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LIPSCHITZ INTERIOR REGULARITY FOR THE VISCOSITY AND WEAK SOLUTIONS OF THE PSEUDO p -LAPLACIAN EQUATION

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(Submitted by: Roger Temam)

Abstract. We consider the pseudo- p -Laplacian operator:

$$\tilde{\Delta}_p u = \sum_{i=1}^N \partial_i (|\partial_i u|^{p-2} \partial_i u) = (p-1) \sum_{i=1}^N |\partial_i u|^{p-2} \partial_{ii} u \quad \text{for } p > 2.$$

We prove interior regularity results for the viscosity (resp. weak) solutions in the unit ball B_1 of $\tilde{\Delta}_p u = (p-1)f$ for $f \in \mathcal{C}(\overline{B_1})$ (resp. $f \in L^\infty(B_1)$). Firstly the Hölder local regularity for any exponent $\gamma < 1$, recovering in that way a known result about weak solutions. In a second time we prove the Lipschitz local regularity.

1. INTRODUCTION

This paper is devoted to the local Lipschitz regularity for viscosity solutions of the equation

$$\sum_{i=1}^N \partial_i (|\partial_i u|^{p-2} \partial_i u) = (p-1)f. \quad (1.1)$$

The operator on the left hand side is known as the pseudo- p -Laplace operator, and the equation above is the Euler Lagrange equation associated to the energy functional

$$\frac{1}{p} \int \sum_{i=1}^N |\partial_i u|^p + (p-1) \int f u.$$

Even if this equation seems very similar to the usual p -Laplace equation (existence of solutions, comparison theorems), the usual methods to prove regularity results cannot easily be adapted here. This is mainly due to the fact that the operator degenerates on non bounded sets in \mathbb{R}^N .

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Several directions have been taken by different people, and except for Belloni and Kawohl in [2], all the solutions they studied were weak solutions, i.e. such that $u \in W_{loc}^{1,p}$ and the equation (1.1) is intended in the distribution sense.

The first regularity results for this type of equations may be found in the pioneer paper of Uraltseva and Urdaletova, [19]. Among other results they prove that, for f replaced by $f(x, u)$ with some specific conditions of growth with respect to u , and for $p > 3$, then the solutions are Lipschitz continuous. The Lipschitz regularity in the case $p < 2$, with a right hand side $f \in L^\infty$ can be derived from the techniques used by Fonseca, Fusco and Marcellini in [13]. One must point out that when $p < 2$ the notion of viscosity solution cannot easily be defined, since the operator has no meaning on points \bar{x} for which some test function φ satisfies $\partial_i \varphi(\bar{x}) = 0$. Therefore it is not immediate to obtain Lipschitz regularity for the solutions using viscosity arguments as here.

In [5] the authors studied (among other things) the Lipschitz regularity for equations as

$$\begin{cases} -\mu \sum_i \partial_i ((|\nabla u|^{p-1} - \delta)_+ \frac{\partial_i u}{|\nabla u|}) - \sum_i \partial_i (\partial_i u |^{p_i-2} \partial_i u) = f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega. \end{cases}$$

Here, the p_i are > 1 , $p > 1$ and $\mu > 0$. They prove a Lipschitz regularity result under some bounded slope condition on φ . The case where $\mu = 0$, even when all the p_i are equal to each other is not covered by their proofs.

The degenerate case $p \geq 2$ has been much explored, and almost all the techniques involved are variational. In particular, the regularity results are obtained using Moser's iteration method.

Among the recent regularity results, let us cite the paper of Brasco and Carlier [7], which proves that for the widely degenerate anisotropic equation, arising in congested optimal transport

$$\sum_{i=1}^N \partial_i \left((|\partial_i u| - \delta_i)_+^{p-1} \frac{\partial_i u}{|\partial_i u|} \right) = (p-1)f. \quad (1.2)$$

(here the δ_i are non negative numbers, and $f \in L_{loc}^\infty$), the solutions are in $W_{loc}^{1,q}$ for any $q < \infty$. In particular it implies the Hölder's regularity of the solutions for any exponent $\gamma < 1$, by means of the Sobolev Morrey's embedding.

In [2] Belloni and Kawohl are interested in the first eigenvalue for the Dirichlet problem of the pseudo p -Laplace operator. They prove existence and uniqueness of the first eigenfunction, up to a multiplicative constant.

The most complete and strongest results about regularity concerns the widely degenerate anisotropic equation in (1.2). In [6], Bousquet, Brasco and Julin prove the following Lipschitz regularity result:

If $N = 2$, for any $p \geq 2$, and for $f \in W_{loc}^{1,p'}$, ($\frac{1}{p} + \frac{1}{p'} = 1$), or if $N \geq 3$, $p \geq 4$, and $f \in W_{loc}^{1,\infty}$, then every weak solution of (1.2) is locally Lipschitz continuous.

In particular, their results include the case where $\delta_i = 0$ for all i , under the regularity assumption on f above. Once more the techniques involved are variational.

In the present paper, we consider \mathcal{C} -viscosity solutions of (1.1). In fact the result for viscosity solutions will be a corollary of the stronger result

Theorem 1.1. *For any $p > 2$, for any $f, g \in \mathcal{C}(\overline{B_1})$ and for any u , USC and v LSC which satisfy in the viscosity sense*

$$\sum_i |\partial_i u|^{p-2} \partial_{ii} u \geq f \text{ and } \sum_i |\partial_i v|^{p-2} \partial_{ii} v \leq g$$

then for any $r < 1$, there exists c_r such that for any $(x, y) \in B_r^2$

$$u(x) - v(y) \leq \sup(u - v) + c_r |x - y|.$$

In particular the Lipschitz regularity result holds true for u if u is both a sub and a super-solution of the equation, even with some right hand sides different. Furthermore the Lipschitz continuity for solutions of (1.1) requires only that the right hand side be continuous.

The second important advantage of the methods here employed is that they can be applied to study regularity of Fully Non Linear Operators on the model of the pseudo p -Laplace operator, but not under divergence form. This will be done in [4].

We will derive the local Lipschitz regularity for $W_{loc}^{1,p}$ solutions and $f \in L_{loc}^\infty$, from the one for viscosity solutions and f continuous.

We hope that the method here employed could be used to treat the case $p \leq 4$ and $N \geq 3$, not covered at this day, to my knowledge, by the results of Bousquet, Brasco and Julin, [6] for the widely degenerate equation (1.2), as well as to weaken the regularity of f in [6], but of course the high degeneracy of (1.2) brings additional technical difficulties.

A further question we ask is: does the \mathcal{C}^1 or $\mathcal{C}^{1,\beta}$ regularity holds, as in the case of the classical p -Laplacian, [18], [12]? A first step would consist in proving the \mathcal{C}^1 regularity when the right hand side is zero and then deduce from it the case $f \neq 0$ by methods as in [15], [3], but even in the case $f \equiv 0$ i have no intuition about the truthfulness of this result. The usual methods in the theory of viscosity solutions, ([8], [17]), cannot directly be applied to the present case, one of the key argument of their proofs being the uniform ellipticity of the operator. Likewise, the methods of Figalli and Colombo [9] to prove the regularity outside of the degeneracy set of the operator, suppose that this set is bounded, which is not the case here. One must find new arguments.

Another probable extension of the results included here consists in proving that the Lipschitz regularity still holds when $f \in L^k$ for $k > N$, using L^k -viscosity solutions ([11]) in place of \mathcal{C} -viscosity solutions. This could be the object of a future work.

As we said before in particular we deduce from Theorem 1.1 the following result:

Theorem 1.2. *For any $p > 2$ and for all $r < 1$, there exists C depending on (p, N, r) such that for any u a \mathcal{C} -viscosity (respectively weak), bounded solution in B_1 of (1.1), with $f \in \mathcal{C}(\overline{B_1})$ (respectively $f \in L^\infty(B_1)$),*

$$\text{Lip}_{B_r} u \leq C(|u|_{L^\infty(B_1)} + |f|_{L^\infty(B_1)}^{\frac{1}{p-1}}).$$

The plan of this paper is as follows. In Section 2, we recall some basic facts about viscosity solutions. We give the material for deducing from the Lipschitz regularity result for viscosity solutions and a right hand side continuous, that the same holds true for weak solutions and $f \in L_{loc}^\infty$. In Section 3 we prove Lipschitz regularity estimates between viscosity sub- and super-solutions, ie the content of Theorem 1.1.

2. WEAK SOLUTIONS AND VISCOSITY SOLUTIONS

2.1. About viscosity solutions. Notations. In all the paper B_r denotes the open ball of center 0 and radius r . x (respectively $\begin{pmatrix} x \\ y \end{pmatrix}$) denotes a vector column in \mathbb{R}^N (respectively a column in $\mathbb{R}^N \times \mathbb{R}^N$), while ${}^t x$, (respectively $({}^t x, {}^t y)$) denotes a vector line in \mathbb{R}^N (respectively a vector line in $\mathbb{R}^N \times \mathbb{R}^N$). For x and $y \in \mathbb{R}^N$, we denote the scalar product of x and y by $\langle x, y \rangle$ or ${}^t x y$ or $x \cdot y$. For $x \in \mathbb{R}^N$, $|x|$ denotes the euclidian norm $|x| = (x \cdot x)^{\frac{1}{2}} = (\sum_{i=1}^N |x_i|^2)^{\frac{1}{2}}$. S is the space of symmetric matrices on \mathbb{R}^N .

For $A \in S$, we define the norm $|A| = \sup_{x \in \mathbb{R}^N, |x|=1} |{}^t x A x|$ or equivalently $|A| = \sup_{1 \leq i \leq N} |\lambda_i(A)|$ where the $\lambda_i(A)$ are the eigenvalues of A . We recall that $X \leq Y$ when X and Y are in S , means that $Y - X \geq 0$ i.e for all $x \in \mathbb{R}^N$, ${}^t x(Y - X)x \geq 0$, or equivalently $\inf_{1 \leq i \leq N} \lambda_i(Y - X) \geq 0$.

