Self-triggered Continuous Discrete Observer with Updated Sampling Period
Vincent Andrieu, Madiha Nadri, Ulysse Serres, Jean-Claude Vivalda

To cite this version:

HAL Id: hal-01218649
https://hal.archives-ouvertes.fr/hal-01218649
Submitted on 21 Oct 2015

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers. L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
Self-triggered Continuous Discrete Observer with Updated Sampling Period

Vincent Andrieu a, c Madiha Nadri a Ulysse Serres a Jean-Claude Vivalda b

a Université de Lyon, F-69622, Lyon, France; Université Lyon 1, Villeurbanne; CNRS, UMR 5007, LAGEP, 43 bd du 11 novembre, 69100 Villeurbanne, France.
https://sites.google.com/site/vincentandrieu/, nadri@lagep.univ-lyon1.fr, ulysse.serres@univ-lyon1.fr.
b Inria, CORIDA, Villers-lès-Nancy, F- 54600, France
Université de Lorraine, IECL, UMR 7502, Vandœuvre-lès-Nancy, F-54506, France
CNRS, IECL, UMR 7502, Vandœuvre-lès-Nancy, F-54506, France
e-mail: jean-claude.vivalda@inria.fr
c Fachbereich C - Mathematik und Naturwissenschaften, Bergische Universität Wuppertal, Gaußstraße 20, 42097 Wuppertal, Germany.

Abstract

This paper deals with the design of high gain observers for a class of continuous-time dynamical systems with discrete-time measurements. Different approaches based on high gain techniques have been followed in the literature to tackle this problem. Contrary to these works, the measurement sampling time is considered to be variable. Moreover, the new idea of the proposed work is that the use of the output measurements by the observer follows an event based on an extended observer state component. Assuming that the vector fields related to the considered system are globally Lipschitz, the asymptotic convergence of the observation error is established. As an application of this approach, a state estimation problem of an academic bioprocess is studied, and its simulation results are discussed.

Key words: Nonlinear systems, sampled-data, continuous-discrete time observers, high gain observer, updated sampling-time, self-triggered observer.

1 Introduction

Estimating the state of a partially measured dynamical system is a classical problem in control theory. An algorithm that solves this problem is an asymptotically convergent observer. When the measurement is available only at some discrete-time instant, a continuous-discrete time observer has to be designed. The study of this type of algorithm can be traced back to Jazwinski who introduced the continuous-discrete Kalman filter to solve a filtering problem for stochastic continuous-discrete time systems (see [10]). Inspired by this approach, the continuous discrete high-gain observer has been studied in [7]. Since then, different approaches have been investigated. The robustness of an observer with respect to time discretization was studied in [5] (see also [17]). In [15], a Newton observer is provided which estimates the state at time $t_k$ from $N$ consecutive measurements of outputs and inputs; in [6], the authors show how this method can be implemented in the case where the sampled system is not known analytically. In [11] observers were designed from an output predictor (see also related works in [1]). Some other approaches based on time delayed techniques have also been considered in [19]. Recently, a new continuous-discrete observer design methodology for Lipschitz nonlinear systems based on reachability analysis was presented in [8] (see also [14]).

In this note, we also consider the design of a continuous discrete time observer. However, in opposition to these results, we consider the case in which the sampling time is variable and used as a tuning parameter. More precisely, we consider that the quantity $t_{k+1} - t_k$ is a part of the design of the continuous discrete observer. Hence, in the proposed algorithm, the measurement time is computed online. In fact, the use of sensors follows an event
based on an extended observer state component. This may be related to the event-triggered control methodology (see for instance [20,21]).

In high-gain designs, the asymptotic convergence of the estimate to the state is obtained by dominating the Lipschitz nonlinearities with high-gain techniques. However, there is a trade-off between the high-gain parameter and the measurement step size. This can lead to restrictive design conditions on the sampling measurement time (see also [16]). Inspired by [4], the extra observer state component estimates the local Lipschitz constant (roughly speaking \(\frac{|x_n - x_k|}{|x_n - x_k|}\)) in order to maximize the measurement sampling interval.

The paper is organized as follows. The class of systems considered and the structure of the estimation algorithm are given in Section 2. The main result and its proof are given in Section 3. Section 4 contains an illustrative example. Finally, Section 5 is devoted to the conclusion.

2 Problem statement and structure of the observer

2.1 Class of systems considered

In this work we consider the problem of designing an observer for nonlinear systems that are diffeomorphic to the following form:

\[ \dot{x} = Ax + f(x, u), \]

where the state \(x\) is in \(\mathbb{R}^n\), \(u : \mathbb{R} \to \mathbb{R}^p\) is a known input in the space of essentially bounded measurable functions from \(\mathbb{R}_+\) to \(\mathbb{R}^p\) (denoted \(L^\infty(\mathbb{R}_+, \mathbb{R}^p)\)), \(A\) is a matrix in \(\mathbb{R}^{n \times n}\) and \(f : \mathbb{R}^n \times \mathbb{R}^p\) is a locally Lipschitz vector field both having the following triangular structure:

\[
A = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 1 & \cdots & 0 \\
0 & \cdots & 0 & 1 & \cdots \\
0 & \cdots & \cdots & \cdots & 0
\end{bmatrix},
\]

\[
f(x, u) = \begin{bmatrix}
f_1(x_1, u) \\
\vdots \\
f_n(x, u)
\end{bmatrix}.
\]

The measured output is given as a sequence of values \((y_k)_{k \geq 0}\) in \(\mathbb{R}\):

\[ y_k = Cx(t_k), \]

where \((t_k)_{k \geq 0}\) is a sequence of times to be selected and \(C = [1 0 \cdots 0]\) is in \(\mathbb{R}^n\). In this paper, we shall denote by \(\langle \cdot, \cdot \rangle\) the canonical scalar product in \(\mathbb{R}^n\) and by \(\| \cdot \|\) the induced Euclidean norm; we shall use the same notation for the corresponding induced matrix norm. Also, we use the symbol \(^t\) to denote the transposition operation.

