega)},$$

for all $v_1, v_2 \in \mathcal{V}$.

Proof. Since g is continuous, the map $v \mapsto \int_\Omega g(v)$ is continuous from $L^\infty(\Omega)$ to $L^\infty(\Omega)$. It follows that there exist a constant $\alpha > 0$ and a neighbourhood $\mathcal{V}$ of v such that

$$\left| \int_\Omega g(\widetilde{v}) \right| \geq \alpha \quad \text{for all} \ \ \widetilde{v} \in \mathcal{V}. \tag{3}$$

Without loss of generality, we may choose

$$\mathcal{V} := \{\widetilde{v} \in L^\infty(\Omega) : \|\widetilde{v} - v\|_{L^\infty(\Omega)} \leq \varepsilon\},$$

for a constant $\varepsilon > 0$ small enough. We set

$$\bar{c} := \|v\|_{L^\infty(\Omega)} + \varepsilon, \quad f(s) := g(s)p(s),$$

and

$$K := \max\Big\{ \sup_{[-\bar{c},\bar{c}]} |f(s)|, \ \sup_{[-\bar{c},\bar{c}]} |g(s)|, \ \sup_{[-\bar{c},\bar{c}]} |f'(s)|, \ \sup_{[-\bar{c},\bar{c}]} |g'(s)| \Big\}.$$

Then the following properties holds and will be used later: For all $v_1, v_2 \in \mathcal{V}$,

$$\|f(v_1) - f(v_2)\|_{L^\infty(\Omega)} \leq K\|v_1 - v_2\|_{L^\infty(\Omega)}, \tag{4}$$

and

$$\|g(v_1) - g(v_2)\|_{L^\infty(\Omega)} \leq K\|v_1 - v_2\|_{L^\infty(\Omega)}.$$

We have

$$F(v_1) - F(v_2) = [f(v_1) - f(v_2)] - \left[g(v_1) \frac{\int_\Omega f(v_1)}{\int_\Omega g(v_1)} - g(v_2) \frac{\int_\Omega f(v_2)}{\int_\Omega g(v_2)} \right]$$

$$= [f(v_1) - f(v_2)] - \frac{g(v_1) \int_\Omega f(v_1) \int_\Omega g(v_2) - g(v_2) \int_\Omega f(v_2) \int_\Omega g(v_1)}{\int_\Omega g(v_1) \int_\Omega g(v_2)}$$

$$=: A_1 - \frac{A_2}{A_3},$$

where

$$A_1 := f(v_1) - f(v_2),$$

$$A_2 := g(v_1) \int_\Omega f(v_1) \int_\Omega g(v_2) - g(v_2) \int_\Omega f(v_2) \int_\Omega g(v_1),$$

and

$$A_3 := \int_\Omega g(v_1) \int_\Omega g(v_2).$$

In the sequel, we estimate A_1, A_2 and A_3. First the inequality (4) yields

$$\|A_1\|_{L^\infty(\Omega)} \le K \|v_1 - v_2\|_{L^\infty(\Omega)}. \tag{5}$$

Next we write A_2 as

$$A_2 = g(v_1) \int_\Omega f(v_1) \int_\Omega g(v_2) - g(v_2) \int_\Omega f(v_1) \int_\Omega g(v_2)$$

$$+ g(v_2) \int_\Omega f(v_1) \int_\Omega g(v_2) - g(v_2) \int_\Omega f(v_2) \int_\Omega g(v_2)$$

$$+ g(v_2) \int_\Omega f(v_2) \int_\Omega g(v_2) - g(v_2) \int_\Omega f(v_2) \int_\Omega g(v_1),$$

or equivalently,

$$A_2 = [g(v_1) - g(v_2)] \int_\Omega f(v_1) \int_\Omega g(v_2)$$

$$+ g(v_2) \int_\Omega [f(v_1) - f(v_2)] \int_\Omega g(v_2)$$

$$+ g(v_2) \int_\Omega f(v_2) \int_\Omega [g(v_2) - g(v_1)],$$

which in turn implies that

$$\|A_2\|_{L^\infty(\Omega)} \le 3K^3 |\Omega|^2 \|v_1 - v_2\|_{L^\infty(\Omega)}. \tag{6}$$

As for the term A_3, we apply (3) to obtain

$$|A_3| \ge \alpha^2 > 0. \tag{7}$$

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Combining (5), (6) and (7), we deduce that

$$\|F(v_1) - F(v_2)\|_{L^\infty(\Omega)} \leq \left(K + \frac{3K^3|\Omega|^2}{\alpha^2}\right)\|v_1 - v_2\|_{L^\infty(\Omega)}.$$

This completes the proof of Lemma 2.1. □

Proposition 2.2. *Let $u_0 \in L^\infty(\Omega)$ satisfy $\int_\Omega g(u_0) \neq 0$. Then Problem (P) has a unique local-in-time solution. Moreover, we have*

$$\int_\Omega u(x,t)\,dx = \int_\Omega u_0(x)\,dx \quad for \ all \ \ t \in [0, T_{max}(u_0)), \tag{8}$$

where $T_{max}(u_0)$ denotes the maximal time interval of the existence of solution.

