Moral hazard under ambiguity

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October 26, 2015

Abstract

In this paper, we extend the Holmström and Milgrom problem [30] by adding uncertainty about the volatility of the output for both the agent and the principal. We study more precisely the impact of the "Nature" playing against the Agent and the Principal by choosing the worst possible volatility of the output. We solve the first-best and the second-best problems associated with this framework and we show that optimal contracts are in a class of contracts similar to [9, 10], linear with respect to the output *and* its quadratic variation. We compare our results with the classical problem in [30].

Key words: Risk-sharing, moral hazard, principal-agent, second-order BSDEs, volatility uncertainty.

AMS 2000 subject classifications: 91B40, 93E20.

JEL subject classifications: C61, C73, D82, J33, M52.

1 Introduction

By and large, it has now become common knowledge among the economists, that almost everything in economics was to a certain degree a matter of incentives: incentives to work hard, to produce, to study, to invest, to consume reasonably... At the heart of the importance of incentives, lies the fact that, to quote B. Salanié [53] "asymmetries of information are pervasive in economic relationships, that is to say, customers know more about their tastes than firms, firms know more about their costs than the government, and all agents take actions that are at least partly unobservable". Starting from the 70s, the theory of contracts evolved from this acknowledgment and the fact that such situations could not be reproduced using the general equilibrium theory. In the corresponding typical situation, a principal (who takes the initiative of the contract) is (potentially) imperfectly informed about the actions of an agent (who accepts or rejects the contract). The goal is to design a contract that maximizes the utility of the principal while that of the agent is held to a given level. Of course, the form of the optimal contracts typically depends on whether these actions are observable/contractible or not, and on whether there are characteristics of the agent that are unknown to the principal. There are three main types of such problems: the first best case, or risk sharing, in which both parties have the same information; the second best case, or moral hazard, in which the action of the agent is hidden or not contractible; the third best case or adverse selection,

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in which the type of the agent is hidden. We will not study this last problem, and refer the interested reader to, among others, [8, 11, 14, 54, 65]. These problems are fundamentally linked to designing optimal incentives, and are therefore present in a very large number of situations. Beyond the obvious application to the optimal remuneration of an employee, one can for instance think on how regulators with imperfect information and limited policy instruments can motivate banks to operate entirely in the social interest, on how a company can optimally compensate its executives, on how banks achieve optimal securitization of mortgage loans or on how investors should pay their portfolio managers (see Bolton and Dewatripont [3] or Laffont and Martimort [32] for many more examples).

Early studies of the risk-sharing problem can be found, among others, in Borch [4], Wilson [68] or Ross [52]. Since then, a large literature has emerged, solving very general risk-sharing problems, for instance in a framework with several agents and recursive utilities (see Duffie et al. [20] or Dumas et al. [21], or for studying optimal compensation of portfolio managers (see Ou-Yang [41] or Cadenillas et al. [6]). From the mathematical point of view, these problems can usually be tackled using either their dual formulation or the so-called stochastic maximum principle, which can characterize the optimal choices of the principal and the agent through coupled systems of Forward Backward Stochastic Differential Equations (FBSDEs in the sequel) (see the very nice monograph [15] by Cvitanić and Zhang for a systematic presentation). One of the main findings in almost all of these works, is that one can find an optimal contract which is linear in the terminal value of the output managed by the agent (a result already obtained in [52]) and possibly some benchmark to which his performance is compared. In specific cases, one can even have Markovian optimal contracts which are given as a (possibly linear) functional of the terminal value of the output (see in particular [6] for details).

Concerning the so-called moral hazard problem, the first paper on continuous-time Principal-Agent problems is the seminal paper by Holmström and Milgrom [30]. They consider principal and agent with exponential utility functions and find that the optimal contract is linear. Their work was generalized by Schättler and Sung [57, 58], Sung [62, 63], Müller [36, 37], and Hellwig and Schmidt [29], using a dynamic programming and martingales approach, which is classical in stochastic control theory (see also the survey paper by Sung [64] for more references). The papers by Williams [67] and Cvitanić, Wan and Zhang [12, 13] use the stochastic maximum principle and FBSDEs to characterize the optimal compensation for more general utility functions. More recently, Djehiche and Hegelsson [18, 19] have also used this approach. A more recent seminal paper in moral hazard setting is Sannikov [55], who finds a tractable model for solving the problem with a random time of retiring the agent and with continuous payments, rather than a lump-sum payment at the terminal time . Since then, a growing literature extending the above models has emerged, be it to include output processes with jumps [2, 7, 43, 69], imperfect information and learning [1, 16, 26, 25, 28, 49], asset pricing [42], executive compensation [27], or mortgage contracts [44] (see also the illuminating survey paper [56] for more references).

Compared to the first-best problem, the moral hazard case corresponds to a Stackelberg-like game between the principal and the agent, in the sense that the principal will start by trying to compute the reaction function of the agent to a given contract (that is to say the optimal action chosen by the agent given the contract), and use this action to maximize his utility over all admissible contracts[1]. This approach does not always work, because it may be hard to solve the agent's

[1]For a recent different approach, see Miller and Yang [35]. For each possible agent's control process, they

stochastic control problem given an arbitrary payoff, possibly non-Markovian, and it may also be hard for the principal to maximize over all such contracts. Furthermore, the agent's optimal control, if it even exists, depends on the given contract in a highly nonlinear manner, rendering the principal's optimization problem even harder and obscure. For these reasons, and as mentioned above, in its most general form the problem was also approached in the literature by the stochastic version of the Pontryagin maximal principle. Nonetheless, none of these standard approaches can solve the problem when the agent also controls the diffusion coefficient of the output, and not just the drift. Building upon this gap in the literature, Cvitanić, Possamaï and Touzi [9, 10] have very recently developed a general approach of the problem through dynamic programming and so-called BSDEs and 2BSDEs, showing that under mild conditions, the problem of the principal could always be rewritten in an equivalent way as a standard stochastic control problem involving two state variables, namely the output itself but also the continuation utility (or value function) of the agent, a property which was pointed out by Sannikov in the specific setting of [55], and which was already well-known by the economists, even in discrete-time models, see for instance Spear and Srivastrava [61]. An important finding of [9], in the context of a delegated portfolio management problem which generalizes Holmström and Milgrom problem [30] to a context where the agent can control the volatility of the (multidimensional) output process, is that in both the first-best and moral hazard problems, the optimal contracts become path-dependent, as they crucially use the quadratic variation of the output process (see also [33] for a related problem).

Our goal in this paper is to study yet another generalization of the Holmström and Milgrom problem [30], to a setting where the agent only controls the drift of the output, but where the twist is that both the principal and the agent may have some uncertainty about the volatility of the output, and only believe that it lies in some given interval of $\mathbb{R}_+$. This is the so-called situation of volatility ambiguity which has received a lot of attention recently, both in the mathematical finance community, since the seminal paper by Denis and Martini [17], and in the economics literature, see for instance [23, 24]. From the mathematical point of view, everything happens as if both the principal and the agent have a "worst-case" approach to the contracting problem, in the sense that they act as if "Nature" was playing against them by choosing the worst possible volatility of the output. Mathematically, this means that the principal and the agent utility criterion incorporates the fact that they are playing a zero-sum game agains "Nature". Furthermore, we put no restrictions on the beliefs that the agent and the principal have with respect to the likely volatility scenario, in the sense that their volatility intervals can be different.

Under this framework, which has not been studied so far in the literature[2], we start by solving the risk-sharing problem. Surprisingly, this problem is much more involved than in the classical case, since it takes a very unusual form, as a supremum of a sum of two infimum over different sets. Nonetheless, we solve it completely by first focusing on a sub-class of contracts similar to the ones obtained in [9, 10], and then using calculus of variations and convex analysis to argue that the

characterize contracts that are incentive compatible for it.

[2] After the completion of this paper, we have been made aware of the paper in preparation [66] by Sung, where the author studies a problem similar to ours. The main difference between the two papers is that [66] does not consider the risk-sharing problem, and that when it comes to the moral hazard case, we solve the maximization problem of the principal over all feasible contracts (the only restriction being integrability), while [66] concentrates on a subset of contracts similar to our class $\mathfrak{C}^{SB}$ (see (4.10)), without showing that this restriction is without loss of generality. Furthermore, [66] imposes as an admissibility condition that the (random) volatility of the output process is Lipschitz in ω for the sup topology, a restriction which is not present at all in our study, and whose interpretation is, in our view, not clear at all.

optimal contracts in the sub-class are actually optimal in the class of all admissible contracts. We also highlight a very surprising effect, since in the case where the volatility intervals of the principal and the agent are completely disjoint, the principal can actually reach utility 0 using an appropriate sequence of contracts (we remind the reader that the exponential utility is $-\exp(-R_p x)$, so that it is bounded from above by 0).

Next, we concentrate on the second-best problem. Our first contribution is to use the theory of second-order BSDEs developed by Soner, Touzi and Zhang [60], and more precisely the recent wellposedness results obtained by Possamaï, Tan and Zhou [47], to obtain a probabilistic representation of the value function of the agent, for any sufficiently integrable contract. In particular, this representation gives an easy access to the optimal action chosen by the agent. Then, following the ideas of [9, 10], we concentrate our attention on a sub-class of contracts, for which the principal problem can be solved by hand, and then show using appropriate bounds that this restriction is actually without loss of generality. We emphasize that though this approach is similar in spirit to the one used in [10], we cannot use their method of proof, since our problem is fundamentally different, because the agent himself does not control the volatility of the output, but rather endures it. Our arguments are actually quite less involved and only require to obtain tight enough upper and lower bounds for the value function of the principal. For simplicity, we present the arguments in the case of a quadratic cost of effort for the agent. Once more, some of our results are quite surprising, since in the case where the volatility intervals of the principal and the agent are completely disjoint, the principal can actually reach utility 0, so that there is no loss of utility due to moral hazard. This is a completely different situation from the classical problem [30] where the second-best problem never coincides with the first-best one.

The rest of the paper is organized as follows. We introduce the model and the contracting problem in Section 2. Then Section 3 is devoted to the risk-sharing problem, while Section 4 treats the moral hazard case. We next present some possible extensions in Section 5, and finally conclude our study in Section 6.

2 The model

2.1 The stochastic basis

We start by giving all the necessary notations and definitions allowing us to consider the so-called "weak" formulation of our problem.

Let $\Omega := \{\omega \in C([0,T],\mathbb{R}) : \omega_0 = 0\}$ be the canonical space equipped with the uniform norm $||\omega||^T_\infty := \sup_{0\leq t\leq T}|\omega_t|$. $\mathcal{F}$ will always be a fixed σ-field on Ω which contains all our filtrations. We then denote B the canonical process, $\mathbb{P}_0$ the Wiener measure, $\mathbb{F} := \{\mathcal{F}_t\}_{0\leq t\leq T}$ the filtration generated by B and $\mathbb{F}^+ := \{\mathcal{F}^+_t, 0 \leq t \leq T\}$, the right limit of $\mathbb{F}$ where $\mathcal{F}^+_t := \cap_{s>t}\mathcal{F}_s$. We will denote by $\mathbf{M}(\Omega)$ the set of all probability measures on Ω. We also recall the so-called universal filtration $\mathbb{F}^\star := \{\mathcal{F}^\star_t\}_{0\leq t\leq T}$ defined as follows

$$\mathcal{F}^\star_t := \bigcap_{\mathbb{P}\in\mathbf{M}(\Omega)} \mathcal{F}^{\mathbb{P}}_t,$$

where $\mathcal{F}^{\mathbb{P}}_t$ is the usual completion under $\mathbb{P}$.

For any subset E of a finite dimensional space and any filtration $\mathbb{X}$ on $(\Omega,\mathcal{F})$, we denote by $\mathbb{H}^0(E,\mathbb{X})$ the set of all $\mathbb{X}$-progressively measurable processes with values in E. Moreover for all

$p > 0$ and for all $\mathbb{P} \in \mathbf{M}(\Omega)$, we denote by $\mathbb{H}^p(\mathbb{P}, E, \mathbb{X})$ the subset of $\mathbb{H}^0(E, \mathbb{X})$ whose elements H satisfy $\mathbb{E}^{\mathbb{P}}\left[\int_0^T |H_t|^p\, dt\right] < +\infty$. The localized versions of these spaces are denoted by $\mathbb{H}^p_{\text{loc}}(\mathbb{P}, E, \mathbb{X})$.

For any subset $\mathcal{P} \subset \mathbf{M}(\Omega)$, a $\mathcal{P}-$polar set is a $\mathbb{P}-$negligible set for all $\mathbb{P} \in \mathcal{P}$, and considering a $\mathcal{P}-$polar set, we say that a property holds $\mathcal{P}-$quasi-surely if it holds outside of this $\mathcal{P}-$polar set. Finally, we introduce the following filtration $\mathbb{G}^{\mathcal{P}} := \{\mathcal{G}_t^{\mathcal{P}}\}_{0 \le t \le T}$ which will be useful in the sequel

$$\mathcal{G}_t^{\mathcal{P}} := \mathcal{F}_t^\star \vee \mathcal{N}^{\mathcal{P}},\ t \le T,$$

where $\mathcal{N}^{\mathcal{P}}$ is the collection of $\mathcal{P}-$polar sets, and its right-continuous limit, denoted $\mathbb{G}^{\mathcal{P},+}$.

For all $\alpha \in \mathbb{H}^1_{\text{loc}}(\mathbb{P}_0, \mathbb{R}_+^\star, \mathbb{F})$, we define the following probability measure on $(\Omega, \mathcal{F})$

$$\mathbb{P}^\alpha := \mathbb{P}_0 \circ (X_\cdot^\alpha)^{-1} \text{ where } X_t^\alpha := \int_0^t \alpha_s^{1/2} dB_s,\ t \in [0,T],\ \mathbb{P}_0 - a.s.$$

We denote by $\mathcal{P}_S$ the collection of all such probability measures on $(\Omega, \mathcal{F})$. We recall from [31] that the quadratic variation process $\langle B \rangle$ is universally defined under any $\mathbb{P} \in \mathcal{P}_S$, and takes values in the set of all non-decreasing continuous functions from $\mathbb{R}_+$ to $\mathbb{R}_+^\star$. We will denote its pathwise density with respect to the Lebesgue measure by $\widehat{\alpha}$. Finally we recall from [59] that every $\mathbb{P} \in \mathcal{P}_S$ satisfies the Blumenthal zero-one law and the martingale representation property.

By definition, for any $\mathbb{P} \in \mathcal{P}_S$

$$W_t^{\mathbb{P}} := \int_0^t \widehat{\alpha}_s^{-1/2} dB_s,\ \mathbb{P} - a.s.,$$

is a $(\mathbb{P}, \mathbb{F})-$Brownian motion. Notice that the probability measures in $\mathbb{P} \in \mathcal{P}_S$ satisfy

$$\mathbb{F}^{\mathbb{P}} = (\mathbb{F}^{W^{\mathbb{P}}})^{\mathbb{P}}, \tag{2.1}$$

where $\mathbb{F}^{W^{\mathbb{P}}}$ is the natural (raw) filtration of the process $W^{\mathbb{P}}$.

Moreover, using the result of [39][3], there actually exists an aggregated version of this family, which we denote by W, which is $\mathbb{F}^\star$-adapted and a $(\mathbb{P}, \mathbb{F}^{\mathbb{P}})-$Brownian motion for every $\mathbb{P} \in \mathcal{P}_S$.

Our focus in this paper will be on the following subsets of $\mathcal{P}_S$.

Definition 2.1. *(i) $\mathcal{P}_m$ is the sub-class of $\mathcal{P}_S$ consisting of all $\mathbb{P} \in \mathcal{P}_S$ such that the canonical process B is a $\mathbb{P}-$uniformly integrable martingale, with respect to $\mathbb{F}$.*

(ii) For any $0 < \underline{\alpha} \le \overline{\alpha}$, $\mathcal{P}_{[\underline{\alpha}, \overline{\alpha}]}$ is the sub-set of $\mathcal{P}_m$ consisting of all the measures $\mathbb{P} \in \mathcal{P}_m$ such that

$$\underline{\alpha} \le \widehat{\alpha} \le \overline{\alpha},\ \mathbb{P} - a.s.$$

The actions of the agent will be considered as $\mathbb{F}$-predictable processes a taking values in the compact set $[0, a_{\max}]$ (for every ω). We denote this set by $\mathcal{A}$. Next, for any subset $\mathcal{P} \subset \mathcal{P}_S$ and any $a \in \mathcal{A}$, we define

$$\mathcal{P}^a := \left\{ \mathbb{Q},\ \text{s.t.} \frac{d\mathbb{Q}}{d\mathbb{P}} = \mathcal{E}\left(\int_0^T a_s \widehat{\alpha}_s^{-1/2} dW_s \right),\ \mathbb{P} - a.s.,\ \text{for some } \mathbb{P} \in \mathcal{P} \right\}.$$

[3] We emphasize that this result actually requires to assume that certain set-theoretic axioms holds, which we do implicitly here. For instance, it is sufficient to work under the usual ZFC framework, and assume in addition the continuum hypothesis.

We also denote $\mathcal{P}^{\mathcal{A}} := \cup_{a\in\mathcal{A}}\mathcal{P}^a$. In particular, for every $\mathbb{P} \in \mathcal{P}^{\mathcal{A}}$ there exists a unique pair $(\alpha^{\mathbb{P}}, a^{\mathbb{P}}) \in \mathbb{H}^1_{\text{loc}}(\mathbb{P}_0, \mathbb{R}^\star_+, \mathbb{F}) \times \mathcal{A}$ such that

$$B_t = \int_0^t a_s^{\mathbb{P}} ds + \int_0^t (\alpha_s^{\mathbb{P}})^{1/2} dW_s^{a^{\mathbb{P}}}, \ \mathbb{P} - a.s., \tag{2.2}$$

where $dW_s^{a^{\mathbb{P}}} := dW_s - (\alpha_s^{\mathbb{P}})^{-1/2} a_s^{\mathbb{P}} ds, \ \mathbb{P}-a.s.$ is a $(\mathbb{P}^a, \mathbb{F}^{\mathbb{P}^a})-$Brownian motion by Girsanov theorem. More precisely, for any $\mathbb{P} \in \mathcal{P}^{\mathcal{A}}$, we must have

$$\frac{d\mathbb{P}}{d\mathbb{P}^\alpha} = \mathcal{E}\left(\int_0^T a_s \widehat{\alpha}_s^{-1/2} dB_s\right),$$

for some $(\alpha, a) \in \mathbb{H}^1_{\text{loc}}(\mathbb{P}_0, \mathbb{R}^\star_+, \mathbb{F}) \times \mathcal{A}$ and the following equalities hold

$$a^{\mathbb{P}}(B.) = a(B.) \text{ and } \alpha^{\mathbb{P}}(B.) = \alpha(W.), \ dt \times \mathbb{P} - a.e.$$

For simplicity, we will therefore sometimes denote a probability measure $\mathbb{P} \in \mathcal{P}_S^{\mathcal{A}}$ by $\mathbb{P}_a^\alpha$. For any subset of $\mathcal{P} \subset \mathcal{P}_m$, we also denote for any $(t, \mathbb{P}) \in [0, T] \times \mathcal{P}$

$$\mathcal{P}(\mathbb{P}, t^+) := \left\{\mathbb{P} \in \mathcal{P}, \ \mathbb{P}' = \mathbb{P}, \text{ on } \mathcal{F}_t^+\right\}.$$

2.2 The contracting problem in finite horizon

We consider a generalization of the classical problem of Holmström and Milgrom [30] and fix a given time horizon $T > 0$. Here the agent and the principal both observe the outcome process B, but the principal may not observe the action chosen by the agent[4], and both of them have a "worst-case" approach to the contract, in the sense that they act as if "Nature" was playing against them by choosing the worst possible volatility of the output. More precisely, a contract will be a $\mathcal{F}_T-$measurable random variable, corresponding to the salary received by the agent at time T only, and the utility of the agent is then, given a contract ξ, a recommended level of effort $a \in \mathcal{A}$ and an ambiguity set $\mathcal{P}_A \subset \mathcal{P}_m$

$$u_0^A(\xi, a) := \inf_{\mathbb{P}\in\mathcal{P}_A^a} \mathbb{E}^{\mathbb{P}}\left[\mathcal{U}_A\left(\xi - \int_0^T k(a_s)ds\right)\right],$$

where $\mathcal{U}_A(x) := -\exp(-R_A x)$ is the utility function of the Agent and $k(x)$ is his cost function, which, as usual is assumed to be increasing, strictly convex and superlinear.

The value function of the agent at time 0 is therefore

$$U_0^A(\xi) := \sup_{a\in\mathcal{A}} \inf_{\mathbb{P}\in\mathcal{P}_A^a} \mathbb{E}^{\mathbb{P}}\left[\mathcal{U}_A\left(\xi - \int_0^T k(a_s)ds\right)\right].$$

Similarly, the utility of the Principal, having an ambiguity set $\mathcal{P}_P \subset \mathcal{P}_m$, when offering a contract ξ and a recommended level of effort $a \in \mathcal{A}$ is

$$u_0^P(\xi, a) := \inf_{\mathbb{P}\in\mathcal{P}_P^a} \mathbb{E}^{\mathbb{P}}\left[\mathcal{U}_P\left(B_T - \xi\right)\right], \tag{2.3}$$

where $\mathcal{U}_P(x) := -\exp(-R_P x)$ is the utility function of the Principal. From now on, we assume that

$$\mathcal{P}_P = \mathcal{P}_{[\underline{\alpha}^P, \overline{\alpha}^P]} \text{ and } \mathcal{P}_A = \mathcal{P}_{[\underline{\alpha}^A, \overline{\alpha}^A]},$$

[4]He observes it in the risk-sharing problem of Section 3, but not in the moral hazard case of Section 4.

for some $0 \leq \underline{\alpha}^P \leq \overline{\alpha}^P$ and $0 \leq \underline{\alpha}^A \leq \overline{\alpha}^A$.

Let $R < 0$ denote the reservation utility of the agent. The problem of the Principal is then to offer a contract ξ as well as a recommended level of effort a so as to maximize his utility (2.3), subject to the constraints

$$u_0^A(\xi, a) \geq R \tag{2.4}$$
$$u_0^A(\xi, a) = U_0^A(\xi). \tag{2.5}$$

The first constraint is the so-called participation constraint, while the second-one is the usual incentive compatibility condition, stating that the recommended level of effort a should be the optimal response of the agent, given the contract ξ.

Furthermore, we will denote by $\mathcal{C}$ the set of admissible contracts, that is to say the set of $\mathcal{F}_T-$measurable random variables such that

$$\sup_{\mathbb{P} \in \mathcal{P}_A^{\mathcal{A}} \cup \mathcal{P}_P^{\mathcal{A}}} \mathbb{E}^{\mathbb{P}} \left[\exp\left(p \left| \xi \right| \right) \right] < +\infty, \text{ for any } p \geq 0,$$

and we emphasize immediately that we will have to restrict a bit more the admissible contracts when solving the second-best problem.