We consider the case in which the vector field \(f : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n\) satisfies the following assumption.

**Assumption 1** The function \(f = (f_1, \ldots, f_n)^t\) is such that the following incremental bound is satisfied for all \((x, e, u) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^p\),

\[ |f_j(x + e, u) - f_j(x, u)| \leq c(x, u) \sum_{i=1}^{j} |e_i|, \tag{3} \]

where \(c : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}_+\) is a continuous function which satisfies the following bound

\[ c(x, u) \leq \Gamma(u), \quad \forall (x, u) \in \mathbb{R}^n \times \mathbb{R}^p, \tag{4} \]

where \(\Gamma : \mathbb{R}^p \to \mathbb{R}_+\).

Compared to the preliminary version of this work presented in [2], now a larger class of nonlinear systems is addressed. Indeed, general upper triangular systems are now allowed.

Note that in the case in which we know a bound on the input \(u\), we come back to the globally Lipschitz context. However, even in this case, we believe that employing a tighter bound in term of a state-dependent function \(c\) implies that the sensors are less used than they would be if we were considering directly the Lipschitz bound.

2.2 Updated sampling time observer

The continuous-discrete time observer with updated sampling period is given by

\[
\begin{align*}
\dot{x}(t) &= A\dot{x}(t) + f(x(t), u(t)), \quad t \in [t_k, t_{k+1}) \\
\dot{x}(t_{k+1}) &= \dot{x}(t_{k+1}) + \delta_k L_t^k(t_{k+1}) K(C\dot{x}(t_{k+1}) - y_{k+1})
\end{align*}
\]

where \(K\) in \(\mathbb{R}^n\) is a gain matrix. The matrix function \(L : \mathbb{R}_+ \to \mathbb{R}^{n \times n}\) is defined as \(L(t) = \text{diag}(L_1(t), \ldots, L_n(t))\) where \(L : \mathbb{R}_+ \to \mathbb{R}\) is given as a solution to the following system of continuous discrete differential equations

\[
\begin{align*}
\dot{L}(t) &= a_2 L(t) M(t) c(x(t), u(t)), \quad t \in [t_k, t_{k+1}) \\
M(t) &= a_3 M(t) c(x(t), u(t)), \quad t \in [t_k, t_{k+1}) \\
L(t_{k+1}) &= L(t_{k+1})(1 - a_1 \alpha) + a_1 \alpha \\
M(t_{k+1}) &= 1,
\end{align*}
\]

initiated from \(L(0) \geq 1\) and with \(a_1 \alpha < 1\). We have for all \(k\),

\[ y_k = Cx(t_k), \]

\[1\] The solution \(\dot{x}(t)\) is a right-continuous function. Given a right-continuous function \(\phi : \mathbb{R} \to \mathbb{R}^n\), the notation \(\phi(t^-)\) stands for \(\phi(t^-) = \lim_{h \to 0, h < 0} \phi(t + h)\).
where the $t_k$'s, $k$ in $\mathbb{N}$ are given by the following relations,

$$t_0 = 0, \quad t_{k+1} = t_k + \delta_k,$$

$$\delta_k = \min\{s \in \mathbb{R}_+ \mid sL((t_k + s)^-) = \alpha\}, \quad (7)$$

where $\alpha$, $a_1$, $a_2$ and $a_3$ are positive real numbers to be chosen.

### 2.3 About the updating time period

To understand the motivation of this update law note that a first order approximation gives

$$L(t_{k+1}) = L(t_k) + a_2L(t_k)c(\hat{x}(t_k), u(t_k))\delta_k + o(\delta_k).$$

Hence, taking into account that $\alpha = \delta_kL(t_{k+1})$, it yields,

$$\frac{L(t_{k+1}) - L(t_k)}{\delta_k} = L(t_k)[a_1(1 - L(t_k)) + a_2c(\hat{x}(t_k), u(t_k))] + o(1).$$

We recognize here the same update law structure than the one introduced in [18, equation (24)] which was motivated by a Riccati equation.

The sampling time interval which depends on $L$ is well defined as this is shown in the following proposition.

#### Proposition 1 (Sequence $(\delta_k)_{k \in \mathbb{N}}$ well defined)

If $u$ is in $L^\infty(\mathbb{R}_+, \mathbb{R}^p)$ then there exists a positive real number $\delta_{\text{min}}$ depending on the initial condition $L(0)$ such that for all $k$ in $\mathbb{N}$ there exists $\delta_k$ such that $\delta_{\text{min}} \leq \delta_k \leq \alpha$.

**Proof.** First of all, note that $L$ is not decreasing in every time interval $[t_k, t_{k+1})$. Moreover, when there is a jump (i.e., when there exists a $k$ such that $t = t_k$), we see that $L(t_k) \geq 1$ if $L(t_k^-) \geq 1$. Hence, since $L(0) > 0$ we get $L(t) \geq 1$ for every $t > 0$. With (7), this implies that for all $k$, $\delta_k$ is well defined and we have $\delta_k \leq \alpha$ for all $k$. The other inequality in the proposition is deduced from the following lemma whose proof is postponed to Appendix A.1.

#### Lemma 1 (Boundedness of $L$)

If $u$ is in $L^\infty(\mathbb{R}_+, \mathbb{R}^p)$ then, there exists $\ell_\infty$ (depending on the initial conditions for system (1) and its observer (5)-(6)) such that $1 \leq L(t) \leq \ell_\infty$ for every $t \geq 0$.