Proof. Since F is locally Lipschitz continuous in $L^\infty(\Omega)$, the local existence follows from the standard theory of ordinary differential equations. We now prove (8). Integrating the differential equation in Problem (P) from 0 to t, we obtain

$$u(t) - u_0 = \int_0^t u_t(s)\,ds = \int_0^t F(u(s))\,ds.$$

It follows that

$$\int_\Omega u(x,t)\,dx - \int_\Omega u_0(x)\,dx = \int_0^t \int_\Omega F(u)\,dxds = 0,$$

where the last identity holds since

$$\int_\Omega F(u)\,dx = 0.$$

This completes the proof of the lemma. □

Lemma 2.3. *If $T_{max}(u_0) < \infty$ and $\limsup_{t\uparrow T_{max}(u_0)} \|F(u(t))\|_{L^\infty(\Omega)} < \infty$, then $u(T_{max}(u_0)-) := \lim_{t\uparrow T_{max}(u_0)} u(t)$ exists in $L^\infty(\Omega)$ and*

$$\int_\Omega g(u(T_{max}(u_0)-)) = 0.$$

Proof. For simplicity we write T_{max} instead of $T_{max}(u_0)$. Set

$$M := \limsup_{t\uparrow T_{max}} \|F(u(t))\|_{L^\infty(\Omega)} < \infty.$$

Then there exists $0 < T < T_{max}$ such that

$$\|F(u(t))\|_{L^\infty(\Omega)} \leq 2M \quad for \ all \ \ t \in [T, T_{max}).$$

Consequently, for any $t, t' \in [T, T_{max})$, with $t < t'$, we have

$$\|u(t) - u(t')\|_{L^\infty(\Omega)} \leq \int_t^{t'} \|F(u(s))\|\,ds \leq 2M|t - t'|.$$

Thus $\{u(t)\}$ is a Cauchy sequence so that the limit $u(T_{max}-) := \lim_{t\uparrow T_{max}} u(t)$ exists in $L^\infty(\Omega)$. If $\int_\Omega g(u(T_{max}-)) \neq 0$, then, by Lemma 2.2, we can extend the solution on $[T_{max}, T_{max} + \delta)$, with some $\delta > 0$, which contradicts the definition to T_{max}. This completes the proof of the lemma. □

2.2 Global solution

In this subsection, we fix $u_0 \in L^\infty(\Omega)$ satisfying $\int_\Omega g(u_0) \neq 0$ and denote by $[0, T_{max})$ the maximal time interval of the existence of solution. Set

$$\lambda(t) = \frac{\displaystyle\int_\Omega g(u)p(u)}{\displaystyle\int_\Omega g(u)} \quad \text{for all} \ \ t \in [0, T_{max}), \tag{9}$$

and study solutions $Y(t; s)$ of the following auxiliary problem:

$$(ODE) \quad \begin{cases} \dot{Y} = g(Y)p(Y) - g(Y)\lambda(t), & t > 0, \\ Y(0) = s, \end{cases} \tag{10}$$

where $\dot{Y} := dY/dt$. We remark that the function u satisfies

$$u(x,t) = Y(t; u_0(x)) \ \ \text{for a.e.} \ \ x \in \Omega \ \ \text{and all} \ \ t \in [0, T_{max}). \tag{11}$$

Lemma 2.4. *Let $\widetilde{s} < s$ and let $0 < T < T_{max}$. Assume that Problem (ODE) possesses the solutions $Y(t; \widetilde{s}), Y(t; s) \in C^1([0, T])$, respectively. Then*

$$Y(t; \widetilde{s}) < Y(t; s) \ \ \text{for all} \ \ t \in [0, T]. \tag{12}$$

Proof. Since $Y(0; \widetilde{s}) = \widetilde{s} < s = Y(0; s)$, the assertion follows immediately from the backward uniqueness of solution of (ODE). □

Theorem 2.5. *Assume that one of the hypotheses $(\mathbf{H_1}), (\mathbf{H_2}), (\mathbf{H_3})$ holds. Then Problem (P) possesses a global solution $u \in C^1([0, \infty); L^\infty(\Omega))$. Moreover:*

(i) *If $(\mathbf{H_1})$ holds, then for all $t \geq 0$,*

$$1 \leq u(x,t) \leq \text{ess sup}_\Omega \, u_0 \ \ \text{a.e.} \ \ x \in \Omega. \tag{13}$$

