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BIRD’S-EYEVIEW OF THE SOLAR DYNAMO

Bertrand C. Barrois

A model of solar and planetary dynamos is constructed by thinning an expansion of the magnetic fields and fluid flows in vector spherical harmonics. The stripped-down model deals with the poloidal dipole field, the toroidal quadrupole field, and differential rotation, which are represented by tensors and coupled so as to conserve total energy. It imitates the mathematical structure of the fundamental equations, which feature a combination of linear, bilinear, and autonomous terms. Interactions with omitted modes motivate the inclusion of driving and damping terms. The dynamo mechanism involves hairpin formation and regeneration, the latter being attributed to a sum over transient eddies and resulting distortions of the magnetic field, summarized by a single term that respects all physical symmetries. The model predicts oscillation and spontaneous alignment with the axis of rotation. It is amenable to calibration against the results of detailed simulations.

The dynamo mechanism by which turbulent convection in the outer core of the Earth and the outer layers of the Sun sustains the magnetic field deserves the appellation “MHD Hell”. It poses all the difficulties of computational fluid dynamics, but resists experimental validation.

Recent computational approaches involving supercomputer simulations produce large volumes of detail but relatively little insight into the correlations that count. They bear out the adage, “You cannot see the forest for the trees.” The goal of this paper is to map the forest at lower resolution from higher altitude.

After surveying the mathematical framework in which dynamo models must operate, we construct a ruthlessly thinned model that focuses on half-a-dozen key variables. It proves adequate to predict (1) oscillatory behavior, and (2) alignment with the axis of rotation. It identifies the key correlations and is amenable to calibration against the results of detailed simulations.

One might hope that such a God’s-eye view of MHD Hell will serve to guide the computational investigators in their search for the needle in a haystack, if only by furnishing a description of the needle. But enough with the mixed metaphors. Let’s get down to business.

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Mathematical Framework

At a fundamental level, MHD heat transport is governed by three interlocking equations, with a combination of linear, bilinear, and constant terms on the right hand side:

\[
\begin{align*}
\frac{\partial \mathcal{H}}{\partial t} &= \mathcal{X}_H \nabla^2 \mathcal{H} - (\mathbf{V} \cdot \nabla)(\mathcal{H} + \mathcal{H}) + \frac{\partial}{\partial \mathcal{P}} \left( \mathbf{g} \cdot \mathbf{V} \right) \mathcal{H} + \frac{1}{\mathcal{C}_P} J^2 + \frac{1}{2} \mathcal{X}_V \mathcal{S}^2 + (\text{sources}) \\
\frac{\partial \mathcal{V}}{\partial t} &= \mathcal{X}_V \nabla^2 \mathcal{V} - (\mathbf{V} \cdot \nabla) \mathcal{V} + 2 \mathcal{\Omega} \times \mathcal{V} - \frac{1}{\mathcal{P}} \nabla \mathcal{P} - \mathbf{g} \frac{\partial}{\partial \mathcal{P}} \mathcal{H} + \frac{1}{\mathcal{P}} \mathbf{J} \times \mathbf{B} \\
\frac{\partial \mathcal{B}}{\partial t} &= \mathcal{X}_B \nabla^2 \mathcal{B} + \text{curl}(\mathbf{V} \times \mathcal{B}) = \frac{1}{\mathcal{P}\mathcal{\Omega}} \nabla^2 \mathcal{B} - (\mathbf{V} \cdot \nabla) \mathcal{B} + (\mathbf{B} \cdot \nabla) \mathcal{V} - \mathcal{B} (\nabla \cdot \mathbf{V}) \\
\mathbf{J} &= \frac{1}{\mathcal{\mu}} \text{curl}(\mathcal{B}) = \mathcal{\sigma}(\mathcal{E} + \mathbf{V} \times \mathbf{B})
\end{align*}
\]