Taking $\delta_{\text{min}} = \frac{\alpha}{\ell_\infty}$, the above lemma yields the result of Proposition 1. □

Note that if we know a bound on $u$, the function $c(\hat{x}, u)$ in (6) could simply be replaced by a constant depending on the function $\Gamma$. Note however that in this case, $\dot{L}$ becomes larger which reduces the duration of each sampling period $(\delta_k)_{k \in \mathbb{N}}$. Consequently, the sensors are more frequently employed which is something we would like to avoid.

Finally note that when $c = 0$, for all $t$ in $[t_k, t_{k+1})$, $\dot{L}(t) = 0$ and the solution $L(t) = 1$ is attractive and invariant along the solution of the continuous discrete dynamics of $L$. In this case, it yields $\lim_{t \to +\infty} \delta_k = \alpha$. This implies that asymptotically, the sampling becomes uniform.

### 3 Observer convergence

With the property given above in hand, we are now able to state our main result.

#### Theorem 2 (Self-triggered continuous-discrete time observer)

There exist a gain matrix $K$ and $\alpha_m > 0$ such that for all $\alpha$ in $(0, \alpha_m]$, there exist positive numbers $a_1, a_2$ and $a_3$ such that for every essentially bounded input functions the estimation error obtained using the observer (5)-(6) converges asymptotically toward zero. More precisely, for every initial condition $(x(0), \hat{x}(0))$ in $\mathbb{R}^n \times \mathbb{R}^n$ and $L(0) \geq 1$, for every input function $u$ in $L^\infty(\mathbb{R}_+, \mathbb{R}^p)$ the associated solution to system (1), (5)-(6) satisfies $\lim_{t \to +\infty} ||x(t) - \hat{x}(t)|| = 0$.

**Proof.** Let $D$ be the diagonal matrix in $\mathbb{R}^{n \times n}$ defined by $D = \text{diag}(1, 2, \ldots, n)$. Let $P$ be a symmetric positive definite matrix in $\mathbb{R}^{n \times n}$ and $K$ a vector in $\mathbb{R}^n$ such that the following inequality is satisfied (see [18, equation (14)] or [13, equation (18)] or [3])

$$p_1I \leq P \leq p_2I, \quad (8)$$

$I$ being the identity matrix, and

$$(A + KC)'P + P(A + KC) \leq -I, \quad (9)$$

with $p_1, \ldots, p_4$ positive real numbers. Let $\varepsilon \triangleq \hat{x} - x$ be the estimation error; $\varepsilon$ satisfies the following differential equation (cf. equations (1)-(5)) for all $t \in [t_k, t_{k+1})$

$$\dot{\varepsilon}(t) = Ae(t) + \Delta(\varepsilon(t), e(t), u(t)), \quad (10)$$

where the function $\Delta : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ is defined as

$$\Delta(\hat{x}, e, u) = f(\hat{x}, u) - f(\hat{x} - e, u), \quad \forall (\hat{x}, e, u).$$

From Assumption 1 (i.e., inequality (3)), this function satisfies $|\Delta_j(\hat{x}, e, u)| \leq c(\hat{x}, u) \sum_{i=1}^l |e_i|$ for all $(\hat{x}, e, u)$.

In the sequel, and using the results presented in [12] (see also [4]) we consider the scaled observation error

2 In the design, $K$ is selected in (8)-(9). Then we set

$$a_1 = \frac{1}{2p_2p_4}, \quad a_m = \text{selected sufficiently small such that}$$

$$a_1\alpha_m < 1 \quad \text{and} \quad (17) \text{holds}. \quad \text{Finally, we set} \quad a_2 = 2n^2 \quad \text{and} \quad$$

$$a_2 \geq 2N_1(\alpha) + N_2(\alpha), \quad \text{where} \quad \alpha \leq \alpha_m \quad \text{and} \quad N_1 \quad \text{and} \quad N_2 \quad \text{are given in the proof of Lemma 5}.$$
defined for all $t$ by $E(t) = \mathcal{L}(t)^{-1} e(t)$. Also, to simplify the presentation, we introduce the notations

$$
L_k^- = L(t_k^-), \quad L_k^+ = L(t_k^+), \quad L_k = L(t_k),
$$

and

$$
E_k = E(t_k), \quad e_k = e(t_k).
$$

If we integrate equation (10) on the interval $[t_k, t_k + \tau)$ with $\tau < \delta_k$, we get

$$
e(t_k + \tau) = \exp(A \tau) e(t_k) + \int_0^\tau \exp(A(\tau - s)) \times \Delta(\dot{x}(t_k + s), e(t_k + s), u(t_k + s)) \, ds.
$$

Moreover, from (5), we get

$$
e_{k+1} = (I + \delta_k L(t_{k+1}^-) K) e((t_k + \delta_k)^-).
$$

In the remaining part of the proof, we shall show that the Lyapunov function $V(E(t_k)) = E(t_k)P E(t_k)$ is decreasing toward zero along the solution to the system. In order to evaluate the Lyapunov function, let us first mention the following algebraic properties of the matrix function $\mathcal{L}_k$:

$$(\mathcal{L}_{k+1}^-)^{-1} (I + \delta_k \mathcal{L}_{k+1}^- K) = (I + \delta_k \mathcal{L}_{k+1}^- K) (\mathcal{L}_{k+1}^-)^{-1} = (I + \alpha K) (\mathcal{L}_{k+1}^-)^{-1},$$

where the last equality has been obtained from (7). Moreover, since for all $k$, $(\mathcal{L}_k^-)^{-1} A = L_k^- a (\mathcal{L}_k^-)^{-1}$, it yields for all $k$ and all $i \geq 1$,

$$(\mathcal{L}_k^-)^{-1} A^i = L_k^- A (\mathcal{L}_k^-)^{-1} A^{i-1} = (L_k^- A)^i (\mathcal{L}_k^-)^{-1},$$

and

$$(\mathcal{L}_k^-)^{-1} \exp(A s) = (\mathcal{L}_k^-)^{-1} \sum_{i=0}^{+\infty} \frac{A^i s^i}{i!} = \exp(L_k^- A s) (\mathcal{L}_k^-)^{-1}.$$ 