(ii) *If $(\mathbf{H_2})$ holds, then for all $t \geq 0$,*

$$0 \leq u(x,t) \leq 1 \ \ \text{a.e. in} \ \ \Omega. \tag{14}$$

(iii) *If $(\mathbf{H_3})$ holds, then for all $t \geq 0$,*

$$\text{ess inf}_\Omega \, u_0 \leq u(x,t) \leq 0 \ \ \text{a.e.} \ \ x \in \Omega.$$

Proof. For simplicity, we set

$$a := \text{ess inf}_\Omega \, u_0, \qquad b := \text{ess sup}_\Omega \, u_0.$$

We only prove (i) and (ii). The proof of (iii) is similar to that of (i).

(i) First, we show that (13) holds as long as the solution u exists and then deduce the global existence from Lemma 2.3. Let $Y(t;s)$ be the solution of (ODE). We remark that $b \geq 1$ and that $Y(t,1) \equiv 1$ for all $t \in [0, T_{max})$. The monotonicity of $Y(t;s)$ in s implies that as long as $u, Y(t;b)$ both exist

$$1 \equiv Y(t,1) \leq Y(t; u_0(x)) = u(x,t) \leq Y(t;b) \quad a.e. \ \ x \in \Omega. \qquad (15)$$

The first inequality above implies the first inequality of (13) as long as the solution u exists. It remains to prove the second inequality of (13). To that purpose, it suffices to show that

$$Y(t;b) \leq b \qquad (16)$$

as long as the solution $Y(t;b)$ exists. In view of (15), we have $1 \leq u(x,t) \leq Y(t;b)$. Then the definition of g and the monotonicity of p imply that

$$g(Y(t;b)) \leq 0, \quad g(u(x,t)) \leq 0, \quad p(Y(t,b)) \geq p(u(x,t)),$$

for a.e. $x \in \Omega$. As a consequence,

$$\dot{Y}(t;b) = g(Y(t,b))(p(Y(t;b)) - \lambda(t)) \leq 0.$$

Hence

$$Y(t,b) \leq Y(0;b) = b,$$

which completes the proof of (16). Thus (13) is satisfied as long as the solution u exists.

Next we show that the solution u exists globally. Suppose, by contradiction, that $T_{max} < \infty$. We have for all $t \in [0, T_{max})$,

$$|\lambda(t)| \leq \frac{\int_\Omega |g(u)p(u)|}{|\int_\Omega g(u)|} = \frac{\int_\Omega |g(u)| \, |p(u)|}{\int_\Omega |g(u)|} \leq \max\{|p(1)|, |p(b)|\}.$$

It follows that there exists $C > 0$ such that $\|F(u(t))\|_{L^\infty(\Omega)} \leq C$ for all $t \in [0, T_{max})$. By Lemma 2.3, $u(T_{max}-) := \lim_{t \uparrow T_{max}} u(t)$ exists in $L^\infty(\Omega)$ and

$$\int_\Omega g(u(T_{max}-)) = 0.$$

Since $u(x,t) \geq 1$ for a.e $x \in \Omega, t \in [0, T_{max})$, $u(x, T_{max}-) \geq 1$ for a.e. $x \in \Omega$. Hence $\int_\Omega g(u(T_{max}-)) = 0$ if and only if $u(x, T_{max}-) \equiv 1$. The mass conservation property (cf. (8)) yields $\int_\Omega u_0 = |\Omega|$. Hence $u_0(x) = 1$ for a.e $x \in \Omega$. This contradicts Hypothesis ($\mathbf{H_1}$) so that $T_{max} = \infty$.

(ii) Since $Y(t,1) \equiv 1, Y(t,0) \equiv 0$, we deduce that

$$0 \equiv Y(t,0) \leq Y(t; u_0(x)) = u(x,t) \leq Y(t,1) \equiv 1 \quad a.e. \ \ x \in \Omega.$$

This implies (14) as long as the solution u exists. We now prove that $T_{max} = \infty$. Indeed, suppose, by contradiction, that $T_{max} < \infty$. Since $0 \leq u(x,t) \leq 1$