<table>
<thead>
<tr>
<th>Term</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\mathcal{H})</td>
<td>time-average enthalpy profile *</td>
</tr>
<tr>
<td>(\mathcal{H})</td>
<td>enthalpy fluctuation *</td>
</tr>
<tr>
<td>(\mathcal{P})</td>
<td>pressure fluctuation</td>
</tr>
<tr>
<td>(\mathcal{V})</td>
<td>velocity of fluid</td>
</tr>
<tr>
<td>(\mathcal{S})</td>
<td>shear-rate tensor</td>
</tr>
<tr>
<td>(\mathcal{B})</td>
<td>magnetic field</td>
</tr>
<tr>
<td>(\mathcal{J})</td>
<td>electrical current</td>
</tr>
<tr>
<td>(\mathbf{g})</td>
<td>acceleration of gravity</td>
</tr>
<tr>
<td>(\mathcal{\Omega})</td>
<td>angular velocity</td>
</tr>
<tr>
<td>(\mathcal{C}_P)</td>
<td>specific heat *</td>
</tr>
<tr>
<td>(\mathcal{X}_H)</td>
<td>heat diffusion</td>
</tr>
<tr>
<td>(\mathcal{X}_V)</td>
<td>kinematic viscosity</td>
</tr>
<tr>
<td>(\mathcal{X}_B)</td>
<td>magnetic diffusion</td>
</tr>
<tr>
<td>(\mathcal{\mu})</td>
<td>magnetic permeability</td>
</tr>
<tr>
<td>(\mathcal{\sigma})</td>
<td>electrical conductivity</td>
</tr>
<tr>
<td>(\mathcal{\rho})</td>
<td>density of fluid</td>
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<tr>
<td>(\mathcal{\alpha})</td>
<td>thermal expansion</td>
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* Per unit mass

Convective instability is induced by a superadiabatic gradient: \(\nabla \mathcal{H} = \left| \mathcal{C}_P \nabla \mathcal{T} \right| > g\). In the Sun, the convective shell is heated from below and cooled from above. In the Earth, the power source is thought to be the settling of heavier elements, which releases gravitational potential energy, but the effect is much the same. The heat transport equation therefore contains a time-independent source term, as well as the quadratic redeposition terms representing viscous and ohmic dissipation.

The evolution of the magnetic field is simply governed by the induction equation, whereas fluid flow is influenced by Coriolis, buoyant, and magnetic forces. The conservation of energy may be verified as an exercise in double-entry bookkeeping. The energy flows are shown in Figure 1. At the three-way junction that diverts energy into electromagnetic fields, \(\mathbf{V} \cdot (\mathbf{J} \times \mathbf{B}) - \mathbf{J} \cdot \mathbf{E} + \mathcal{\sigma}(\mathcal{E} + \mathbf{V} \times \mathbf{B})^2 = 0\).
Detailed simulations may be implemented either on a spatial grid or by expansion of the fields and flows in a basis set of vector spherical harmonics (VSH).²

- Expansions in VSH respect spherical symmetry explicitly, but every harmonic interacts with every other harmonic. The radial dependence is arbitrary and may be chosen for convenience. (Glatzmaier 1995; Clune 1999)

- Simulations on a grid are computationally more efficient, because the equations are local. (But the Navier-Stokes equation is not quite as local as it looks, because the pressure must be calculated by solving Poisson’s equation -- a tedious step, even with Fast Fourier Transforms.)

Thinned models deal with a truncated set of VSH. Although any two harmonics may interact, selection rules restrict the outcomes:

- Clebsch-Gordan rules for vectors as well as scalars: \((l', m')(l'', m'') \rightarrow (l''', m''')\) processes require that \(m' + m'' = m'''\) and \(|l' - l''| \leq l''' \leq |l' + l''|\).