Hence, employing the previous algebraic equalities (13) and (14), together with relation (11), we get, when left multiplying (12) by $(\mathcal{L}_{k+1}^-)^{-1}$,

$$(\mathcal{L}_{k+1}^-)^{-1} e_{k+1} = Q(\alpha) (\mathcal{L}_{k+1}^-)^{-1} L_k E_k + R,$$

with

$$Q(\alpha) = (I + \alpha K) \exp(A \alpha),$$

and

$$R = (I + \alpha K) \int_{t_k}^{t_k+\delta_k} \exp(AL_{k+1}^- (\delta_k - s)) (\mathcal{L}_{k+1}^-)^{-1} \times \Delta(\dot{x}(t_k + s), e(t_k + s), u(t_k + s)) \, ds.
$$

Note that, since we have $E_{k+1} = \Psi (\mathcal{L}_{k+1}^-)^{-1} e_{k+1}$ with $\Psi = (\mathcal{L}_{k+1}^-)^{-1} L_{k+1}$, it yields

$$V(E_{k+1}) = V(E_k) + T_1 + T_2,$$

with

$$T_1 = V(\Psi Q(\alpha)(\mathcal{L}_{k+1}^-)^{-1} L_k E_k) - V(E_k),$$

$$T_2 = 2E_k' (\mathcal{L}_{k+1}^-)^{-1} L_{k+1} Q(\alpha)' \Psi P \Psi R + R' \Psi P \Psi R.$$

The remaining part of the proof is done in three steps. The first two ones are devoted to upper bound the two terms $T_1$ and $T_2$ and the last one is devoted to the Lyapunov analysis. The fact that the Lyapunov function is decreasing is due to the term $T_1$ which will be shown to be negative. The second term is handled by robustness.

**Step 1 : Upper bounding $T_1$**

**Lemma 3** Let $a_1 = \frac{1}{2^p P^q}$. There exists $\alpha_m > 0$ sufficiently small such that for all $\alpha \in [0, \alpha_m)$

$$T_1 \leq - \left( \frac{a_2 P \delta_m}{a_3} \left[ \exp \left( a_3 \int_{t_k}^{t_k+\delta_k} c(r) \, dr \right) - 1 \right] + \frac{a_4 P}{a_5} \right) \times \left\| (\mathcal{L}_{k+1}^-)^{-1} e_{k} \right\|^2,
$$

where $c(r) = c(\dot{x}(r), u(r))$.

The proof of Lemma 3 uses the following lemma whose proof is given in Appendix.

**Lemma 4** Taking $a_1$ sufficiently small, there exists $\alpha_m > 0$ sufficiently small such that for all $\alpha < \alpha_m$, we have

$$Q(\alpha)' \Psi P \Psi Q(\alpha) \leq P - \alpha \frac{1}{4P} P,
$$

**Proof of Lemma 3.** We have, for all $v$ in $\mathbb{R}^n$

$$\nu' L_k (\mathcal{L}_{k+1}^-)^{-1} P (\mathcal{L}_{k+1}^-)^{-1} L_k v - \nu' P v
=$$

$$\nu' \left( \int_{t_k}^{t_k+\delta_k} L_k \frac{d}{ds} (\mathcal{L}(s)^{-1}) L_k v \right) + L_k \mathcal{L}(s)^{-1} P \frac{d}{ds} (\mathcal{L}(s)^{-1}) L_k v.
$$

However, we have for all $s$ in $[t_k, t_{k+1})$

$$\frac{d}{ds} (\mathcal{L}(s)^{-1}) = - \frac{L(s)}{L(s)} D \mathcal{L}(s)^{-1}.$$
Consequently, it yields
\[
v' L_k (L_{k+1})^{-1} P (L_{k+1})^{-1} L_k v - v' P v = -v' \left( \int_{t_k}^{t_{k+1}} \frac{\dot{L}(s)}{L(s)} \mathcal{L}(s)^{-1} [PD + PD] \mathcal{L}(s)^{-1} \mathcal{L}_k ds \right) v.
\]

Bearing in mind that \( L \geq 1 \) and \( \dot{L} \geq 0 \) and taking into account the bounds on \( PD + PD \) in (9) and \( P \) in (8), we get
\[
v' L_k (L_{k+1})^{-1} P (L_{k+1})^{-1} L_k v - v' P v \leq -p_3 v' \left( \int_{t_k}^{t_{k+1}} a_2 c(s) \exp \left( a_3 \int_{t_k}^{s} c(r) dr \right) \times \mathcal{L}_k \mathcal{L}(s)^{-1} P \mathcal{L}(s)^{-1} \mathcal{L}_k ds \right) v.
\]

Note that since \( L_k \leq L(s) \leq L_{k+1} \), we finally get
\[
v' L_k (L_{k+1})^{-1} P (L_{k+1})^{-1} L_k v - v' P v \leq -\frac{a_2 p_3 p_1}{a_3} \left[ \exp \left( a_3 \int_{t_k}^{t_{k+1}} c(r) dr \right) - 1 \right] \times \| (L_{k+1})^{-1} L_k v \|^2.
\]

Consequently, the bound (16) is obtained from the previous inequality with \( v = E_k \) and from inequality (17) in Lemma 4 together with (8).