Let us recall the definition of \mathcal{C} -viscosity solutions for Elliptic Second Order Differential Operators.

Let F be continuous on $\Omega \times \mathbb{R} \times \mathbb{R}^N \times S$, where Ω denotes an open subset in \mathbb{R}^N . We consider the "equation"

$$F(x, u, Du, D^2u) = 0.$$

Definition 2.1. u , lower-semicontinuous (LSC) in Ω is a \mathcal{C} -viscosity super-solution of $F(x, u, Du, D^2u) = 0$ in Ω if, for any $x_o \in \Omega$ and any φ , \mathcal{C}^2 around x_o which satisfies $(u - \varphi)(x) \geq (u - \varphi)(x_o) = 0$ in a neighborhood of x_o , one has $F(x_o, \varphi(x_o), D\varphi(x_o), D^2\varphi(x_o)) \leq 0$.

u upper-semicontinuous (USC), is a \mathcal{C} -viscosity sub-solution of

$F(x, u, Du, D^2u) = 0$ in Ω if for any $x_o \in \Omega$ and any φ , \mathcal{C}^2 around x_o which satisfies $(u - \varphi)(x) \leq (u - \varphi)(x_o) = 0$ in a neighborhood of x_o , one has $F(x_o, \varphi(x_o), D\varphi(x_o), D^2\varphi(x_o)) \geq 0$.

u is a solution if it is both a super- and a sub-solution.

It is classical in the theory of Second Order Elliptic Equations that one can work with semi-jets, and closed semi-jets in place of \mathcal{C}^2 functions. For the convenience of the reader we recall their definition.

Definition 2.2. Let u be an upper semi-continuous function in a neighbourhood of \bar{x} . Then we define the super-jet $(q, X) \in \mathbb{R}^N \times S$ and we note $(q, X) \in J^{2,+}u(\bar{x})$ if there exists $r > 0$ such that for all $x \in B_r(\bar{x})$,

$$u(x) \leq u(\bar{x}) + \langle q, x - \bar{x} \rangle + \frac{1}{2} {}^t(x - \bar{x})X(x - \bar{x}) + o(|x - \bar{x}|^2).$$

Let u be a lower semi-continuous function in a neighbourhood of \bar{x} . Then we define the sub-jet $(q, X) \in \mathbb{R}^N \times S$ and we note $(q, X) \in J^{2,-}u(\bar{x})$ if there exists $r > 0$ such that for all $x \in B_r(\bar{x})$,

$$u(x) \geq u(\bar{x}) + \langle q, x - \bar{x} \rangle + \frac{1}{2} {}^t(x - \bar{x})X(x - \bar{x}) + o(|x - \bar{x}|^2).$$

We also define the "closed semi-jets"

$$\begin{aligned} \bar{J}^{2,\pm}u(\bar{x}) &= \{(q, X), \exists (x_n, q_n, X_n), (q_n, X_n) \in J^{2,\pm}u(x_n) \\ &\text{and } (x_n, q_n, X_n) \rightarrow (\bar{x}, q, X)\}. \end{aligned}$$

We refer to the survey of Ishii [16], and to [10] for more complete results about semi-jets. The link between semi-jets and test functions for sub- and super-solutions is the following:

u , USC is a sub-solution if and only if for any \bar{x} and for any $(q, X) \in \bar{J}^{2,+}u(\bar{x})$, then

$$F(\bar{x}, u(\bar{x}), q, X) \geq 0 \quad (2.1)$$

and the same analogous with obvious changes is valid for super-solutions.

Let us now recall Lemma 9 in [16] and one of its consequences for the proofs in the present paper

Lemma 2.3. *Suppose that A is a symmetric matrix on \mathbb{R}^{2N} and that $U \in USC(\mathbb{R}^N)$, $V \in USC(\mathbb{R}^N)$ satisfy $U(0) = V(0)$ and for all $(x, y) \in (\mathbb{R}^N)^2$*

$$U(x) + V(y) \leq \frac{1}{2}({}^t x, {}^t y)A \begin{pmatrix} x \\ y \end{pmatrix}.$$

Then for all $\iota > 0$ there exist $X_\iota^U \in S$, $X_\iota^V \in S$ such that

$$(0, X_\iota^U) \in \bar{J}^{2,+}U(0), \quad (0, X_\iota^V) \in \bar{J}^{2,+}V(0)$$

and

$$-\left(\frac{1}{\iota} + |A|\right) \begin{pmatrix} \text{Id} & 0 \\ 0 & \text{Id} \end{pmatrix} \leq \begin{pmatrix} X_\iota^U & 0 \\ 0 & X_\iota^V \end{pmatrix} \leq (A + \iota A^2). \quad (2.2)$$

The proof of Lemma 2.3 uses the approximation of U and V by Sup and Inf convolution. This lemma has the following consequence for the results of this paper.

Lemma 2.4. *Suppose that u and v are respectively USC and LSC and satisfy for some constant $M > 1$ and for some function Φ which is \mathcal{C}^2 around (\bar{x}, \bar{y})*

$$\begin{aligned} & u(x) - v(y) - M|x - x_o|^2 - M|y - x_o|^2 - \Phi(x, y) \\ & \leq u(\bar{x}) - v(\bar{y}) - M|\bar{x} - x_o|^2 - M|\bar{y} - x_o|^2 - \Phi(\bar{x}, \bar{y}). \end{aligned}$$

Then for any ι , there exist X_ι, Y_ι such that

$$\begin{aligned} & (D_1\Phi(\bar{x}, \bar{y}) + 2M(\bar{x} - x_o), X_\iota) \in \bar{J}^{2,+}u(\bar{x}), \\ & (-D_2\Phi(\bar{x}, \bar{y}) - 2M(\bar{y} - x_o), -Y_\iota) \in \bar{J}^{2,-}v(\bar{y}) \end{aligned}$$

with

$$\begin{aligned} -\left(\frac{1}{\iota} + |A| + 1\right) \begin{pmatrix} \text{Id} & 0 \\ 0 & \text{Id} \end{pmatrix} & \leq \begin{pmatrix} X_\iota - 2M\text{Id} & 0 \\ 0 & Y_\iota - 2M\text{Id} \end{pmatrix} \\ & \leq (A + \iota A^2) + \begin{pmatrix} \text{Id} & 0 \\ 0 & \text{Id} \end{pmatrix} \end{aligned} \quad (2.3)$$

and $A = D^2\Phi(\bar{x}, \bar{y})$.

Proof. By Taylor's formula at the order 2 for Φ , for all $\epsilon > 0$ there exists $r > 0$ such that for $|x - \bar{x}|^2 + |\bar{y} - y|^2 \leq r^2$,

$$\begin{aligned} & u(x) - \langle D_1\Phi(\bar{x}, \bar{y}) + 2M(\bar{x} - x_o), x - \bar{x} \rangle - v(y) \\ & \quad - \langle D_2\phi(\bar{x}, \bar{y}) + 2M(\bar{y} - x_o), y - \bar{y} \rangle - u(\bar{x}) + v(\bar{y}) \\ & \leq \frac{1}{2}({}^t(x - \bar{x}), {}^t(y - \bar{y})) (D^2\Phi(\bar{x}, \bar{y}) + \epsilon Id) \begin{pmatrix} x - \bar{x} \\ y - \bar{y} \end{pmatrix} + M(|x - \bar{x}|^2 + |y - \bar{y}|^2). \end{aligned}$$

We define

$$U(x) = u(x + \bar{x}) - \langle D_1\Phi(\bar{x}, \bar{y}) + 2M(\bar{x} - x_o), x \rangle - u(\bar{x}) - M|x|^2,$$

$$V(y) = -v(y + \bar{y}) - \langle D_2\Phi(\bar{x}, \bar{y}) + 2M(\bar{y} - x_o), y \rangle + v(\bar{y}) - M|y|^2$$

in the closed ball $|x - \bar{x}|^2 + |\bar{y} - y|^2 \leq r^2$, extend U and V by some convenient negative constants in the complementary (see [16] for details) and apply Lemma 2.3. Note that $(0, X_\iota^U) \in \bar{J}^{2,+}U(0)$, $(0, X_\iota^V) \in \bar{J}^{2,-}V(0)$ is equivalent to

$$(D_1\Phi(\bar{x}, \bar{y}) + 2M(\bar{x} - x_o), X_\iota^U + 2MId) \in \bar{J}^{2,+}u(\bar{x})$$

and

$$(-D_2\Phi(\bar{x}, \bar{y}) - 2M(\bar{y} - x_o), -X_\iota^V - 2MId) \in \bar{J}^{2,-}v(\bar{y}).$$

Hence, using (2.2), one obtains that for any ι there exists (X_ι, Y_ι) such that

$$(D_1\Phi(\bar{x}, \bar{y}) + 2M(\bar{x} - x_o), X_\iota) \in \bar{J}^{2,+}u(\bar{x}),$$

$$(-D_2\Phi(\bar{x}, \bar{y}) - 2M(\bar{y} - x_o), -Y_\iota) \in \bar{J}^{2,-}v(\bar{y})$$

and taking ϵ such that $2\epsilon\iota|A| + \epsilon + \iota(\epsilon)^2 < 1$, one obtains (2.3). \square

In the sequel we will use Lemma 2.4 with $\Phi(x, y)$ of the form

$$\Phi(x, y) = Mg(x - y) = M\omega(|x - y|),$$

where g is defined on \mathbb{R}^N and ω on \mathbb{R}^+ will be defined later and then noting $H_1(x) = D^2g(x)$, defining $\iota = \frac{1}{4M|H_1(x)|}$, $\tilde{H}(x) = H_1(x) + 2\iota H_1^2(x)$,

$$A = M \begin{pmatrix} H_1(\bar{x} - \bar{y}) & -H_1(\bar{x} - \bar{y}) \\ -H_1(\bar{x} - \bar{y}) & H_1(\bar{x} - \bar{y}) \end{pmatrix},$$

and

$$A + \iota A^2 = M \begin{pmatrix} \tilde{H}(\bar{x} - \bar{y}) & -\tilde{H}(\bar{x} - \bar{y}) \\ -\tilde{H}(\bar{x} - \bar{y}) & \tilde{H}(\bar{x} - \bar{y}) \end{pmatrix}.$$

Note that $|A| = 2M|H_1(\bar{x} - \bar{y})|$.