---

² These are divergence-free vector fields that transform as ordinary spherical harmonics under rotations. They are generated by repeatedly taking the curl of a purely radial vector field. **Poloidal** ("S") patterns have a radial component, and their tangential components form irrotational patterns. **Toroidal** ("T") patterns have no radial component, and their tangential components form swirling patterns. We shall use idiosyncratic but self-explanatory notation: BS(L,M) poloidal field, BT(L,M) toroidal field, VS(L,M) poloidal flow, and VT(L,M) toroidal flow. Since density varies with depth, the divergence-free pattern VS(L,M) actually represents \(\rho V\). For a good introduction, see (Backus 1996).
Parity rules: As vectors, \( S(L,M) \) and \( T(L,M) \) have opposite parity. It follows that \( TT \to T \), \( SS \to T \), \( ST \to S \) and \( TS \to S \) processes require \( L' + L'' + L''' = \text{odd} \); whereas \( SS \to S \), \( ST \to T \) and \( TS \to T \) processes require \( L' + L'' + L''' = \text{even} \).

- TT\( \to S \) processes are always forbidden, for lack of a radial component.
- There are further selection rules for hydrodynamic processes in a flat (or very thin) layer with hard walls. They exploit the fact that the flow patterns would have definite parity under exchange of the upper and lower boundaries, VT being \textit{even}, VS \textit{odd}. Magnetic boundary conditions are never so symmetric.

One may finesse the need for detailed simulation of heat transport by taking a free ride on convective instability analysis. Given the background temperature profile, one may use the linearized equations off heat transport to calculate the exponential growth rates of convective modes.

**Mathematical Prototype**

The thinned model should copy the key features of the fundamental equations. The prototype will be a system of coupled ODEs, with bilinear terms on the right hand side, and coefficients chosen to conserve the sum of squares, as well as a Liouville condition:

\[
\frac{d}{dt} X^i = \frac{1}{2} \sum_{jk} f_{jk}^i X^j X^k
\]

\[
f_{jk}^i + f_{ki}^j + f_{ij}^k = 0 \Rightarrow \frac{d}{dt} \sum_i (X^i)^2 = 0
\]

\[
\sum_i f_{ij}^i = 0 \Rightarrow \sum_i \frac{\partial X^i}{\partial X^i} = 0
\]

The trajectories of such a system never leave the hypersphere, which they almost always cover ergodically and uniformly. However, when linear driver/damper terms are added, the trajectories tend to approach an attractor set of fractal character.

Turbulent transport processes that mimic diffusion may be represented by linear damping terms, whereas interactions with omitted variables motivate the inclusion of additive noise terms with appropriate time-dependence. Poloidal flows driven by buoyancy may be modeled with linear drivers.
Thinned Model of the Dynamo

Dramatis personae:

- $\Omega T(1,0)$ = overall rotation rate
- $BS(1,0)$ = poloidal dipole field
- $BT(2,0)$ = toroidal quadrupole field
- $VT(3,0)$ = differential rotation
- $VS(L,M)$ = a representative poloidal eddy
- $VT(L\pm1,M)$ = its toroidal Coriolis sidekick
- $BS(L\pm1,M)$ = transient distortion of the main field
- $BT(L,M)$ = transient distortion of the main field

The equations of the model must capture four key effects:

- Coriolis effect: A linear process twists $VS(L,M)$ eddies to $VT(L\pm1,M)$. This process would be forbidden by parity selection rules in a flat layer with hard walls, but the convective shells of the Earth and Sun are neither thin nor flat.

- Hairpin formation: $VT(3,0)$ distorts $BS(1,0)$ to generate $BT(2,0)$. This process does not draw energy from $BS(1,0)$.

- Regeneration: A complex black-box process by which transient eddies and their sidekicks act in tandem on $BT(2,0)$ to regenerate $BS(1,0)$. Figure 2 lifts the lid of the black box to show how regeneration might work.

- Reactive forces: A further process, required by conservation of energy, by which $BT(2,0)$ and $BS(1,0)$ may either drive or oppose $VT(3,0)$.