**Step 2 : Upperbounding \( T_2 \)**

**Lemma 5.** There exist two continuous functions \( N_1 \) and \( N_2 \) such that the following inequality holds
\[
\begin{align*}
T_2 \leq & \left( (L_{k+1})^{-1} c_k \right)^2 \\
& \times \left[ N_1(\alpha) \left[ \exp \left( \int_0^{\delta_k} n c(t_k + r) dr \right) - 1 \right] + N_2(\alpha) \left[ \exp \left( \int_0^{\delta_k} n c(t_k + r) dr \right) - 1 \right] \right] \right)^2. 
\end{align*}
\]

**Proof.** In order to prove inequality (18), let us first analyze the term \( R \) given by equation (15). First, we seek for an upper bound of the norm of \( (L_{k+1})^{-1} \Delta(\dot{x}(t_k + s), e(t_k + s), u(t_k + s)) \), we have
\[
\| (L_{k+1})^{-1} \Delta(\dot{x}(t_k + s), e(t_k + s), u(t_k + s)) \|
\leq \left( \sum_{j=1}^{n} (L_{k+1}^{-1})^{-2j} c^2(t_k + s) \left( \sum_{i=1}^{j} |e_i(t_k + s)| \right) \right) \frac{1}{2}
\leq \left( \sum_{j=1}^{n} c^2(t_k + s) \left( \sum_{i=1}^{j} (L_{k+1}^{-1})^{-j} |e_i(t_k + s)| \right) \right) \frac{1}{2}.
\]

Since, \( L_{k+1}^{-1} \geq 1 \), we have \( (L_{k+1}^{-1})^{-j} \leq (L_{k+1}^{-1})^{-1} \) whenever \( 1 \leq i \leq j \). It yields
\[
\| (L_{k+1}^{-1})^{-1} \Delta(\dot{x}(t_k + s), e(t_k + s), u(t_k + s)) \|
\leq \left( \sum_{j=1}^{n} c^2(t_k + s) \left( \sum_{i=1}^{j} (L_{k+1}^{-1})^{-j} |e_i(t_k + s)| \right) \right) \frac{1}{2}
\leq \left( \sum_{j=1}^{n} c^2(t_k + s) n \left( (L_{k+1}^{-1})^{-1} e(t_k + s) \right) \| \right) \frac{1}{2}.
\]

From formula (15) and inequality (19), we get
\[
\| R \| \leq \| I + \alpha KC \| \int_{0}^{\delta_k} \exp \left( \| A \| L_{k+1}(\delta_k - s) \right) \times n c(t_k + s) \| (L_{k+1}^{-1})^{-1} e(t_k + s) \| ds.
\]

We have for all \( s \) in \( [0, \delta_k] \),
\[
(L_{k+1}^{-1})^{-1} e(t_k + s) = L_{k+1}^{-1} A (L_{k+1}^{-1})^{-1} e(t_k + s) + (L_{k+1}^{-1})^{-1} \Delta(t_k + s).
\]

Denoting by \( w(s) \) the expression \( (L_{k+1}^{-1})^{-1} e(t_k + s) \), this gives
\[
\frac{d}{ds} \| w(s) \| = \frac{\langle \dot{w}(s), w(s) \rangle}{\| w(s) \|}
\leq \| L_{k+1}^{-1} A (L_{k+1}^{-1})^{-1} e(t_k + s) \| + \| (L_{k+1}^{-1})^{-1} \Delta(\dot{x}(t_k + s), e(t_k + s), u(t_k + s)) \|
\leq (L_{k+1}^{-1} A + n c(t_k + s)) \| (L_{k+1}^{-1} e(t_k + s) \|.
\]
Hence, we finally obtain
\[
\left\| (L_{k+1}^{-1}) e(t_k + s) \right\| 
\leq \exp \left( \int_0^s L_{k+1}^\top ||A|| + nc(t_k + r)dr \right) 
\times \left\| (L_{k+1}^{-1})^{-1} e_k \right\|. \tag{21}
\]
Consequently, according to (20) and (21), we get
\[
\| R \| \leq \| I + \alpha KC \| \int_0^{\delta_k} \exp \left( ||A||L_{k+1}^{-1} \delta_k - s \right) 
\times nc(t_k + s) \exp \left( \int_0^s L_{k+1}^\top ||A|| + nc(t_k + r)dr \right) 
\times \left\| (L_{k+1}^{-1})^{-1} e_k \right\| ds
\]
\[
= \| I + \alpha KC \| \exp (||A||\alpha) \left\| (L_{k+1}^{-1})^{-1} e_k \right\| 
\times \int_0^{\delta_k} nc(t_k + s) \exp \left( \int_0^s nc(t_k + r)dr \right) ds
\]
\[
= \| I + \alpha KC \| \exp (||A||\alpha) \left\| (L_{k+1}^{-1})^{-1} e_k \right\| 
\times \exp \left( \int_0^{\delta_k} nc(t_k + r)dr \right) - 1.
\]
Hence, employing Lemma 6 this gives the existence of two continuous function \(N_1\) and \(N_2\) such that
\[
2E_k(L_{k+1}^{-1})^{-1} L_k Q(\alpha)/\Psi P \Psi R
\leq \left\| (L_{k+1}^{-1})^{-1} e_k \right\|^2 \left[ N_1(\alpha) \left( \exp \left( \int_0^{\delta_k} nc(t_k + r)dr \right) - 1 \right) \right].
\]
with
\[
N_1(\alpha) = 2\|Q(\alpha)\| \| I + \alpha KC \| \exp (||A||\alpha) \frac{\|P\|}{(1 - a_1\alpha)^2n}.
\]
Moreover,
\[
R^\top \Psi P \Psi R
\leq \left\| (L_{k+1}^{-1})^{-1} e_k \right\|^2 \left[ N_2(\alpha) \left( \exp \left( \int_0^{\delta_k} nc(t_k + r)dr \right) - 1 \right) \right]^2
\]
where
\[
N_2(\alpha) = \| I + \alpha KC \| \exp (2||A||\alpha) \frac{\|P\|}{(1 - a_1\alpha)^2n}.
\]
The two previous inequalities imply that (18) holds. □