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for a.e. $x \in \Omega$, and all $t \in [0, T_{max})$, $g(u(x,t)) \geq 0$ for a.e. $x \in \Omega$, and all $t \in [0, T_{max})$. Therefore

$$|\lambda(t)| \leq \frac{\int_\Omega |g(u)p(u)|}{|\int_\Omega g(u)|} = \frac{\int_\Omega g(u) |p(u)|}{\int_\Omega g(u)} \leq \max\{|p(0)|, |p(1)|\},$$

for all $t \in [0, T_{max})$. It follows that there exists $C > 0$ such that $\|F(u(t))\|_{L^\infty(\Omega)} \leq C$ for all $t \in [0, T_{max})$. By Lemma 2.3, $u(T_{max}-) := \lim_{t \uparrow T_{max}} u(t)$ exists in $L^\infty(\Omega)$ and

$$\int_\Omega g(u(T_{max}-)) = 0.$$

This implies that $u(T_{max}-)$ only takes two values 0 and 1. Or equivalently, $Y(T_{max}-; u_0(x))$ only takes two values 0 and 1. Thus the backward uniqueness of the solution of the initial value problem (ODE) implies that $u_0(x)$ only takes two values 0 and 1; hence $\int_\Omega g(u_0) = 0$. This contradicts Hypothesis $(\mathbf{H_2})$ so that $T_{max} = \infty$. $\qquad\square$

The result below follows from the proof of Theorem 2.5.

Corollary 2.6. *Assume that one of the hypotheses* $(\mathbf{H_1}), (\mathbf{H_2}), (\mathbf{H_3})$ *holds and let* $\lambda(t)$ *be defined by* (9). *Then there exists* $C > 0$ *such that*

$$|\lambda(t)| \leq C$$

for all $t \in [0, \infty)$.

3 Boundedness of the solution and one-dimensional associated problem $(P^\sharp)$

All the results in this section are similar to those of [2, Section 3]. We recall and state some important results. Let w be a function from Ω to $\mathbb{R}$ and let $\Omega^\sharp := (0, |\Omega|) \subset \mathbb{R}$. The distribution function of w is given by

$$\mu_w(s) := |\{x \in \Omega : w(x) > s\}|.$$

Definition 3.1. *The (one-dimensional) decreasing rearrangement of* w, *denoted by* $w^\sharp$, *is defined on* $\overline{\Omega}^\sharp = [0, |\Omega|]$ *by*

$$\begin{cases} w^\sharp(0) := \text{ess sup}(w) \\ w^\sharp(y) = \inf\{s : \mu_w(s) < y\}, \quad y > 0. \end{cases} \tag{17}$$

Remark 3.2. The function $w^\sharp$ is nonincreasing on $\Omega^\sharp$ and we have $\mu_w(s) = \mu_{w^\sharp}(s)$ for all $s \in \mathbb{R}$. Moreover, if $a \leq w(x) \leq b$ a.e. $x \in \Omega$, then

$$a \leq w^\sharp(y) \leq b \text{ for all } y \in \Omega^\sharp.$$

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Theorem 3.3. *Let one of the hypotheses* $(\mathbf{H_1}), (\mathbf{H_2}), (\mathbf{H_3})$ *hold. We define*

$$u^\sharp(y,t) := (u(t))^\sharp(y) \quad on \ \ \Omega^\sharp \times [0,+\infty). \tag{18}$$

Then $u^\sharp$ *is the unique solution in* $C^1([0,\infty); L^\infty(\Omega^\sharp))$ *of Problem* $(P^\sharp)$

$$(P^\sharp) \quad \begin{cases} \dfrac{dv}{dt} = g(v)p(v) - g(v)\dfrac{\displaystyle\int_\Omega g(v)p(v)}{\displaystyle\int_\Omega g(v)} & t > 0, \\[1em] v(0) = u_0^\sharp. \end{cases}$$

Moreover, for all $t \geq 0$,

$$u^\sharp(y,t) = Y(t; u_0^\sharp(y)) \quad for \ a.e. \ \ y \in \Omega^\sharp, \tag{19}$$

and the assertions (i), (ii), (iii) of Theorem 2.5 hold for the function $u^\sharp$.

Lemma 3.4 ([2, Lemma 3.7]). *Let* u *be the solution of* (P) *with* $u_0 \in L^\infty(\Omega)$ *and let* $u^\sharp$ *be as in* (18). *Then*

$$\|u^\sharp(t) - u^\sharp(\tau)\|_{L^1(\Omega^\sharp)} = \|u(t) - u(\tau)\|_{L^1(\Omega)}, \tag{20}$$

for any $t, \tau \in [0,\infty)$.

Corollary 3.5 ([2, Corollary 3.9]). *Let* $\{t_n\}$ *be a sequence of positive numbers such that* $t_n \to \infty$ *as* $n \to \infty$. *Then the following statements are equivalent*

(a) $u^\sharp(t_n) \to \psi$ *in* $L^1(\Omega^\sharp)$ *as* $n \to \infty$ *for some* $\psi \in L^1(\Omega^\sharp)$;

(b) $u(t_n) \to \varphi$ *in* $L^1(\Omega)$ *as* $n \to \infty$ *for some* $\varphi \in L^1(\Omega)$ *with* $\varphi^\sharp = \psi$.