3 The first-best: a problem of calculus of variations

In this section, we start by solving the first-best problem for the principal, since it will serve as our main benchmark and has not been considered, as far as we know, in the pre-existing literature. Moreover, we will see that the derivation is a lot more complicated than in the classical setting. So much so that, quite surprisingly compared with the classical Holmström and Milgrom [30] problem, the optimal contracts are in general not linear with respect to the final value of the output B_T, and are even path-dependent.

Recall that for any contract $\xi \in \mathcal{C}$ and for any recommended effort level $a \in \mathcal{A}$

$$u_0^{P,FB}(\xi, a) = \inf_{\mathbb{P} \in \mathcal{P}_P^a} \mathbb{E}^{\mathbb{P}} \left[\mathcal{U}_P \left(B_T - \xi \right) \right].$$

The value function of the principal is then

$$U_0^{P,FB} := \sup_{\xi \in \mathcal{C}} \sup_{a \in \mathcal{A}} \left\{ u_0^P(\xi, a) \right\}, \tag{3.1}$$

where the following participation constraint is satisfied

$$\inf_{\mathbb{P} \in \mathcal{P}_A^a} \mathbb{E}^{\mathbb{P}} \left[\mathcal{U}_A \left(\xi - \int_0^T k(a_s) ds \right) \right] \geq R. \tag{3.2}$$

The value function of the principal defined by (3.1) can be then rewritten

$$U_0^{P,FB} := \sup_{\xi \in \mathcal{C}} \sup_{a \in \mathcal{A}} \left\{ \inf_{\mathbb{P} \in \mathcal{P}_P^a} \mathbb{E}^{\mathbb{P}} \left[\mathcal{U}_P \left(B_T - \xi \right) \right] + \rho \inf_{\mathbb{P} \in \mathcal{P}_A^a} \mathbb{E}^{\mathbb{P}} \left[\mathcal{U}_A \left(\xi - \int_0^T k(a_s) ds \right) \right] \right\}, \tag{3.3}$$

where the Lagrange multiplier $\rho > 0$ is here to ensure that the participation constraint (3.2) holds.

3.1 A particular sub-class of non-linear contracts

It has been pointed out recently in [9] that in contracting problems where the agent can control the volatility of the output process, non-linear contracts involving the quadratic variation $\langle B\rangle$ of the output appeared naturally. Though the problem we consider here is different, since it can be interpreted as if the agent was actually playing a game against another "player", who is the one controlling the volatility of the output, we expect that these non-linear and path-dependent contracts will play an important role. We therefore introduce the set

$$\mathcal{Q} := \left\{ \xi \in \mathcal{C},\ \xi = zB_T + \frac{\gamma}{2}\langle B\rangle_T + \delta,\ (z,\gamma,\delta) \in \mathbb{R}^3 \right\}.$$

From now on, noticing that any contract ξ in $\mathcal{Q}$ is uniquely defined by the corresponding triplet of reals (z,γ,δ), we will abuse notations and make the identification $\xi \equiv (z,\gamma,\delta)$.

3.1.1 Degeneracy for disjoints $\mathcal{P}_P$ and $\mathcal{P}_A$

Our first result shows that if the sets of ambiguity of the principal and the agent are completely disjoint, then there are sequences of contracts in $\mathcal{Q}$ such that the Principal can attain the universal upper bound 0 of his utility, while ensuring that the agent still receives his reservation utility R.

Theorem 3.1. *(i) Assume that* $\overline{\alpha}^P < \underline{\alpha}^A$. *Then, there exist a sequence of contracts* $(\xi^n)_{n\in\mathbb{N}^\star}$ *and a recommended effort* $a^\star := a^{max}$, *with*

$$\xi^n := \frac{1}{2}n\langle B\rangle_T - \frac{T}{2}n\underline{\alpha}^A + \delta^\star,\ \delta^\star := Tk(a^{max}) - \frac{\log(-R)}{R_A},$$

such that $\lim_{n\to+\infty} u_0^{P,FB}(\xi^n, a^{\max}) = 0$ *and* $u_0^A(\xi^n, a^{\max}) = R$, *for any* $n \geq 1$.

(ii) Assume that $\underline{\alpha}^P > \overline{\alpha}^A$. *Then, there exist a sequence of contracts* $(\xi^n)_{n\in\mathbb{N}^\star}$ *and a recommended effort* $a^\star := a^{max}$ *where*

$$\xi^n := -\frac{1}{2}n\langle B\rangle_T + \frac{T}{2}n\overline{\alpha}^A + \delta^\star,\ \delta^\star := Tk(a^{max}) - \frac{\log(-R)}{R_A}$$

such that $\lim_{n\to+\infty} u_0^{P,FB}(\xi^n, a^{\max}) = 0$ *and* $u_0^A(\xi^n, a^{\max}) = R$, *for any* $n \geq 1$.

Before proving this result, let us comment on it. We will see during the proof that when the sets of uncertainty for the principal and the agent are completely disjoint, the principal can use the quadratic variation component in the contract in order to make appear in the exponential a term which he can make arbitrarily large, but which is not seen at all by the agent in his utility, as it is constructed so that it disappears under the worst-case probability measure the agent. This is therefore the combination of this difference between the worst-case measures of the principal and the agent, as well as the fact that their uncertainty sets are disjoints which make the problem degenerate. This is, from our point view, quite a surprising result, all the more since we will prove later on that this phenomenon also happens in the second-best problem.

Proof. (i) **First case:** $\underline{\alpha}^{\mathbf{A}} > \overline{\alpha}^{\mathbf{P}}$**.** Let

$$\xi^n = \frac{1}{2}n\langle B\rangle_T - \frac{T}{2}n\underline{\alpha}^A + \delta^\star,\ n \in \mathbb{N}^\star,$$

with $\delta^\star = Tk(a^{\max}) - \frac{\log(-R)}{R_A}$. We aim at showing that the sequence of contracts (ξ^n) is a maximizing sequence of contracts, allowing the principal to reach utility 0, when recommending in addition the level of effort $a^{\max}$. We have

$$\begin{aligned}
u_0^P(\xi^n, a^{\max}) &= \inf_{\mathbb{P}\in\mathcal{P}_P^{a^{\max}}} \mathbb{E}^{\mathbb{P}}\left[\mathcal{U}_P\left(B_T - \xi^n\right)\right] \\
&= \inf_{\mathbb{P}\in\mathcal{P}_P^{a^{\max}}} \mathbb{E}^{\mathbb{P}}\left[-e^{-R_P\left(B_T - \frac{1}{2}n\int_0^T \alpha_s^{\mathbb{P}}ds + \frac{T}{2}n\underline{\alpha}^A - \delta^\star\right)}\right] \\
&= e^{-R_P(\frac{T}{2}n\underline{\alpha}^A - \delta^\star)} \inf_{\mathbb{P}\in\mathcal{P}_P^{a^{\max}}} \mathbb{E}^{\mathbb{P}}\left[-e^{-R_P\left(B_T - \frac{1}{2}n\int_0^T \alpha_s^{\mathbb{P}}ds\right)}\right] \\
&= e^{-R_P(\frac{T}{2}n\underline{\alpha}^A - \delta^\star)} \inf_{\mathbb{P}\in\mathcal{P}_P^{a^{\max}}} \mathbb{E}^{\mathbb{P}}\left[-e^{-R_P\left(\int_0^T (\alpha_s^{\mathbb{P}})^{1/2}dW_s^{a^{\max}} + Ta^{\max} - \frac{1}{2}n\int_0^T \alpha_s ds\right)}\right] \\
&= -e^{-R_P\left(Ta^{\max} + \frac{T}{2}n\underline{\alpha}^A - \delta^\star\right)} \sup_{\mathbb{P}\in\mathcal{P}_P^{a^{\max}}} \mathbb{E}^{\mathbb{P}}\bigg[\mathcal{E}\left(-R_P\int_0^T (\alpha_s^{\mathbb{P}})^{1/2}dW_s^{a^{\max}}\right) \\
&\qquad\qquad\qquad\qquad \times \exp\left(\frac{1}{2}R_P^2\int_0^T \alpha_s^{\mathbb{P}}ds + \frac{R_P}{2}n\int_0^T \alpha_s^{\mathbb{P}}ds\right)\bigg] \\
&= -\exp\left(-R_P\left(a^{\max}T - \delta^\star + \frac{T}{2}n(\underline{\alpha}^A - \overline{\alpha}^P) - \frac{1}{2}R_P T\overline{\alpha}^P\right)\right),
\end{aligned}$$

where we have used the fact that for any $\mathbb{P}\in\mathcal{P}_P^{a^{\max}}$, we have

$$\exp\left(\frac{1}{2}R_P^2\int_0^T \alpha_s^{\mathbb{P}}ds + \frac{R_P}{2}n\int_0^T \alpha_s^{\mathbb{P}}ds\right) \leq \exp\left(\frac{T}{2}R_P^2\overline{\alpha}^P + \frac{R_P}{2}nT\overline{\alpha}^P\right),\ \mathbb{P}-a.s.,$$

and that the stochastic exponential appearing above is clearly a $\mathbb{P}-$martingale for any $\mathbb{P}\in\mathcal{P}_P^{a^{\max}}$, so that the value of the supremum is clear and attained for the measure $\mathbb{P}^{\overline{\alpha}^P}_{a^{\max}}$.

Hence, we obtain $u_0^P(\xi^n, a^{\max}) \longrightarrow 0$ when $n\to+\infty$. Since $U_0^P \leq 0$, we deduce that the sequence (ξ^n) approaches the best utility for the principal when n goes to $+\infty$. It remains to prove that for any $n\in\mathbb{N}^\star$, ξ^n is admissible, *i.e.*, ξ^n satisfies

$$\inf_{\mathbb{P}\in\mathcal{P}_A^{a^{\max}}} \mathbb{E}^{\mathbb{P}}\left[\mathcal{U}_A\left(\xi^n - K_T^{a^{\max})}\right)\right] \geq R,\ n\in\mathbb{N}^\star.$$

Indeed,

$$\begin{aligned}
&\inf_{\mathbb{P}\in\mathcal{P}_A^{a^{\max}}} \mathbb{E}^{\mathbb{P}}\left[\mathcal{U}_A\left(\xi^n - Tk(a^{\max})\right)\right] \\
&= \inf_{\mathbb{P}\in\mathcal{P}_A^{a^{\max}}} \mathbb{E}^{\mathbb{P}}\left[-\exp\left(-R_A\left(\xi^n - Tk(a^{\max})\right)\right)\right] \\
&= \inf_{\mathbb{P}\in\mathcal{P}_A^{a^{\max}}} \mathbb{E}^{\mathbb{P}}\left[-\exp\left(-R_A\left(\frac{1}{2}n\int_0^T \alpha_s^{\mathbb{P}}ds - \frac{T}{2}n\underline{\alpha}^A + \delta^\star - Tk(a^{\max})\right)\right)\right] \\
&= -\exp\left(-R_A\left(\delta^\star - Tk(a^{\max}) - \frac{T}{2}n\underline{\alpha}^A\right)\right) \sup_{\mathbb{P}\in\mathcal{P}_A^{a^{\max}}} \mathbb{E}^{\mathbb{P}}\left[\exp\left(-\frac{R_A n}{2}\int_0^T \alpha_s^{\mathbb{P}}ds\right)\right] \\
&= -e^{-R_A(\delta^\star - Tk(a^{\max}))} = R.
\end{aligned}$$

(ii) **Second case: $\overline{\alpha}^{\mathbf{A}} < \underline{\alpha}^{\mathbf{P}}$.** Similarly, let

$$\xi^n = -\frac{n}{2}\langle B\rangle_T + \frac{T}{2}n\overline{\alpha}^A + \delta^\star,\ n\in\mathbb{N}^\star,\ \delta^\star = Tk(a^{\max}) - \frac{\log(-R)}{R_A}.$$

Assume that $n > R_P$, the same computations as above lead to

$$\begin{aligned}
u_0^P(\xi^n, a^{\max}) &= \inf_{\mathbb{P}\in\mathcal{P}_P^a} \mathbb{E}^{\mathbb{P}}\left[\mathcal{U}_P\left(B_T - \xi^n\right)\right] \\
&= -e^{-R_P\left(Ta^{\max} - \frac{T}{2}n\overline{\alpha}^A - \delta^\star\right)} \sup_{\mathbb{P}\in\mathcal{P}_P^{a^{\max}}} \mathbb{E}^{\mathbb{P}}\left[\mathcal{E}\left(-R_P\int_0^T (\alpha_s^{\mathbb{P}})^{1/2} dW_s^{a^{\max}}\right)\right. \\
&\qquad\qquad \left.\times \exp\left(\frac{1}{2}R_P^2\int_0^T \alpha_s^{\mathbb{P}} ds - \frac{R_P}{2}n\int_0^T \alpha_s^{\mathbb{P}} ds\right)\right] \\
&= -\exp\left(-R_P\left(Ta^{\max} - \delta^\star + \frac{T}{2}n(\underline{\alpha}^P - \overline{\alpha}^A) - \frac{1}{2}R_P T\overline{\alpha}^P\right)\right).
\end{aligned}$$

Hence, we obtain again $u_0^P(\xi^n, a^{\max}) \longrightarrow 0$ when $n \to +\infty$. Next, we have

$$\begin{aligned}
&\inf_{\mathbb{P}\in\mathcal{P}_A^{a^{\max}}} \mathbb{E}^{\mathbb{P}}\left[\mathcal{U}_A\left(\xi^n - Tk(a^{\max})\right)\right] \\
&= \inf_{\mathbb{P}\in\mathcal{P}_A^{a^{\max}}} \mathbb{E}^{\mathbb{P}}\left[-\exp\left(-R_A\left(\xi^n - Tk(a^{\max})\right)\right)\right] \\
&= \inf_{\mathbb{P}\in\mathcal{P}_A^{a^{\max}}} \mathbb{E}^{\mathbb{P}}\left[-\exp\left(-R_A\left(-\frac{1}{2}n\int_0^T \alpha_s ds + \frac{T}{2}n\overline{\alpha}^A + \delta^\star - Tk(a^{\max})\right)\right)\right] \\
&= -\exp\left(-R_A\left(\delta^\star - Tk(a^{\max}) + \frac{T}{2}n\overline{\alpha}^A\right)\right) \times \sup_{\mathbb{P}\in\mathcal{P}_A^a} \mathbb{E}^{\mathbb{P}}\left[\exp\left(\frac{R_A n}{2}\int_0^T \alpha_s ds\right)\right] \\
&= -e^{-R_A(\delta^\star - Tk(a^{\max}))} = R.
\end{aligned}$$

□

3.1.2 Optimal contracts in $\mathcal{Q}$ with intersecting uncertainty sets

We study now non-degenerate cases. Let us define the subset $\mathcal{A}_{\text{det}} \subset \mathcal{A}$ of actions which are deterministic. The following maps, defined for any $(a, z, \gamma, \delta, \alpha_P, \alpha_A) \in \mathcal{A}\times\mathbb{R}^3\times[\underline{\alpha}^P, \overline{\alpha}^P]\times[\underline{\alpha}^A, \overline{\alpha}^A]$, will play an important role in what follows

$$F(a, z, \gamma, \delta, \alpha_P, \alpha_A) := \Gamma_P(a, z, \gamma, \delta, \alpha_P) + \rho\Gamma_A(a, z, \gamma, \delta, \alpha_A), \tag{3.4}$$

where

$$\begin{aligned}
\Gamma_P(a, z, \gamma, \delta, \alpha_P) &:= -\exp\left(R_P\left(\delta - (1-z)\int_0^T a_s ds + \left(\frac{R_P(1-z)^2}{2} + \frac{\gamma}{2}\right)\alpha_P T\right)\right), \\
\Gamma_A(a, z, \gamma, \delta, \alpha_A) &:= -\exp\left(R_A\left(\int_0^T k(a_s)ds - z\int_0^T a_s ds - \delta + \left(\frac{R_A z^2}{2} - \frac{\gamma}{2}\right)\alpha_A T\right)\right).
\end{aligned}$$

We also define

$$\begin{aligned}
G(a, z, \gamma, \alpha_P, \alpha_A) := &-\rho^{\frac{R_P}{R_A+R_P}} \frac{R_A + R_P}{R_P}\left(\frac{R_A}{R_P}\right)^{-\frac{R_A}{R_A+R_P}} e^{\frac{R_A R_P}{R_A+R_P}\int_0^T (k(a_s) - a_s)ds} \\
&\times e^{\frac{R_A R_P}{R_A+R_P}\left(\frac{\gamma}{2}T(\alpha_P - \alpha_A) + \frac{T}{2}\left(\alpha_P R_P(1-z)^2 + \alpha_A R_A z^2\right)\right)}.
\end{aligned}$$

When $\alpha_P = \alpha_A$, by noticing that $G(a, z, \gamma, \alpha_P, \alpha_A)$ does not depend on γ we will simply write without any ambiguity

$$G(a, z, \alpha_P) := G(a, z, \gamma, \alpha_P, \alpha_P).$$

To alleviate later computations, we partition the set $\mathcal{Q}$ into

$$\begin{aligned}
\mathcal{Q}^{\underline{\gamma}} &:= \left\{\xi \equiv (z,\gamma,\delta) \in \mathcal{Q},\ \gamma < -R_P(1-z)^2\right\},\\
\mathcal{Q}^{|\gamma|} &:= \left\{\xi \equiv (z,\gamma,\delta) \in \mathcal{Q},\ -R_P(1-z)^2 < \gamma < R_A z^2\right\},\\
\mathcal{Q}^{d} &:= \left\{\xi \equiv (z,\gamma,\delta) \in \mathcal{Q},\ -R_P(1-z)^2 = \gamma\right\},\\
\mathcal{Q}^{u} &:= \left\{\xi \equiv (z,\gamma,\delta) \in \mathcal{Q},\ \gamma = R_A z^2\right\},\\
\mathcal{Q}^{\overline{\gamma}} &:= \left\{\xi \equiv (z,\gamma,\delta) \in \mathcal{Q},\ \gamma > R_A z^2\right\},
\end{aligned}$$

and define for any $(a,\xi) \in \mathcal{A} \times \mathcal{C}$

$$\widetilde{u}_0^{P,FB}(a,\xi) := \inf_{\mathbb{P}\in\mathcal{P}_P^a} \mathbb{E}^{\mathbb{P}}\left[\mathcal{U}_P\left(B_T - \xi\right)\right] + \rho \inf_{\mathbb{P}\in\mathcal{P}_A^a} \mathbb{E}^{\mathbb{P}}\left[\mathcal{U}_A\left(\xi - \int_0^T k(a_s)ds\right)\right].$$

The following lemma computes the principal utility $\widetilde{u}_0^{P,FB}(a,\xi)$ for a recommended level of effort in $\mathcal{A}_{\text{det}}$ and any contract $\xi \in \mathcal{Q}$. Its proof is relegated to the Appendix.

Lemma 3.1. *Let us fix some $a \in \mathcal{A}_{det}$ and some $\xi \in \mathcal{Q}$, with $\xi \equiv (z,\gamma,\delta)$.*

(i) If $\xi \in \mathcal{Q}^{\underline{\gamma}}$

$$\widetilde{u}_0^{P,FB}(a,\xi) = F(a,z,\gamma,\delta,\underline{\alpha}^P,\overline{\alpha}^A).$$

(ii) a) *If $\xi \in \mathcal{Q}^d$, then for any $\mathbb{P} \in \mathcal{P}_P^a$,*

$$\mathbb{E}^{\mathbb{P}}\left[\mathcal{U}_P\left(B_T - \xi\right)\right] = \inf_{\mathbb{P}\in\mathcal{P}_P^a} \mathbb{E}^{\mathbb{P}}\left[\mathcal{U}_P\left(B_T - \xi\right)\right],$$

and in particular

$$\widetilde{u}_0^{P,FB}(a,\xi) = F(a,z,\gamma,\delta,\alpha_P,\overline{\alpha}^A),\ \textit{for any}\ \alpha_P \in [\underline{\alpha}^P,\overline{\alpha}^P].$$

b) *If $\xi \in \mathcal{Q}^{|\gamma|}$,*

$$\widetilde{u}_0^{P,FB}(a,\xi) = F(a,z,\gamma,\delta,\overline{\alpha}^P,\overline{\alpha}^A).$$

c) *If $\xi \in \mathcal{Q}^u$, then for any $\mathbb{P} \in \mathcal{P}_A^a$,*

$$\mathbb{E}^{\mathbb{P}}\left[\mathcal{U}_A\left(\xi - \int_0^T k(a_s)ds\right)\right] = \inf_{\mathbb{P}\in\mathcal{P}_A^a} \mathbb{E}^{\mathbb{P}}\left[\mathcal{U}_A\left(\xi - \int_0^T k(a_s)ds\right)\right],$$

and in particular

$$\widetilde{u}_0^{P,FB}(a,\xi) = F(a,z,\gamma,\delta,\overline{\alpha}^P,\alpha_A),\ \textit{for any}\ \alpha_A \in [\underline{\alpha}^A,\overline{\alpha}^A].$$

(iii) If $\xi \in \mathcal{Q}^{\overline{\gamma}}$,

$$\widetilde{u}_0^{P,FB}(a,\xi) = F(a,z,\gamma,\delta,\overline{\alpha}^P,\underline{\alpha}^A).$$

The next lemma computes the supremum of F with respect to δ. Its proof is also relegated to the Appendix.

Lemma 3.2. *For any* $(a, z, \gamma, \alpha_P, \alpha_A) \in \mathcal{A} \times \mathbb{R} \times \mathbb{R} \times [\underline{\alpha}^P, \overline{\alpha}^P] \times [\underline{\alpha}^A, \overline{\alpha}^A]$ *we have*

$$\sup_{\delta \in \mathbb{R}} F(a, z, \gamma, \delta, \alpha_P, \alpha_A) = F(a, z, \gamma, \delta^\star(z, \gamma, \alpha_P, \alpha_A), \alpha_P, \alpha_A) = G(a, z, \gamma, \alpha_P, \alpha_A),$$

where

$$\delta^\star(z, \gamma, \alpha_P, \alpha_A) := \frac{1}{R_A + R_P}\bigg[\log\left(\frac{\rho R_A}{R_P}\right) + \int_0^T ((R_P(1-z) - R_A z)a_s + R_A k(a_s))\, ds - \frac{R_P}{2}(R_P(1-z)^2 + \gamma)\alpha_P T + \frac{R_A}{2}\left(R_A z^2 - \gamma\right)\alpha_A T\bigg].$$

The following lemma gives the optimal contracts and efforts in $\mathcal{A}_{\text{det}}$ and each subset of our partition of $\mathcal{Q}$ for the principal problem, for a fixed Lagrange multiplier ρ.

Lemma 3.3. *Let* $a^\star$ *be the constant minimiser of the strictly convex map* $a \longmapsto k(a) - a$ *and define* $z^\star := \frac{R_P}{R_A + R_P}$.