In all the situations later, (2.3) has the following consequence for $X := X_t$ and $Y := Y_t$,

$$|X - (2M + 1) \text{Id}| + |Y - (2M + 1) \text{Id}| \leq 6M|D^2g(\bar{x} - \bar{y})|. \quad (2.4)$$

In the rest of the paper we will consider the operators

$$F(x, u, q, X) = \sum_i |q_i|^{p-2} X_{ii} - f(x) := F(q, X) - f(x),$$

where $q_i = \vec{q} \cdot e_i$ and $X_{ij} = {}^t e_i X e_j$, e_i is some given orthonormal basis in \mathbb{R}^N , f is continuous. In the sequel we suppose known that the weak solutions are continuous, see for example [7]. This permits to use the above definition of viscosity solutions, and not its generalization to bounded functions u which makes use of lower semi-continuous or upper semi-continuous envelope of u , see [16].

2.2. Weak solutions are Viscosity solutions. In this section we want to show how one can deduce the Lipschitz regularity result for weak solutions, from the regularity result for viscosity solutions, i.e the half part of Theorem 1.2.

We begin to recall the following comparison theorem for weak solutions,

Theorem 2.5. *Suppose that Ω is a bounded C^1 open subset in \mathbb{R}^N , that u and v are in $W^{1,p}(\Omega)$ and satisfy in the distribution sense $\tilde{\Delta}_p u \geq \tilde{\Delta}_p v$. Suppose that $u \leq v$ on $\partial\Omega$, then $u \leq v$ in Ω .*

Proof. Since $(u - v)^+ \in W_0^{1,p}(\Omega)$, there exists $\varphi_\epsilon \in \mathcal{D}(\Omega)$, which converges to $(u - v)^+$ in $W^{1,p}(\Omega)$, φ_ϵ can be chosen ≥ 0 . Multiply the difference $\tilde{\Delta}_p u - \tilde{\Delta}_p v$ by φ_ϵ and use the definition of the derivative in the distribution sense. One obtains

$$- \sum_{i=1}^N \int_{\Omega} (|\partial_i u|^{p-2} \partial_i u - |\partial_i v|^{p-2} \partial_i v) \partial_i \varphi_\epsilon = \int_{\Omega} (\tilde{\Delta}_p u - \tilde{\Delta}_p v) \varphi_\epsilon \geq 0.$$

The left hand side tends to

$$- \sum_{i=1}^N \int_{\Omega} (|\partial_i u|^{p-2} \partial_i u - |\partial_i v|^{p-2} \partial_i v) \partial_i (u - v)^+ \leq 0.$$

Therefore,

$$\int_{u-v \geq 0} \sum_{1 \leq i \leq N} (|\partial_i u|^{p-2} \partial_i u - |\partial_i v|^{p-2} \partial_i v) \partial_i (u - v) = 0,$$

hence, $\partial_i((u - v)^+) = 0$ for all i , finally $u \leq v$ in Ω . \square

Let us take a few lines to motivate the following propositions. We want to have a Lipschitz estimate depending on the L^∞ norm of u and f , when u is a weak solution. A natural idea is to regularize f , hence use $f_\epsilon \in \mathcal{C}(\overline{B_1})$ which tends to f in L^∞ weakly, define u_ϵ which is a solution of the Dirichlet problem associated to (1.1) with a right hand side f_ϵ and the boundary data u on ∂B_1 . In order to apply the results in Section 3, we will use the fact that u_ϵ is also a viscosity solution, (see [2] or Proposition 2.7 below). Since the uniform Lipschitz estimate in Section 3 for u_ϵ , depends on the L^∞ norms of u_ϵ and f_ϵ , we need, in order to pass to the limit and obtain some estimate depending on the L^∞ norms of u and f , to "control" the L^∞ norm of u_ϵ by those of u , this is what we do in Proposition 2.6 now.

Proposition 2.6. *There exists some constant C depending only on (p, N) , such that for any $f \in L^\infty(B_1)$ and $v \in L^\infty(\partial B_1) \cap W^{1-\frac{1}{p}, p}(\partial B_1)$, if u is a weak solution of the Dirichlet problem*

$$\begin{cases} \tilde{\Delta}_p u = (p-1)f & \text{in } B_1 \\ u = v & \text{on } \partial B_1, \end{cases}$$

then

$$|u|_{L^\infty(B_1)} \leq C(|f|_{L^\infty(B_1)}^{\frac{1}{p-1}} + |v|_{L^\infty(\partial B_1)}). \quad (2.5)$$

Proof. Let d denote the distance to the boundary, say $d(x) = 1 - |x|$, and let h be defined by

$$h(x) = |v|_{L^\infty(\partial B_1)} + M \left(1 - \frac{1}{(1+d(x))} \right).$$

We want to prove that as soon as M is large as $|f|_{L^\infty}^{\frac{1}{p-1}}$, and depending on (N, p) , h is a weak super-solution of (1.1) in B_1 , more precisely $h \in W^{1,p}(B_1)$ and $\tilde{\Delta}_p h \leq -(p-1)|f|_\infty$, in the distribution sense.

Note that h is \mathcal{C}^2 except at zero, $|Dh| \leq M$, and for $x \neq 0$, $\tilde{\Delta}_p h(x) \leq -(p-1)|f|_\infty$. Suppose for a while that this last assertion has been proven, and let us derive from it that h is a weak super-solution. Let $\varphi \in \mathcal{D}(B_1)$, $\varphi \geq 0$,

$$\begin{aligned} \langle \tilde{\Delta}_p h, \varphi \rangle &= - \int_{B_1} \sum_{i=1}^N |\partial_i h|^{p-2} \partial_i h \partial_i \varphi = \lim_{\epsilon \rightarrow 0} - \int_{B_1 \setminus \overline{B(0, \epsilon)}} \sum_{i=1}^N |\partial_i h|^{p-2} \partial_i h \partial_i \varphi \\ &= \lim_{\epsilon \rightarrow 0} \int_{B_1 \setminus \overline{B(0, \epsilon)}} (p-1) \sum_{i=1}^N |\partial_i h|^{p-2} \partial_{ii} h \varphi - \int_{\partial B(0, \epsilon)} \sum_{i=1}^N |\partial_i h|^{p-2} \partial_i h n_i \varphi \end{aligned}$$

$$\begin{aligned}
&\leq -(p-1) \lim_{\epsilon \rightarrow 0} |f|_\infty \int_{B_1 \setminus \overline{B(0,\epsilon)}} \varphi + M^{p-1} \lim_{\epsilon \rightarrow 0} \int_{\partial B(0,\epsilon)} \varphi \\
&\leq -(p-1) |f|_\infty \int_{B_1} \varphi.
\end{aligned}$$

There remains to prove that for $x \neq 0$, and for M chosen conveniently, $\tilde{\Delta}_p h(x) \leq -(p-1)|f|_\infty$. One has for $x \neq 0$

$$Dh = M \frac{Dd}{(1+d)^2}, \quad D^2h = \frac{-2M(Dd \otimes Dd) + (1+d)D^2d}{(1+d)^3}$$

and then since $D_{ii}d \leq 0$, by the concavity of d ,

$$\begin{aligned}
(p-1) \sum_{i=1}^N |\partial_i h|^{p-2} \partial_{ii} h &\leq -2M^{p-1}(p-1) \frac{\sum_{i=1}^N |\partial_i d|^p}{(1+d)^{2p+1}} \\
&\leq -2^{-2p} M^{p-1} (p-1) N^{1-\frac{p}{2}}.
\end{aligned}$$

Then we choose M so that $M^{p-1} 2^{-2p} N^{1-\frac{p}{2}} > |f|_\infty$ and get $\tilde{\Delta}_p h(x) \leq -(p-1)|f|_\infty$. Using Theorem 2.5 one gets that $u \leq h$ in B_1 , which is the desired conclusion. Replacing h by $-h$, one sees that $\tilde{\Delta}_p(-h) \geq (p-1)|f|_\infty$ and then $-h$ is a sub-solution of (1.1), hence $u \geq -h$ in Ω . \square

We now recall the following local result, [2].

Proposition 2.7. *Suppose that u is a weak ($W^{1,p}$) solution of $\tilde{\Delta}_p u = (p-1)f$ in some open set \mathcal{O} with $f \in \mathcal{C}(\mathcal{O})$, then u is a \mathcal{C} -viscosity solution of the same equation in \mathcal{O} .*

Proof. The proof is made in [2], but it is written here for the reader's convenience.