The following proposition follows by similar results in [2, Lemma 3.5 and Proposition 3.10].

Proposition 3.6. *Let one of the hypotheses* $(\mathbf{H_1}), (\mathbf{H_2}), (\mathbf{H_3})$ *hold. Then* $\{u(t) : t \geq 0\}$ *is relatively compact in* $L^1(\Omega)$ *and the set* $\{u^\sharp(t) : t \geq 0\}$ *is relatively compact in* $L^1(\Omega^\sharp)$.

4 Lyapunov functional and ω-limit set for (P)

We define three Lyapunov functionals according to whether the initial function satisfies either Hypothesis $(\mathbf{H_1}), (\mathbf{H_2})$ or $(\mathbf{H_3})$. More precisely, we define for $i = 1, 2, 3$ the functional E_i by

$$E_i(u) = (-1)^{i+1} \int_\Omega \mathcal{P}(u), \tag{21}$$

where

$$\mathcal{P}(s) = \int_0^s p(\tau)\,d\tau.$$

Lemma 4.1 (Lyapunov functional). *Assume that the hypotheses* $(\mathbf{H_i})$ *holds either for $i = 1$, or for $i = 2$, or for $i = 3$. Then*

(i) *There exists $C > 0$ such that for all $\tau_2 > \tau_1 \geq 0$,*

$$E_i(u(\tau_2)) - E_i(u(\tau_1)) = (-1)^{i+1} \int_{\tau_1}^{\tau_2} \int_{\Omega} g(u)(p(u) - \lambda(t))^2 \, dxdt$$

$$\leq -C \int_{\tau_1}^{\tau_2} \int_{\Omega} |u_t|^2 \, dxdt \leq 0.$$

(ii) *$E_i(u(\cdot))$ is continuous and non increasing on $[0, \infty)$, and the limit $E_{i\infty} := \lim_{t\to\infty} E_i(u(t))$ exists.*

Remark 4.2. Note that the solution orbit $\{u(t) : t \geq 0\}$ is uniquely defined by the initial function u_0. Hence $E_{i\infty}$—the limit of the Lyapunov functional a long the solution orbit—is also uniquely defined by the initial function. We will use the quantity $E_{i\infty}$ to make a constrain for all elements in the ω-limit set (see Proposition 4.3 (ii) below) then use it to characterize the ω-limit set.

Proof of Lemma 4.1. (i) We only give the proof for the case $i = 1$. We have

$$\frac{d}{dt} E_1(u(t)) = \frac{d}{dt} \int_{\Omega} \mathcal{P}(u) \, dx = \int_{\Omega} p(u) u_t \, dx.$$

Since

$$\int_{\Omega} u_t \, dx = 0,$$

it follows that

$$\frac{d}{dt} E_1(u(t)) = \int_{\Omega} (p(u) - \lambda(t)) u_t \, dx$$

$$= \int_{\Omega} g(u)(p(u) - \lambda(t))^2 \, dx. \tag{22}$$

Set

$$C = -\frac{1}{\min_{s \in [1, \text{ess sup } u_0]} g(s)} > 0.$$

Then, since for all $t \geq 0$,

$$1 \leq u(t) \leq \text{ess sup } u_0 \quad \text{a.e. in } \Omega,$$

we have, for all $t \geq 0$,

$$-\frac{1}{C} \leq g(u(t)) \leq 0 \quad \text{a.e. in } \Omega.$$

As a consequence, for all $t \geq 0$,

$$g(u(t)) \leq -C g^2(u(t)) \quad \text{a.e in } \Omega.$$

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Substituting this inequality into (22) yields

$$\frac{d}{dt}E_1(u(t)) \leq -C \int_\Omega g^2(u)(p(u) - \lambda(t))^2 \, dx$$

$$= -C \int_\Omega |u_t|^2 \, dx \leq 0,$$

which proves (i).

(ii) As a consequence of (i), $E_i(u(\cdot))$ is continuous and nonincreasing. Moreover, E_i is bounded from below. Therefore there exists the limit of $E_i(u(t))$ as $t \to \infty$, which completes the proof of (ii). $\qquad\square$

Proposition 4.3. *Assume that one of the hypotheses* $(\mathbf{H_i})$, $(i = 1, 2, 3)$ *holds. Then*

(i) $\omega(u_0)$ *a nonempty set in* $L^1(\Omega)$.

(ii) *Let* $E_{i\infty}$ *be given in Lemma 4.1, we have*

$$E_i(\varphi) = E_{i\infty} \quad \text{for all} \quad \varphi \in \omega(u_0). \tag{23}$$

In other words, $E_i(\cdot)$ *is constant on* $\omega(u_0)$.