Figure 2. Black-Box Mechanism for Regeneration
The process of hairpin formation is uncontroversial and is commonly known as the omega effect, but regeneration is more controversial. Mean-field theories of regeneration posit a quasi-local correlation between transient fields and flows that induces a JT(1,0) current loop. The literature mutters mystically of alpha and beta effects (e.g., Schrinner 2008). To avoid confusion with these preempted terms, we shall refer to regeneration as the zeta effect, making the non-local black-box mechanism an omega-zetadynamo.

The black-box mechanism for regeneration posits correlations among three kinds of transient variables: convective eddies, their sidekicks, and resulting distortions of the fields. These correlations can only be determined via detailed simulations.

The overall sign of the black-box mechanism is still uncertain. Since stimulus and response should be positively correlated, the sign comes down to the couplings, which we now write in shorthand, with steady main-field components barred:

\[ +f(\tilde{\Omega}T: VS \rightarrow VT) f(VS : \tilde{B}T \rightarrow BS) f(VT : BS \rightarrow \tilde{B}S) \langle VT BS \rangle \]

\[ +f(\tilde{\Omega}T: VS \rightarrow VT) f(VT : \tilde{B}T \rightarrow BT) f(VS : BT \rightarrow \tilde{B}S) \langle VS BT \rangle \]

The black-box mechanism is consistent with Cowling’s Theorem, which maintains that no axisymmetric flow can sustain an axisymmetric field. Per Clebsch-Gordan rules, if \( L' + L'' + L''' = \text{odd} \), as it is here, then \((-M',-M'',-M''') = -(M',M'',M''')\), hence the axisymmetric process with \( M' = M'' = M''' = 0 \) is forbidden.

**Cyclical Behavior**

Conservation of energy demands that the coefficients of bilinear terms obey the cyclic sum condition. The energy-conserving processes involving axisymmetric variables are paired as follows:

\[ \frac{d}{dt} BT(2,0) = -\omega VT(3,0) BS(1,0) \]
\[ \frac{d}{dt} VT(3,0) = +\omega BT(2,0) BS(1,0) \]

---

3 By standard definition, \( \langle \mathbf{V} \times \mathbf{B} \rangle = \alpha \mathbf{B} + \beta \text{curl}(\mathbf{B}) \). The coefficient, is odd under parity (\( P \)) symmetry. It cannot legitimately be derived from any power of the rotation parameter \( \Omega \), which is even under parity, but it can be related to the helicity of turbulence. Helicity is also odd under parity, but simulations find local helicity to be an odd function of latitude, thereby resolving the paradox. Time reversal invariance is violated by dissipative systems.
Since VT(3,0) does not vary appreciably over the solar cycle, let us simply treat it as constant. We now throw in a one-line summary of the black box, which steals energy from convectively driven eddies:

\[
\frac{d}{dt} BT(2,0) = -\omega VT(3,0) BS(1,0)
\]

\[
\frac{d}{dt} BS(1,0) = +\zeta \Omega T(1,0) BT(2,0)
\]

If \( W \equiv \omega VT(3,0) \) and \( Z \equiv \zeta \Omega T(1,0) \) have the same sign,\(^4\) then these equations predict oscillatory behavior, as actually observed in the Sun. (On the other hand, if these quasi-constant coefficients were to start out with opposite signs, the equations would predict runaway exponential growth of the fields, but the reactive force would put the brakes on differential rotation and quickly reverse it.)

But before we count this as triumph, we must deal with a paradox. So far, the equations predict that BT(2,0) and BS(1,0) will oscillate exactly 90° out of phase, so that their product averages to zero and cannot possibly drive VT(3,0). This defect can be cured by including linear damping terms. If BS(1,0) is more strongly damped than BT(2,0), then it lags < 90° behind BT(2,0), and their correlation has the correct sign to drive VT(3,0).