**Step 3: Lyapunov analysis**

With the two bounds obtained for \(T_1\) and \(T_2\) in Lemmas 3 and 5, we finally get
\[
V(E_{k+1} - V(E_k) \leq \left\| (L_{k+1}^{-1})^{-1} e_k \right\|^2 \left[ N_1(\alpha) [e^\beta - 1] + N_2(\alpha) [e^\beta - 1] - \frac{a_2p_1}{p_1} \right],
\]
where \(\beta = \alpha \int_0^{\delta_k} c(t_k + r)dr\). Note that for all \(\alpha\), thanks to a good choice of \(a_3\) and \(a_2\) it yields that the right-hand member in the previous inequality is negative for every \(\beta\). For example, if we take \(a_3 = 2n\), the previous inequality becomes
\[
V(E_{k+1} - V(E_k) \leq \left\| (L_{k+1}^{-1})^{-1} e_k \right\|^2 \left[ - \frac{p_1}{4p_2} \right]
+ \left[ e^\beta - 1 \right] \left[ - \frac{a_2p_1}{2n} \right] \left[ e^\beta + 1 \right] - N_1(\alpha) + N_2(\alpha)[e^\beta - 1] \right]
\leq \left\| (L_{k+1}^{-1})^{-1} e_k \right\|^2 \left[ - \frac{p_1}{4p_2} \right]
+ \left[ e^\beta - 1 \right] \left[ e^\beta + 1 \right] \left[ - \frac{a_2p_1}{2n} \right] + N_1(\alpha) + N_2(\alpha) \right].
\]
If \(a_2 \geq 2n \frac{N_1(\alpha) + N_2(\alpha)}{p_1} \) it yields
\[
V(E_{k+1} - V(E_k) \leq \left\| (L_{k+1}^{-1})^{-1} e_k \right\|^2.
\]
The function \(V\) being positive definite, it yields that
\[
\lim_{k \to \infty} \left\| (L_{k+1}^{-1})^{-1} e_k \right\| = 0.
\]
The function \(L\) being upper and lower bounded (by Lemma 1), this implies that the error \(e_k\) goes to zero. With (11), we get the result. □

**Remark 1** In the proof of Lemma 4 (see Appendix), we explain how one can obtain an estimation of \(o_{\infty}\). An interesting question would be to find the optimal value of \(\theta\) in \((0, \alpha_{\infty})\) to maximize the measurement step-size. This is a difficult question that requires some further analysis and depends on the bound on \(L(\cdot)\). Indeed, from the equality \(\alpha = \delta_k L(t_{k+1})\), we wish to select \(\alpha\) large. However, at the same time, a large \(\alpha\) implies a large parameter \(a_2\) which implies also large \(L(t_{k+1})\). Hence, a nonlinear optimization has to be carried out.

**4 Illustrative example**

In this section, the performance of the proposed observer is illustrated through a bioreactor. In most cases, a cheap and reliable instrumentation required for real-time measurement of key variables of such processes (biomass, substrate) is not available. Nevertheless, biomass measurement can be obtained using off-line analysis (sampled measurements) which requires time and staff investment. The proposed approach allows to reduce the
measurement cost and consequently, the monitoring cost is also reduced.

The bioprocess considered is an academic bioreactor which consists of a microbial culture which involves a biomass $X$ growing on a substrate $S$. The bioprocess is supposed to be continuous with a scalar dilution rate $D$ and an input substrate concentration $S_{\text{in}}$ (which is assumed to be constant). Under these conditions and using the Contois model, the dynamical model of the process is given by

$$
\begin{aligned}
\dot{X} &= -\frac{K_1S}{K_2X + S}X - DX \\
\dot{S} &= -\frac{K_1S}{K_2X + S}X - D(S - S_{\text{in}}),
\end{aligned}
$$

(22)

where the $K_i$’s $(i=1,2,3)$ are positive constants. Our objective is the on-line estimation of the substrate concentrations $S$ through sampled biomass measurements. In the case where the output is assumed to be time-continuous, the authors in [9] gave a stationary high gain observer. In the sequel, the same hypothesis as in [9] and the same notations are used.

Consider the state vector $z = [X,S]'$, the input $u = D$ and the output $y(t_k) = X(t_k)$. Under the constraint $0 < u_{\text{min}} \leq u \leq u_{\text{max}} \leq K_1$, the authors in [9], determined a compact domain $M_z \in \mathbb{R}^2$ which is invariant under the normal form (1). In the sequel, we choose $K_3 = 1$ which means that there is no change of volume when the substrate transforms into microorganisms, the two other values being $K_1 = K_2 = 1$ (notice that $K_1 = 1$ up to a change of time unit). Let

$$
M_z = \{ z \in \mathbb{R}^2 : X \geq \epsilon_1, S \geq \epsilon_2, X + S \leq 1 \},
$$

where $\epsilon_1 = \frac{(1 - u_{\text{max}})x_2}{S_{\text{in}}u_{\text{max}}}$.

Then, using the change of coordinates $z \in M_z \rightarrow x = \Phi(z)$, defined as

$$
\Phi(z) = \left[ X, \frac{SX}{X + S} \right]',
$$

system (22) takes the normal form (1) with $n = 2$, and $f_1(x,u) = -ux_1$,

$$
f_2(x,u) = S_{\text{in}}u - \left( 1 + u + \frac{2S_{\text{in}}u}{x_1} \right)x_2 + \left( \frac{2 - u}{x_1} + \frac{S_{\text{in}}u}{x_1^2} \right)x_2^2,
$$

and $x$ evolving in $M_z = \Phi(M_z)$.