We do the super-solution case. Take any $x_o \in \mathcal{O}$ and some \mathcal{C}^2 function φ such that $(u - \varphi)(x) \geq (u - \varphi)(x_o) = 0$ in a neighbourhood of x_o . We can assume the inequality to be strict for $x \neq x_o$, by replacing φ by $x \mapsto \varphi(x) - |x - x_o|^4$. Assume by contradiction that for some $\epsilon > 0$

$$(p-1) \sum_{i=1}^N |\partial_i \varphi(x_o)|^{p-2} \partial_{ii} \varphi(x_o) \geq (p-1)f(x_o) + \epsilon,$$

then by continuity this is also true (up to changing ϵ) in a neighbourhood $B_r(x_o)$, with $\overline{B_r(x_o)} \subset \mathcal{O}$. Let $m = \inf_{\partial B_r(x_o)} (u - \varphi)$ and $\phi = \varphi + \frac{m}{2}$ which

satisfies $u - \phi > 0$ on $\partial B_r(x_o)$ and $\phi(x_o) > u(x_o)$. We also have

$$(p-1) \sum_{i=1}^N |\partial_i \phi(x)|^{p-2} \partial_{ii} \phi(x) \geq (p-1)f(x) + \epsilon \quad \text{in } B_r(x_o).$$

Multiplying by $(\phi - u)^+$, and integrating over B_r , we obtain

$$- \int_{B_r(x_o) \cap \{\phi - u > 0\}} \sum_{i=1}^N |\partial_i \phi|^{p-2} \partial_i \phi \partial_i (\phi - u) \geq (p-1) \int_{B_r(x_o)} (f + \epsilon)(\phi - u)^+.$$

On the other hand, since u is a weak solution,

$$\int_{B_r(x_o) \cap \{\phi - u > 0\}} \sum_{i=1}^N |\partial_i u|^{p-2} \partial_i u \partial_i (\phi - u) = (p-1) \int_{B_r(x_o)} f(\phi - u)^+.$$

Adding the two equations, one obtains

$$- \int_{B_r(x_o)} \sum_{i=1}^N (|\partial_i \phi|^{p-2} \partial_i \phi - |\partial_i u|^{p-2} \partial_i u) \partial_i (\phi - u)^+ \geq (p-1)\epsilon \int_{B_r(x_o)} (\phi - u)^+.$$

This is a contradiction, because the right hand side is positive, while the left hand side is non positive. This proves that u is a viscosity super-solution. We could do the same with obvious changes to prove that a weak solution is a viscosity sub-solution. \square

We derive from this the result in Theorem 1.2 for weak solutions, once we know it for viscosity solutions.

Suppose that $f \in L^\infty(B_1)$ and that u is a weak solution of $\tilde{\Delta}_p u = (p-1)f$ in B_1 , $u \in L^\infty(B_1)$. Let $f_\epsilon \in C(\overline{B_1})$, $f_\epsilon \rightharpoonup f$ in $L^\infty(B_1)$ weakly, $|f_\epsilon|_\infty \rightarrow |f|_\infty$, and let u_ϵ be the unique weak solution of

$$\begin{cases} \tilde{\Delta}_p u_\epsilon = (p-1)f_\epsilon & \text{in } B_1 \\ u_\epsilon = u & \text{on } \partial B_1. \end{cases}$$

which makes sense since $u|_{\partial B_1} \in W^{1-\frac{1}{p}, p}(\partial B_1)$. It is equivalent to say that u_ϵ satisfies $J_\epsilon(u_\epsilon) = \inf_{v-u \in W_o^{1,p}(B_1)} J_\epsilon(v)$ where $J_\epsilon(v) = \frac{1}{p} \int_{B_1} \sum_1^N |\partial_i v|^p + (p-1) \int_{B_1} f_\epsilon v$, $J(v) = \frac{1}{p} \int_{B_1} \sum_1^N |\partial_i v|^p + (p-1) \int_{B_1} f v$. It is clear that $(u_\epsilon)_\epsilon$ is bounded in $W^{1,p}(B_1)$, and that $\limsup_{\epsilon \rightarrow 0} \inf_{\{v-u \in W_o^{1,p}\}} J_\epsilon(v) \leq \inf_{\{v-u \in W_o^{1,p}\}} J(v)$. Hence one can extract from it a subsequence which converges weakly in $W^{1,p}(B_1)$ to some \bar{u} . Note that by Poincaré's inequality, since $u_\epsilon - \bar{u} \in W_o^{1,p}$, $u_\epsilon - \bar{u}$ tends to zero in L^p strongly. Hence \bar{u} satisfies $J(\bar{u}) \leq \liminf J_\epsilon(u_\epsilon)$. Finally \bar{u} is a minimizer for J , with $\bar{u} = u$ on ∂B_1 ,

hence by uniqueness $\bar{u} = u$ and the convergence of u_ϵ to u is strong in $W^{1,p}$. By the results in Section 3, for a viscosity solution and a right hand side continuous and bounded, for all r there exists some constant $C(N, p, r)$ such that for any ϵ , since u_ϵ is a \mathcal{C} -viscosity solution in B_1 ,

$$\text{Lip}_{B_r}(u_\epsilon) \leq C(N, p, r)(|u_\epsilon|_{L^\infty(B_1)} + |f_\epsilon|_{L^\infty(B_1)}^{\frac{1}{p-1}}).$$

Using (2.5) for u_ϵ and passing to the limit, one gets the estimate $\text{Lip}_{B_r}(u) \leq C(N, p, r)(|u|_{L^\infty(B_1)} + |f|_{L^\infty(B_1)}^{\frac{1}{p-1}})$.

Remark 2.8. We also have the alternative result.

There exists some constant $C(N, p, r)$ such that for any u a weak solution of (1.1) in B_1 , $\text{Lip}_{B_r}(u) \leq C(N, p, r)(|u|_{W^{1,p}(B_1)} + |f|_{L^\infty(B_1)}^{\frac{1}{p-1}})$.

This result can directly be deduced from the previous one by using the L_{loc}^∞ estimate

$$|u|_{L^\infty(B_{\frac{1+x}{2}})} \leq C(|u|_{W^{1,p}(B_1)} + |f|_\infty^{\frac{1}{p-1}})$$

which can be derived from the results in [14], [7].

3. PROOF OF THEOREM 1.1

From now we assume that f and g are continuous in B_1 and that u satisfies in the viscosity sense

$$\sum_i |\partial_i u|^{p-2} \partial_{ii} u \geq f$$

and v satisfies in the viscosity sense

$$\sum_i |\partial_i v|^{p-2} \partial_{ii} v \leq g.$$

The notations c , and c_i will always denote some positive constants which depend only on r, p, N , the Hölder's exponent γ when it intervenes and of universal constants. Proving in a first time some Hölder's estimate for any exponent γ is necessary to get the Lipschitz estimate.

3.1. Material for the proofs. In all the section, ω denotes some continuous function on \mathbb{R}^+ , such that $\omega(0) = 0$, ω is \mathcal{C}^2 on $\mathbb{R}^{+\star}$ and $\omega(s) > 0$, $\omega'(s) > 0$ and $\omega''(s) < 0$ on $]0, 1[$. Let g be the radial function

$$g(x) = \omega(|x|).$$

Then for $|x| < 1$, $Dg(x) = \omega'(|x|)\frac{x}{|x|}$, and

$$D^2g(x) = \left(\omega''(|x|) - \frac{\omega'(|x|)}{|x|} \right) \frac{x \otimes x}{|x|^2} + \frac{\omega'(|x|)}{|x|} \text{Id}.$$

We denote by $H_1(x)$ the symmetric matrix with entries $\partial_{ij}g(x)$, and for $\iota \leq \frac{1}{4|H_1(x)|}$, we define $\tilde{H} = H_1 + 2\iota H_1^2$. With that choice of ι there exist constants $\alpha_H \in [\frac{1}{2}, \frac{3}{2}]$, $\beta_H \geq \frac{1}{2}$ such that

$$\tilde{H}(x) = \left(\beta_H \omega''(|x|) - \alpha_H \frac{\omega'(|x|)}{|x|} \right) \frac{x \otimes x}{|x|^2} + \alpha_H \frac{\omega'(|x|)}{|x|} \text{Id}. \quad (3.1)$$

When $p > 4$, and for any $|x| < 1$, $x \neq 0$, we will use a number $\epsilon > 0$, (that we will make precise depending on the Hölder and Lipschitz cases) for which we define $I(x, \epsilon) = \{i \in [1, N], |x_i| \geq |x|^{1+\epsilon}\}$. When no ambiguity arises, we will denote it I for simplicity. We define the vector

$$w = \begin{cases} \sum_1^N |x_i|^{\frac{2-p}{2}} x_i e_i & \text{if } p \leq 4 \\ \sum_{i \in I} |x_i|^{\frac{2-p}{2}} x_i e_i & \text{if } p > 4, \text{ when } I(x, \epsilon) \neq \emptyset. \end{cases} \quad (3.2)$$

Note that if $p \leq 4$,

$$|w|^2 \leq |x|^{4-p} N^{\frac{p-2}{2}}, \quad (3.3)$$

while if $p > 4$

$$|w|^2 \leq \#I(x, \epsilon) |x|^{(4-p)(1+\epsilon)}. \quad (3.4)$$

Furthermore,

$$|x|^2 - \sum_{i \in I} |x_i|^2 = \sum_1^N |x_i|^2 - \sum_{i \in I} |x_i|^2 \leq N |x|^{2+2\epsilon}.$$

We also define the diagonal matrix $\Theta(x)$ with entries

$\Theta_{ij}(x) = \left| \frac{\omega'(|x|x_i)}{|x|} \right|^{\frac{p-2}{2}} \delta_i^j$, where δ_i^j denotes the Kronecker symbol, and the matrix

$$H(x) = \Theta(x) \tilde{H}(x) \Theta(x).$$

Proposition 3.1. *1) If $p \leq 4$, for all $x \neq 0$, $|x| < 1$, $H(x)$ has at least one eigenvalue less than*

$$N^{1-\frac{p}{2}} \beta_H \omega''(|x|) (\omega'(|x|))^{p-2}. \quad (3.5)$$

2) If $p > 4$, for all $x \neq 0$, $|x| < 1$, for any $\epsilon > 0$ such that $I(x, \epsilon) \neq \emptyset$, and such that

$$\beta_H \omega''(|x|)(1 - N|x|^{2\epsilon}) + \alpha_H N|x|^{2\epsilon} \frac{\omega'(|x|)}{|x|} \leq \frac{\omega''(|x|)}{4} < 0, \quad (3.6)$$

then $H(x)$ possesses at least one eigenvalue less than

$$\frac{1 - N|x|^{2\epsilon}}{\#I(x, \epsilon)} (\omega'(|x|))^{p-2} |x|^{(p-4)\epsilon} \frac{\omega''(|x|)}{4}. \quad (3.7)$$