The famous Butterfly Diagram furnishes visible evidence that this is precisely what happens. Figure 3 reveals the latitudes at which sunspots occur at various phases of the solar cycle. By one interpretation,\(^5\) sunspots occur predominantly at the latitudes where tangential fields are strongest: BT(2,0) at mid-latitudes, BS(1,0) at the equator. If the phase lag were ±90°, then no forward-backward asymmetry would be expected. The observed asymmetry indicates that BS(1,0) is lagging roughly 45° behind BT(2,0).

---

\(^4\) Differential rotation on the Sun is seen to be prograde at the equator, retrograde at the poles. Under our sign convention, \( \omega > 0 \) as well as \( \Omega T(1,0) > 0 \) and \( VT(3,0) > 0 \). The overall sign of our black-box \( \zeta \)-effect is still uncertain. If we get it wrong, then we would wrongly predict VT(3,0) < 0, but we would still predict oscillatory behavior.

\(^5\) A different interpretation of the Butterfly Diagram attributes latitude migration to a slow meridional flow (e.g., Hathaway 2003).
**Amplification by Hyperbolic Sloshing**

Having included damping terms, we must also include driving terms, which might be modeled as additive noise, or more exotically, as hyperbolic sloshing. The latter mechanism is intriguing because it explains how a convectively driven eddy can pump up main field components that are not directly driven.

Consider any three components coupled so as to conserve energy:

\[
\begin{align*}
\frac{d}{dt} A &= \alpha BC \\
\frac{d}{dt} B &= \beta AC \quad \text{where} \quad \alpha + \beta + \gamma = 0 \\
\frac{d}{dt} C &= \gamma AB
\end{align*}
\]

If \( C(t) \) is directly driven, it may be considered an autonomous process that is not influenced by donating energy to \( A \) and/or \( B \). If \( \alpha \beta > 0 \), then \( (A, B) \) is driven along a hyperbolic path, \( A^2 / \alpha - B^2 / \beta = \text{const} \), and either large-amplitude oscillations or the cumulative effect of a random walk will push \( (A, B) \) far out onto an asymptote, where \( dA/dB \approx A/B \approx \pm \sqrt{\alpha / \beta} \). We then include drawdown terms:

\[
\begin{align*}
\frac{d}{dt} A &= \alpha B \ c(t) - gA \\
\frac{d}{dt} B &= \beta A \ c(t) - hB
\end{align*}
\]

These terms will pull \( (A, B) \) so as to raise the \( A \)-intercept (or lower the \( B \)-intercept) of the path, provided that \( |gA/hB| < |dA/dB| \), which implies \( g < h \). Exponential growth can continue until \( c(t) \) is suppressed. (See Figure 4.)

\[\text{Figure 4. Pumping by hyperbolic sloshing}\]

As a rule, hyperbolic sloshing tends to pump up the most durable component ("Methuselah mode") of a system. Paradoxically, other components must be killed off. (Methuselah’s wife had to die, for him to live so long.)
The *sine qua non* of hyperbolic sloshing is the requirement that $\alpha\beta > 0$, and the geomagnetic dynamo furnishes a real-life example. Radial flows driven by buoyancy can induce hyperbolic sloshing that may pump up the main magnetic field. Rising plumes create divergence zones as they approach the surface, thereby stretching tangential field lines longitudinally, squeezing them transversely, and amplifying the field strength, say by a factor of $\exp(+a)$. Falling plumes elsewhere do the opposite, but the overall effect amounts to amplification because $\frac{1}{2}\exp(+a) + \frac{1}{2}\exp(-a) \geq 1$.

In the jargon of vector spherical harmonics, we may say that the real poloidal flow $VS(L,M)+VS(L,-M)$ distorts the toroidal quadrupole field $BT(2,0)$ to $BS(L\pm1,M)$ and back. A similar mechanism can amplify the poloidal dipole field $BS(1,0)$ via $BT(L,M)$.