(i) Optimal contracts in $\mathcal{Q}^{\underline{\gamma}}$.

a) If $\underline{\alpha}^P < \overline{\alpha}^A$,

$$\sup_{a \in \mathcal{A}_{det}} \sup_{\xi \in \mathcal{Q}^{\underline{\gamma}}} \widetilde{u}_0^{P,FB}(a, \xi) = F(a^\star, z^\star, \gamma^\star, \delta^\star, \underline{\alpha}^P, \overline{\alpha}^A),$$

where $\gamma^\star := -R_P(1 - z^\star)^2$ *and*

$$\delta^\star := \frac{1}{R_A + R_P}\left[\log\left(\rho \frac{R_A}{R_P}\right) + R_A T k(a^\star) ds + \frac{R_A^2 R_P T}{2(R_A + R_P)}\overline{\alpha}^A\right].$$

b) If $\underline{\alpha}^P = \overline{\alpha}^A =: \tilde{\alpha}$,

$$\sup_{a \in \mathcal{A}_{det}} \sup_{\xi \in \mathcal{Q}^{\underline{\gamma}}} \widetilde{u}_0^{P,FB}(a, \xi) = F(a^\star, z^\star, \gamma^\star, \delta^\star, \tilde{\alpha}, \tilde{\alpha}),$$

for any $\gamma^\star < -R_P(1 - z^\star)^2$ *and*

$$\delta^\star := \frac{1}{R_A + R_P}\left[\log\left(\rho \frac{R_A}{R_P}\right) + R_A T k(a^\star)\right] - \frac{\gamma^\star}{2}\tilde{\alpha} T.$$

In these two cases

$$\sup_{a \in \mathcal{A}_{det}} \sup_{\xi \in \mathcal{Q}^{\underline{\gamma}}} \widetilde{u}_0^{P,FB}(a, \xi) = G(a^\star, z^\star, \overline{\alpha}^A).$$

(ii) Optimal contracts in $\mathcal{Q}^d$. *For any* $\alpha_P \in [\underline{\alpha}^P, \overline{\alpha}^P]$,

$$\sup_{a \in \mathcal{A}_{det}} \sup_{\xi \in \mathcal{Q}^d} \widetilde{u}_0^{P,FB}(a, \xi) = F(a^\star, z^\star, \gamma^\star, \delta^\star, \alpha_P, \overline{\alpha}^A) = G(a^\star, z^\star, \overline{\alpha}^A),$$

with $\gamma^\star := -R_P(1 - z^\star)^2$ *and*

$$\delta^\star := \frac{1}{R_A + R_P}\left[\log\left(\rho \frac{R_A}{R_P}\right) + R_A T k(a^\star) + \frac{R_A^2 R_P T}{2(R_A + R_P)}\overline{\alpha}^A\right].$$

(iii) Optimal contracts in $\mathcal{Q}^{|\gamma|}$.

a) If $\overline{\alpha}^P < \overline{\alpha}^A$,

$$\sup_{a\in\mathcal{A}_{det}} \sup_{\xi\in\mathcal{Q}^{|\gamma|}} \widetilde{u}_0^{P,FB}(a,\xi) = F(a^\star, z^\star, \gamma^\star, \delta^\star, \overline{\alpha}^P, \overline{\alpha}^A) = G(a^\star, z^\star, \overline{\alpha}^P),$$

where $\gamma^\star := R_A|z^\star|^2$ *and*

$$\delta^\star := \frac{1}{R_A+R_P}\left[\log\left(\rho\frac{R_A}{R_P}\right) + R_ATk(a^\star)\right] - \frac{\gamma^\star}{2}\overline{\alpha}^PT.$$

b) If $\overline{\alpha}^P = \overline{\alpha}^A =: \overline{\alpha}$,

$$\sup_{a\in\mathcal{A}_{det}} \sup_{\xi\in\mathcal{Q}^{|\gamma|}} \widetilde{u}_0^{P,FB}(a,\xi) = F(a^\star, z^\star, \gamma^\star, \delta^\star, \overline{\alpha}, \overline{\alpha}) = G(a^\star, z^\star, \overline{\alpha}),$$

for any $\gamma^\star \in (-R_P(1-z^\star)^2, R_A|z^\star|^2)$ *and*

$$\delta^\star := \frac{1}{R_A+R_P}\left[\log\left(\rho\frac{R_A}{R_P}\right) + R_ATk(a^\star)\right] - \frac{\gamma^\star}{2}\overline{\alpha}T.$$

c) If $\overline{\alpha}^P > \overline{\alpha}^A$,

$$\sup_{a\in\mathcal{A}_{det}} \sup_{\xi\in\mathcal{Q}^{|\gamma|}} \widetilde{u}_0^{P,FB}(a,\xi) = F(a^\star, z^\star, \gamma^\star, \delta^\star, \overline{\alpha}^P, \overline{\alpha}^A) = G(a^\star, z^\star, \overline{\alpha}^A),$$

where $\gamma^\star := -R_P(1-z^\star)^2$ *and*

$$\delta^\star := \frac{1}{R_A+R_P}\left[\log\left(\rho\frac{R_A}{R_P}\right) + R_ATk(a^\star) + \frac{R_A^2R_PT}{2(R_A+R_P)}\overline{\alpha}^A\right].$$

(iv) Optimal contracts in $\mathcal{Q}^u$*. For any* $\alpha_A \in [\underline{\alpha}^A, \overline{\alpha}^A]$,

$$\sup_{a\in\mathcal{A}_{det}} \sup_{\xi\in\mathcal{Q}^{u}} \widetilde{u}_0^{P,FB}(a,\xi) = F(a^\star, z^\star, \gamma^\star, \delta^\star, \overline{\alpha}^P, \alpha_A) = G(a^\star, z^\star, \overline{\alpha}^P),$$

with $\gamma^\star := R_A|z^\star|^2$ *and*

$$\delta^\star := \frac{1}{R_A+R_P}\left[\log\left(\rho\frac{R_A}{R_P}\right) + R_ATk(a^\star) - \frac{R_AR_P^2T}{2(R_A+R_P)}\overline{\alpha}^P\right].$$

(v) Optimal contracts in $\mathcal{Q}^{\overline{\gamma}}$.

a) If $\overline{\alpha}^P = \underline{\alpha}^A =: \check{\alpha}$,

$$\sup_{a\in\mathcal{A}_{det}} \sup_{\xi\in\mathcal{Q}^{\overline{\gamma}}} \widetilde{u}_0^{P,FB}(a,\xi) = F(a^\star, z^\star, \gamma^\star, \delta^\star, \check{\alpha}, \check{\alpha}),$$

for any $\gamma^\star > R_A|z^\star|^2$ *and*

$$\delta^\star := \frac{1}{R_A+R_P}\left[\log\left(\rho\frac{R_A}{R_P}\right) + R_ATk(a^\star)\right] - \frac{\gamma^\star}{2}\check{\alpha}T.$$

b) If $\overline{\alpha}^P > \underline{\alpha}^A$,

$$\sup_{a\in\mathcal{A}_{det}} \sup_{\xi\in\mathcal{Q}^{\overline{\gamma}}} \widetilde{u}_0^{P,FB}(a,\xi) = F(a^\star, z^\star, \gamma^\star, \delta^\star, \overline{\alpha}^P, \underline{\alpha}^A),$$

with $\gamma^\star := R_A|z^\star|^2$ *and*

$$\delta^\star := \frac{1}{R_A+R_P}\left[\log\left(\rho\frac{R_A}{R_P}\right) + R_ATk(a^\star) - \frac{R_AR_P^2T}{2(R_A+R_P)}\overline{\alpha}^P\right].$$

In these two cases

$$\sup_{a\in\mathcal{A}_{det}} \sup_{\xi\in\mathcal{Q}^{\underline{\gamma}}} \widetilde{u}_0^{P,FB}(a,\xi) = G(a^\star, z^\star, \overline{\alpha}^P).$$

3.1.3 Are contracts in $\mathcal{Q}$ optimal?

The question now is obviously whether the optimal contracts in $\mathcal{Q}$ that we have derived in the previous section are optimal among all the admissible contracts in $\mathcal{C}$. We will show in this section that this is actually always true. We insist on the fact that such a situation is different from the original Holmstrom-Milgrom [30] problem, where the first-best contract was linear in B_T, and is thus much closer to its recent generalization in [9] where the agent is allowed to control the volatility of the output, where optimal contracts are shown to be linear in B_T and its quadratic variation $\langle B\rangle_T$. Nonetheless, in the setting of [9], moral hazard arises from the dimension of the output process, while it comes from the worst-case attitude of both the principal and the agent in our framework

Let us consider the so-called Morse-Transue space on $(\Omega,\mathcal{F},\mathbb{P}_0)$ (we refer the reader to the monographs [50, 51] for more details), defined by

$$M^\phi := \left\{\xi := \Omega \longrightarrow \mathbb{R},\ \text{measurable},\ \mathbb{E}^{\mathbb{P}_0}\left[\phi(a\xi)\right] < +\infty,\ \text{for any } a\geq 0\right\},$$

where ϕ is the Young function

$$\phi(x) := \exp(|x|)-1.$$

Then, if M^ϕ is endowed with the norm

$$\|\xi\|_\phi := \sup\left\{\mathbb{E}^{\mathbb{P}_0}[\xi g],\ \text{with } \mathbb{E}^{\mathbb{P}_0}[\phi(g)]\leq 1\right\},$$

it becomes a (non-reflexive) Banach space. For any $a\in\mathcal{A}$ and $(\alpha_P,\alpha_A)\in[\underline{\alpha}^P,\overline{\alpha}^P]\times[\underline{\alpha}^A,\overline{\alpha}^P]$, we consider the map $\Xi_a^{\alpha_P,\alpha_A}: M^\phi\longrightarrow\mathbb{R}$ defined by

$$\Xi_a^{\alpha_P,\alpha_A}(\xi) := \mathbb{E}^{\mathbb{P}_0}\Big[e^{-R_P\left(\int_0^T a_s(X_\cdot^{a,\alpha_P})ds+\alpha_P^{\frac12}B_T-\xi(X_\cdot^{a,P})\right)}+\rho e^{-R_A\left(\xi(X_\cdot^{a,\alpha_A})-\int_0^T k(a_s(X_\cdot^{a,\alpha_A}))ds\right)}\Big],$$

with

$$X_\cdot^{a,\alpha_P}(B_\cdot) := \int_0^\cdot a_s(B_\cdot)ds+(\alpha_P)^{\frac12}B_\cdot,\ X_\cdot^{a,\alpha_A}(B_\cdot) := \int_0^\cdot a_s(B_\cdot)ds+(\alpha_A)^{\frac12}B_\cdot.$$

Notice that if $\xi\equiv(z,\gamma,\delta)\in\mathcal{Q}$, and $a\in\mathcal{A}_{\rm det}$

$$\Xi_a^{\alpha_P,\alpha_A}(\xi) = -F(a,z,\gamma,\delta,\alpha_P,\alpha_A). \tag{3.5}$$

It can be readily checked that $\Xi_a^{\alpha_P,\alpha_A}$ is a strictly convex mapping in ξ, which is in addition proper and continuous. However, since M^ϕ is not reflexive, we cannot affirm that its minimum is attained. Nonetheless, we can still use the characterization of a minimizer in terms of Gâteaux derivative. Indeed, a random variable ξ which minimizes $\Xi_a^{\alpha_P,\alpha_A}$ necessarily satisfies the following property

$$\widetilde{D}\Xi_a^{\alpha_P,\alpha_A}(\xi)[h-\xi]\geq 0, \tag{3.6}$$

for any $h\in M^\phi$, where $\widetilde{D}\Xi_a^{\alpha_P,\alpha_A}$ denotes the Gâteaux derivative of $\Xi_a^{\alpha_P,\alpha_A}$ given by

$$\widetilde{D}\Xi_a^{\alpha_P,\alpha_A}(\xi)[h] = \mathbb{E}^{\mathbb{P}_0}\Big[R_P h(X_\cdot^{a,\alpha_P})e^{-R_P\left(\int_0^T a_s(X_\cdot^{a,\alpha_P})ds+\alpha_P{}^{\frac12}B_T-\xi(X_\cdot^{a,\alpha_P})\right)} \\ - R_A h(X_\cdot^{a,\alpha_A})\rho e^{-R_A\left(\xi(X_\cdot^{a,A})-\int_0^T k(a_s(X_\cdot^{a,\alpha_A}))ds\right)}\Big].$$

The following lemma studies when (3.6) holds for contracts having the form of the optimal contracts in $\mathcal{Q}$. Its proof is postponed to the Appendix.

Lemma 3.4. *Fix some $a \in \mathcal{A}$ and let $\xi_a := z^\star B_T + \frac{\gamma^\star}{2}\langle B\rangle_T + \delta^\star(a)$ where $\gamma^\star \in \mathbb{R}$, and*

$$z^\star := \frac{R_P}{R_A + R_P},\ \delta^\star(a) := \frac{1}{R_A+R_P}\left(\log\left(\rho\frac{R_A}{R_P}\right) + R_A\int_0^T k(a_s)ds\right) + \lambda,\ \lambda \in \mathbb{R}.$$

Then,

- *if $R_A = R_P$, Property* (3.6) *is satisfied for ξ_a if $\alpha_P = \alpha_A$.*
- *if $R_A \neq R_P$, Property* (3.6) *is satisfied for ξ_a if $\alpha_P = \alpha_A =: \alpha$ and the following condition holds*

$$\frac{\gamma^\star}{2}\alpha T + \lambda = 0. \tag{3.7}$$

We can now give our main result, which states that the optimal contract in the first best problem belongs to $\mathcal{Q}$.

Theorem 3.2. *We have*

(i) Assume that $\underline{\alpha}^A = \overline{\alpha}^P$. Then, the set

$$\overline{\mathcal{Q}^{\overline{\gamma}}} := \Big\{\xi^\star \equiv (z^\star, \gamma^\star, \delta^\star) \in \mathcal{Q},\ z^\star = \frac{R_P}{R_A+R_P},\ \gamma^\star \geq R_A|z^\star|^2,$$
$$\delta^\star = Tk(a^\star) - \frac{R_P}{R_A+R_P}Ta^\star + \frac{\overline{\alpha}^P T}{2}\left(\frac{R_A R_P^2}{(R_A+R_P)^2} - \gamma^\star\right) - \frac{1}{R_A}\log(-R)\Big\},$$

is the subset of optimal contracts in $\mathcal{Q}$ for the first best problem (3.1) *with the optimal recommended level effort*

$$a^\star := \mathit{argmax}\,(k(a) - a)\,.$$

(ii) Assume that $\underline{\alpha}^A < \overline{\alpha}^P < \overline{\alpha}^A$. Then, an optimal contract is given by

$$\xi^\star := z^\star B_T + \frac{\gamma^\star}{2}\langle B\rangle_T + \delta^\star,$$

where $\gamma^\star = R_A(z^\star)^2$, and

$$z^\star := \frac{R_P}{R_A+R_P},\ \delta^\star := Tk(a^\star) - \frac{R_P}{R_A+R_P}Ta^\star - \frac{1}{R_A}\log(-R).$$

(iii) Assume that $\overline{\alpha}^A = \overline{\alpha}^P$. Then, the set

$$\overline{\mathcal{Q}^{|\gamma|}} := \Big\{\xi^\star \equiv (z^\star, \gamma^\star, \delta^\star) \in \mathcal{Q},\ z^\star = \frac{R_P}{R_A+R_P},\ \gamma^\star \in [-R_P(1-z^\star)^2, R_A|z^\star|^2],$$
$$\delta^\star = Tk(a^\star) - \frac{R_P}{R_A+R_P}Ta^\star + \frac{\overline{\alpha}^P T}{2}\left(\frac{R_A R_P^2}{(R_A+R_P)^2} - \gamma^\star\right) - \frac{1}{R_A}\log(-R)\Big\},$$

is the subset of optimal contracts in $\mathcal{Q}$ for the first best problem (3.1) *with the optimal recommended level effort*

$$a^\star := \mathit{argmax}\,(k(a) - a)\,.$$

(iv) Assume that $\underline{\alpha}^P = \overline{\alpha}^A$. Then, the set

$$\overline{\mathcal{Q}^{\underline{\gamma}}} := \Big\{\xi^\star \equiv (z^\star, \gamma^\star, \delta^\star) \in \mathcal{Q},\ z^\star = \frac{R_P}{R_A+R_P},\ \gamma^\star \leq -R_P(1-z^\star)^2,$$
$$\delta^\star = Tk(a^\star) - \frac{R_P}{R_A+R_P}Ta^\star + \frac{\underline{\alpha}^P T}{2}\left(\frac{R_A R_P^2}{(R_A+R_P)^2} - \gamma^\star\right) - \frac{1}{R_A}\log(-R)\Big\},$$

is the subset of optimal contracts in $\mathcal{Q}$ for the first best problem (3.1) *with the optimal recommended level effort*

$$a^\star := argmax\,(k(a) - a)\,.$$

(*v*) *Assume that* $\underline{\alpha}^P < \overline{\alpha}^A < \overline{\alpha}^P$. *Then, an optimal contract is given by*

$$\xi^\star := z^\star B_T + \frac{\gamma^\star}{2}\langle B\rangle_T + \delta^\star,$$

where $\gamma^\star = -R_P|1 - z^\star|^2$, *and*

$$z^\star := \frac{R_P}{R_A + R_P},\ \delta^\star := Tk(a^\star) - \frac{R_P}{R_A + R_P}Ta^\star + \frac{\overline{\alpha}^A T}{2}\frac{R_A R_P}{R_A + R_P} - \frac{1}{R_A}\log(-R).$$

Proof. **We begin by proving** (*i*)**.** Assume that $\underline{\alpha}^A = \overline{\alpha}^P$. First notice that

$$\begin{aligned}U_0^{P,FB} &\leq \sup_{\xi\in\mathcal{C}}\sup_{a\in\mathcal{A}}\left\{\mathbb{E}^{\mathbb{P}_a^{\overline{\alpha}^P}}\left[\mathcal{U}_P\left(B_T - \xi\right)\right] + \rho\mathbb{E}^{\mathbb{P}_a^{\underline{\alpha}^A}}\left[\mathcal{U}_A\left(\xi - \int_0^T k(a_s)ds\right)\right]\right\}\\ &= -\inf_{\xi\in\mathcal{C}}\inf_{a\in\mathcal{A}}\Xi_a^{\overline{\alpha}^P,\underline{\alpha}^A}(\xi),\end{aligned}$$

where we have used the fact that by definition, the law of B under $\mathbb{P}_a^\alpha$ is equal to the law of $X^{a,\alpha}$ under $\mathbb{P}_0$. Let us then define for any $a \in \mathcal{A}$

$$\xi_a := z^\star B_T + \frac{\gamma^\star}{2}\langle B\rangle_T + \delta^\star(a),$$

where $\gamma^\star \in [R_A(z^\star)^2, +\infty)$, and

$$z^\star := \frac{R_P}{R_A + R_P},\ \delta^\star(a) := \frac{1}{R_A + R_P}\left(\log\left(\rho\frac{R_A}{R_P}\right) + R_A\int_0^T k(a_s)ds\right) - \frac{\gamma^\star}{2}\overline{\alpha}^P T.$$

Then by Lemma 3.4, we know that

$$\inf_{\xi\in\mathcal{C}}\inf_{a\in\mathcal{A}}\Xi_a^{\overline{\alpha}^P,\underline{\alpha}^A}(\xi) = \inf_{a\in\mathcal{A}}\Xi_a^{\overline{\alpha}^P,\underline{\alpha}^A}(\xi_a).$$

We then have

$$\begin{aligned}\Xi_a^{\overline{\alpha}^P,\underline{\alpha}^A}(\xi_a) =& \rho^{\frac{R_P}{R_A+R_P}}\left(\frac{R_A}{R_P}\right)^{-\frac{R_A}{R_A+R_P}}\frac{R_P + R_A}{R_P}e^{\overline{\alpha}^P T\frac{R_A^2 R_P^2}{2(R_A+R_P)^2}}\\ &\times\mathbb{E}^{\mathbb{P}_0}\left[\mathcal{E}\left(-\frac{R_A R_P}{R_A + R_P}(\overline{\alpha}^P)^{1/2}B_T\right)e^{\frac{R_A R_P}{R_A+R_P}\int_0^T(k(a_s(X_\cdot^{a,\overline{\alpha}^P})) - a_s(X_\cdot^{a,\overline{\alpha}^P}))ds}\right],\end{aligned}$$

so that we clearly have

$$\inf_{a\in\mathcal{A}}\Xi_a^{\overline{\alpha}^P,\underline{\alpha}^A}(\xi_a) = \Xi_{a^\star}^{\overline{\alpha}^P,\underline{\alpha}^A}(\xi_{a^\star}).$$

Thus we have obtained

$$U_0^{P,FB} \leq -\Xi_{a^\star}^{\overline{\alpha}^P,\underline{\alpha}^A}(\xi_{a^\star}).$$

Conversely, we have from Lemma 3.3 (*iv*) and (*v*) *a.*,

$$U_0^{P,FB} \geq \sup_{\xi\in\overline{\mathcal{Q}^{\overline{\gamma}}}}\sup_{a\in\mathcal{A}_{det}}\widetilde{u}_0^{P,FB}(a,\xi) = G(a^\star, z^\star, \overline{\alpha}^P) = -\Xi_{a^\star}^{\overline{\alpha}^P,\underline{\alpha}^A}(\xi_{a^\star}).$$

Therefore

$$U_0^{P,FB} = -\Xi_{a^\star}^{\overline{\alpha}^P,\underline{\alpha}^A}(\xi_{a^\star}).$$

Finally, it remains to choose ρ so as to satisfy the participation constraint of the agent. Some calculations show that it suffices to take ρ such that

$$\frac{1}{R_A+R_P}\log\left(\rho\frac{R_A}{R_P}\right) = -\frac{1}{R_A}\log(-R) + \frac{R_P}{R_A+R_P}T\left[k(a^\star) - a^\star + \overline{\alpha}^P T\frac{R_AR_P}{2(R_A+R_P)}\right].$$

Thus,

$$\delta(a^\star) = Tk(a^\star) - \frac{R_P}{R_A+R_P}Ta^\star + \frac{\overline{\alpha}^P T}{2}\left(\frac{R_AR_P^2}{(R_A+R_P)^2} - \gamma^\star\right) - \frac{1}{R_A}\log(-R).$$

We now turn to (ii)**.** Since we have $\underline{\alpha}^A < \overline{\alpha}^P < \overline{\alpha}^A$, we deduce that

$$\begin{aligned}U_0^{P,FB} &\leq \sup_{\xi\in\mathcal{C}}\sup_{a\in\mathcal{A}}\left\{\mathbb{E}^{\mathbb{P}_a^{\overline{\alpha}^P}}\left[\mathcal{U}_P\left(B_T-\xi\right)\right] + \rho\mathbb{E}^{\mathbb{P}_a^{\overline{\alpha}^P}}\left[\mathcal{U}_A\left(\xi - \int_0^T k(a_s)ds\right)\right]\right\}\\ &= -\inf_{\xi\in\mathcal{C}}\inf_{a\in\mathcal{A}}\Xi_a^{\overline{\alpha}^P,\overline{\alpha}^P}(\xi).\end{aligned}$$