Proof. Using the definitions of H_1 and H one has

$$\begin{aligned} H_{ij}(x) &= \left(\frac{\omega'(|x|)}{|x|} \right)^{p-2} \left(\beta_H \omega''(|x|) - \alpha_H \frac{\omega'(|x|)}{|x|} \right) |x_i|^{\frac{p-2}{2}} |x_j|^{\frac{p-2}{2}} \frac{x_i x_j}{|x|^2} \\ &+ \alpha_H \left(\frac{\omega'(|x|)}{|x|} \right)^{p-1} |x_i|^{p-2} \delta_i^j. \end{aligned}$$

Let w be defined above,

- For $p \leq 4$, using the definition of w in (3.2)

$${}^t w H(x) w = \beta_H \left(\frac{\omega'(|x|)}{|x|} \right)^{p-2} \omega''(|x|) |x|^2,$$

and then using estimate (3.3)

$$\frac{{}^t w H(x) w}{|w|_2^2} \leq \beta_H N^{\frac{2-p}{2}} (\omega'(|x|))^{p-2} \omega''(|x|) \leq \frac{1}{2} N^{\frac{2-p}{2}} (\omega'(|x|))^{p-2} \omega''(|x|),$$

-while if $p > 4$,

$$\begin{aligned} {}^t w H(x) w &= \left(\frac{\omega'(|x|)}{|x|} \right)^{p-2} \beta_H \omega''(|x|) \frac{(\sum_{i \in I} |x_i|^2)^2}{|x|^2} \\ &+ \alpha_H \left(\frac{\omega'(|x|)}{|x|} \right)^{p-1} \left(-\frac{(\sum_{i \in I} |x_i|^2)^2}{|x|^2} + \sum_{i \in I} |x_i|^2 \right), \end{aligned}$$

and then if $p > 4$, and $I(x, \epsilon) \neq \emptyset$, using (3.6), (3.1) and (3.4)

$$\begin{aligned} &\frac{{}^t w H(x) w}{|w|_2^2} \\ &\leq \left(\frac{\omega'(|x|)}{|x|} \right)^{p-2} \frac{\sum_{i \in I} |x_i|^2}{\#I(x, \epsilon) |x|^{(4-p)(1+\epsilon)}} [\beta_H \omega''(|x|)(1 - N|x|^{2\epsilon}) \\ &+ \alpha_H N \omega'(|x|) |x|^{-1+2\epsilon}] \end{aligned}$$

$$\begin{aligned}
&\leq \frac{(1 - N|x|^{2\epsilon})}{\#I(x, \epsilon)} (\omega'(|x|))^{p-2} |x|^{(p-4)\epsilon} \left[\frac{1}{2} \omega''(|x|) (1 - N|x|^{2\epsilon}) \right. \\
&+ \left. N \frac{3}{2} \omega'(|x|) |x|^{-1+2\epsilon} \right] \\
&\leq \frac{1 - N|x|^{2\epsilon}}{\#I(x, \epsilon)} (\omega'(|x|))^{p-2} |x|^{(p-4)\epsilon} \frac{\omega''(|x|)}{4}.
\end{aligned}$$

□

We derive now from these last observations and from Proposition 3.1 the following

Proposition 3.2. *Suppose that ω , g , $\Theta(x)$, and $H_1(x)$ are as in Proposition 3.1 and that for some $M > 1$, for $\iota = \frac{1}{4M|H_1(x)|}$ and $\tilde{H}(x) = H_1(x) + 2\iota H_1^2(x)$, (X, Y) satisfy*

$$\begin{aligned}
-6M|H_1(x)| \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} &\leq \begin{pmatrix} X - (2M+1)\text{Id} & 0 \\ 0 & Y - (2M+1)\text{Id} \end{pmatrix} \\
&\leq M \begin{pmatrix} \tilde{H}(x) & -\tilde{H}(x) \\ -\tilde{H}(x) & \tilde{H}(x) \end{pmatrix}. \tag{3.8}
\end{aligned}$$

1) Then

$$\Theta(x)(X + Y - 2(2M+1)\text{Id})\Theta(x) \leq 0,$$

consequently $M^{p-2}\Theta(x)(X + Y)\Theta(x) \leq 2(2M+1)M^{p-2}|\Theta(x)|^2\text{Id}$, and then for any $i \in [1, N]$

$$\lambda_i(M^{p-2}\Theta(x)(X + Y)\Theta(x)) \leq 2(2M+1)M^{p-2}|\Theta(x)|^2 \leq 6M^{p-1}|\Theta(x)|^2. \tag{3.9}$$

2) The smallest eigenvalue satisfies

$$\text{If } p \leq 4, \tag{3.10}$$

$$\lambda_1(M^{p-2}\Theta(X + Y - 2(2M+1)\text{Id})\Theta(x)) \leq 2M^{p-1}N^{\frac{2-p}{2}} (\omega'(|x|))^{p-2} \omega''(|x|).$$

While if $p \geq 4$

$$\begin{aligned}
&\lambda_1(M^{p-2}\Theta(x)(X + Y - 2(2M+1)\text{Id})\Theta(x)) \\
&\leq M^{p-1} \frac{(1 - N|x|^{2\epsilon})}{\#I(x, \epsilon)} (\omega'(|x|))^{p-2} |x|^{(p-4)\epsilon} \omega''(|x|). \tag{3.11}
\end{aligned}$$

Proof. Multiplying equation (3.8) by $\begin{pmatrix} \Theta(x) & 0 \\ 0 & \Theta(x) \end{pmatrix}$ on the right and on the left one gets

$$\begin{aligned}
& \begin{pmatrix} \Theta(x) & 0 \\ 0 & \Theta(x) \end{pmatrix} \begin{pmatrix} X - (2M+1)\text{Id} & 0 \\ 0 & Y - (2M+1)\text{Id} \end{pmatrix} \begin{pmatrix} \Theta(x) & 0 \\ 0 & \Theta(x) \end{pmatrix} \\
& \leq M \begin{pmatrix} \Theta(x) & 0 \\ 0 & \Theta(x) \end{pmatrix} \begin{pmatrix} \tilde{H}(x) & -\tilde{H}(x) \\ -\tilde{H}(x) & \tilde{H}(x) \end{pmatrix} \begin{pmatrix} \Theta(x) & 0 \\ 0 & \Theta(x) \end{pmatrix} \\
& = M \begin{pmatrix} H(x) & -H(x) \\ -H(x) & H(x) \end{pmatrix}
\end{aligned}$$

where

$$H(x) = \Theta(x)\tilde{H}(x)\Theta(x) = \Theta(x)(H_1(x) + \frac{1}{2M|H_1(x)|}H_1^2(x))\Theta(x).$$

To prove 1) let \vec{v} be any vector then multiplying by $({}^t\vec{v}, {}^t\vec{v})$ on the left of the previous inequalities and by $(\frac{\vec{v}}{v})$ on the right one gets that ${}^t\vec{v}(\Theta(x)(X + Y - 2(2M+1)\text{Id})\Theta(x))\vec{v} \leq 0$ which yields the desired result. To prove 2) using (3.5) let e be an eigenvector for $H(x)$, for some eigenvalue less than $\beta_H N^{\frac{2-p}{2}} (\omega'(|x|))^{p-2} \omega''(|x|)$, let us multiply the right hand side of the previous inequality by $({}^te, -{}^te)$ on the left and by $(\begin{smallmatrix} e \\ -e \end{smallmatrix})$ on the right one gets

$${}^te \Theta(x)(X + Y - 2(2M+1)\text{Id})\Theta(x) e \leq 4{}^te H(x) e$$

and then using (3.10) one gets

$$\lambda_1(M^{p-2}\Theta(x)(X + Y - 2(2M+1)\text{Id})\Theta(x)) \leq 2M^{p-1}N^{\frac{2-p}{2}}\omega''(|x|)\omega'(|x|)^{p-2}$$

which yields the result in the case $p \leq 4$. In the case $p \geq 4$ we argue in the same manner by replacing $\beta_H N^{\frac{2-p}{2}} (\omega'(|x|))^{p-2} \omega''(|x|)$ by the right hand side of (3.7). By the conclusion (3.7) in Proposition 3.2

$$\begin{aligned}
& \lambda_1(M^{p-2}\Theta(x)(X + Y - 2(2M+1)\text{Id})\Theta(x)) \\
& \leq 4M^{p-1} \frac{(1 - N|x|^{2\epsilon})}{\#I(x, \epsilon)} (\omega'(|x|))^{p-2} |x|^{(p-4)\epsilon} \left(\frac{\omega''(|x|)}{2} (1 - N|x|^{2\epsilon}) \right. \\
& \quad \left. + \frac{3N\omega'(|x|)}{2} |x|^{-1+2\epsilon} \right) \\
& \leq M^{p-1} \omega''(|x|)\omega'(|x|)^{p-2} |x|^{(p-4)\epsilon}.
\end{aligned}$$

□

In the sequel we will use Proposition 3.2 in the following context. For $x_o \in B_r$, let M be a constant to be defined later and ω an increasing function which, near zero, behaves in the Hölder's case as $\omega(s) = s^\gamma$ and in the

Lipschitz case like $\omega(s) = s$. We define, borrowing ideas from [15], [1], [4], the function

$$\psi(x, y) = u(x) - v(y) - \sup(u - v) - M\omega(|x - y|) - M|x - x_o|^2 - M|y - x_o|^2. \quad (3.12)$$

If there exists M independent on $x_o \in B_r$ such that $\psi(x, y) \leq 0$ in B_1^2 then the desired result holds. Indeed taking first $x = x_o$ and using $|x_o - y| \leq 2$ one gets

$$u(x_o) - v(y) \leq \sup(u - v) + M(1 + 2^{2-\gamma})\omega(|x_o - y|)$$

and by taking secondly $y = x_o$

$$u(x) - v(x_o) \leq \sup(u - v) + M(1 + 2^{2-\gamma})\omega(|x_o - x|).$$

In fact, it is sufficient to prove the following.