Moreover, the function $f_2$ can easily be extended to a global Lipschitz $C^1$ function on the whole domain $\mathbb{R}^2 \times \mathcal{M}_u$, where $\mathcal{M}_u = [u_{\text{min}}, u_{\text{max}}]$. For all $(\hat{x}, x, u) \in \mathbb{R}^2 \times \mathcal{M}_z \times \mathcal{M}_u$, we can write

$$
|f_1(x,u) - f_1(\hat{x}, u)| \leq c_11(u)|x_1 - \hat{x}_1|
$$

$$
|f_2(x,u) - f_2(\hat{x}, u)| \leq |f_2(x_1,x_2,u) - f_2(\hat{x}_1, \hat{x}_2,u)|
$$

$$
+ |f_2(x_1, \hat{x}_2, u) - f_2(\hat{x}_1, \hat{x}_2, u)|
$$

$$
\leq c_{21}(x_1, \hat{x}_2, u)|x_2 - \hat{x}_2| + c_{22}(x_1, \hat{x}_1, x_2, u)|x_1 - \hat{x}_1|,
$$

where

$$
c_{11}(u) = -u,
$$

$$
c_{21}(x, \hat{x}, u) = - \left( 1 + u + \frac{2S_{\text{in}}u}{x_1} \right) + \left( \frac{2 - u}{x_1} + \frac{S_{\text{in}}u}{x_1^2} \right)(x_2 + \hat{x}_2),
$$

and

$$
c_{22}(x, \hat{x}, u) = \frac{2S_{\text{in}}u\hat{x}_2 + (2 - u)x_2^2}{x_1 x_2} + \frac{S_{\text{in}}u \hat{x}_1 x_2 - S_{\text{in}}u x_2^2 \hat{x}_1}{x_1 x_2^2}.
$$

Setting

$$
c(\hat{x}, u) = \min\{L(u), \max_{x \in \mathcal{M}_z} \{c_{11}(u), c_{21}(x, \hat{x}, u), c_{22}(x, \hat{x}, u)\} \},
$$

where $L(u)$ is the global Lipschitz constant, we obtain

$$
|f_1(x,u) - f_1(\hat{x}, u)| \leq c(\hat{x}, u)|e_1|,
$$

$$
|f_2(x,u) - f_2(\hat{x}, u)| \leq c(\hat{x}, u)[|e_1| + |e_2|],
$$

for all $(\hat{x}, x, u) \in \mathbb{R}^2 \times \mathcal{M}_z \times \mathcal{M}_u$. Now, it suffices to use (5)-(7) to give the updated sampling time observer. The observer parameters have been selected through a trial and error procedure as follows:

$$
K = [-2, -1]', \quad \alpha = .9, \quad a_1 = 1, \quad a_2 = .1, \quad a_3 = .1.
$$

4.1 Simulation results

For the simulation test\footnote{Note that this is not exactly Assumption 1 since we restrict ourself to $x$ in $M_z$. However, it is not a problem since the state trajectory remains in this set.} the output has been corrupted by an additive noisy signal as shown in Figure 1. The observer simulation was performed under similar operating conditions as the model ($K_i = 1$) and $S_{\text{in}} = 0.1$, and $u : [0, 40] \rightarrow \mathbb{R}$ is displayed in Figure 1.

Figure 2 displays the calculated values of the sampling-time $\delta_k$. It may be noticed that the sampling-time suggested by the proposed approach is relatively small when

\footnote{The Matlab files can be downloaded from https://sites.google.com/site/vincentandrieu/}
Fig. 1. Input $u(t) = D(t)$ and output $y(t_k) = X(t_k)$ with measurement noise.

Fig. 2. Updated sampling time $\delta_k$.

Fig. 3. $S$ given by the model (22) compared to $\hat{S}$ given by system (5)-(7).

Fig. 4. Graph of the gain function $L$.

the estimated dynamics speed is important and take a large value when the dynamics speed is close to zero. This behavior is quite natural: when the system is not much excited, the state variables vary slowly and we can wait a little bit more time between two measurements; moreover, the designed gain $L$ of the observer can be chosen small.

Figure 3 illustrates the impact of the measurement noise on the observer performances. We can see that the observer behavior with respect to the measurement noise is satisfactory.

5 Conclusion

In this paper, a high gain observer for continuous-discrete time systems in the observability normal form has been designed. The problem of observer synthesis for these systems is related to the sampling time of the output measurement which is always uniform and should be small to guarantee the observer convergence. To overcome this constraint which increases the control cost, a high gain updated sampling-time observer has been proposed. The principal advantage of this observer is that it may reduce the use of the output measurement. The obtained results have been illustrated in the biological process and demonstrated good performances.

Acknowledgements

We acknowledge the PEPS for the support to the Project “SOSSYAL” (Stratégie d’Observation optimale pour la Synthèse doSservateurs hYbrides, Application en biotechnoLogie) and the ANR LIMICOS contract number 12 BS03 005 01.

References

A. Proofs of Lemmas

A.1 Proof of Lemma 1

Assuming that the input \( u \) is an essentially bounded time function (with unknown bound), thanks to (4) we get that the function \( t \mapsto c(\bar{x}(t), u(t)) \) is essentially upper bounded on the time of existence of the solution. Let \( c_m \) be an essential upper bound of \( c(\bar{x}(t), u(t)) \). Note that by integrating equation (6b) with the previous upper bound on the interval \([t_k, t_{k+1}]\), it yields \( M(t) \leq e^{\alpha c_m (t-t_k)} \) for every \( t \in [t_k, t_{k+1}] \), reporting this inequality in (6a) it yields for all \( k \) and \( t \in [t_k, t_{k+1}] \)

\[
L(t) \leq \kappa(t-t_k)L(t_k), \quad \text{for every } t \in [t_k, t_{k+1}) \quad (A.1)
\]

where \( \kappa \) is an increasing function such that \( \kappa(0) = 1 \) defined as

\[
\kappa(s) = \exp(a_2 c_m s \exp(a_3 c_m s)).
\]