Let us then define for any $a\in\mathcal{A}$

$$\xi_a := z^\star B_T + \frac{\gamma^\star}{2}\langle B\rangle_T + \delta^\star(a),$$

where $\gamma^\star = R_A(z^\star)^2$, and

$$z^\star := \frac{R_P}{R_A+R_P},\ \delta^\star(a) := \frac{1}{R_A+R_P}\left(\log\left(\rho\frac{R_A}{R_P}\right) + R_A\int_0^T k(a_s)ds\right) - \frac{\gamma^\star}{2}\overline{\alpha}^P T.$$

Then by Lemma 3.4, we know that

$$\inf_{\xi\in\mathcal{C}}\inf_{a\in\mathcal{A}}\Xi_a^{\overline{\alpha}^P,\overline{\alpha}^P}(\xi) = \inf_{a\in\mathcal{A}}\Xi_a^{\overline{\alpha}^P,\overline{\alpha}^P}(\xi_a).$$

We then have

$$\begin{aligned}\Xi_a^{\overline{\alpha}^P,\overline{\alpha}^P}(\xi_a) &= \left(\rho\frac{R_A}{R_P}\right)^{\frac{R_P}{R_A+R_P}}\left(1+\frac{R_P}{R_A}\right)e^{\overline{\alpha}^P T\frac{R_A^2R_P^2}{2(R_A+R_P)^2}}\\ &\quad\times\mathbb{E}^{\mathbb{P}_0}\left[\mathcal{E}\left(-\frac{R_AR_P}{R_A+R_P}(\overline{\alpha}^P)^{1/2}B_T\right)e^{\frac{R_AR_P}{R_A+R_P}\int_0^T(k(a_s(X_\cdot^{a,\overline{\alpha}^P}))-a_s(X_\cdot^{a,\overline{\alpha}^P}))ds}\right],\end{aligned}$$

so that we clearly have

$$\inf_{a\in\mathcal{A}}\Xi_a^{\overline{\alpha}^P,\overline{\alpha}^P}(\xi_a) = \Xi_{a^\star}^{\overline{\alpha}^P,\overline{\alpha}^P}(\xi_{a^\star}).$$

Thus we have obtained

$$U_0^{P,FB} \leq -\Xi_{a^\star}^{\overline{\alpha}^P,\overline{\alpha}^P}(\xi_{a^\star}).$$

Conversely, using Lemma 3.3(iv) we have

$$U_0^{P,FB} \geq \sup_{a\in\mathcal{A}_{\rm det}}\sup_{\xi\in\mathcal{Q}^u}\widetilde{u}_0^{P,FB}(a,\xi) = G(a^\star,z^\star,\overline{\alpha}^P) = -\Xi_{a^\star}^{\overline{\alpha}^P,\overline{\alpha}^P}(\xi_{a^\star}).$$

Therefore

$$U_0^{P,FB} = -\Xi_{a^\star}^{\overline{\alpha}^P,\overline{\alpha}^P}(\xi_{a^\star}).$$

Finally, it remains to choose ρ so as to satisfy the participation constraint of the agent. Some calculations show that it suffices to take ρ such that

$$\frac{1}{R_A+R_P}\log\left(\rho\frac{R_A}{R_P}\right)=-\frac{1}{R_A}\log(-R)+\frac{R_P}{R_A+R_P}T\left[k(a^\star)-a^\star+\overline{\alpha}^PT\frac{R_AR_P}{2(R_A+R_P)}\right].$$

Thus,

$$\begin{aligned}\delta(a^\star)&=Tk(a^\star)-\frac{R_P}{R_A+R_P}Ta^\star+\frac{\overline{\alpha}^PT}{2}\left(\frac{R_AR_P^2}{(R_A+R_P)^2}-\gamma^\star\right)-\frac{1}{R_A}\log(-R)\\&=Tk(a^\star)-\frac{R_P}{R_A+R_P}Ta^\star-\frac{1}{R_A}\log(-R).\end{aligned}$$

We now turn to (v). Since we have $\underline{\alpha}^P<\overline{\alpha}^A<\overline{\alpha}^P$, we deduce that

$$\begin{aligned}U_0^{P,FB}&\leq\sup_{\xi\in\mathcal{C}}\sup_{a\in\mathcal{A}}\left\{\mathbb{E}^{\mathbb{P}_a^{\overline{\alpha}^A}}\left[\mathcal{U}_P\left(B_T-\xi\right)\right]+\rho\mathbb{E}^{\mathbb{P}_a^{\overline{\alpha}^A}}\left[\mathcal{U}_A\left(\xi-\int_0^Tk(a_s)ds\right)\right]\right\}\\&=-\inf_{\xi\in\mathcal{C}}\inf_{a\in\mathcal{A}}\Xi_a^{\overline{\alpha}^A,\overline{\alpha}^A}(\xi).\end{aligned}$$

Let us then define for any $a\in\mathcal{A}$

$$\xi_a:=z^\star B_T+\frac{\gamma^\star}{2}\langle B\rangle_T+\delta^\star(a),$$

where $\gamma^\star=-R_P|1-z^\star|^2$, and

$$z^\star:=\frac{R_P}{R_A+R_P},\ \delta^\star(a):=\frac{1}{R_A+R_P}\left(\log\left(\rho\frac{R_A}{R_P}\right)+R_A\int_0^Tk(a_s)ds\right)-\frac{\gamma^\star}{2}\overline{\alpha}^AT.$$

Then by Lemma 3.4, we know that

$$\inf_{\xi\in\mathcal{C}}\inf_{a\in\mathcal{A}}\Xi_a^{\overline{\alpha}^A,\overline{\alpha}^A}(\xi)=\inf_{a\in\mathcal{A}}\Xi_a^{\overline{\alpha}^A,\overline{\alpha}^A}(\xi_a).$$

We then have

$$\begin{aligned}\Xi_a^{\overline{\alpha}^A,\overline{\alpha}^A}(\xi_a)&=\left(\rho\frac{R_A}{R_P}\right)^{\frac{R_P}{R_A+R_P}}\left(1+\frac{R_P}{R_A}\right)e^{\overline{\alpha}^AT\frac{R_A^2R_P^2}{2(R_A+R_P)^2}}\\&\quad\times\mathbb{E}^{\mathbb{P}_0}\left[\mathcal{E}\left(-\frac{R_AR_P}{R_A+R_P}(\overline{\alpha}^A)^{1/2}B_T\right)e^{\frac{R_AR_P}{R_A+R_P}\int_0^T(k(a_s(X_\cdot^{a,\overline{\alpha}^A}))-a_s(X_\cdot^{a,\overline{\alpha}^A}))ds}\right],\end{aligned}$$

so that we clearly have

$$\inf_{a\in\mathcal{A}}\Xi_a^{\overline{\alpha}^A,\overline{\alpha}^A}(\xi_a)=\Xi_{a^\star}^{\overline{\alpha}^A,\overline{\alpha}^A}(\xi_{a^\star}).$$

Thus we have obtained

$$U_0^{P,FB}\leq-\Xi_{a^\star}^{\overline{\alpha}^A,\overline{\alpha}^A}(\xi_{a^\star}).$$

Conversely, using Lemma 3.3 (ii) we have

$$U_0^{P,FB}\geq\sup_{a\in\mathcal{A}_{\det}}\sup_{\xi\in\mathcal{Q}^d}\widetilde{u}_0^{P,FB}(a,\xi)=G(a^\star,z^\star,\overline{\alpha}^A)=-\Xi_{a^\star}^{\overline{\alpha}^A,\overline{\alpha}^A}(\xi_{a^\star}).$$

Therefore

$$U_0^{P,FB}=-\Xi_{a^\star}^{\overline{\alpha}^A,\overline{\alpha}^A}(\xi_{a^\star}).$$

Finally, it remains to choose ρ so as to satisfy the participation constraint of the agent. Some calculations show that it suffices to take ρ such that

$$\frac{1}{R_A+R_P}\log\left(\rho\frac{R_A}{R_P}\right) = -\frac{1}{R_A}\log(-R) + \frac{R_P}{R_A+R_P}T\left[k(a^\star) - a^\star + \overline{\alpha}^A T\frac{R_A R_P}{2(R_A+R_P)}\right].$$

Thus,

$$\begin{aligned}\delta(a^\star) &= Tk(a^\star) - \frac{R_P}{R_A+R_P}Ta^\star + \frac{\overline{\alpha}^A T}{2}\left(\frac{R_A R_P^2}{(R_A+R_P)^2} - \gamma^\star\right) - \frac{1}{R_A}\log(-R)\\ &= Tk(a^\star) - \frac{R_P}{R_A+R_P}Ta^\star + \frac{\overline{\alpha}^A T}{2}\frac{R_A R_P}{R_A+R_P} - \frac{1}{R_A}\log(-R).\end{aligned}$$

We now prove (iii)**.** Assume that $\overline{\alpha}^A = \overline{\alpha}^P$, and notice that

$$\begin{aligned}U_0^{P,FB} &\leq \sup_{\xi\in\mathcal{C}}\sup_{a\in\mathcal{A}}\left\{\mathbb{E}^{\mathbb{P}_a^{\overline{\alpha}^P}}\left[\mathcal{U}_P\left(B_T-\xi\right)\right] + \rho\mathbb{E}^{\mathbb{P}_a^{\overline{\alpha}^A}}\left[\mathcal{U}_A\left(\xi - \int_0^T k(a_s)ds\right)\right]\right\}\\ &= -\inf_{\xi\in\mathcal{C}}\inf_{a\in\mathcal{A}}\Xi_a^{\overline{\alpha}^P,\overline{\alpha}^A}(\xi),\end{aligned}$$

where we have used the fact that by definition, the law of B under $\mathbb{P}_a^\alpha$ is equal to the law of $X^{a,\alpha}$ under $\mathbb{P}_0$. Let us then define for any $a\in\mathcal{A}$

$$\xi_a := z^\star B_T + \frac{\gamma^\star}{2}\langle B\rangle_T + \delta^\star(a),$$

where $\gamma^\star\in[-R_P|1-z^\star|^2, R_A|z^\star|^2]$, and

$$z^\star := \frac{R_P}{R_A+R_P},\ \delta^\star(a) := \frac{1}{R_A+R_P}\left(\log\left(\rho\frac{R_A}{R_P}\right) + R_A\int_0^T k(a_s)ds\right) - \frac{\gamma^\star}{2}\overline{\alpha}^P T.$$

Then by Lemma 3.4, we know that

$$\inf_{\xi\in\mathcal{C}}\inf_{a\in\mathcal{A}}\Xi_a^{\overline{\alpha}^P,\overline{\alpha}^A}(\xi) = \inf_{a\in\mathcal{A}}\Xi_a^{\overline{\alpha}^P,\overline{\alpha}^A}(\xi_a).$$

We then have

$$\begin{aligned}\Xi_a^{\overline{\alpha}^P,\overline{\alpha}^A}(\xi_a) &= \left(\rho\frac{R_A}{R_P}\right)^{\frac{R_P}{R_A+R_P}}\left(1+\frac{R_P}{R_A}\right)e^{\overline{\alpha}^P T\frac{R_A^2R_P^2}{2(R_A+R_P)^2}}\\ &\quad\times\mathbb{E}^{\mathbb{P}_0}\left[\mathcal{E}\left(-\frac{R_AR_P}{R_A+R_P}(\overline{\alpha}^P)^{1/2}B_T\right)e^{\frac{R_AR_P}{R_A+R_P}\int_0^T(k(a_s(X_\cdot^{a,\overline{\alpha}^P}))-a_s(X_\cdot^{a,\overline{\alpha}^P}))ds}\right],\end{aligned}$$

so that we clearly have

$$\inf_{a\in\mathcal{A}}\Xi_a^{\overline{\alpha}^P,\overline{\alpha}^A}(\xi_a) = \Xi_{a^\star}^{\overline{\alpha}^P,\overline{\alpha}^A}(\xi_{a^\star}).$$

Thus we have obtained

$$U_0^{P,FB} \leq -\Xi_{a^\star}^{\overline{\alpha}^P,\overline{\alpha}^A}(\xi_{a^\star}).$$

Conversely, we have from Lemma 3.3 (ii), $(iii)b)$, (iv),

$$U_0^{P,FB} \geq \sup_{\xi\in\mathcal{Q}^{|\gamma|}}\sup_{a\in\mathcal{A}_{det}}\widetilde{u}_0^{P,FB}(a,\xi) = G(a^\star,z^\star,\overline{\alpha}^P) = -\Xi_{a^\star}^{\overline{\alpha}^P,\overline{\alpha}^A}(\xi_{a^\star}).$$

Therefore
$$U_0^{P,FB} = -\Xi_{a^\star}^{\overline{\alpha}^P,\overline{\alpha}^A}(\xi_{a^\star}).$$
Finally, it remains to choose ρ so as to satisfy the participation constraint of the agent. Some calculations show that it suffices to take ρ such that
$$\frac{1}{R_A+R_P}\log\left(\rho\frac{R_A}{R_P}\right) = -\frac{1}{R_A}\log(-R) + \frac{R_P}{R_A+R_P}T\left[k(a^\star) - a^\star + \overline{\alpha}^P T\frac{R_AR_P}{2(R_A+R_P)}\right].$$
Thus,
$$\delta(a^\star) = Tk(a^\star) - \frac{R_P}{R_A+R_P}Ta^\star + \frac{\overline{\alpha}^P T}{2}\left(\frac{R_AR_P^2}{(R_A+R_P)^2} - \gamma^\star\right) - \frac{1}{R_A}\log(-R).$$
We finally prove (iv)**.** Assume that $\underline{\alpha}^P = \overline{\alpha}^A$, and notice that
$$\begin{aligned}U_0^{P,FB} &\leq \sup_{\xi\in\mathcal{C}}\sup_{a\in\mathcal{A}}\left\{\mathbb{E}^{\mathbb{P}_a^{\underline{\alpha}^P}}\left[\mathcal{U}_P\left(B_T-\xi\right)\right] + \rho\mathbb{E}^{\mathbb{P}_a^{\overline{\alpha}^A}}\left[\mathcal{U}_A\left(\xi - \int_0^T k(a_s)ds\right)\right]\right\}\\ &= -\inf_{\xi\in\mathcal{C}}\inf_{a\in\mathcal{A}}\Xi_a^{\underline{\alpha}^P,\overline{\alpha}^A}(\xi),\end{aligned}$$
where we have used the fact that by definition, the law of B under $\mathbb{P}_a^\alpha$ is equal to the law of $X^{a,\alpha}$ under $\mathbb{P}_0$. Let us then define for any $a\in\mathcal{A}$
$$\xi_a := z^\star B_T + \frac{\gamma^\star}{2}\langle B\rangle_T + \delta^\star(a),$$
where $\gamma^\star \in (-\infty, -R_P|1-z^\star|^2]$, and
$$z^\star := \frac{R_P}{R_A+R_P},\ \delta^\star(a) := \frac{1}{R_A+R_P}\left(\log\left(\rho\frac{R_A}{R_P}\right) + R_A\int_0^T k(a_s)ds\right) - \frac{\gamma^\star}{2}\underline{\alpha}^P T.$$
Then by Lemma 3.4, we know that
$$\inf_{\xi\in\mathcal{C}}\inf_{a\in\mathcal{A}}\Xi_a^{\underline{\alpha}^P,\overline{\alpha}^A}(\xi) = \inf_{a\in\mathcal{A}}\Xi_a^{\underline{\alpha}^P,\overline{\alpha}^A}(\xi_a).$$
We then have
$$\begin{aligned}\Xi_a^{\underline{\alpha}^P,\overline{\alpha}^A}(\xi_a) = &\left(\rho\frac{R_A}{R_P}\right)^{\frac{R_P}{R_A+R_P}}\left(1+\frac{R_P}{R_A}\right)e^{\underline{\alpha}^P T\frac{R_A^2R_P^2}{2(R_A+R_P)^2}}\\ &\times\mathbb{E}^{\mathbb{P}_0}\left[\mathcal{E}\left(-\frac{R_AR_P}{R_A+R_P}(\underline{\alpha}^P)^{1/2}B_T\right)e^{\frac{R_AR_P}{R_A+R_P}\int_0^T(k(a_s(X_\cdot^{a,\underline{\alpha}^P}))-a_s(X_\cdot^{a,\underline{\alpha}^P}))ds}\right],\end{aligned}$$
so that we clearly have
$$\inf_{a\in\mathcal{A}}\Xi_a^{\underline{\alpha}^P,\overline{\alpha}^A}(\xi_a) = \Xi_{a^\star}^{\underline{\alpha}^P,\overline{\alpha}^A}(\xi_{a^\star}).$$
Thus we have obtained
$$U_0^{P,FB} \leq -\Xi_{a^\star}^{\underline{\alpha}^P,\overline{\alpha}^A}(\xi_{a^\star}).$$
Conversely, we have from Lemma 3.3 (i) $b.$, (ii) and (iii) $c.$,
$$U_0^{P,FB} \geq \sup_{\xi\in\overline{\mathcal{Q}^\gamma}}\sup_{a\in\mathcal{A}_{det}}\widetilde{u}_0^{P,FB}(a,\xi) = G(a^\star,z^\star,\underline{\alpha}^P) = -\Xi_{a^\star}^{\underline{\alpha}^P,\overline{\alpha}^A}(\xi_{a^\star}).$$
Therefore
$$U_0^{P,FB} = -\Xi_{a^\star}^{\underline{\alpha}^P,\overline{\alpha}^A}(\xi_{a^\star}).$$

Finally, it remains to choose ρ so as to satisfy the participation constraint of the agent. Some calculations show that it suffices to take ρ such that

$$\frac{1}{R_A+R_P}\log\left(\rho\frac{R_A}{R_P}\right) = -\frac{1}{R_A}\log(-R) + \frac{R_P}{R_A+R_P}T\left[k(a^\star) - a^\star + \underline{\alpha}^P T\frac{R_AR_P}{2(R_A+R_P)}\right].$$

Thus,

$$\delta(a^\star) = Tk(a^\star) - \frac{R_P}{R_A+R_P}Ta^\star + \frac{\underline{\alpha}^P T}{2}\left(\frac{R_AR_P^2}{(R_A+R_P)^2} - \gamma^\star\right) - \frac{1}{R_A}\log(-R).$$

□

3.2 Comments and comparison with the case without ambiguity

All the results obtained above are summarized in Section 6. Using Theorem 3.2, we recover the classical result that when $\underline{\alpha}^P = \overline{\alpha}^P = \underline{\alpha}_A = \overline{\alpha}^A =: \alpha$ (that is to say when there is no ambiguity), the optimal first-best contract is given by

$$z^\star B_T + \frac{R_AR_P^2\alpha}{2(R_A+R_P)^2}T + Tk(a^\star) - \frac{R_P}{R_A+R_P}Ta^\star - \frac{1}{R_A}\log(-R), \tag{3.8}$$

which provides the principal with utility

$$-(-R)^{-\frac{R_P}{R_A}}\exp\left(R_PT\left(k(a^*) - a^* + \frac{\alpha}{2}\frac{R_AR_P}{R_A+R_P}\right)\right). \tag{3.9}$$

Therefore, as mentioned above, the first main difference with the ambiguity case is that in our framework, one has in general to rely on path-dependent contracts using the quadratic variation of the output. There is nonetheless an exception. Indeed, in the case where $\overline{\alpha}^A = \overline{\alpha}^P$, the choice $\gamma^\star = 0$ is allowed, so that there is a linear optimal contract in this case (which coincides with (3.8) above), and in this case only. Furthermore, in the three cases $\overline{\alpha}^A = \overline{\alpha}^P$, $\overline{\alpha}^A = \underline{\alpha}^P$, $\overline{\alpha}^P = \underline{\alpha}^A$, we have identified uncountably many optimal contracts in the class $\mathcal{Q}$. This is really different from the case without ambiguity, where the optimal contract is basically unique.

Finally, let us compare the utility the the principal can get out of the problem (since the agent always receives his reservation utility, there is nothing to compare for him). Again by Theorem 3.2, whenever we have $\underline{\alpha}^A \leq \overline{\alpha}^P \leq \overline{\alpha}^A$, the principal receives

$$-(-R)^{-\frac{R_P}{R_A}}\exp\left(R_PT\left(k(a^*) - a^* + \frac{\overline{\alpha}^P}{2}\frac{R_AR_P}{R_A+R_P}\right)\right),$$

which is always less than (3.9), for any $\alpha \in [\underline{\alpha}^P, \overline{\alpha}^P]$, which means hat, as intuition would dictate, the principal is worse-off compared to the case where he would not have any aversion to ambiguity.

Then, when we have $\underline{\alpha}^P \leq \overline{\alpha}^A \leq \overline{\alpha}^P$, the principal gets

$$-(-R)^{-\frac{R_P}{R_A}}\exp\left(R_PT\left(k(a^*) - a^* + \frac{\overline{\alpha}^A}{2}\frac{R_AR_P}{R_A+R_P}\right)\right),$$

which is actually larger than (3.9) if $\alpha \geq \overline{\alpha}^A$. In other words, compared to a situation where the principal would have no ambiguity, but were more pessimistic than the agent and believed in a level of volatility higher than $\overline{\alpha}^A$, the ambiguity averse principal actually obtains a larger utility.

The situation is the same, though even more extreme, when $\overline{\alpha}^P < \underline{\alpha}^A$ or $\overline{\alpha}^A < \underline{\alpha}^P$, since the principal can reach utility 0 and is therefore always better off compared to the case without ambiguity. We believe that such results are the most striking consequences of our new modelization of the contracting problem.

4 Moral hazard and second-best problem

We now study the so-called second best problem, corresponding to a Stackelberg-like equilibrium between the principal and the agent. Now, the principal has no control (or cannot observe) the effort level chosen by the agent. Hence, his strategy is to first compute the best-reaction function of the agent to a given contract, and to determine his corresponding optimal effort (if it exists) and then use this in his own utility function to maximize over all the contracts. Obviously, the above approach can only work if the principal can actually find the optimal effort of the agent. Therefore, the set of admissible contracts in the second best setting must at least be reduced to the contracts ξ such that there exists (possibly several) $a^\star \in \mathcal{A}$ with

$$R \leq U_0^A(\xi) = \inf_{\mathbb{P} \in \mathcal{P}_A^{a^\star}} \mathbb{E}^{\mathbb{P}}\left[\mathcal{U}_A\left(\xi - \int_0^T k(a_s^\star)ds\right)\right].$$

As we will see below, this set of contracts is actually equal to $\mathcal{C}$, so that the above restriction is without generality.