(iii) *Any element* $\varphi \in \omega(u_0)$ *either satisfies* $\int_\Omega g(\varphi) \, dx = 0$ *or is a stationary solution of Problem* (P).

Proof. (i) follows from Proposition 3.6. (ii) Let $\varphi \in \omega(u_0)$ and let $t_n \to \infty$ be a sequence such that

$$u(t_n) \to \varphi \quad \text{in} \quad L^1(\Omega) \quad \text{as} \quad n \to \infty.$$

Since $\{u(x, t_n)\}$ is uniformly bounded, the convergence $u(t_n) \to \varphi$ implies $E_i(u(t_n)) \to E_i(\varphi)$ as $n \to \infty$. Hence in view of Lemma 4.1 (ii) we have

$$E_i(\varphi) = \lim_{n\to\infty} E_i(u(t_n)) = E_{i\infty}.$$

(iii) Assume that

$$\int_\Omega g(\varphi(x)) \, dx \neq 0;$$

we show below that φ is a stationary solution of Problem (P). Let $\{t_n\}$ be a sequence such that $t_n \to \infty$ and

$$u(t_n) \to \varphi \quad \text{in} \quad L^1(\Omega) \quad \text{as} \quad n \to \infty. \tag{24}$$

It follows from Lemma 4.1 that

$$\int_0^{+\infty} \int_\Omega |u_t|^2 \, dx dt \leq \frac{1}{C}(E_i(u_0) - \lim_{t\to\infty} E_i(u(t))) < +\infty.$$

Thus

$$\lim_{n\to\infty} \int_{t_n}^{t_n+1} \int_\Omega |u_t|^2 \, dx dt = 0.$$

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It follows that for $t \in [0, 1]$,

$$\|u(t_n + t) - \varphi\|_{L^1(\Omega)} \le \|u(t_n + t) - u(t_n)\|_{L^1(\Omega)} + \|u(t_n) - \varphi\|_{L^1(\Omega)}$$

$$\le \int_{t_n}^{t_n+t} \|u_t\|_{L^1(\Omega)} + \|u(t_n) - \varphi\|_{L^1(\Omega)}$$

$$\le t^{\frac{1}{2}} |\Omega|^{\frac{1}{2}} \left(\int_{t_n}^{t_n+t} \int_{\Omega} |u_t|^2 \, dxdt \right)^{\frac{1}{2}} + \|u(t_n) - \varphi\|_{L^1(\Omega)} \to 0$$

as $n \to \infty$. Hence for all $t \in [0, 1]$,

$$u(t_n + t) \to \varphi \quad \text{in} \quad L^1(\Omega) \quad \text{as} \quad n \to \infty.$$

Set $f(s) = g(s)p(s)$; then the uniform boundedness of u (cf. Theorem 2.5) implies that for all $t \in [0, 1]$,

$$f(u(t_n + t)) \to f(\varphi), \quad g(u(t_n + t)) \to g(\varphi) \quad \text{in} \quad L^1(\Omega).$$

as $n \to \infty$. Note that $\int_\Omega g(\varphi) \ne 0$ implies

$$\lambda(t_n + t) := \frac{\int_\Omega f(u(t_n + t))}{\int_\Omega g(u(t_n + t))} \to \frac{\int_\Omega f(\varphi)}{\int_\Omega g(\varphi)} \quad \text{as} \quad n \to \infty.$$

It follows that for all $t \in [0, 1]$

$$\int_\Omega F(u(t_n + t)) \, dx \to \int_\Omega F(\varphi) \, dx \quad \text{as} \quad n \to \infty,$$

where F is defined by (1). On the other hand, the uniform boundedness of $u(x, t)$ and Corollary 2.6 imply that $F(u(x, t))$ is uniformly bounded. Thus

$$\int_\Omega F^2(u(t_n + t)) \, dx \to \int_\Omega F^2(\varphi) \, dx \quad \text{as} \quad n \to \infty,$$

for all $t \in [0, 1]$. By Lebesgue's dominated convergence theorem, we have

$$\int_0^1 \int_\Omega F^2(u(t_n + t)) \, dxdt \to \int_0^1 \int_\Omega F^2(\varphi) \, dxdt \quad \text{as} \quad n \to \infty.$$

Since

$$\int_0^1 \int_\Omega F^2(u(t_n + t)) \, dxdt = \int_{t_n}^{t_n+1} \int_\Omega F^2(u(t)) \, dxdt = \int_{t_n}^{t_n+1} \int_\Omega |u_t|^2 \, dxdt \to 0,$$

as $n \to \infty$, it follows that

$$\int_0^1 \int_\Omega F^2(\varphi) \, dxdt = 0.$$

This yields

$$F(\varphi) = 0 \quad \text{a.e. in} \quad \Omega,$$

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or equivalently,

$$g(\varphi) \left[p(\varphi) - \frac{\displaystyle\int_\Omega g(\varphi)p(\varphi)}{\displaystyle\int_\Omega g(\varphi)} \right] = 0 \quad \text{a.e. in} \ \ \Omega.$$

Therefore φ is a stationary solution of Problem (P). $\qquad\square$

In two following theorems, we obtain a more precise form and constraints of the elements in the ω-limit set.