**Periodic vs. Erratic Behavior**

With linear damping terms and additive noise drivers included, the system of coupled equations reads as follows:

\[
\frac{d}{dt} \begin{bmatrix} B_S \\ B_T \end{bmatrix} = \begin{bmatrix} -g_S & +Z \\ -W & -g_T \end{bmatrix} \begin{bmatrix} B_S \\ B_T \end{bmatrix} + \begin{bmatrix} N_S(t) \\ N_T(t) \end{bmatrix}
\]

The system resonates at $\pm\sqrt{WZ - g_S^2}$ with bandwidth $g_\Sigma$, where $g_\Sigma \equiv \frac{1}{2}(g_S + g_T)$ and $g_\Delta \equiv \frac{1}{2}(g_S - g_T)$. We may distinguish three kinds of behavior:

- **Weakly damped (“narrowband”) oscillations**: Reversals are obviously periodic, but erratic variations of amplitude may be mistaken for beats. A good model for the Sun.
- **Strongly damped (“wideband”) oscillations**: Erratic reversals obscure resonant periodicity. A possible model for the Earth.
- **Suppressed oscillation**: $WZ < g_\Delta^2$. Another possible model for the Earth.

The $\langle BB \rangle$ covariance matrix could in principle be calculated from the unknown $\langle NN \rangle$ matrix, which must in practice be extracted from the results of detailed simulations.

**Alignment with the Axis of Rotation**

We must now repent of a mortal sin, one that could send us straight to actual Hell. In formulating the thinned model, we singled out main-field components aligned with the axis of rotation. If we wish to investigate how and why the main field aligns with the
axis, we must respect rotational symmetry and treat all components evenhandedly. We may do this explicitly by using tensor representations of BS, BT, and VT:

- BS(1, ...) is denoted $S_i$
- BT(2, ...) is denoted $T_{ij}$ (symmetric traceless)
- VT(3, ...) is denoted $D_{ijk}$ (symmetric traceless)

Using fancy brackets to denote symmetrization and trace removal, we may rewrite the equations as follows:

$$\frac{d}{dt} S_k = + \zeta \left[ T_{ki} \Omega_i \right] - g_S S_k + N_S(t)$$
$$\frac{d}{dt} T_{ij} = - \omega \left[ D_{ijk} S_k \right] + g_T T_{ij} + N_T(t)$$
$$\frac{d}{dt} D_{ijk} = \omega \left[ T_{ij} S_k \right] - g_D D_{ijk} + N_D(t)$$

If these equations were driven by isotropic additive noise, the main field would not naturally align. However, there may be third-order effects.

Magnetic fields tend to oppose transverse fluid flows. Since strong tangential fields act as a barrier to radial convection currents, we argue that the latitude-dependent magnitude of VT(2,0) modulates convection. Since the Coriolis effect is also latitude-dependent, there could well be extra $\alpha(\Omega^3)$ terms, which are found to promote alignment even if they are relatively weak:

$$\frac{d}{dt} D_{ijk} = \left[ - \omega T_{ij} S_k \pm ... T_{in} T_{mj} \Omega_k \pm ... S_j S_i \Omega_k \right]$$

**Afterword**

There are countless demons in MHD Hell, to wit, the many unsolved problems regarding dynamo phenomena, a few of which the author hopes to address in future publications:

- Calibration of coefficients using detailed simulations
- Validation of the black-box mechanism for regeneration
- Erratic reversals of Earth’s field vice regular oscillations
- Partition of energy among BS, BT, VS, VT components

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6 And conversely, transverse fluid flows tend to distort magnetic field lines. The opposing force is $F = \sigma(E + V \times B) \times B$, but kinetic energy may either be transferred to the magnetic field or dissipated as heat. Elastic transfers set the stage for Alfvén waves. The magnetic term is inherently dissipative, but the electric term of the force equation tends to offset it, either due to prompt charge separation (e.g., if currents diverge or dead-end on a boundary) or due to induction by the changing magnetic field.
- Validity of the Kolmogorov spectrum in convective systems (Barrois 2016a,b)
- Dimensional scaling rules for fast and slow-rotation regimes (Barrois 2016c)

**Selected References**


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