4.1 The agent's problem

Let us start by looking at the dynamic version of the value function of the agent. Fix some $a \in \mathcal{A}$. The fact that there is an infimum with respect to a family of non-dominated measures makes things a bit technical, however, we refer to the papers [40, 38] for the proofs that, for any $\mathcal{F}_T-$measurable contract $\xi \in \mathcal{C}$, one can define a process, which we denote by $u_t^A(\xi, a)$ (denoted by Y_t in [38]), which is càdlàg, $\mathbb{G}^{\mathcal{P}_A,+}-$adapted (recall that for any $a \in \mathcal{A}$, $\mathbb{G}^{\mathcal{P}_A^a} = \mathbb{G}^{\mathcal{P}_A}$, since the polar sets of $\mathcal{P}_A^a$ are the same as the polar sets of $\mathcal{P}_A$) and such that

$$u_t^A(\xi, a) = \underset{\mathbb{P}' \in \mathcal{P}_A^a(\mathbb{P}, t^+)}{\operatorname{essinf}^{\mathbb{P}}} \mathbb{E}^{\mathbb{P}'}\left[\mathcal{U}_A\left(\xi - \int_t^T k(a_s)ds\right)\middle| \mathcal{F}_t\right],\ \mathbb{P}-a.s.,\ \text{for all } \mathbb{P} \in \mathcal{P}_A^a. \tag{4.1}$$

Notice that since $\xi \in \mathcal{C}$, it has exponential moments of any order, so that since in addition the effort process a is bounded, we have that $u^A(\xi, a)$ has moments of any order, in the sense that

$$\sup_{\mathbb{P} \in \mathcal{P}_{\mathcal{A}}^a} \mathbb{E}^{\mathbb{P}}\left[\sup_{0 \leq t \leq T} \left|u_t^A(\xi, a)\right|^p\right] < +\infty,\ \text{for all } p \geq 0, \tag{4.2}$$

where we have used the generalized Doob inequality for sublinear expectations given in Proposition A.1 in [45].

Moreover, by [38] (see in particular step 2 in the proof of Theorem 2.3), $e^{R_A \int_0^t k(a_s)ds} u_t^A(\xi, a)$ is a $(\mathbb{P}, \mathbb{G}^{\mathcal{P}_A,+})-$submartingale for every $\mathbb{P} \in \mathcal{P}_A^a$, and by step 3 in the proof of Theorem 2.3 in [38], there is a $\mathbb{G}^{\mathcal{P}_A}-$predictable process $\widetilde{Z}$, and a family of non-decreasing processes $(\widetilde{K}^{\mathbb{P}})_{\mathbb{P} \in \mathcal{P}_A^a}$, which are $\mathbb{F}^{\mathbb{P}}-$predictable, such that, for all $\mathbb{P} \in \mathcal{P}_A^a$

$$e^{R_A \int_0^t k(a_s)ds} u_t^A(\xi, a) = e^{R_A \int_0^T k(a_s)ds} \mathcal{U}_A(\xi) - \int_t^T \widetilde{Z}_s \widehat{\alpha}_s^{\frac{1}{2}} dW_s^a - \widetilde{K}_T^{\mathbb{P}} + \widetilde{K}_t^{\mathbb{P}},\ \mathbb{P}-a.s.$$

Notice also that since every probability measure in $\mathcal{P}_A$ is equivalent, by definition, to a probability measure in $\mathcal{P}_A^a$ (and conversely), the above also holds $\mathbb{P}-a.s.$, for any $\mathbb{P} \in \mathcal{P}_A$, with the convention that we will still denote by $\widetilde{K}^{\mathbb{P}}$ the non-decreasing process associated to $\mathbb{P} \in \mathcal{P}_A^a$ or $\mathcal{P}_A$. Moreover, using again the aggregation result of [39], we can actually aggregate the family $\widetilde{K}^{\mathbb{P}}$ into a universal process, which is $\mathbb{G}^{\mathcal{P}_A}-$predictable, and which we denote by $\widetilde{K}$.

We deduce that

$$u_t^A(\xi,a)=\mathcal{U}_A(\xi)-\int_t^T R_A u_s^A(\xi,a)\left(a_s Z_s^a-k(a_s)\right)ds+\int_t^T R_A u_s^A(\xi,a) Z_s^a\widehat{\alpha}_s^{1/2}dW_s$$
$$+\int_t^T R_A u_s^A(\xi,a)dK_s^a,\ \mathbb{P}-a.s.,\ \text{for all } \mathbb{P}\in\mathcal{P}_A,$$

where

$$Z_t^a:=-\frac{e^{-R_A\int_0^t k(a_s)ds}}{R_A u_t(\xi,a)}\widetilde{Z}_t,\quad K_t^a:=-\int_0^t \frac{e^{-R_A\int_0^s k(a_r)dr}}{R_A u_t^A(\xi,a)}d\widetilde{K}_r.$$

Define

$$Y_t^a:=-\frac{\ln\left(-u_t^A(\xi,a)\right)}{R_A}.$$

We therefore have for all $\mathbb{P}\in\mathcal{P}_A$

$$Y_t^a=\xi-\int_t^T\left(\frac{R_A}{2}\left|Z_s^a\right|^2\widehat{\alpha}_s+k(a_s)-a_sZ_s^a\right)ds-\int_t^T Z_s^a\widehat{\alpha}_s^{1/2}dW_s-\int_t^T dK_s^a,\ \mathbb{P}-a.s.$$

Now notice that by (4.2), we immediately have

$$\sup_{\mathbb{P}\in\mathcal{P}_A}\mathbb{E}^{\mathbb{P}}\left[\exp\left(p\sup_{0\leq t\leq T}|Y_t^a|\right)\right]<+\infty,\ \text{for every } p\geq 0.$$

Moreover, remember that by (4.1), we also have for every $\mathbb{P}\in\mathcal{P}_A^a$, by the exact same arguments as above applied under any fixed measure $\mathbb{P}\in\mathcal{P}_A$, that

$$Y_t^a=\underset{\mathbb{P}'\in\mathcal{P}_A^a(\mathbb{P},t^+)}{\operatorname{essinf}^{\mathbb{P}}}\ \mathcal{Y}_t^{\mathbb{P}',a},\ \mathbb{P}-a.s.,\tag{4.3}$$

where for any $\mathbb{P}\in\mathcal{P}_A^a$, $(\mathcal{Y}^{\mathbb{P},a},\mathcal{Z}^{\mathbb{P},a})$ is the unique[5] solution to the following BSDE defined under $\mathbb{P}$

$$\mathcal{Y}_t^{\mathbb{P},a}=\xi-\int_t^T\left(\frac{R_A}{2}\left|\mathcal{Z}_s^{\mathbb{P},a}\right|^2\widehat{\alpha}_s+k(a_s)-a_s\mathcal{Z}_s^{\mathbb{P},a}\right)ds-\int_t^T\mathcal{Z}_s^{\mathbb{P},a}\widehat{\alpha}_s^{1/2}dW_s,\ \mathbb{P}-a.s.$$

Then, using (4.2), we can follow the proof of Lemma 3.1 in [48][6] to obtain that Z^a actually belongs to the BMO space defined in [48] (see Section 2.3.2). Then, we can follow exactly the proof of Theorem 6.1 in [48] to obtain with (4.3), that for any $\mathbb{P}\in\mathcal{P}_A^a$

$$K_t^a=\underset{\mathbb{P}'\in\mathcal{P}_A^a(\mathbb{P},t^+)}{\operatorname{essinf}^{\mathbb{P}}}\ \mathbb{E}^{\mathbb{P}'}\left[K_T^a\middle|\mathcal{F}_t\right],\ \mathbb{P}-a.s.$$

Therefore, (Y_t^a,Z_t^a) is the unique solution to the (quadratic-linear) 2BSDE with terminal condition ξ and generator $R_A/2z^2\widehat{\alpha}_s+k(a_s)-a_sz$ (see for instance Definition 2.3 of [48]).

[5]Wellposedness is clear here, since we have easily that

$$\mathcal{Y}_t^{\mathbb{P},a}=-\frac{\log\left(-u_t^{A,\mathbb{P}}(\xi,a)\right)}{R_A},\ \mathbb{P}-a.s.,$$

where

$$u_t^{A,\mathbb{P}}(\xi,a):=\mathbb{E}^{\mathbb{P}}\left[\mathcal{U}_A\left(\xi-\int_t^T k(a_s)ds\right)\middle|\mathcal{F}_t\right],\ \mathbb{P}-a.s.$$

[6]In this result, ξ and Y^a are assumed to be bounded, but the proof generalizes easily to our setting where Y^a satisfies (4.2).

Let us now denote by (Y, Z, K) the maximal solution to the following 2BSDE

$$Y_t = \xi - \int_t^T \left(\frac{R_A}{2} |Z_s|^2 \widehat{\alpha}_s + \inf_{a \in [0, a_{\max}]} \{k(a) - aZ_s\} \right) ds - \int_t^T Z_s \widehat{\alpha}_s^{1/2} dW_s - \int_t^T dK_s. \tag{4.4}$$

Before proceeding, let us explain why such a 2BSDE does indeed admit a maximal solution. First of all, the corresponding quadratic BSDEs admit a maximal solution, because, since the infimum in the generator is over a compact set, the generator of the BSDE is bounded from above by a function with linear growth in z. The existence of a maximal solution is then direct from Proposition 4 of [5]. Furthermore, since this maximal solution is obtained as a monotone approximation of Lipschitz BSDEs, it satisfies a comparison theorem. Hence, we can apply first Proposition 2.1 of [47] to obtain the existence of a maximal solution of the 2BSDE, in the sense of Definition 4.1 of [47], and then use Remark 4.1 of [47] to aggregate the family of non-decreasing processes into K (we remind the reader that all the measures in $\mathcal{P}_A$ satisfy the predictable martingale representation property).

In particular, we have the following representation for any $\mathbb{P} \in \mathcal{P}_A^a$,

$$Y_t = \operatorname*{essinf}^{\mathbb{P}}_{\mathbb{P}' \in \mathcal{P}_A^a(\mathbb{P}, t^+)} \mathcal{Y}_t^{\mathbb{P}'}, \ \mathbb{P} - a.s., \tag{4.5}$$

where for any $\mathbb{P} \in \mathcal{P}_A^a$, $(\mathcal{Y}^{\mathbb{P}}, \mathcal{Z}^{\mathbb{P}})$ is the maximal solution of the quadratic BSDE

$$\mathcal{Y}_t^{\mathbb{P}} = \xi - \int_t^T \left(\frac{R_A}{2} \left| \mathcal{Z}_s^{\mathbb{P}} \right|^2 \widehat{\alpha}_s + \inf_{a \in [0, a_{\max}]} \left\{ k(a) - a\mathcal{Z}_s^{\mathbb{P}} \right\} \right) ds - \int_t^T \mathcal{Z}_s^{\mathbb{P}} \widehat{\alpha}_s^{1/2} dW_s, \ \mathbb{P} - a.s.$$

Now it is a classical result dating back to [22] (see also [34] for a similar result using 2BSDEs) that, using the comparison theorem satisfied by the maximal solution of the 2BSDEs (which automatically inherited from the one satisfied by the BSDEs), that

$$Y_0 = \sup_{a \in \mathcal{A}} Y_0^a = \sup_{a \in \mathcal{A}} \inf_{\mathbb{P} \in \mathcal{P}_A^a} \mathcal{Y}_0^{\mathbb{P}, a} = \sup_{a \in \mathcal{A}} \inf_{\mathbb{P} \in \mathcal{P}_A^a} \left\{ -\frac{1}{R_A} \log \left(-\mathbb{E}^{\mathbb{P}} \left[\mathcal{U}_A \left(\xi - \int_0^T k(a_s) ds \right) \right] \right) \right\},$$

so that

$$U_0^A(\xi) = -\exp(-R_A Y_0).$$

Furthermore, it is then clear, since the function k is strictly convex that there is some $a^\star(Z_\cdot) \in \mathcal{A}$ such that

$$\inf_{a \in [0, a_{\max}]} \{k(a) - aZ_s\} = k(a^\star(Z_s)) - a^\star(Z_s) Z_s, \ s \in [0, T].$$

This implies that

$$Y_0 = \inf_{\mathbb{P} \in \mathcal{P}_A^{a^\star(Z_\cdot)}} \mathcal{Y}_0^{\mathbb{P}, a^\star(Z_\cdot)}.$$

We have thus proved the following result.

Proposition 4.1. *For any $\xi \in \mathcal{C}$, the value function of the agent verifies*

$$U_0^A(\xi) = -\exp(-R_A Y_0),$$

and the optimal effort of the agent is given by $a^\star(Z_s)$ which satisfies

$$\inf_{a \in [0, a_{\max}]} \{k(a) - aZ_s\} = k(a^\star(Z_s)) - a^\star(Z_s) Z_s, \ s \in [0, T],$$

where (Y, Z) is the maximal solution to (4.4). Furthermore, $\xi \in \mathcal{C}$ if and only if

$$Y_0 \geq -\frac{\log(-R)}{R_A} =: R_0. \tag{4.6}$$

We have thus solved the problem of the agent for any $\xi \in \mathcal{C}$. Along the way, we showed that any $\xi \in \mathcal{C}$ had the following decomposition

$$\xi = Y_0 + \int_0^T \left(\frac{R_A}{2} |Z_s|^2 \widehat{\alpha}_s + \inf_{a\in[0,a_{\max}]} \{k(a) - aZ_s\} \right) ds + \int_0^T Z_s \widehat{\alpha}_s^{1/2} dW_s + \int_0^T dK_s, \tag{4.7}$$

for some $\mathbb{G}^{\mathcal{P}_A}-$predictable process Z, $Y_0 \in \mathbb{R}$ and some non-decreasing process K satisfying the same minimality condition as in (4.3).

4.2 The principal's problem

From Equality (4.7) in the previous section, we know that we can associate to each contract ξ a $\mathbb{G}^{\mathcal{P}_A}-$predictable process Z^ξ, which determines completely the optimal effort of the agent $a^\star(Z^\xi_\cdot)$ given by Proposition 4.1. We also denote by Y^ξ and K^ξ the other components of the maximal solution to the 2BSDE (4.4). For technical reasons, we need to assume some integrability on Z^ξ. More precisely, we define

$$\mathcal{C}^{SB} := \left\{ \xi \in \mathcal{C}^{SB},\ \mathcal{E}\left(-R_P \int_0^\cdot \widehat{\alpha}_s^{\frac{1}{2}} (1 - Z_s^\xi) dW_s^{a^\star(Z^\xi_\cdot)} \right) \text{ is a } \mathbb{P}\text{−martingale, } \forall \mathbb{P} \in \mathcal{P}_P^{a^\star(Z^\xi_\cdot)} \right\}.$$

The principal's problem then becomes

$$\begin{aligned} U_0^P &= \sup_{\xi\in\mathcal{C}^{SB}} \inf_{\mathbb{P}\in\mathcal{P}_P^{a^\star(Z^\xi_\cdot)}} \mathbb{E}^{\mathbb{P}} \left[\mathcal{U}_P \left(B_T - \xi \right) \right] \\ &= \sup_{\xi\in\mathcal{C}^{SB}} \inf_{\mathbb{P}\in\mathcal{P}_P^{a^\star(Z^\xi_\cdot)}} \mathbb{E}^{\mathbb{P}} \left[-\mathcal{E}\left(-R_P \int_0^T \widehat{\alpha}_s^{\frac{1}{2}} (1 - Z_s^\xi) dW_s^{a^\star(Z^\xi_\cdot)} \right) e^{R_P\left(K_T + Y_0^\xi - \int_0^T f(Z_s^\xi, \widehat{\alpha}_s) ds \right)} \right], \end{aligned} \tag{4.8}$$

where

$$f(z,\alpha) := a^\star(z) - k(a^\star(z)) - \frac{\alpha}{2}\left(R_A z^2 + R_P (1-z)^2 \right).$$

In order to continue our computations, we actually need to have more information on the non-decreasing process K. Using similar intuitions as the ones given in [9, 10], we expect that when the contract ξ is sufficiently "smooth", we can find a $\mathbb{G}^{\mathcal{P}^A}-$predictable process Γ such that

$$K_t = \int_0^t \left(\frac{1}{2} \widehat{\alpha}_s \Gamma_s - \inf_{\alpha\in[\underline{\alpha}_A, \overline{\alpha}^A]} \left\{ \frac{1}{2} \alpha \Gamma_s \right\} \right) ds. \tag{4.9}$$

However, in general, such a decomposition for K is not true for every $\xi \in \mathcal{C}^{SB}$. We will therefore start by solving the principal problem for a particular sub-class of contracts in $\mathcal{C}^{SB}$ such that the process Γ exists, and then show that the principal's value function is not actually affected by this restriction.

4.2.1 A sub-optimal problem for the principal

Building upon (4.7) and (4.9) we consider the class $\mathfrak{C}^{SB} \subset \mathcal{C}^{SB}$ of contracts ξ admitting the decomposition

$$\begin{aligned} \xi = Y_0 &+ \int_0^T \left(\frac{1}{2} \widehat{\alpha}_s \Gamma_s - \inf_{\alpha\in[\underline{\alpha}_A, \overline{\alpha}^A]} \left\{ \frac{1}{2} \alpha \Gamma_s \right\} + \frac{R_A}{2} |Z_s|^2 \widehat{\alpha}_s + \inf_{a\in[0,a_{\max}]} \{k(a) - aZ_s\} \right) ds \\ &+ \int_0^T Z_s \widehat{\alpha}_s^{1/2} dW_s, \end{aligned} \tag{4.10}$$

for some $Y_0 \geq R_0$, some $\mathbb{G}^{\mathcal{P}_A}-$predictable process Z such that

$$\mathcal{E}\left(-R_P \int_0^{\cdot} \widehat{\alpha}_s^{\frac{1}{2}}(1-Z_s)dW_s^{a^\star(Z.)}\right) \text{ is a } \mathbb{P}-\text{martingale}, \forall \mathbb{P} \in \mathcal{P}_P^{a^\star(Z.)},$$

and some $\mathbb{G}^{\mathcal{P}_A}-$predictable process Γ. The set of corresponding processes (Z,Γ) will be denoted by $\mathfrak{U}$. A contract $\xi \in \mathfrak{C}^{SB}$ will be identified with the triplet (Y_0, Z, Γ).

We define now the following sub-optimal principal problem

$$\overline{U}_0^P := \sup_{\xi \in \mathfrak{C}^{SB}} \inf_{\mathbb{P} \in \mathcal{P}_P^{a^\star(Z.)}} \mathbb{E}^{\mathbb{P}}\left[\mathcal{U}_P\left(B_T - \xi\right)\right].$$

Let us also denote for any $\xi \in \mathfrak{C}^{SB}$

$$\overline{U}_0^P(\xi) := \inf_{\mathbb{P} \in \mathcal{P}_P^{a^\star(Z.)}} \mathbb{E}^{\mathbb{P}}\left[\mathcal{U}_P\left(B_T - \xi\right)\right].$$

By the above calculations, we immediately have that

$$\overline{U}_0^P := \sup_{(Z,\Gamma)\in\mathfrak{U}} \inf_{\mathbb{P} \in \mathcal{P}_P^{a^\star(Z.)}} \mathbb{E}^{\mathbb{P}}\left[-\mathcal{E}\left(-R_P \int_0^T \widehat{\alpha}_s^{1/2}(1-Z_s)dW_s^{a^\star(Z.)}\right) e^{R_P\left(R_0 - \int_0^T H(\widehat{\alpha}_s, Z_s, \Gamma_s)ds\right)}\right],$$

where

$$H(\alpha, z, \gamma) := a^\star(z) - k(a^\star(z)) - \frac{\alpha}{2}\left(R_A z^2 + R_P(1-z)^2\right) - \frac{1}{2}\alpha\gamma + \inf_{\alpha \in [\underline{\alpha}_A, \overline{\alpha}^A]}\left\{\frac{1}{2}\alpha\gamma\right\}. \tag{4.11}$$

In order to pursue the computations, we need to specify a form for the cost function k. Namely, we will assume in what follows that

Assumption 4.1. *The cost function of the agent is quadratic, defined, for some $k > 0$, by*

$$k(a) := k\frac{a^2}{2},\ a \geq 0.$$

We deduce from Proposition 4.1 that the agent chooses the control $a^\star(z) = \frac{z}{k}$. Hence, Equality (4.11) can be rewritten

$$\begin{aligned} H(\alpha, z, \gamma) &= \frac{z}{k} - \frac{z^2}{2k} - \frac{\alpha}{2}\left(R_A z^2 + R_P(1-z)^2\right) - \frac{1}{2}\alpha\gamma + \inf_{\alpha \in [\underline{\alpha}_A, \overline{\alpha}^A]}\left\{\frac{1}{2}\alpha\gamma\right\} \\ &=: H^z(\alpha, z) + H^\gamma(\alpha, \gamma), \end{aligned} \tag{4.12}$$

where

$$H^z(\alpha, z) := \frac{z}{k} - \frac{z^2}{2k} - \frac{\alpha}{2}\left(R_A z^2 + R_P(1-z)^2\right)$$

and

$$H^\gamma(\alpha, \gamma) := -\frac{1}{2}\alpha\gamma + \inf_{\alpha \in [\underline{\alpha}_A, \overline{\alpha}^A]}\left\{\frac{1}{2}\alpha\gamma\right\}.$$

Notice that for any $\alpha \geq 0$

$$H(\overline{\alpha}^A, z, 0) = H(\alpha, z, -R_A z^2 - R_P(1-z)^2). \tag{4.13}$$

The following lemma studies computes the maximum of the map $(z,\gamma) \longmapsto H(\alpha, z, \gamma)$, depending on the value of $\alpha \in \mathbb{R}_+$.

Lemma 4.1. *We distinguish three cases.*

(i) If $\underline{\alpha}^A \leq \alpha \leq \overline{\alpha}^A$*, then* $(z,\gamma) \longmapsto H(\alpha,z,\gamma)$ *admits a (global) maximum at*

$$z^\star(\alpha) := \frac{1+k\alpha R_P}{1+\alpha k(R_A+R_P)},\ \gamma^\star := 0. \tag{4.14}$$

(ii) If $\alpha < \underline{\alpha}^A$*, then* $\gamma \longmapsto H(\alpha,z,\gamma)$ *is increasing and attains its maximum at* $\gamma^\star = +\infty$*, for which* $H(\alpha,z,\gamma^\star) = +\infty$.

(iii) If $\overline{\alpha}^A < \alpha$*, then* $\gamma \longmapsto H(\alpha,z,\gamma)$ *is decreasing and attains its maximum at* $\gamma^\star = -\infty$*, for which* $H(\alpha,z,\gamma^\star) = +\infty$.

Proof. We have

$$\frac{\partial H}{\partial z}(\alpha,z,\gamma) = \frac{\partial H^z}{\partial z}(\alpha,z) = \frac{1}{k} - \frac{z}{k} - \alpha(R_A z - R_P(1-z)),$$

so that

$$\frac{\partial H}{\partial z}(\alpha,z,\gamma) = 0 \Longleftrightarrow z = z^\star(\alpha) := \frac{1+k\alpha R_P}{1+\alpha k(R_A+R_P)}.$$

Since $z \longmapsto H^z(\alpha,z)$ is concave for any $\alpha \geq 0$, we deduce that the maximum of H^z is attained at $z^\star(\alpha)$.