Theorem 4.4. *Assume that one of hypotheses* $(\mathbf{H_i}), (i = 1, 2, 3)$ *holds and let* $\varphi \in \omega(u_0)$. *Then:*

(i) *If* $(\mathbf{H_1})$ *holds, then* $1 \leq \varphi \leq \text{ess sup}_\Omega\, u_0$ *and* φ *is a step function. More precisely,*
$$\varphi = \mu\chi_{A_1} + \chi_{\Omega\setminus A_1},$$
where $\mu > 1$, $A_1 \subset \Omega, |A_1| \neq 0$.

(ii) *If* $(\mathbf{H_2})$ *holds, then* $0 \leq \varphi \leq 1$ *and* φ *is a step function. More precisely,*
$$\varphi = \chi_{A_1} + \nu\chi_{A_2},$$
where $0 < \nu < 1$, $A_1, A_2 \subset \Omega$, *with* $A_1 \cup A_2 \subset \Omega$ *and* $A_1 \cap A_2 = \emptyset$.

(iii) *If* $(\mathbf{H_3})$ *holds, then* $\text{ess inf}_\Omega\, u_0 \leq \varphi \leq 0$ *and* φ *is a step function. More precisely,*
$$\varphi = \xi\chi_{A_1},$$
where $\xi < 0$, $A_1 \subset \Omega, |A_1| \neq 0$.

Proof. We only prove (i); the proofs of (ii) and (iii) are similar. Since for all $t \geq 0$,
$$1 \leq u(t) \leq \text{ess sup}_\Omega\, u_0 \quad \text{a.e. in} \ \ \Omega,$$
it follows that
$$1 \leq \varphi \leq \text{ess sup}_\Omega\, u_0 \quad \text{a.e. in} \ \ \Omega.$$

Note that since $\int_\Omega \varphi = \int_\Omega u_0 > |\Omega|$, $\varphi \not\equiv 1$. Therefore $\int_\Omega g(\varphi) < 0$. It follows from Proposition 4.3 (iii) that φ is a stationary solution of (P), namely

$$g(\varphi) \left[p(\varphi) - \frac{\displaystyle\int_\Omega p(\varphi)g(\varphi)}{\displaystyle\int_\Omega g(\varphi)} \right] = 0 \qquad \text{a.e. in} \ \ \Omega,$$

which together with the monotonicity of p yields

$$\varphi = \mu\chi_{A_1} + \chi_{\Omega\setminus A_1},$$

for some constant $\mu > 1$ and $A_1 \subset \Omega$. Moreover $|A_1| \neq 0$ since $\varphi \not\equiv 1$. $\qquad\square$

Theorem 4.5. *Assume that one of hypotheses* $(\mathbf{H_i}), (i = 1, 2, 3)$ *holds. Let* $\varphi \in \omega(u_0)$ *and let* $E_{i\infty}$ *be given in Lemma 4.1 (ii). We set* $m_0 := \int_\Omega u_0$. *Then*

(i) *If* $(\mathbf{H_1})$ *holds, then*

$$\mu|A_1| + |\Omega| - |A_1| = m_0, \quad \mathcal{P}(\mu)|A_1| + \mathcal{P}(1)(|\Omega| - |A_1|) = E_{1\infty}.$$

(ii) *If* $(\mathbf{H_2})$ *holds, then*

$$|A_1| + \nu|A_2| = m_0, \quad \mathcal{P}(1)|A_1| + \mathcal{P}(\nu)|A_2| = -E_{2\infty}.$$

(iii) *If* $(\mathbf{H_3})$ *holds, then*

$$\xi|A_1| = m_0, \quad \mathcal{P}(\xi)|A_1| = E_{3\infty}.$$

Proof. We only prove (i). The other cases can be proven in a similar way. We apply (23) for $i = 1$ to obtain

$$\int_\Omega \mathcal{P}(\varphi) = E_{1\infty}. \tag{25}$$

Hence (i) follows from (25), the mass conservation property and Theorem 4.4. □

5 Large time behavior of the solution of (P)

In this section, we only suppose Hypotheses $(\mathbf{H_1}), (\mathbf{H_3})$.