Hence, from (6c), and (A.1) with \( t = t_{k+1}^l \), we get

\[
L(t_{k+1}) \leq (1 - a_1 \alpha)\kappa(\delta_k) L(t_k) + a_1 \alpha.
\]

Note moreover that we have \( \dot{L}(s) \geq 0 \) for all \( s \) in \([t_k, t_{k+1}^l] \), and so \( L(t_{k+1}) \geq L(t_k) \). Hence, since we have \( L(t_{k+1})\delta_k = \alpha \) we get \( \delta_k \leq \frac{\alpha}{L(t_k)} \) which gives

\[
\frac{L(t_{k+1})}{L(t_k)} \leq (1 - a_1 \alpha)\kappa \left( \frac{\alpha}{L(t_k)} \right) + \frac{a_1 \alpha}{L(t_k)} \quad (A.2)
\]

To see that the sequence \( (L(t_k))_{k \geq 0} \) is bounded, let us introduce \( \varphi \) the function defined on the interval \((0, +\infty)\) as

\[
\varphi(t) = (1 - a_1 \alpha)\kappa \left( \frac{\alpha}{t} \right) + \frac{a_1 \alpha}{t}.
\]

Notice that \( \varphi \) is decreasing on this interval, that \( \lim_{t \to 0} \varphi(t) = +\infty \) and that \( \lim_{t \to +\infty} \varphi(t) = 1 - a_1 \alpha < 1 \); so there exists a unique \( \ell_1 \in (0, +\infty) \) such that

\[
\varphi(\ell_1) = 1.
\]

Assume now that \( L(t_k) \leq \ell_1 \) for every \( k \geq 0 \), then we can say that the sequence \( (L(t_k))_{k \geq 0} \) is bounded. If \( L(t_k) \geq \ell_1 \) for every \( k \geq 0 \), the inequality (A.2) implies that

\[
L(t_{k+1}) \leq L(t_k)\varphi(L(t_k)) \leq L(t_k)\varphi(\ell_1) \leq L(t_k) \quad (\text{because } L(t_k) \geq \ell_1)
\]

\[
= L(t_k)
\]
Lemma 6: \( \Psi \) satisfies the following property for all \( a_1 \) and \( \alpha \) such that \( a_1 \alpha < 1 \)

\[
\Psi P \Psi \leq \psi_0(\alpha) P \psi_0(\alpha),
\]

where

\[
\psi_0(\alpha) = \text{diag}\left( \frac{1}{1 - a_1 \alpha}, \ldots, \frac{1}{(1 - a_1 \alpha)^n} \right).
\]

Given \( v \) in \( S^{n-1} = \{ v \in \mathbb{R}^n \mid \|v\| = 1 \} \), consider the function

\[
\nu(\alpha, v) = v'Q(\alpha)\psi_0(\alpha)P\psi_0(\alpha)Q(\alpha)v.
\]

We have

\[
\nu(0, v) = v'Pv,
\]

\[
\frac{\partial \nu}{\partial \alpha}(0, v) = v' [P[A + KC + a_1 D] + [A + KC + a_1 D]' P] v,
\]

so using the inequalities in (9) and setting \( a_1 = \frac{1}{2p_2} \), we get

\[
\frac{\partial \nu}{\partial \alpha}(0, v) \leq v' \left( a_1 p_4 P - \frac{1}{p_2} P \right) v
\]

\[
= -\frac{1}{2p_2} v' P v.
\]

Now, we can write

\[
\nu(\alpha, v) = v' P v + \alpha \frac{\partial \nu}{\partial \alpha}(0, v) + \rho(\alpha, v)
\]

with \( \lim_{\alpha \to 0} \frac{\rho(\alpha, v)}{\alpha} = 0 \). This equality together with (A.4) imply that

\[
\nu(\alpha, v) \leq v' P v \left[ 1 - \alpha \frac{1}{4p_2} \right] + \rho(\alpha, v).
\]

The vector \( v \) being in a compact set and the function \( \rho \) being continuous, there exists \( \alpha_m \) such that for all \( \alpha \in [0, \alpha_m) \) we have \( \rho(\alpha, v) \leq \alpha \frac{1}{4p_2} v' P v \) for all \( v \). This gives

\[
\nu(\alpha, v) \leq v' P v \left[ 1 - \alpha \frac{1}{4p_2} \right], \forall \alpha \in [0, \alpha_m), \forall v \in S^{n-1}.
\]

This property being true for every \( v \), this ends the proof of Lemma 4.

A.3 Proof of Lemma 6

Consider the matrix function defined as

\[
P(s) = \text{diag}(s, \ldots, s^n) P \text{diag}(s, \ldots, s^n).
\]

Note that for all \( v \in \mathbb{R}^n \)

\[
\frac{d}{ds} v' P(s) v
\]

\[
= \frac{1}{2} v' \text{diag}(s, \ldots, s^n) (D' P + PD) \text{diag}(s, \ldots, s^n) v > 0.
\]

Hence, \( P \) is an increasing function. Furthermore, we have

\[
\Psi P \Psi \leq \mathcal{L}_{k+1}^{-1} \mathcal{L}_{k+1}^{-1} \mathcal{P} \mathcal{L}_{k+1}^{-1} \mathcal{L}_{k+1}^{-1}
\]

\[
= \mathcal{P} \left( \frac{L_{k+1}}{L_{k+1}} \frac{1}{(1-a_1 \alpha) + a_1 \alpha} \right),
\]

so as \( \frac{L_{k+1}}{L_{k+1}} \frac{1}{(1-a_1 \alpha) + a_1 \alpha} \leq \frac{1}{1-a_1 \alpha} \), we get the inequality of Lemma 6: \( \Psi P \Psi \leq \mathcal{P} \left( \frac{1}{1-a_1 \alpha} \right) \).