There exists δ depending on (r, p, N) and M depending on the same variables, such that for $|x - y| \leq \delta$, $\psi(x, y) \leq 0$. Then, assuming in addition that

$$M > 1 + \frac{4(|u|_\infty + |v|_\infty)}{\omega(\delta)},$$

one gets $\psi(x, y) \leq 0$ anywhere in B_1^2 . Suppose then that

$$M(1 - r)^2 > 8(|u|_\infty + |v|_\infty), \text{ and } M > 1 + \frac{4(|u|_\infty + |v|_\infty)}{\omega(\delta)}. \quad (3.13)$$

Assume by contradiction that the supremum of ψ is positive. Then it is achieved on some (\bar{x}, \bar{y}) which satisfy $|\bar{x} - x_o|, |\bar{y} - x_o| < \frac{1-r}{2}$. In particular $(\bar{x}, \bar{y}) \in B_{\frac{1+r}{2}}^2$, hence \bar{x} and \bar{y} are in the interior of B_1 , furthermore $|\bar{x} - \bar{y}| < \delta$.

Then by the consequences of Lemma 2.4 there exist $(q^x, X) \in \bar{J}^{2,+}u(\bar{x})$ and $(q^y, -Y) \in \bar{J}^{2,-}v(\bar{y})$ with $q^x = M\omega'(|\bar{x} - \bar{y}|)\frac{\bar{x} - \bar{y}}{|\bar{x} - \bar{y}|} + 2M(\bar{x} - x_o)$ and $q^y = M\omega'(|\bar{x} - \bar{y}|)\frac{\bar{x} - \bar{y}}{|\bar{x} - \bar{y}|} - 2M(\bar{y} - x_o)$. Furthermore with the notations above,

ie $g(x) = \omega(|x|)$, $H_1(\bar{x} - \bar{y}) = D^2g(|\bar{x} - \bar{y}|)$, $(\Theta)_{ij} = \left| \omega'(|\bar{x} - \bar{y}|)\frac{\bar{x}_i - \bar{y}_i}{|\bar{x} - \bar{y}|} \right|^{\frac{p-2}{2}} \delta_i^j$ and $\tilde{H}(\bar{x} - \bar{y}) = (H_1(\bar{x} - \bar{y}) + 2\iota H_1^2(\bar{x} - \bar{y}))$, $(\iota = \frac{1}{4M|H_1(\bar{x} - \bar{y})|})$

$$\begin{aligned} -6M|H_1(\bar{x} - \bar{y})| \begin{pmatrix} \text{Id} & 0 \\ 0 & \text{Id} \end{pmatrix} &\leq \begin{pmatrix} X - (2M + 1)\text{Id} & 0 \\ 0 & Y - (2M + 1)\text{Id} \end{pmatrix} \\ &\leq M \begin{pmatrix} \tilde{H}(\bar{x} - \bar{y}) & -\tilde{H}(\bar{x} - \bar{y}) \\ -\tilde{H}(\bar{x} - \bar{y}) & \tilde{H}(\bar{x} - \bar{y}) \end{pmatrix} \end{aligned} \quad (3.14)$$

Finally, defining $q = M\omega'(|\bar{x} - \bar{y}|) \frac{\bar{x} - \bar{y}}{|\bar{x} - \bar{y}|}$, in each of the cases below we will prove the following :

Claims. *There exists $\hat{\tau} > 0$, depending on p (and on γ in the Hölder's case), and some constant $c > 0$ depending on (r, p, N) (and on γ in the Hölder's case) such that for $\delta < 1$ depending on (r, p, N) , and for $|\bar{x} - \bar{y}| < \delta$ the matrix $M^{p-2}\Theta(\bar{x} - \bar{y})(X + Y)\Theta(\bar{x} - \bar{y})$ has one eigenvalue λ_1 such that*

$$\lambda_1(M^{p-2}\Theta(\bar{x} - \bar{y})(X + Y)\Theta(\bar{x} - \bar{y})) \leq -cM^{p-1}|\bar{x} - \bar{y}|^{-\hat{\tau}}. \quad (3.15)$$

$\exists c_1$, and $\tau_1 < \tau$, both depending on (r, p, N) such that for all $i \geq 2$

$$\lambda_i(M^{p-2}\Theta(\bar{x} - \bar{y})(X + Y)\Theta(\bar{x} - \bar{y})) \leq c_1M^{p-1}|\bar{x} - \bar{y}|^{-\tau_1}. \quad (3.16)$$

There exist $\tau_2 < \hat{\tau}$, and c_2 both depending on (r, p, N) so that

$$\|q^x\|^{p-2} - \|q\|^{p-2}\|X\| + \|q^y\|^{p-2} - \|q\|^{p-2}\|Y\| \leq c_2M^{p-1}|\bar{x} - \bar{y}|^{-\tau_2}. \quad (3.17)$$

All these claims imply that taking δ small enough depending on c, c_1, c_2 , there exists c_3 such that

$$F(q^x, X) - F(q^y, -Y) \leq -c_3M^{p-1}|\bar{x} - \bar{y}|^{-\hat{\tau}}.$$

Indeed

$$\begin{aligned} F(q^x, X) &= \sum_{i=1}^{i=N} |q_i^x|^{p-2} X_{ii} \leq \sum_{i=1}^N |q_i|^{p-2} X_{ii} + \|q^x\|^{p-2} - \|q\|^{p-2}\|X\| \\ &\leq \sum_{i=1}^N |q_i|^{p-2} (-Y)_{ii} + M^{p-2} \sum_{i=1}^N \lambda_i(\Theta(\bar{x} - \bar{y})(X + Y)\Theta(\bar{x} - \bar{y})) \\ &\quad + \|q^x\|^{p-2} - \|q\|^{p-2}\|X\| \\ &\leq \sum_i |q_i|^{p-2} (-Y)_{ii} + M^{p-2} \lambda_1(\Theta(\bar{x} - \bar{y})(X + Y)\Theta(\bar{x} - \bar{y})) \\ &\quad + M^{p-2} \sum_{i \geq 2} \lambda_i(\Theta(\bar{x} - \bar{y})(X + Y)\Theta(\bar{x} - \bar{y})) + \|q^x\|^{p-2} - \|q\|^{p-2}\|X\| \\ &\leq \sum_i |q_i^y|^{p-2} (-Y)_{ii} - cM^{p-1}|\bar{x} - \bar{y}|^{-\hat{\tau}} + (N-1)c_1M^{p-1}|\bar{x} - \bar{y}|^{-\tau_1} \\ &\quad + \|q^y\|^{p-2} - \|q\|^{p-2}\|Y\| + \|q^x\|^{p-2} - \|q\|^{p-2}\|X\| \\ &\leq F(q^y, -Y) - cM^{p-1}|\bar{x} - \bar{y}|^{-\hat{\tau}} + (N-1)c_1M^{p-1}|\bar{x} - \bar{y}|^{-\tau_1} \\ &\quad + c_2M^{p-1}|\bar{x} - \bar{y}|^{-\tau_2} \\ &\leq F(q^y, -Y) - c_3M^{p-1}|\bar{x} - \bar{y}|^{-\hat{\tau}}. \end{aligned}$$

as soon as $c_1(N-1)\delta^{\hat{\tau}-\tau_1} + c_2\delta^{\hat{\tau}-\tau_2} < \frac{c}{2}$.

Then one can conclude using the alternative Definition 2.1 of viscosity sub- and super-solutions, since

$$\begin{aligned} f(\bar{x}) &\leq F(q^x, X) \\ &\leq F(q^y, -Y) - c_3 M^{p-1} |\bar{x} - \bar{y}|^{-\hat{\tau}} \\ &\leq g(\bar{y}) - c_3 M^{p-1} |\bar{x} - \bar{y}|^{-\hat{\tau}}. \end{aligned}$$

This is clearly false as soon as δ is small enough since f and g are bounded.

So in order to get the Hölder and Lipschitz regularity in the case $p \leq 4$ and $p \geq 4$ it is sufficient to prove (3.15), (3.16) and (3.17) in any cases, note that the cases $p \geq 4$ will also require to check (3.6).

Note that (3.17) will always be a consequence of (3.18) below and of (2.4):

For any $\theta \in]0, \inf(1, (p-2)]$, and for any $Z, T \in \mathbb{R}^N$,

$$\||Z|^{p-2} - |T|^{p-2}\| \leq \sup(1, p-2) |Z - T|^\theta (|Z| + |T|)^{p-2-\theta}. \quad (3.18)$$

This is obtained for $p-2 \leq 1$, from the inequality $\||Z|^{p-2} - |T|^{p-2}\| \leq |Z - T|^{p-2}$, and for $p \geq 3$, using the mean values 's Theorem.