Theorem 5.1. (i) *Let* $(\mathbf{H_1})$ *hold. Then* $\omega(u_0)$ *only contains one element, denoted by* φ. *Moreover* φ *is a step function of the form*

$$\varphi = \mu\chi_{A_1} + \chi_{\Omega\backslash A_1} \quad with \quad A_1 \subset \Omega, \quad and \quad \mu > 1.$$

(ii) *Let* $(\mathbf{H_3})$ *hold. Then* $\omega(u_0)$ *only contains one element, denoted by* φ. *Moreover,* φ *is a step function of the form*

$$\varphi = \xi\chi_{A_1} \quad with \quad A_1 \subset \Omega, \quad and \quad \xi < 0.$$

Proof. We only prove (i). First we prove that $\omega(u_0^\sharp)$ only contains one element. Note that (cf. Corollary 3.5) any element of $\omega(u_0^\sharp)$ has the form $\varphi^\sharp$ with $\varphi \in \omega(u_0)$. Since $\varphi^\sharp$ is non-increasing, Theorem 4.4 implies that there exist $0 < a_1 \le |\Omega|, \mu > 1$ such that

$$\varphi^\sharp = \mu\chi_{(0,a_1)} + \chi_{(a_1,|\Omega|)}.$$

It follows from (23) and a standard property in the rearrangement theory (cf. [2, Proposition 3.1 (iii)]) that

$$\int_{\Omega^\sharp} \mathcal{P}(\varphi^\sharp) = \int_\Omega \mathcal{P}(\varphi) = E_{1\infty}.$$

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Hence using the mass conservation property and recalling that $m_0 := \int_\Omega u_0$, we obtain

$$\begin{cases} \mu a_1 + |\Omega| - a_1 = m_0 \\ \mathcal{P}(\mu)a_1 + \mathcal{P}(1)(|\Omega| - a_1) = E_{1\infty}, \end{cases}$$

or equivalently,

$$\begin{cases} (\mu - 1)a_1 = m_0 - |\Omega| \\ (\mathcal{P}(\mu) - \mathcal{P}(1))a_1 = E_{1\infty} - \mathcal{P}(1)|\Omega|. \end{cases} \tag{26}$$

Since we know the existence of a function $\varphi^\sharp$, we also know that the system (26) possesses a solution. We show below that it is unique. Indeed, we deduce from (26) that

$$\frac{\mathcal{P}(\mu) - \mathcal{P}(1)}{\mu - 1} = \frac{E_{1\infty} - \mathcal{P}(1)|\Omega|}{m_0 - |\Omega|}. \tag{27}$$

Set

$$\mathcal{G}(s) = \frac{\mathcal{P}(s) - \mathcal{P}(1)}{s - 1}.$$

Then (27) becomes

$$\mathcal{G}(\mu) = \frac{E_{1\infty} - \mathcal{P}(1)|\Omega|}{m_0 - |\Omega|}. \tag{28}$$

We use the monotonicity of p to deduce that

$$\begin{aligned} \mathcal{G}'(s) &= \frac{p(s)(s-1) - (\mathcal{P}(s) - \mathcal{P}(1))}{(s-1)^2} \\ &= \frac{\displaystyle\int_1^s [p(s) - p(\tau)]d\tau}{(s-1)^2} > 0 \quad \text{for} \quad s > 1. \end{aligned}$$

Hence $\mathcal{G}$ is strictly increasing on $(1, \infty)$. It follows that the equation (28) admits at most one solution $\mu > 1$. Furthermore, in view of Remark 4.2, the right-hand-side of (28) is uniquely defined by the initial function u_0. Thus μ is uniquely defined by the initial function. Therefore also a_1 is uniquely determined. The knowledge of the constants μ and a_1 completely determines the stationary solution $\varphi^\sharp$, so that $\omega(u_0^\sharp)$ only contains one element.

Next we show that $\omega(u_0)$ only contains one element. Since $\omega(u_0^\sharp)$ only contains one element, $u^\sharp(t)$ converges to $\varphi^\sharp$ as $t \to \infty$. Consequently, $u^\sharp(t)$ is a Cauchy sequence in $L^1(\Omega^\sharp)$. By Lemma 3.4, $u(t)$ is also a Cauchy sequence in $L^1(\Omega)$. This implies that $u(t)$ converges as $t \to \infty$ and hence $\omega(u_0)$ only contains one element. $\square$

The following result is an immediate consequence of Theorem 5.1 and the uniform boundedness of u.

Corollary 5.2. *Let* $(\mathbf{H_i})$ *hold for* $i = 1$ *or* 3. *Then for all* $p \in [1, \infty)$,

$$u(t) \to \varphi \qquad \text{in} \quad L^p(\Omega) \quad \text{as} \quad t \to \infty,$$

where φ *is given in Theorem 5.1.*

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References

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