Furthermore, for any $\gamma \neq 0$

$$\frac{\partial H}{\partial \gamma}(\alpha,z,\gamma) = \frac{\partial H^\gamma}{\partial \gamma}(\alpha,\gamma) = \frac{1}{2}(\underline{\alpha}^A - \alpha)\mathbf{1}_{\gamma>0} + \frac{1}{2}(\overline{\alpha}^A - \alpha)\mathbf{1}_{\gamma<0}$$

If $\underline{\alpha}^A \leq \alpha \leq \overline{\alpha}^A$, then $(z,\gamma) \longmapsto H(\alpha,z,\gamma)$ admits a global maximum at $(z^\star(\alpha),0)$ which proves (i). (ii) and (iii) are clear. □

We can now state the main result of this section, which gives the optimal contracts for the second-best problem, when contracts are restricted to the class $\mathfrak{C}^{SB}$.

Theorem 4.1. *Let Assumption 4.1 hold. Define for any* $\alpha \geq 0$

$$z^\star(\alpha) := \frac{1+k\alpha R_P}{1+\alpha k(R_A+R_P)}.$$

(i) If $\underline{\alpha}^A \leq \overline{\alpha}^P \leq \overline{\alpha}^A$*, then an admissible optimal contract is given by*

$$\xi^\star \equiv (R_0, z^\star(\overline{\alpha}^P), 0),$$

where R_0 *is defined by Relation* (4.6)*. In this case,*

$$\overline{U}_0^P = -\exp\left(-R_P(TH(\overline{\alpha}^P, z^\star(\overline{\alpha}^P), 0) - R_0)\right).$$

(ii) If $\underline{\alpha}^P \leq \overline{\alpha}^A \leq \overline{\alpha}^P$*, then an admissible optimal contract is given by*

$$\xi^\star \equiv (R_0, z^\star(\overline{\alpha}^A), \gamma^\star),$$

where R_0 *is defined by Relation* (4.6) *and*

$$\gamma^\star := -R_A(z^\star(\overline{\alpha}^A))^2 - R_P(1-z^\star(\overline{\alpha}^A))^2.$$

In this case,

$$\overline{U}_0^P = -\exp\left(-R_P(TH(\overline{\alpha}^A, z^\star(\overline{\alpha}^A), \gamma^\star) - R_0)\right).$$

(iii) Assume that $\overline{\alpha}^P < \underline{\alpha}^A$*. Let* $\xi^n \equiv (R_0, 0, n)$*, then*

$$\overline{U}_0^P = \lim_{n\to+\infty} \overline{U}_0^P(\xi^n) = 0.$$

(iv) Assume that $\overline{\alpha}^A < \underline{\alpha}^P$*. Let* $\xi^n \equiv (R_0, 0, -n)$*, then*

$$\overline{U}_0^P = \lim_{n\to+\infty} \overline{U}_0^P(\xi^n) = 0.$$

Proof. We recall Definition (4.14) for any $\alpha \geq 0$

$$z^\star(\alpha) := \frac{1 + k\alpha R_P}{1 + \alpha k(R_A + R_P)}.$$

We begin with the proof of (i)**.** Assume that $\underline{\alpha}^A \leq \overline{\alpha}^P \leq \overline{\alpha}^A$, then

$$\overline{U}_0^P \geq \inf_{\mathbb{P}\in\mathcal{P}_P^{a^\star(z^\star(\widehat{\alpha}.))}} \mathbb{E}^{\mathbb{P}}\left[-\mathcal{E}\left(-R_P\int_0^T \widehat{\alpha}_s^{\frac{1}{2}}(1 - z^\star(\widehat{\alpha}_s))dW_s^{a^\star(z^\star)}\right) e^{R_P\left(R_0 - \int_0^T H(\widehat{\alpha}_s, z^\star(\widehat{\alpha}_s),0)ds\right)}\right].$$

Then we have for any $\alpha \geq 0$

$$\begin{aligned}
H(\alpha, z^\star(\alpha), 0) &= -\frac{\alpha}{2}R_P + \frac{z^\star(\alpha)}{k}(1 + \alpha k R_P) - \frac{|z^\star(\alpha)|^2}{2k}(1 + \alpha k(R_A + R_P)) \\
&= -\frac{\alpha}{2}R_P + \frac{(1 + \alpha k R_P)^2}{k(1 + \alpha k(R_A + R_P))} - \frac{(1 + \alpha k R_P)^2}{2k(1 + \alpha k(R_A + R_P))} \\
&= -\frac{\alpha}{2}R_P + \frac{(1 + \alpha k R_P)^2}{2k(1 + \alpha k(R_A + R_P))}.
\end{aligned}$$

Hence,

$$\frac{\partial H}{\partial \alpha}(\alpha, z^\star(\alpha), 0) = \frac{-R_A\left(1 + 2k\alpha R_P + k^2\alpha^2 R_P(R_A + R_P)\right)}{2(1 + \alpha k(R_A + R_P))^2} \leq 0, \ \forall \alpha \in [\underline{\alpha}^P, \overline{\alpha}^P].$$

Therefore,

$$\overline{U}_0^P \geq -e^{R_P R_0} e^{-R_P\int_0^T H(\overline{\alpha}^P, z^\star(\overline{\alpha}^P),0)ds}.$$

Indeed, $z^\star(\widehat{\alpha}_s)$ is bounded so that the stochastic exponential is trivially a true martingale. We now turn to the converse inequality, we have

$$\overline{U}_0^P \leq \sup_{(Z,\Gamma)\in\mathfrak{U}} \mathbb{E}^{\mathbb{P}^{\overline{\alpha}^P}_{a^\star(Z.)}}\left[-\mathcal{E}\left(-R_P\int_0^T (\overline{\alpha}^P)^{1/2}(1 - Z_s)dW_s^{a^\star(Z.)}\right) e^{R_P\left(R_0 - \int_0^T H(\overline{\alpha}^P, Z_s, \Gamma_s)ds\right)}\right].$$

According to Lemma 4.1(i), we obtain

$$\overline{U}_0^P \leq -e^{R_P R_0} e^{-R_P T H(\overline{\alpha}^P, z^\star(\overline{\alpha}^P),0)}.$$

Hence, if $\underline{\alpha}^A \leq \overline{\alpha}^P \leq \overline{\alpha}^A$, then $\overline{U}_0^P = -e^{R_P R_0}e^{-R_P\int_0^T H(\overline{\alpha}^P, z^\star(\overline{\alpha}^P),0)ds}$. We now prove that the contract $\xi^\star \equiv (R_0, z^\star(\overline{\alpha}^P), 0) \in \mathfrak{C}^{SB}$ is indeed optimal. We have

$$\begin{aligned}
&\inf_{\mathbb{P}\in\mathcal{P}_P^{a^\star(z^\star(\overline{\alpha}^P))}} \mathbb{E}^{\mathbb{P}}\left[-\mathcal{E}\left(-R_P\int_0^T \widehat{\alpha}_s^{\frac{1}{2}}(1 - z^\star(\overline{\alpha}^P))dW_s^{a^\star(z^\star(\overline{\alpha}^P))}\right) e^{R_P\left(R_0 - \int_0^T H(\widehat{\alpha}_s, z^\star(\overline{\alpha}^P),0)ds\right)}\right] \\
&= \overline{U}_0^P,
\end{aligned}$$

since by definition (4.12) of H, $\alpha \longmapsto H(\alpha, z^\star(\overline{\alpha}^P), 0$ is decreasing, so that the above infimum is attained for the measure $\mathbb{P}^{\overline{\alpha}^P}_{a^\star(z^\star(\overline{\alpha}^P))}$.

We now turn to the proof of (ii). Assume that $\underline{\alpha}^P \leq \overline{\alpha}^A \leq \overline{\alpha}^P$. On the one hand

$$\inf_{\mathbb{P}\in\mathcal{P}_P^{a^\star(z^\star(\overline{\alpha}^A))}} \mathbb{E}^{\mathbb{P}}\left[-\mathcal{E}\left(-R_P\int_0^T \widehat{\alpha}_s^{\frac{1}{2}}(1-z^\star(\overline{\alpha}^A))dW_s^{a^\star(z^\star(\overline{\alpha}^A))}\right)e^{R_P\left(R_0-\int_0^T H(\widehat{\alpha}_s, z^\star(\overline{\alpha}^A),\gamma^\star)ds\right)}\right]$$
$$\leq \overline{U}_0^P,$$

where $\gamma^\star := -R_A(z^\star(\overline{\alpha}^A))^2 - R_P(1-z^\star(\overline{\alpha}^A))^2$. Thus, using Relation (4.13), we have

$$\overline{U}_0^P \geq -e^{R_P R_0}e^{-R_P T H(\overline{\alpha}^A, z^\star(\overline{\alpha}^A),0)}.$$

On the other hand, since $\overline{\alpha}^A \in [\underline{\alpha}^P, \overline{\alpha}^P]$

$$\overline{U}_0^P \leq e^{R_P R_0}\sup_{(Z,\Gamma)\in\mathfrak{U}} \mathbb{E}^{\mathbb{P}^{\overline{\alpha}^A}_{a^\star(Z.)}}\left[-\mathcal{E}\left(-R_P\int_0^T \overline{\alpha}^{A^{1/2}}(1-Z_s)dW_s^{a^\star(Z.)}\right)e^{-R_P\int_0^T H(\overline{\alpha}^A, Z_s,\Gamma_s)ds}\right].$$

By using Lemma 4.1(i), we obtain

$$\overline{U}_0^P \leq -e^{R_P R_0}e^{-R_P T H(\overline{\alpha}^A, z^\star(\overline{\alpha}^A),0)}.$$

We deduce that

$$\overline{U}_0^P = -e^{R_P R_0}e^{-R_P T H(\overline{\alpha}^A, z^\star(\overline{\alpha}^A),0)}.$$

We consider now a contract $\xi^\star \equiv (R_0, z^\star(\overline{\alpha}^A), \gamma^\star)$ and we show that $\overline{U}_0^P(\xi^\star) = \overline{U}_0^P$. We have

$$\inf_{\mathbb{P}\in\mathcal{P}_P^{a^\star(z^\star(\overline{\alpha}^A)}} \mathbb{E}^{\mathbb{P}}\left[-\mathcal{E}\left(-R_P\int_0^T \widehat{\alpha}_s^{1/2}(1-z^\star(\overline{\alpha}^A))dW_s^{a^\star(z^\star(\overline{\alpha}^A))}\right)e^{R_P\left(R_0-\int_0^T H(\widehat{\alpha}_s, z^\star(\overline{\alpha}^A),\gamma^\star)ds\right)}\right]$$
$$= -e^{R_P R_0}e^{-R_P T H(\overline{\alpha}^A, z^\star(\overline{\alpha}^A),0)} = \overline{U}_0^P,$$

since $H(\alpha, z^\star(\overline{\alpha}^A), \gamma^\star)$ is actually independent of α.

We now prove (iii). Assume that $\overline{\alpha}^P < \underline{\alpha}^A$. First notice that $\overline{U}_0^P \leq 0$. Let $\xi^n \equiv (R_0, 0, n)$.

$$\overline{U}_0^P \geq \overline{U}_0^P(\xi^n) = e^{R_P R_0}\inf_{\mathbb{P}\in\mathcal{P}_P} \mathbb{E}^{\mathbb{P}}\left[-\mathcal{E}\left(-R_P\int_0^T \widehat{\alpha}_s^{1/2}dW_s\right)e^{-R_P\int_0^T H(\widehat{\alpha}_s,0,n)ds}\right].$$

Notice that $H(\alpha, 0, n) = \frac{n}{2}(\underline{\alpha}^A - \alpha)$. Hence,

$$\overline{U}_0^P \geq -e^{R_P R_0}e^{-R_P \frac{nT}{2}(\underline{\alpha}^A - \overline{\alpha}^P)}$$

and since $\overline{\alpha}^P < \underline{\alpha}^A$, taking the limit when n goes to $+\infty$ we deduce that

$$\overline{U}_0^P \geq \lim_{n\to+\infty} \overline{U}_0^P(\xi^n) = 0.$$

We finally prove (iv). Assume that $\underline{\alpha}^P > \overline{\alpha}^A$. The proof is similar to the previous case by choosing $\xi^n \equiv (R_0, 0, -n)$. □

4.3 The principal sub-optimal problem is equal to U_0^P

In this section, we prove the following result.

Theorem 4.2. *Let Assumption 4.1 hold. Then*

$$U_0^P = \overline{U}_0^P.$$

Proof. First of all, notice that when $\overline{\alpha}^P < \underline{\alpha}^A$ or $\overline{\alpha}^A < \underline{\alpha}^P$, we have $\overline{U}_0^P = 0$, so that $U_0^P = 0$ as well. Let us now assume that $\underline{\alpha}^A \leq \overline{\alpha}^P \leq \overline{\alpha}^A$. By (4.8), since $K_T \geq 0$, we easily have

$$U_0^P \leq \sup_{\xi \in \mathcal{C}^{SB}} \mathbb{E}^{\mathbb{P}^{\overline{\alpha}^P}_{a^\star(Z^\xi_\cdot)}} \left[-\mathcal{E}\left(-R_P \int_0^T (\overline{\alpha}^P)^{\frac{1}{2}} (1 - Z_s^\xi) dW_s^{a^\star(Z^\xi_\cdot)} \right) e^{R_P \left(R_0 - \int_0^T f(Z_s^\xi, \overline{\alpha}^P) ds \right)} \right].$$

Then, we easily have that the map $z \longmapsto f(z, \overline{\alpha}^P)$ attains its maximum at $z^\star(\overline{\alpha}^P)$, where it is actually equal to $H(\overline{\alpha}^P, z^\star(\overline{\alpha}^P), 0)$. Thus

$$U_0^P \leq e^{R_P R_0} e^{-R_P T H(\overline{\alpha}^P, z^\star(\overline{\alpha}^P), 0)} = \overline{U}_0^P,$$

by Theorem 4.1(i).

Assume now that $\underline{\alpha}^P \leq \overline{\alpha}^A \leq \overline{\alpha}^P$. Then, we also have by (4.8) that, with the same arguments

$$\begin{aligned} U_0^P &\leq \sup_{\xi \in \mathcal{C}^{SB}} \mathbb{E}^{\mathbb{P}^{\overline{\alpha}^P}_{a^\star(Z^\xi_\cdot)}} \left[-\mathcal{E}\left(-R_P \int_0^T (\overline{\alpha}^A)^{\frac{1}{2}} (1 - Z_s^\xi) dW_s^{a^\star(Z^\xi_\cdot)} \right) e^{R_P \left(R_0 - \int_0^T f(Z_s^\xi, \overline{\alpha}^A) ds \right)} \right] \\ &\leq e^{R_P R_0} e^{-R_P T H(\overline{\alpha}^A, z^\star(\overline{\alpha}^A), 0)} = \overline{U}_0^P, \end{aligned}$$

by Theorem 4.1(ii). □

4.4 Comments

The comparison with the case without ambiguity is actually very similar to the first best problem. First, notice that when $\overline{\alpha}^P \in [\underline{\alpha}^A, \overline{\alpha}^A]$, an optimal contract can be chosen to be linear in the terminal value of the output, and it is actually the exact same contract as the optimal one for a principal who would only believe in a constant volatility process equal to $\overline{\alpha}^P$. Since the utility of the principal is then a decreasing function of the volatility, this means that the principal always gets less utility than in a context without ambiguity.

However, as soon as $\overline{\alpha}^A \in [\underline{\alpha}^P, \overline{\alpha}^P]$, the second-best optimal contract makes use of the quadratic variation of the output and is therefore path-dependent. Besides, as in the first-best case, the principal may get an higher utility level than in the case without ambiguity.

Finally, in the degenerated cases (iii) and (iv) of Theorem 4.1, we have seen that the optimal effort for the agent is equal to 0 since $z^* = 0$ and $a^*(z^*) = 0$, on the contrary to the first-best problem where, in the same case, the optimal level of effort for the agent, chosen by the principal to obtained his best utility 0, was $a_{\max}$. Hence, to solve the second-best problem, the agent does not provide any effort and attains his reservation utility. It can be explained by the fact that in the second-best problem, an optimal contract is a Stackelberg equilibrium, where the principal has to anticipate the reaction of the agent given an admissible contract, unlike the first-best problem for which the principal chooses the level of effort for the agent.

5 Possible extensions

In this section, we examine several potential generalizations of the problem at hand, and we try to explain how to tackle it in each case.

5.1 More general dynamics

The first possible extension would be to consider an output with more general dynamics. Typically, one could have a general model where

$$B_t = \int_0^t b_s(B_s, a_s^{\mathbb{P}}, \alpha_s^{\mathbb{P}})ds + \int_0^t (\alpha_s^{\mathbb{P}})^{1/2} dW_s^{a^{\mathbb{P}}},\ \mathbb{P} - a.s.,$$

that is to say that the impact of the effort choice of the agent on the drift of the output is now non-linear, and the value of this drift may also depend on the current value of the output itself, which could model some synergy effects. Furthermore, b should be assumed to have linear growth in a and to be continuous.

Furthermore, the cost function k could also take the form $k_s(B_s, a_s^{\mathbb{P}})$.

In the first-best problem, if the map b actually only depends on a, and not on B and α, and if k does not depend on B, then it is not difficult to see that our approach will still work, albeit with more complicated computations. Notably, the optimal effort of the agent will either be $a_{\max}$ or any (deterministic) minimizer of $a \longmapsto k_s(a) - b_s(a)$ (which exist since k is superlinear). It is however not clear to us how to handle the general dynamics.

In the second-best problem, the representation of the value function of the agent in terms of 2BSDEs will always work, provided that one can indeed check that it is wellposed (which requires obviously some assumptions on k and b). Then, following [10], we can always associate to the suboptimal value function of the principal $\overline{U}_0^P$ an Hamilton-Jacobi-Bellman-Isaacs (HJBI) equation in the two state variables B and $Y^{Z,\Gamma}$ defined by

$$\begin{aligned} Y_t^{Z,\Gamma} := R_0 &+ \int_0^t \left(\frac{R_A}{2} |Z_s|^2 \widehat{\alpha}_s + \inf_{a \in [0, a_{\max}]} \{ k_s(B_s, a) - b_s(B_s, a) Z_s \} \right) ds \\ &+ \int_0^t \left(\frac{1}{2} \widehat{\alpha}_s \Gamma_s - \inf_{\alpha \in [\underline{\alpha}_A, \overline{\alpha}^A]} \left\{ \frac{1}{2} \alpha \Gamma_s \right\} \right) ds + \int_0^t Z_s \widehat{\alpha}_s^{1/2} dW_s, \end{aligned}$$

which one can then interpret as controlled diffusions processes, with controls (Z, Γ), chosen by the principal, and α, chosen by the "Nature". The HJBI equation then writes

$$\begin{cases} \partial_t v(t,x,y) + \sup_{(z,\gamma) \in \mathbb{R}^2} \inf_{\alpha \in [\underline{\alpha}^P, \overline{\alpha}^P]} \Big\{ b_t(x, a_t^\star(x,z)) \partial_x v(t,x,y) + \Big(\frac{R_A}{2} \alpha z^2 + k_t(x, a_t^\star(x,z)) + \frac{1}{2} \alpha \gamma \\ \qquad - \inf_{\tilde{\alpha} \in [\underline{\alpha}_A, \overline{\alpha}^A]} \left\{ \frac{1}{2} \tilde{\alpha} \gamma \right\} \Big) \partial_y v(t,x,y) + \frac{1}{2} \alpha \partial_{xx} v(t,x,y) \\ \qquad + \frac{1}{2} \alpha z^2 \partial_{yy} v(t,x,y) + \alpha z \partial_{xy} v(t,x,y) \Big\},\ (t,x,y) \in [0,T) \times \mathbb{R}^2, \\ v(T,x,y) = \mathcal{U}_P(x-y),\ (x,y) \in \mathbb{R}^2, \end{cases} \tag{5.1}$$

where $a_t^\star(x,z)$ is the unique (for simplicity)[7] minimizer of the map $a \longmapsto k_t(x,a) - b_t(x,a)z$. We then expect to have the link $\overline{U}_0^P = v(0,0,R_0)$. We could next try to study the above equation, and, for instance, find a smooth solution which would allow us to use a verification type argument to identify it with the value function of the principal. We believe nonetheless that such an approach should probably be used on a case by case basis.

5.2 More general utility functions

Another possible generalization would be to go beyond the case of exponential utility functions for the principal and the agent. As usual, if the utility of the agent is separable (that is to say if the cost comes out of the utility), then the 2BSDE characterization of his value function would still hold. The main problem would then be to solve the principal problem. Once again, one could write down an HJBI equation similar to the one above and try to study it.

6 Conclusion and summary table

$$Q := \left\{ \xi \in \mathcal{C},\ \xi = zB_T + \frac{\gamma}{2}\langle B\rangle_T + \delta,\ (z,\gamma,\delta) \in \mathbb{R}^3 \right\}$$

$$\overline{\mathcal{Q}^{\overline{\gamma}}} := \Big\{ \xi^\star \equiv (z^\star,\gamma^\star,\delta^\star) \in \mathcal{Q},\ z^\star = \frac{R_P}{R_A+R_P},\ \gamma^\star \geq R_A|z^\star|^2,$$
$$\delta^\star = Tk(a^\star) - \frac{R_P}{R_A+R_P}Ta^\star + \frac{\overline{\alpha}^P T}{2}\left(\frac{R_A R_P^2}{(R_A+R_P)^2} - \gamma^\star\right) - \frac{1}{R_A}\log(-R)\Big\}$$

$$\overline{\mathcal{Q}^{|\gamma|}} := \Big\{ \xi^\star \equiv (z^\star,\gamma^\star,\delta^\star) \in \mathcal{Q},\ z^\star = \frac{R_P}{R_A+R_P},\ \gamma^\star \in [-R_P(1-z^\star)^2, R_A|z^\star|^2],$$
$$\delta^\star = Tk(a^\star) - \frac{R_P}{R_A+R_P}Ta^\star + \frac{\overline{\alpha}^P T}{2}\left(\frac{R_A R_P^2}{(R_A+R_P)^2} - \gamma^\star\right) - \frac{1}{R_A}\log(-R)\Big\}$$

$$\overline{\mathcal{Q}^{\underline{\gamma}}} := \Big\{ \xi^\star \equiv (z^\star,\gamma^\star,\delta^\star) \in \mathcal{Q},\ z^\star = \frac{R_P}{R_A+R_P},\ \gamma^\star \leq -R_P(1-z^\star)^2,$$
$$\delta^\star = Tk(a^\star) - \frac{R_P}{R_A+R_P}Ta^\star + \frac{\underline{\alpha}^P T}{2}\left(\frac{R_A R_P^2}{(R_A+R_P)^2} - \gamma^\star\right) - \frac{1}{R_A}\log(-R)\Big\}.$$

We set $a^* := \mathrm{argmin}\,\{k(a) - a\}$.