3.2. Proofs of (3.15), (3.16) and (3.17) in the Hölder's case and $p \leq 4$. Let $r < 1$, γ be some number in $]0, 1[$, and $\omega(s) = s^\gamma$. ψ is defined by (3.12) and M will be chosen large later independently on x_o , but depending on $r, \gamma, p, N, |f|_\infty, |g|_\infty, |u|_\infty$ and $|v|_\infty$.

Note that below even if we did not always make it explicit for simplicity, some of the constants depend on $\frac{1}{1-\gamma}$ and then the Lipschitz result cannot be derived immediately from the Hölder's one by letting γ go to 1.

Let us recall that we want to prove that there exists δ depending on (r, p, N) and M depending on the same variables, and on $|f|_\infty, |g|_\infty, |u|_\infty$ and $|v|_\infty$, such that for $|x-y| \leq \delta$, $\psi(x, y) \leq 0$. Recall that we impose (3.13), ie $M(1-r)^2 > 8(|u|_\infty + |v|_\infty)$, and $M > 1 + \frac{4(|u|_\infty + |v|_\infty)}{\omega(\delta)}$.

Let (\bar{x}, \bar{y}) be some couple in $(\bar{B}_1)^2$ on which the supremum is positive and achieved. Clearly $\bar{x} \neq \bar{y}$, and from the assumptions on M , $(\bar{x}, \bar{y}) \in B_{\frac{1+r}{2}} \times B_{\frac{1+r}{2}}$, and $|\bar{x} - \bar{y}| < \delta$. Here,

$$q^x = \gamma M |\bar{x} - \bar{y}|^{\gamma-2} (\bar{x} - \bar{y}) + 2M(\bar{x} - x_o), \quad q^y = \gamma M |\bar{x} - \bar{y}|^{\gamma-2} (\bar{x} - \bar{y}) - 2M(\bar{y} - x_o).$$

and

$$q = M\omega'(|\bar{x} - \bar{y}|) \frac{\bar{x} - \bar{y}}{|\bar{x} - \bar{y}|} = \gamma M |\bar{x} - \bar{y}|^{\gamma-2} (\bar{x} - \bar{y}).$$

In particular

$$|q| = M\gamma |\bar{x} - \bar{y}|^{\gamma-1}$$

and for $\delta^{1-\gamma} < \frac{\gamma}{8}$ one has

$$\frac{|q|}{2} \leq |q^x|, |q^y| \leq \frac{3|q|}{2}.$$

Note that Θ defined in the previous sub-section is also the matrix

$$\Theta_{ij}(\bar{x} - \bar{y}) = \left| \frac{q_i}{M} \right|^{\frac{p-2}{2}} \delta_i^j.$$

Furthermore,

$$\begin{aligned} |H_1(\bar{x} - \bar{y})| &= |D^2g(|\bar{x} - \bar{y}|)| \leq |\omega''(|\bar{x} - \bar{y}|)| + (N-1) \frac{\omega'(|\bar{x} - \bar{y}|)}{|\bar{x} - \bar{y}|} \\ &= \gamma(\gamma + N - 2)|\bar{x} - \bar{y}|^{\gamma-2}, \end{aligned}$$

and so by (2.4) and for δ small enough

$$|X| + |Y| \leq cM|\bar{x} - \bar{y}|^{\gamma-2}. \quad (3.19)$$

Using (3.10) one has

$$\begin{aligned} \lambda_1(M^{p-2}\Theta(\bar{x} - \bar{y})(X + Y - 2(2M + 1)\text{Id})\Theta(\bar{x} - \bar{y})) \\ \leq c(M\gamma)^{p-1}(\gamma - 1)|\bar{x} - \bar{y}|^{(\gamma-1)(p-2)+\gamma-2}. \end{aligned}$$

Hence,

$$\begin{aligned} \lambda_1(M^{p-2}\Theta(\bar{x} - \bar{y})(X + Y)\Theta(\bar{x} - \bar{y})) \\ \leq -cM^{p-1}(1 - \gamma)\gamma^{p-1}|\bar{x} - \bar{y}|^{(\gamma-1)(p-2)+\gamma-2} + 2(2M + 1)M^{p-2}|\Theta(\bar{x} - \bar{y})|^2 \\ \leq -\frac{c}{2}M^{p-1}(1 - \gamma)\gamma^{p-1}|\bar{x} - \bar{y}|^{(\gamma-1)(p-2)+\gamma-2} \end{aligned}$$

as soon as $\delta^{2-\gamma} < \frac{c(1-\gamma)\gamma^{p-1}}{12}$. Hence, (3.15) holds with $\hat{\tau} = (1 - \gamma)(p - 2) + 2 - \gamma$.

On the other hand by (3.9)

$$\begin{aligned} \lambda_i(M^{p-2}\Theta(\bar{x} - \bar{y})(X + Y)\Theta(\bar{x} - \bar{y})) &\leq 2(2M + 1)M^{p-2}\Theta^2(\bar{x} - \bar{y}) \\ &\leq 2(2M + 1)M^{p-2}\gamma^{p-2}|\bar{x} - \bar{y}|^{(\gamma-1)(p-2)} \end{aligned}$$

and then (3.16) holds with $\tau_1 = (1 - \gamma)(p - 2)$.

To check (3.17) let us use (3.18) and (3.19) to get

$$||q^x|^{p-2} - |q|^{p-2}||X| \leq c_2M^{p-1}|\bar{x} - \bar{y}|^{(\gamma-1)(p-2-\theta)+\gamma-2}$$

where $\theta = \inf(1, p - 2) > 0$ and then (3.17) holds with $\tau_2 = (1 - \gamma)(p - 3)^+ + 2 - \gamma$.

3.3. Proof of (3.6), (3.15), (3.16), (3.17) in the Hölder's case and $p \geq 4$. ω is the same as in the Hölder's case and $p \leq 4$. We define

$$\epsilon = \frac{(1-\gamma)}{2(p-4)}, \quad \delta_N = \exp\left(\frac{-\log(2N(4-\gamma)) + \log(1-\gamma)}{2\epsilon}\right), \quad (3.20)$$

and assume $\delta < \delta_N$. One still suppose (3.13).

In particular since there exists $i \in [1, N]$ such that $|\bar{x}_i - \bar{y}_i|^2 \geq \frac{|\bar{x} - \bar{y}|^2}{N} \geq |\bar{x} - \bar{y}|^{2+2\epsilon}$, for $p \geq 4$, using the definition of δ_N in (3.20), $I(\bar{x} - \bar{y}, \epsilon) \neq \emptyset$. Furthermore for $|\bar{x} - \bar{y}| < \delta \leq \delta_N$

$$\sum_{i \in I} |\bar{x}_i - \bar{y}_i|^2 \geq |\bar{x} - \bar{y}|^2 \left(1 - \frac{1-\gamma}{2N(4-\gamma)}\right) \geq \frac{3}{4} |\bar{x} - \bar{y}|^2 \quad (3.21)$$

and

$$\begin{aligned} & \frac{1}{2} \omega''(|\bar{x} - \bar{y}|) (1 - N|\bar{x} - \bar{y}|^{2\epsilon}) + \frac{3N}{2} |\bar{x} - \bar{y}|^{2\epsilon} \frac{\omega'(|\bar{x} - \bar{y}|)}{|\bar{x} - \bar{y}|} \\ & \leq \frac{1}{2} \omega''(|\bar{x} - \bar{y}|) + |\bar{x} - \bar{y}|^{2\epsilon} \left(\frac{N}{2} \gamma(1-\gamma) + \frac{3N}{2} \gamma\right) |\bar{x} - \bar{y}|^{\gamma-2} \\ & \leq \frac{1}{4} \gamma(\gamma-1) (|\bar{x} - \bar{y}|)^{\gamma-2} \leq \frac{\omega''(|\bar{x} - \bar{y}|)}{4}, \end{aligned} \quad (3.22)$$

and then (3.6) is satisfied. Using (3.22), (3.21), and (3.11) one gets

$$\begin{aligned} & \lambda_1(M^{p-2} \Theta(\bar{x} - \bar{y})(X + Y - 2(2M+1)\text{Id}) \Theta(\bar{x} - \bar{y})) \\ & \leq cM^{p-1} \omega''(|\bar{x} - \bar{y}|) \omega'(|\bar{x} - \bar{y}|)^{p-2} |\bar{x} - \bar{y}|^{(p-4)\epsilon} \end{aligned}$$

and then

$$\begin{aligned} & \lambda_1(M^{p-2} \Theta(\bar{x} - \bar{y})(X + Y) \Theta(\bar{x} - \bar{y})) \\ & \leq -cM^{p-1} (1-\gamma) \gamma^{p-1} |\bar{x} - \bar{y}|^{(\gamma-1)(p-2) + \gamma - 2 + (p-4)\epsilon} \\ & \quad + 2(2M+1)M^{p-2} |\Theta(\bar{x} - \bar{y})|^2 \\ & \leq -cM^{p-1} (1-\gamma) \gamma^{p-1} |\bar{x} - \bar{y}|^{(\gamma-1)(p-2) + \gamma - 2 + (p-4)\epsilon} \\ & \quad + (2(2M+1)M^{p-2} |\bar{x} - \bar{y}|^{(\gamma-1)(p-2)}) \\ & \leq -\frac{c}{2} M^{p-1} (1-\gamma) \gamma^{p-1} |\bar{x} - \bar{y}|^{(\gamma-1)(p-2) + \gamma - 2 + (p-4)\epsilon} \end{aligned}$$

as soon as δ is small enough, by the choice of ϵ in (3.20).

Then (3.15) holds with $\hat{\tau} = (1-\gamma)(p-2) + 2 - \gamma - (p-4)\epsilon > 0$, by (3.20) while (3.16) holds with $\tau_1 = (1-\gamma)(p-2)$ since $M^{p-1} |\Theta(\bar{x} - \bar{y})|^2 \leq c_1 M^{p-1} |\bar{x} - \bar{y}|^{(\gamma-1)(p-2)}$ and $(1-\gamma)(p-2) < (1-\gamma)(p-2) + (2-\gamma) - (p-4)\epsilon$.