[7] If the minimizer is not unique, then we assume as usual that the principal has sufficient bargaining power to make the agent choose the best minimizer for him. This means that one has also to take the supremum over all minimizers in the Hamiltonian above.

6.1 Summary table for the first-best problem

	Opt. contracts	Uniqueness	Effort	Worst $\mathcal{P}_P$	Worst $\mathcal{P}_A$
$\overline{\alpha}^P < \underline{\alpha}^A$ Theorem 3.1(i)	nonlinear ξ^n s.t. $U_0^P(\xi^n) \underset{n\to+\infty}{\longrightarrow} 0$	No	a_{max}	$\overline{\alpha}^P$	$\underline{\alpha}^A$
$\underline{\alpha}^A = \overline{\alpha}^P$ Theorem 3.2(i)	$\overline{\mathcal{Q}^{\overline{\gamma}}}$	No	$a^\star$	$\overline{\alpha}^P$	$\underline{\alpha}^A$
$\underline{\alpha}^A < \overline{\alpha}^P < \overline{\alpha}^A$ Theorem 3.2(ii)	$\mathcal{Q}^u$	Yes	$a^\star$	$\overline{\alpha}^P$	$[\underline{\alpha}^A, \overline{\alpha}^A]$
$\overline{\alpha}^P = \overline{\alpha}^A$ Theorem 3.2(iii)	$\overline{\mathcal{Q}^{\lvert\gamma\rvert}}$	No	$a^\star$	$\overline{\alpha}^P$	$\overline{\alpha}^A$
$\underline{\alpha}^P = \overline{\alpha}^A$ Theorem 3.2(iv)	$\overline{\mathcal{Q}^{\underline{\gamma}}}$	No	$a^\star$	$\underline{\alpha}^P$	$\overline{\alpha}^A$
$\underline{\alpha}^P < \overline{\alpha}^A < \overline{\alpha}^P$ Theorem 3.2(v)	$\mathcal{Q}^d$	Yes	$a^\star$	$[\underline{\alpha}^P, \overline{\alpha}^P]$	$\overline{\alpha}^A$
$\overline{\alpha}^A < \underline{\alpha}^P$ Theorem 3.1(ii)	nonlinear ξ^n s.t. $U_0^P(\xi^n) \underset{n\to+\infty}{\longrightarrow} 0$	No	a_{max}	$\underline{\alpha}^P$	$\overline{\alpha}^A$

6.2 Summary table for the second-best problem

For any $\alpha \geq 0$ we set

$$z^*(\alpha) := \frac{1 + k\alpha R_P}{1 + \alpha k(R_A + R_P)},$$

and

$$\gamma^* := -R_A(z^*(\overline{\alpha}^A))^2 - R_P(1 - z^*(\overline{\alpha}^A))^2.$$

For any $z \in \mathbb{R}$,

$$a^*(z) := \underset{a\in[0,a_{\max}]}{\text{argmin}} \{k(a) - az\} = \frac{z}{k}.$$

	Opt. contracts	Uniqueness	Effort	Worst $\mathcal{P}_P$	Worst $\mathcal{P}_A$
$\overline{\alpha}^P < \underline{\alpha}^A$ Theorem 4.1(iii)	Nonlinear ξ^n s.t. $U_0^P(\xi^n) \underset{n\to+\infty}{\to} 0$	No	0	$\overline{\alpha}^P$	$\underline{\alpha}^A$
$\underline{\alpha}^A \leq \overline{\alpha}^P < \overline{\alpha}^A$ Theorem 4.1(i)	Linear contract $(R_0, z^*(\overline{\alpha}^P), 0)$	Yes	$a^*(z^*(\overline{\alpha}^P))$	$\overline{\alpha}^P$	$[\underline{\alpha}^A, \overline{\alpha}^A]$
$\overline{\alpha}^P = \overline{\alpha}^A$ Theorem 4.1(i), (ii)	$(R_0, z^*(\overline{\alpha}^P), 0)$ $(R_0, z^*(\overline{\alpha}^P), \gamma^*)$	No	$a^*(z^*(\overline{\alpha}^P))$	$\overline{\alpha}^P$	$\overline{\alpha}^A$
$\underline{\alpha}^P \leq \overline{\alpha}^A < \overline{\alpha}^P$ Theorem 4.1(ii)	Non linear $(R_0, z^*(\overline{\alpha}^A), \gamma^*)$	Yes	$a^*(z^*(\overline{\alpha}^A))$	$[\underline{\alpha}^P, \overline{\alpha}^P]$	$\overline{\alpha}^A$
$\overline{\alpha}^A < \underline{\alpha}^P$ Theorem 3.1(ii)	nonlinear ξ^n s.t. $U_0^P(\xi^n) \underset{n\to+\infty}{\to} 0$	No	0	$\underline{\alpha}^P$	$\overline{\alpha}^A$

A Appendix

Proof of Lemma 3.1. Let $a \in \mathcal{A}^{\text{det}}$ and $\xi \equiv (z, \gamma, \delta) \in \mathcal{Q}$. We compute on the one hand

$$\inf_{\mathbb{P}\in\mathcal{P}_P^a} \mathbb{E}^{\mathbb{P}}\left[\mathcal{U}_P\left(B_T - \xi\right)\right] = \inf_{\mathbb{P}\in\mathcal{P}_P^a} \mathbb{E}^{\mathbb{P}}\left[-e^{-R_P\left(B_T(1-z)-\frac{\gamma}{2}\int_0^T \alpha_s^{\mathbb{P}}ds-\delta\right)}\right]$$
$$= -e^{R_P(\delta-(1-z)\int_0^T a_s ds)} \sup_{\mathbb{P}\in\mathcal{P}_P^a} \mathbb{E}^{\mathbb{P}}\left[\mathcal{E}\left(-R_P(1-z)\int_0^T (\alpha_s^{\mathbb{P}})^{\frac{1}{2}}dW_s^a\right) e^{\frac{R_P}{2}\left(R_P(1-z)^2+\gamma\right)\int_0^T \alpha_s^{\mathbb{P}}ds}\right].$$

Hence, using the fact that the stochastic exponential appearing above is a true martingale under any $\mathbb{P} \in \mathcal{P}_P^a$, we deduce easily that

$$\inf_{\mathbb{P}\in\mathcal{P}_P^a} \mathbb{E}^{\mathbb{P}}\left[\mathcal{U}_P\left(B_T - \xi\right)\right] = \begin{cases} \Gamma_P(a, z, \gamma, \delta, \underline{\alpha}^P), \text{ if } \gamma < -R_P(1-z)^2 \\ \Gamma_P(a, z, \gamma, \delta, \overline{\alpha}^P), \text{ if } \gamma > -R_P(1-z)^2 \\ \mathbb{E}^{\mathbb{P}}\left[\mathcal{U}_P\left(B_T - \xi\right)\right], \ \forall \mathbb{P} \in \mathcal{P}_P^a \text{ if } \gamma = -R_P(1-z)^2. \end{cases} \tag{A.1}$$

We compute on the other hand

$$\inf_{\mathbb{P}\in\mathcal{P}_A^a} \mathbb{E}^{\mathbb{P}}\left[\mathcal{U}_A\left(\xi - \int_0^T k(a_s)ds\right)\right]$$
$$= \inf_{\mathbb{P}\in\mathcal{P}_A^a} \mathbb{E}^{\mathbb{P}}\left[-e^{-R_A\left(zB_T+\frac{\gamma}{2}\int_0^T \alpha_s^{\mathbb{P}}ds+\delta-\int_0^T k(a_s)ds\right)}\right]$$
$$= -e^{R_A(\int_0^T k(a_s)ds-\delta-z\int_0^T a_s ds)} \sup_{\mathbb{P}\in\mathcal{P}_P^a} \mathbb{E}^{\mathbb{P}}\left[\mathcal{E}\left(-R_A z\int_0^T (\alpha_s^{\mathbb{P}})^{\frac{1}{2}}dW_s^a\right) e^{R_A\left(\frac{R_A z^2}{2}-\frac{\gamma}{2}\right)\int_0^T \alpha_s^{\mathbb{P}}ds}\right].$$

Hence,

$$\inf_{\mathbb{P}\in\mathcal{P}^a_A}\mathbb{E}^{\mathbb{P}}\left[\mathcal{U}_A\left(\xi-\int_0^T k(a_s)ds\right)\right]=\begin{cases}\Gamma_A(a,z,\gamma,\delta,\overline{\alpha}^A), \text{ if } \gamma<R_Az^2,\\ \Gamma_A(a,z,\gamma,\delta,\underline{\alpha}^A), \text{ if } \gamma>R_Az^2,\\ \mathbb{E}^{\mathbb{P}}\left[\mathcal{U}_A\left(\xi-\int_0^T k(a_s)ds\right)\right], \ \forall\mathbb{P}\in\mathcal{P}^a_A \text{ if } \gamma=R_Az^2.\end{cases} \tag{A.2}$$

By combining (A.1) and (A.2) and using the definition (3.4), we conclude the proof of the lemma. □

Proof of Lemma 3.2. Let $(a,z,\gamma,\alpha_P,\alpha_A)\in\mathcal{A}\times\mathbb{R}\times\mathbb{R}\times[\underline{\alpha}^P,\overline{\alpha}^P]\times[\underline{\alpha}^A,\overline{\alpha}^A]$. First notice that the map $\delta\longmapsto F(a,z,\gamma,\delta,\alpha_P,\alpha_A)$ is clearly concave (we remind the reader that $\rho>0$). Using the first order condition for δ, we obtain after some calculations

$$\begin{aligned}&\frac{\partial F}{\partial\delta}(z,\gamma,\delta,\alpha_P,\alpha_A)=0\\ \Longleftrightarrow\ &\Gamma_P(z,\gamma,\delta,\alpha_P)=\rho\frac{R_A}{R_P}\Gamma_A(z,\gamma,\delta,\alpha_A)\\ \Longleftrightarrow\ &\delta=\frac{1}{R_A+R_P}\left[\log\left(\rho\frac{R_A}{R_P}\right)+(R_P(1-z)-R_Az)\int_0^T a_sds+R_A\int_0^T k(a_s)ds\right.\\ &\qquad\left.-\frac{R_P}{2}(R_P(1-z)^2+\gamma)\alpha_PT+\frac{R_A}{2}\left(R_Az^2-\gamma\right)\alpha_AT\right],\end{aligned}$$

which ends the proof. □

Proof of Lemma 3.3. (i) From Lemma 3.1(i) together with Lemma 3.2, we have

$$\begin{aligned}&\sup_{a\in\mathcal{A}_{\rm det}}\sup_{\xi\in\mathcal{Q}^\gamma}\inf_{\mathbb{P}\in\mathcal{P}^a_P}\mathbb{E}^{\mathbb{P}}\left[\mathcal{U}_P\left(B_T-\xi\right)\right]+\rho\inf_{\mathbb{P}\in\mathcal{P}^a_A}\mathbb{E}^{\mathbb{P}}\left[\mathcal{U}_A\left(\xi-\int_0^T k(a_s)ds\right)\right]\\ &=\sup_{a\in\mathcal{A}_{\rm det}}\sup_{z\in\mathbb{R}}\sup_{\gamma<-R_P(1-z)^2}F(a,z,\gamma,\delta^\star(z,\gamma,\underline{\alpha}^P,\overline{\alpha}^A),\underline{\alpha}^P,\overline{\alpha}^A),\end{aligned}$$

where

$$\begin{aligned}\delta^\star(z,\gamma,\underline{\alpha}^P,\overline{\alpha}^A):=\frac{1}{R_A+R_P}&\left[\log\left(\rho\frac{R_A}{R_P}\right)+\int_0^T\left((R_P(1-z)-R_Az)a_s+R_Ak(a_s)\right)ds\right.\\ &\left.-\frac{R_P}{2}(R_P(1-z)^2+\gamma)\underline{\alpha}^PT+\frac{R_A}{2}\left(R_Az^2-\gamma\right)\overline{\alpha}^AT\right],\end{aligned}$$

and where we recall that then

$$\begin{aligned}&F(a,z,\gamma,\delta^\star(z,\gamma,\underline{\alpha}^P,\overline{\alpha}^A),\underline{\alpha}^P,\overline{\alpha}^A)\\ &=-\rho^{\frac{R_P}{R_A+R_P}}\frac{R_A+R_P}{R_P}\left(\frac{R_A}{R_P}\right)^{-\frac{R_A}{R_A+R_P}}e^{\frac{R_AR_P}{R_A+R_P}\left(\int_0^T(k(a_s)-a_s)ds+\frac{\gamma}{2}T(\underline{\alpha}^P-\overline{\alpha}^A)\right)}\\ &\quad\times e^{\frac{R_AR_P}{R_A+R_P}\frac{T}{2}\left(\underline{\alpha}^PR_P(1-z)^2+\overline{\alpha}^AR_Az^2\right)}.\end{aligned}$$

$a)$ Assume that $\underline{\alpha}^P<\overline{\alpha}^A$. Then $\gamma\longmapsto F(a,z,\gamma,\delta^\star(z,\gamma,\underline{\alpha}^P,\overline{\alpha}^A),\underline{\alpha}^P,\overline{\alpha}^A)$ is increasing for $\gamma<-R_P(1-z)^2$ and is thus maximal at $\gamma^\star(z):=-R_P(1-z)^2$. Hence, by setting

$$\begin{aligned}\delta^\star(z):=\frac{1}{R_A+R_P}&\left[\log\left(\rho\frac{R_A}{R_P}\right)+(R_P(1-z)-R_Az)\int_0^T a_sds+R_A\int_0^T k(a_s)ds\right.\\ &\left.+\frac{R_AT}{2}\left(R_Az^2+R_P(1-z)^2\right)\overline{\alpha}^A\right],\end{aligned}$$

we have

$$\sup_{a\in\mathcal{A}_{\text{det}}}\sup_{z\in\mathbb{R}}\sup_{\gamma<-R_P(1-z)^2} F(a,z,\gamma,\delta^\star(z,\gamma,\underline{\alpha}^P,\overline{\alpha}^A),\underline{\alpha}^P,\overline{\alpha}^A)$$
$$=\sup_{a\in\mathcal{A}_{\text{det}}}\sup_{z\in\mathbb{R}} F(a,z,\gamma^\star(z),\delta^\star(z),\underline{\alpha}^P,\overline{\alpha}^A),$$

with

$$F(a,z,\gamma^\star(z),\delta^\star(z),\underline{\alpha}^P,\overline{\alpha}^A)$$
$$=-\rho^{\frac{R_P}{R_A+R_P}}\frac{R_A+R_P}{R_P}\left(\frac{R_A}{R_P}\right)^{-\frac{R_A}{R_A+R_P}} e^{\frac{R_AR_P}{R_A+R_P}\left(\int_0^T(k(a_s)-a_s)ds+\frac{T}{2}\overline{\alpha}^A\left(R_P(1-z)^2+R_Az^2\right)\right)}.$$

Hence, by choosing $z^\star:=\frac{R_P}{R_A+R_P}$, $a^\star$ the constant minimiser of $k(a)-a$, $\gamma^\star=-R_P(1-z^\star)^2$ and

$$\delta^\star:=\delta^\star(z^\star)=\frac{1}{R_A+R_P}\left[\log\left(\rho\frac{R_A}{R_P}\right)+R_ATk(a^\star)+\frac{R_A^2R_PT}{2(R_A+R_P)}\overline{\alpha}^A\right],$$

we have

$$\sup_{a\in\mathcal{A}_{\text{det}}}\sup_{\xi\in\mathcal{Q}^{\underline{\gamma}}}\widetilde{u}_0^{P,FB}(a,\xi)=F(a^\star,z^\star,\gamma^\star,\delta^\star,\underline{\alpha}^P,\overline{\alpha}^A)$$
$$=-\rho^{\frac{R_P}{R_A+R_P}}\frac{R_A+R_P}{R_P}\left(\frac{R_A}{R_P}\right)^{-\frac{R_A}{R_A+R_P}}\exp\left(\frac{R_AR_P}{R_A+R_P}T(k(a^\star)-a^\star)+\frac{T}{2}\frac{R_A^2R_P^2}{(R_A+R_P)^2}\overline{\alpha}^A\right).$$

b) Assume that $\underline{\alpha}^P=\overline{\alpha}^A=:\tilde{\alpha}$. Then $\gamma\longmapsto F(a,z,\gamma,\delta^\star(z,\gamma,\tilde{\alpha},\tilde{\alpha}),\tilde{\alpha},\tilde{\alpha})$ is constant for $\gamma<-R_P(1-z)^2$. Hence for any $\gamma<-R_P(1-z)^2$

$$\sup_{a\in\mathcal{A}_{\text{det}}}\sup_{z\in\mathbb{R}}\sup_{\gamma<-R_P(1-z)^2} F(a,z,\gamma,\delta^\star(z,\gamma,\underline{\alpha}^P,\overline{\alpha}^A),\underline{\alpha}^P,\overline{\alpha}^A)$$
$$=\sup_{a\in\mathcal{A}_{\text{det}}}\sup_{z\in\mathbb{R}} F(a,z,\gamma,\delta^\star(z,\gamma,\tilde{\alpha},\tilde{\alpha}),\tilde{\alpha}),$$

where

$$\delta^\star(z,\gamma,\tilde{\alpha}):=\frac{1}{R_A+R_P}\left[\log\left(\rho\frac{R_A}{R_P}\right)+(R_P(1-z)-R_Az)\int_0^T a_sds+R_A\int_0^T k(a_s)ds\right.$$
$$\left.(R_A^2z^2-R_P^2(1-z)^2)\tilde{\alpha}\frac{T}{2}\right]-\frac{\gamma}{2}\tilde{\alpha}T,$$

and

$$F(a,z,\gamma,\delta^\star(z,\gamma),\tilde{\alpha},\tilde{\alpha})$$
$$=-\rho^{\frac{R_P}{R_A+R_P}}\frac{R_A+R_P}{R_P}\left(\frac{R_A}{R_P}\right)^{-\frac{R_A}{R_A+R_P}} e^{\frac{R_AR_P}{R_A+R_P}\left(\int_0^T(k(a_s)-a_s)ds+\frac{T}{2}\tilde{\alpha}\left(R_P(1-z)^2+R_Az^2\right)\right)}.$$

Hence, by choosing $z^\star:=\frac{R_P}{R_A+R_P}$, $a^\star$ the constant minimiser of $k(a)-a$, for $\gamma^\star$ any value in $(-\infty,-R_P(1-z^\star)^2)$ and

$$\delta^\star:=\delta^\star(z^\star,\gamma^\star)=\frac{1}{R_A+R_P}\left[\log\left(\rho\frac{R_A}{R_P}\right)+R_ATk(a^\star)\right]-\frac{\gamma^\star}{2}\tilde{\alpha}T,$$

we have

$$\sup_{a\in\mathcal{A}_{\rm det}}\sup_{\xi\in\mathcal{Q}^{\underline{\gamma}}}\widetilde{u}_0^{P,FB}(a,\xi)=F(a^\star,z^\star,\gamma^\star,\delta^\star,\tilde{\alpha},\tilde{\alpha})$$
$$=-\rho^{\frac{R_P}{R_A+R_P}}\frac{R_A+R_P}{R_P}\left(\frac{R_A}{R_P}\right)^{-\frac{R_A}{R_A+R_P}}\exp\left(\frac{R_AR_PT}{R_A+R_P}\left(k(a^\star)-a^\star+\frac{R_AR_P}{2(R_A+R_P)}\tilde{\alpha}\right)\right).$$

(ii) From Lemma 3.1$(ii)a)$, together with Lemma 3.2, we have for any $\alpha_P\in[\underline{\alpha}^P,\overline{\alpha}^P]$

$$\sup_{a\in\mathcal{A}_{\rm det}}\sup_{\xi\in\mathcal{Q}^d}\widetilde{u}_0^{P,FB}(a,\xi)=\sup_{a\in\mathcal{A}_{\rm det}}\sup_{z\in\mathbb{R}}F(a,z,\gamma^\star,\delta^\star(z,\gamma^\star,\overline{\alpha}^A),\alpha_P,\overline{\alpha}^A),$$

where $\gamma^\star=-R_P(1-z)^2$, and

$$\delta^\star(z,\gamma^\star,\overline{\alpha}^A):=\frac{1}{R_A+R_P}\bigg[\log\left(\rho\frac{R_A}{R_P}\right)+(R_P(1-z)-R_Az)\int_0^Ta_sds+R_A\int_0^Tk(a_s)ds$$
$$+\frac{R_A}{2}\left(R_Az^2+R_P(1-z)^2\right)\overline{\alpha}^AT\bigg],$$

with also

$$F(a,z,\gamma^\star,\delta^\star(z,\gamma,\overline{\alpha}^A),\alpha_P,\overline{\alpha}^A)$$
$$=-\rho^{\frac{R_P}{R_A+R_P}}\frac{R_A+R_P}{R_P}\left(\frac{R_A}{R_P}\right)^{-\frac{R_A}{R_A+R_P}}e^{\frac{R_AR_P}{R_A+R_P}\left(\int_0^T(k(a_s)-a_s)ds+\frac{T}{2}\overline{\alpha}^A\left(R_P(1-z)^2+R_Az^2\right)\right)},$$

which does not depend on α_P. Hence, by choosing $z^\star:=\frac{R_P}{R_A+R_P}$, $a^\star$ the constant minimiser of $k(a)-a$, $\gamma^\star=-R_P(1-z^\star)^2$ and

$$\delta^\star:=\frac{1}{R_A+R_P}\left[\log\left(\rho\frac{R_A}{R_P}\right)+R_ATk(a^\star)+\frac{R_A^2R_PT}{2(R_A+R_P)}\overline{\alpha}^A\right],$$

we have

$$\sup_{a\in\mathcal{A}_{\rm det}}\sup_{\xi\in\mathcal{Q}^d}\widetilde{u}_0^{P,FB}(a,\xi)=F(a^\star,z^\star,\gamma^\star,\delta^\star,\alpha_P,\overline{\alpha}^A)$$
$$=-\rho^{\frac{R_P}{R_A+R_P}}\frac{R_A+R_P}{R_P}\left(\frac{R_A}{R_P}\right)^{-\frac{R_A}{R_A+R_P}}\exp\left(\frac{R_AR_PT}{R_A+R_P}\left(k(a^\star)-a^\star+\frac{R_AR_P}{2(R_A+R_P)}\overline{\alpha}^A\right)\right).$$

(iii) From Lemma 3.1$(ii)b)$ together with Lemma 3.2, we have

$$\sup_{a\in\mathcal{A}_{\rm det}}\sup_{\xi\in\mathcal{Q}^{|\gamma|}}\widetilde{u}_0^{P,FB}(a,\xi)=\sup_{a\in\mathcal{A}_{\rm det}}\sup_{z\in\mathbb{R}}\sup_{-R_P(1-z)^2<\gamma<R_Az^2}F(a,z,\gamma,\delta^\star(z,\gamma,\overline{\alpha}^P,\overline{\alpha}^A),\overline{\alpha}^P,\overline{\alpha}^A),$$

where

$$\delta^\star(z,\gamma,\overline{\alpha}^P,\overline{\alpha}^A):=\frac{1}{R_A+R_P}\bigg[\log\left(\rho\frac{R_A}{R_P}\right)+\int_0^T\left((R_P(1-z)-R_Az)a_s+R_Ak(a_s)\right)ds$$
$$-\frac{R_P}{2}(R_P(1-z)^2+\gamma)\overline{\alpha}^PT+\frac{R_A}{2}\left(R_Az^2-\gamma\right)\overline{\alpha}^AT\bigg],$$

and with

$$
\begin{aligned}
&F(a,z,\gamma,\delta^\star(z,\gamma,\overline{\alpha}^P,\overline{\alpha}^A),\overline{\alpha}^P,\overline{\alpha}^A)\\
&= -\rho^{\frac{R_P}{R_A+R_P}}\frac{R_A+R_P}{R_P}\left(\frac{R_A}{R_P}\right)^{-\frac{R_A}{R_A+R_P}} e^{\frac{R_AR_P}{R_A+R_P}\left(\int_0^T(k(a_s)-a_s)ds+\frac{\gamma}{2}T(\overline{\alpha}^P-\overline{\alpha}^A)\right)}\\
&\quad\times e^{\frac{R_AR_P}{R_A+R_P}\frac{T}{2}\left(\overline{\alpha}^PR_P(1-z)^2+\overline{\alpha}^AR_Az^2\right)}.
\end{aligned}
$$

a) Assume that $\overline{\alpha}^P<\overline{\alpha}^A$. Then $\gamma\longmapsto F(a,z,\gamma,\delta^\star(z,\gamma,\overline{\alpha}^P,\overline{\alpha}^A),\overline{\alpha}^P,\overline{\alpha}^A)$ is increasing for $-R_P(1-z)^2<\gamma<R_Az^2$ and is maximal at $\gamma^\star(z):=R_Az^2$. Hence, by setting

$$
\begin{aligned}
\delta^\star(z):=\frac{1}{R_A+R_P}\Bigg[&\log\left(\rho\frac{R_A}{R_P}\right)+(R_P(1-z)-R_Az)\int_0^T a_sds+R_A\int_0^T k(a_s)ds\\
&-\frac{R_PT}{2}\left(R_Az^2+R_P(1-z)^2\right)\overline{\alpha}^P\Bigg],
\end{aligned}
$$

we have

$$
\begin{aligned}
&\sup_{a\in\mathcal{A}_{\rm det}}\sup_{z\in\mathbb{R}}\sup_{-R_P(1-z)^2<\gamma<R_Az^2}F(a,z,\gamma,\delta^\star(z,\gamma,\overline{\alpha}^P,\overline{\alpha}^A),\overline{\alpha}^P,\overline{\alpha}^A)\\
&=\sup_{a\in\mathcal{A}_{\rm det}}\sup_{z\in\mathbb{R}}F(a,z,\gamma^\star(z),\delta^\star(z),\overline{\alpha}^P,\overline{\alpha}^A),
\end{aligned}
$$

with

$$
\begin{aligned}
&F(a,z,\gamma^\star(z),\delta^\star(z),\overline{\alpha}^P,\overline{\alpha}^A)\\
&=-\rho^{\frac{R_P}{R_A+R_P}}\frac{R_A+R_P}{R_P}\left(\frac{R_A}{R_P}\right)^{-\frac{R_A}{R_A+R_P}} e^{\frac{R_AR_P}{R_A+R_P}\left(\int_0^T(k(a_s)-a_s)ds+\frac{T}{2}\overline{\alpha}^P\left(R_P(1-z)^2+R_Az^2\right)\right)}.
\end{aligned}
$$

Hence, by choosing $z^\star:=\frac{R_P}{R_A+R_P}$, $a^\star$ the constant minimiser of $k(a)-a$, $\gamma^\star=R_A|z^\star|^2$ and

$$
\delta^\star:=\delta^\star(z^\star)=\frac{1}{R_A+R_P}\left[\log\left(\rho\frac{R_A}{R_P}\right)+R_ATk(a^\star)-\frac{R_AR_P^2T}{2(R_A+R_P)}\overline{\alpha}^P\right],
$$

we have

$$
\begin{aligned}
&\sup_{a\in\mathcal{A}_{\rm det}}\sup_{\xi\in\mathcal{Q}^{|\gamma|}}\widetilde{u}_0^{P,FB}(a,\xi)=F(a^\star,z^\star,\gamma^\star,\delta^\star,\overline{\alpha}^P,\overline{\alpha}^A)\\
&=-\rho^{\frac{R_P}{R_A+R_P}}\frac{R_A+R_P}{R_P}\left(\frac{R_A}{R_P}\right)^{-\frac{R_A}{R_A+R_P}}\exp\left(\frac{R_AR_P}{R_A+R_P}T(k(a^\star)-a^\star)+\frac{T}{2}\frac{R_A^2R_P^2}{(R_A+R_P)^2}\overline{\alpha}^P\right).
\end{aligned}
$$

b) Assume that $\overline{\alpha}^P=\overline{\alpha}^A=:\overline{\alpha}$. Then $\gamma\longmapsto F(a,z,\gamma,\delta^\star(z,\gamma,\overline{\alpha},\overline{\alpha}),\overline{\alpha},\overline{\alpha})$ is constant for $-R_P(1-z)^\star<\gamma<R_Az^2$. Hence for any $\gamma\in(-R_P(1-z)^2,R_Az^2)$

$$
\begin{aligned}
&\sup_{a\in\mathcal{A}_{\rm det}}\sup_{z\in\mathbb{R}}\sup_{\gamma\in(-R_P(1-z)^2,R_Az^2)}F(a,z,\gamma,\delta^\star(z,\gamma,\overline{\alpha}^P,\overline{\alpha}^A),\overline{\alpha}^P,\overline{\alpha}^A)\\
&=\sup_{a\in\mathcal{A}_{\rm det}}\sup_{z\in\mathbb{R}}F(a,z,\gamma,\delta^\star(z,\gamma),\overline{\alpha},\overline{\alpha}),
\end{aligned}
$$

where

$$
\begin{aligned}
\delta^\star(z,\gamma):=\frac{1}{R_A+R_P}\Bigg[&\log\left(\rho\frac{R_A}{R_P}\right)+(R_P(1-z)-R_Az)\int_0^T a_sds+R_A\int_0^T k(a_s)ds\\
&+(R_Az^2-R_P(1-z)^2)\overline{\alpha}\frac{T}{2}\Bigg]-\frac{\gamma}{2}\overline{\alpha}T,
\end{aligned}
$$

and with

$$
\begin{aligned}
&F(a,z,\gamma,\delta^\star(z,\gamma),\overline{\alpha},\overline{\alpha})\\
&= -\rho^{\frac{R_P}{R_A+R_P}}\frac{R_A+R_P}{R_P}\left(\frac{R_A}{R_P}\right)^{-\frac{R_A}{R_A+R_P}} e^{\frac{R_AR_P}{R_A+R_P}\left(\int_0^T(k(a_s)-a_s)ds+\frac{T}{2}\overline{\alpha}\left(R_P(1-z)^2+R_Az^2\right)\right)}.
\end{aligned}
$$

Hence, by choosing $z^\star := \frac{R_P}{R_A+R_P}$, $a^\star$ the constant minimiser of $k(a)-a$, any $\gamma^\star\in(-R_P(1-z)^2,R_Az^2)$ and

$$
\delta^\star := \delta^\star(z^\star,\gamma^\star) = \frac{1}{R_A+R_P}\left[\log\left(\rho\frac{R_A}{R_P}\right)+R_ATk(a^\star)\right]-\frac{\gamma^\star}{2}\overline{\alpha}T,
$$

we have

$$
\begin{aligned}
&\sup_{a\in\mathcal{A}_{\rm det}}\sup_{\xi\in\mathcal{Q}^{\underline{\gamma}}}\widetilde{u}_0^{P,FB}(a,\xi) = F(a^\star,z^\star,\gamma^\star,\delta^\star,\overline{\alpha},\overline{\alpha})\\
&= -\rho^{\frac{R_P}{R_A+R_P}}\frac{R_A+R_P}{R_P}\left(\frac{R_A}{R_P}\right)^{-\frac{R_A}{R_A+R_P}}\exp\left(\frac{R_AR_PT}{R_A+R_P}\left((k(a^\star)-a^\star)+\frac{R_AR_P}{2(R_A+R_P)}\overline{\alpha}\right)\right).
\end{aligned}
$$

c) Assume that $\overline{\alpha}^P>\overline{\alpha}^A$. The proof is exactly the same as in the case $\overline{\alpha}^P<\overline{\alpha}^A$, and we obtain, with $z^\star=\frac{R_P}{R_A+R_P}$, $\gamma^\star=-R_P(1-z^\star)^2$, and

$$
\delta^\star := \frac{1}{R_A+R_P}\left[\log\left(\rho\frac{R_A}{R_P}\right)+R_ATk(a^\star)+\frac{R_A^2R_PT}{2(R_A+R_P)}\overline{\alpha}^A\right],
$$

that

$$
\begin{aligned}
&\sup_{a\in\mathcal{A}_{\rm det}}\sup_{\xi\in\mathcal{Q}^u}\widetilde{u}_0^{P,FB}(a,\xi) = F(a^\star,z^\star,\gamma^\star,\delta^\star,\overline{\alpha}^P,\overline{\alpha}^A)\\
&= -\rho^{\frac{R_P}{R_A+R_P}}\frac{R_A+R_P}{R_P}\left(\frac{R_A}{R_P}\right)^{-\frac{R_A}{R_A+R_P}}\exp\left(\frac{R_AR_P}{R_A+R_P}T(k(a^\star)-a^\star)+\frac{T}{2}\frac{R_A^2R_P^2}{(R_A+R_P)^2}\overline{\alpha}^A\right).
\end{aligned}
$$

(iv) The proof is similar to the case (ii). It suffices to change $\overline{\alpha}^A$ into $\overline{\alpha}^P$ and choose $\gamma^\star=R_A|z^\star|^2$.

(v) From Lemma 3.1(iii) together with Lemma 3.2, we have

$$
\sup_{a\in\mathcal{A}_{\rm det}}\sup_{\xi\in\mathcal{Q}^{\overline{\gamma}}}\widetilde{u}_0^{P,FB}(a,\xi) = \sup_{a\in\mathcal{A}_{\rm det}}\sup_{z\in\mathbb{R}}\sup_{\gamma>R_Az^2} F(a,z,\gamma,\delta^\star(z,\gamma,\overline{\alpha}^P,\underline{\alpha}^A),\overline{\alpha}^P,\underline{\alpha}^A),
$$

where

$$
\begin{aligned}
\delta^\star(z,\gamma,\overline{\alpha}^P,\underline{\alpha}^A) := \frac{1}{R_A+R_P}\Bigg[&\log\left(\rho\frac{R_A}{R_P}\right)+\int_0^T((R_P(1-z)-R_Az)a_s+R_Ak(a_s))\,ds\\
&-\frac{R_P}{2}(R_P(1-z)^2+\gamma)\overline{\alpha}^PT+\frac{R_A}{2}\left(R_Az^2-\gamma\right)\underline{\alpha}^AT\Bigg],
\end{aligned}
$$

and with

$$
\begin{aligned}
&F(a,z,\gamma,\delta^\star(z,\gamma,\overline{\alpha}^P,\underline{\alpha}^A),\overline{\alpha}^P,\underline{\alpha}^A)\\
&= -\rho^{\frac{R_P}{R_A+R_P}}\frac{R_A+R_P}{R_P}\left(\frac{R_A}{R_P}\right)^{-\frac{R_A}{R_A+R_P}} e^{\frac{R_AR_P}{R_A+R_P}\left(\int_0^T(k(a_s)-a_s)ds+\frac{\gamma}{2}T(\overline{\alpha}^P-\underline{\alpha}^A)\right)}\\
&\quad\times e^{\frac{R_AR_P}{R_A+R_P}\frac{T}{2}\left(\overline{\alpha}^PR_P(1-z)^2+\underline{\alpha}^AR_Az^2\right)}.
\end{aligned}
$$

a) Assume that $\overline{\alpha}^P = \underline{\alpha}^A =: \breve{\alpha}$. Then $\gamma \longmapsto F(a,z,\gamma,\delta^\star(z,\gamma,\breve{\alpha},\breve{\alpha}),\breve{\alpha},\breve{\alpha})$ is constant for $\gamma > R_A z^2$. Hence for any $\gamma > R_A z^2$

$$\begin{aligned}&\sup_{a\in\mathcal{A}_{\rm det}} \sup_{z\in\mathbb{R}} \sup_{\gamma > R_A z^2} F(a,z,\gamma,\delta^\star(z,\gamma,\overline{\alpha}^P,\underline{\alpha}^A),\overline{\alpha}^P,\underline{\alpha}^A)\\ &= \sup_{a\in\mathcal{A}_{\rm det}} \sup_{z\in\mathbb{R}} F(a,z,\gamma,\delta^\star(z,\gamma),\breve{\alpha},\breve{\alpha}),\end{aligned}$$

where

$$\begin{aligned}\delta^\star(z,\gamma) := \frac{1}{R_A+R_P}\bigg[\log\left(\rho\frac{R_A}{R_P}\right) + (R_P(1-z) - R_A z)\int_0^T a_s ds + R_A\int_0^T k(a_s)ds\\ +(R_A z^2 - R_P(1-z)^2)\breve{\alpha}\frac{T}{2}\bigg] - \frac{\gamma}{2}\breve{\alpha}T,\end{aligned}$$

and

$$\begin{aligned}&F(a,z,\gamma,\delta^\star(z,\gamma),\breve{\alpha},\breve{\alpha})\\ &= -\rho^{\frac{R_P}{R_A+R_P}}\frac{R_A+R_P}{R_P}\left(\frac{R_A}{R_P}\right)^{-\frac{R_A}{R_A+R_P}} e^{\frac{R_A R_P}{R_A+R_P}\left(\int_0^T (k(a_s)-a_s)ds + \frac{T}{2}\breve{\alpha}\left(R_P(1-z)^2 + R_A z^2\right)\right)}.\end{aligned}$$

Hence, by choosing $z^\star := \frac{R_P}{R_A+R_P}$, $a^\star$ the constant minimiser of $k(a) - a$, any $\gamma^\star > R_A|z^\star|^2$ and

$$\delta^\star := \delta^\star(z^\star,\gamma^\star) = \frac{1}{R_A+R_P}\left[\log\left(\rho\frac{R_A}{R_P}\right) + R_A T k(a^\star)\right] - \frac{\gamma^\star}{2}\breve{\alpha}T,$$

we have

$$\begin{aligned}&\sup_{a\in\mathcal{A}_{\rm det}} \sup_{\xi\in\mathcal{Q}^{\underline{\gamma}}} \widetilde{u}_0^{P,FB}(a,\xi) = F(a^\star,z^\star,\gamma^\star,\delta^\star,\breve{\alpha},\breve{\alpha})\\ &= -\rho^{\frac{R_P}{R_A+R_P}}\frac{R_A+R_P}{R_P}\left(\frac{R_A}{R_P}\right)^{-\frac{R_A}{R_A+R_P}} \exp\left(\frac{R_A R_P T}{R_A+R_P}\left((k(a^\star) - a^\star) + \frac{R_A R_P}{2(R_A+R_P)}\breve{\alpha}\right)\right).\end{aligned}$$

b) Assume that $\overline{\alpha}^P > \underline{\alpha}^A$. Then $\gamma \longmapsto F(a,z,\gamma,\delta^\star(z,\gamma,\underline{\alpha}^P,\overline{\alpha}^A),\underline{\alpha}^P,\overline{\alpha}^A)$ is decreasing for $\gamma > R_A z^2$ and is maximal at $\gamma^\star(z) := R_A z^2$. Hence, by setting

$$\begin{aligned}\delta^\star(z) := \frac{1}{R_A+R_P}\bigg[\log\left(\rho\frac{R_A}{R_P}\right) + (R_P(1-z) - R_A z)\int_0^T a_s ds + R_A\int_0^T k(a_s)ds\\ -\frac{R_P T}{2}\left(R_A z^2 + R_P(1-z)^2\right)\overline{\alpha}^P\bigg],\end{aligned}$$

we have

$$\begin{aligned}&\sup_{a\in\mathcal{A}_{\rm det}} \sup_{z\in\mathbb{R}} \sup_{\gamma > R_A z^2} F(a,z,\gamma,\delta^\star(z,\gamma,\overline{\alpha}^P,\underline{\alpha}^A),\overline{\alpha}^P,\underline{\alpha}^A)\\ &= \sup_{a\in\mathcal{A}_{\rm det}} \sup_{z\in\mathbb{R}} F(a,z,\gamma^\star(z),\delta^\star(z),\overline{\alpha}^P,\underline{\alpha}^A),\end{aligned}$$

with

$$\begin{aligned}&F(a,z,\gamma^\star(z),\delta^\star(z),\overline{\alpha}^P,\underline{\alpha}^A)\\ &= -\rho^{\frac{R_P}{R_A+R_P}}\frac{R_A+R_P}{R_P}\left(\frac{R_A}{R_P}\right)^{-\frac{R_A}{R_A+R_P}} e^{\frac{R_A R_P}{R_A+R_P}\left(\int_0^T (k(a_s)-a_s)ds + \frac{T}{2}\overline{\alpha}^P\left(R_P(1-z)^2 + R_A z^2\right)\right)}.\end{aligned}$$

Hence, by choosing $z^\star := \frac{R_P}{R_A+R_P}$, $a^\star$ the constant minimiser of $k(a)-a$, $\gamma^\star = R_A|z^\star|^2$ and

$$\delta^\star := \delta^\star(z^\star) = \frac{1}{R_A+R_P}\left[\log\left(\rho\frac{R_A}{R_P}\right) + R_A T k(a^\star) - \frac{R_A R_P^2 T}{2(R_A+R_P)}\overline{\alpha}^P\right],$$

we have

$$\begin{aligned}
&\sup_{a\in\mathcal{A}_{\text{det}}}\ \sup_{\xi\in\mathcal{Q}^{\gamma}} \widetilde{u}_0^{P,FB}(a,\xi) = F(a^\star, z^\star, \gamma^\star, \delta^\star, \overline{\alpha}^P, \underline{\alpha}^A) \\
&= -\rho^{\frac{R_P}{R_A+R_P}}\frac{R_A+R_P}{R_P}\left(\frac{R_A}{R_P}\right)^{-\frac{R_A}{R_A+R_P}}\exp\left(\frac{R_A R_P}{R_A+R_P}T(k(a^\star)-a^\star) + \frac{T}{2}\frac{R_A^2 R_P^2}{(R_A+R_P)^2}\overline{\alpha}^P\right).
\end{aligned}$$

□

Proof of Lemma 3.4. For any $a\in\mathcal{A}$, let us define $\xi_a := z^\star B_T + \frac{\gamma^\star}{2}\langle B\rangle_T + \delta^\star(a)$ where $\gamma^\star\in\mathbb{R}$,

$$z^\star = \frac{R_P}{R_A+R_P},\ \delta^\star(a) := \frac{1}{R_A+R_P}\left(\log\left(\rho\frac{R_A}{R_P}\right) + R_A\int_0^T k(a_s)ds\right) + \lambda,\ \lambda\in\mathbb{R}.$$

Then, for any $h\in M^\phi$

$$\begin{aligned}
&\widetilde{D}\Xi_a^{\alpha_P,\alpha_A}(\xi_a)[h] \\
&= \mathbb{E}^{\mathbb{P}_0}\Big[R_P h(X_\cdot^{a,\alpha_P})e^{-R_P\left(\int_0^T a_s(X_\cdot^{a,\alpha_P})ds + \alpha_P^{\frac{1}{2}}B_T - \xi_a(X_\cdot^{a,\alpha_P})\right)} \\
&\quad - R_A h(X_\cdot^{a,\alpha_A})\rho e^{-R_A\left(\xi_a(X_\cdot^{a,A}) - \int_0^T k(a_s(X_\cdot^{a,\alpha_A}))ds\right)}\Big] \\
&= \mathbb{E}^{\mathbb{P}_0}\Big[R_P h(X_\cdot^{a,\alpha_P})e^{-R_P\left(\frac{R_A}{R_A+R_P}\int_0^T a_s(X_\cdot^{a,\alpha_p})ds + \alpha_P^{\frac{1}{2}}\frac{R_A}{R_A+R_P}B_T - \frac{\gamma^\star}{2}\alpha_P T - \delta^\star(a))\right)} \\
&\quad - R_A h(X_\cdot^{a,\alpha_A})\rho e^{-R_A\left(\frac{R_P}{R_A+R_P}\alpha_A^{\frac{1}{2}}B_T + \frac{R_P}{R_A+R_P}\int_0^T a_s(X_\cdot^{a,\alpha_A})ds + \frac{\gamma^\star}{2}\alpha_A T + \delta^\star(a) - \int_0^T k(a_s(X_\cdot^{a,\alpha_A}))ds\right)}\Big] \\
&= R_P\left(\rho\frac{R_A}{R_P}\right)^{\frac{R_P}{R_A+R_P}} \\
&\quad\Bigg(\mathbb{E}^{\mathbb{P}_0^{\alpha_P}}\Big[h(X_\cdot^{a,\alpha_P})e^{-\frac{R_A R_P}{R_A+R_P}\int_0^T(a_s(X_\cdot^{a,\alpha_P}) - k(a_s(X_\cdot^{a,\alpha_P})))ds + R_P\left(\frac{\gamma^\star}{2}\alpha_P T + \lambda\right) + \frac{R_P^2 R_A^2}{2(R_A+R_P)^2}\alpha_P T} \\
&\quad - \mathbb{E}^{\mathbb{P}_0^{\alpha_A}}\Big[h(X_\cdot^{a,\alpha_A})e^{-\frac{R_A R_P}{R_A+R_P}\int_0^T(a_s(X_\cdot^{a,\alpha_A}) - k(a_s(X_\cdot^{a,\alpha_A})))ds + R_A\left(\frac{\gamma^\star}{2}\alpha_A T + \lambda\right) + \frac{R_P^2 R_A^2}{2(R_A+R_P)^2}\alpha_A T}\Bigg),
\end{aligned} \tag{A.3}$$

where

$$\frac{d\mathbb{P}_0^{\alpha_P}}{d\mathbb{P}_0} := \mathcal{E}\left(-\frac{R_P R_A(\alpha_P)^{\frac{1}{2}}}{R_A+R_P}B_T\right),\ \frac{d\mathbb{P}_0^{\alpha_A}}{d\mathbb{P}_0} := \mathcal{E}\left(-\frac{R_P R_A(\alpha_A)^{\frac{1}{2}}}{R_A+R_P}B_T\right).$$

Assume that $\alpha_P = \alpha_A =: \alpha$. Then, if $R_A = R_P$ or if $R_A \neq R_P$ and Property (3.7) holds then we automatically have

$$\widetilde{D}\Xi_a^{\alpha_P,\alpha_A}(\xi)[h-\xi_a] = 0,$$

which proves the first result.

□

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