Finally using (3.18) and (3.19)

$$\begin{aligned} ||q_i^x|^{p-2} - |q_i|^{p-2}||X_{ii}| &\leq (p-2)|q_i^x - q_i|(|q_i^x| + |q_i|)^{p-3} c_1 M |\bar{x} - \bar{y}|^{\gamma-2} \\ &\leq c_2 M^{p-1} |\bar{x} - \bar{y}|^{(\gamma-1)(p-3)+\gamma-2} \end{aligned}$$

and then (3.17) is satisfied with $\tau_2 = (1-\gamma)(p-3) + 2 - \gamma < \hat{\tau}$ by (3.20).

Remark 3.3. Suppose that $u = v$ and $f = g$.

From the previous proof one gets that for any $\gamma \in]0, 1[$, there exists some constant $C_{p,\gamma,N,r}$ such that if u is a solution of (1.1) in B_1 , such that $|u|_\infty \leq 1$ and $|f|_\infty \leq 1$, $|u|_{\mathcal{C}^{0,\gamma}(B_r)} \leq C_{p,\gamma,N,r}$. Let now u be a bounded solution in the ball B_1 , then $v = \frac{u}{|u|_\infty + |f|_\infty^{\frac{1}{p-1}}}$ satisfies the equation, with $|v|_\infty \leq 1$ and the right hand side $\tilde{f} = \frac{f}{(|u|_\infty + |f|_\infty^{\frac{1}{p-1}})^{p-1}}$. Then $|u|_{\mathcal{C}^{0,\gamma}(B_r)} \leq C_{p,\gamma,N,r} (|u|_\infty + |f|_\infty^{\frac{1}{p-1}})$.

3.4. Proof of (3.15), (3.16) and (3.17) in the Lipschitz case and $p \leq 4$. We choose $\tau \in]0, \inf(\frac{1}{2}, \frac{p-2}{2})[$, $\gamma > \frac{\tau}{\inf(\frac{1}{2}, \frac{p-2}{2})}$, $\gamma < 1$, and define

$\omega(s) = s - \omega_o s^{1+\tau}$ where $s < s_o = \left(\frac{1}{(1+\tau)\omega_o}\right)^{\frac{1}{\tau}}$ and ω_o is chosen so that $s_o > 1$. We suppose that $\delta^\tau \omega_o(1+\tau) < \frac{1}{2}$, which ensures that

$$\frac{1}{2} \leq \omega'(s) < 1, \text{ for } s < \delta \quad (3.23)$$

and by the mean value's theorem, for $s < \delta$, $\omega(s) \geq \frac{s}{2}$. Note that ω is globally Lipschitz continuous. We recall that by (3.13) we choose $M(1-r)^2 > 8(|u|_\infty + |v|_\infty)$ and $M > 1 + \frac{8(|u|_\infty + |v|_\infty)}{\delta}$.

Suppose that (\bar{x}, \bar{y}) is a pair on which the supremum of ψ is achieved > 0 . As in the previous subsections, $(\bar{x}, \bar{y}) \in B_{\frac{1+r}{2}}^2$ and $|\bar{x} - \bar{y}| < \delta$. Here $q^x = M\omega'(|\bar{x} - \bar{y}|) \frac{\bar{x} - \bar{y}}{|\bar{x} - \bar{y}|} + 2M(\bar{x} - x_o)$, $q^y = M\omega'(|\bar{x} - \bar{y}|) \frac{\bar{x} - \bar{y}}{|\bar{x} - \bar{y}|} + 2M(\bar{y} - x_o)$, and we also define $q = M\omega'(|\bar{x} - \bar{y}|) \frac{\bar{x} - \bar{y}}{|\bar{x} - \bar{y}|}$. In particular $\frac{M}{2} \leq |q| \leq M$. Note that since the solution has been proven to be Hölder in $B_{\frac{1+r}{2}}$ for all $\gamma < 1$, for some constant $c_{p,\gamma,N,r}$, from $\psi(\bar{x}, \bar{y}) \leq 0$ and $\omega(|\bar{x} - \bar{y}|) \geq 0$ one gets $M|\bar{x} - x_o|^2 + M|\bar{y} - x_o|^2 \leq c_{p,\gamma,N,r} |\bar{x} - \bar{y}|^\gamma$ and then

$$|\bar{x} - x_o| \leq \left(\frac{c_{p,\gamma,N,r} |\bar{x} - \bar{y}|^\gamma}{M} \right)^{\frac{1}{2}}. \quad (3.24)$$

and an analogous estimate holds for $|\bar{y} - x_o|$, then taking δ small enough, by (3.23),

$$\frac{M}{4} \leq |q^x|, |q^y| \leq \frac{5M}{4} \quad (3.25)$$

To prove (3.15) let us observe that here one has

$$(\omega'(|\bar{x} - \bar{y}|))^{p-2} \omega''(|\bar{x} - \bar{y}|) \leq -c|\bar{x} - \bar{y}|^{\tau-1}$$

and using $M^{p-2}|\Theta|^2 \leq M^{p-2}$, then by (3.10), for δ small enough, (3.15) holds with $\hat{\tau} = 1 - \tau$, while (3.16) holds with $\tau_1 = 0$.

To check (3.17), let us observe that here $|D^2g(|\bar{x} - \bar{y}|)| \leq |\omega''(|\bar{x} - \bar{y}|)| + (N-1)\frac{\omega'(|\bar{x} - \bar{y}|)}{|\bar{x} - \bar{y}|} \leq \frac{c}{|\bar{x} - \bar{y}|}$ and then using (3.14)

$$|X - (2M+1)\text{Id}| + |Y - (2M+1)\text{Id}| \leq c\frac{M}{|\bar{x} - \bar{y}|}.$$

Hence, for δ small enough one also has

$$|X| + |Y| \leq c\frac{M}{|\bar{x} - \bar{y}|} \quad (3.26)$$

and then by (3.18), (3.25), (3.24), and (3.26)

$$||q^x|^{p-2} - |q^y|^{p-2}| |X| \leq cM^{p-1}c_{p,\gamma,N,r}^{\frac{\sup(1,p-2)}{2}} |\bar{x} - \bar{y}|^{\frac{\inf(1,p-2)\gamma}{2}} |\bar{x} - \bar{y}|^{-1}$$

hence (3.17) holds with $\tau_2 = 1 - \frac{\inf(1,p-2)\gamma}{2} < 1 - \tau$ by the choice of γ .

3.5. Proofs of (3.6), (3.15), (3.16), (3.17) in the Lipschitz case and $p \geq 4$. We take the same function ψ and ω as in the Lipschitz case and $p \leq 4$, with still $M(1-r)^2 > 8(|u|_\infty + |v|_\infty)$ and $M > 1 + \frac{8(|u|_\infty + |v|_\infty)}{\delta}$. We choose also

$$0 < \tau < \frac{1}{(p-2)}, \quad 1 > \gamma > \tau(p-2), \quad \text{and} \quad \frac{\tau}{2} < \epsilon < \frac{\frac{\gamma}{2} - \tau}{(p-4)}. \quad (3.27)$$

(Recall that it has been proven in Subsection 3.2 that u is γ -Hölder continuous in $B_{\frac{1+r}{2}}$). Let us define ω, s_o, g, ψ , as in the case $p \leq 4$, Θ, \tilde{H} and H are as in the Subsection 3.1. We define

$$\begin{aligned} \delta_N &= \inf \left(\exp\left(\frac{\log(\omega_o(1+\tau)\tau) - \log(2N(\omega_o\tau(1+\tau) + 3))}{2\epsilon - \tau}\right), \right. \\ &\quad \left. \exp\left(\frac{-\log(2\omega_o(1+\tau))}{\tau}\right) \right). \end{aligned} \quad (3.28)$$

Note that $\delta_N \leq \exp\left(\frac{-\log(2N)}{2\epsilon}\right)$. If we suppose that $\sup_{(x,y) \in B_1^2} \psi(x,y) > 0$, it is achieved on some (\bar{x}, \bar{y}) which satisfies $\bar{x} \neq \bar{y}$, $(\bar{x}, \bar{y}) \in B_{\frac{1+r}{2}} \times B_{\frac{1+r}{2}}$

and $|\bar{x} - \bar{y}| \leq \delta_N$. In particular since there exists i such that $|\bar{x}_i - \bar{y}_i|^2 \geq \frac{1}{N}|\bar{x} - \bar{y}|^2 \geq |\bar{x} - \bar{y}|^{2+2\epsilon}$, by (3.28), $I(\bar{x} - \bar{y}, \epsilon) \neq \emptyset$, and

$$\begin{aligned} \sum_{i \in I(\bar{x} - \bar{y}, \epsilon)} |\bar{x}_i - \bar{y}_i|^2 &= |\bar{x} - \bar{y}|^2 - \sum_{i \notin I(\bar{x} - \bar{y}, \epsilon)} |\bar{x}_i - \bar{y}_i|^2 \\ &\geq |\bar{x} - \bar{y}|^2 - N|\bar{x} - \bar{y}|^{2+2\epsilon} \geq \frac{1}{2}|\bar{x} - \bar{y}|^2. \end{aligned}$$

Furthermore, recall that by (3.28), $1 \geq \omega'(|\bar{x} - \bar{y}|) \geq \frac{1}{2}$ and

$$\begin{aligned} \frac{1}{2}\omega''(|\bar{x} - \bar{y}|) &+ \frac{N}{2}\omega_o\tau(1 + \tau)|\bar{x} - \bar{y}|^{\tau-1+2\epsilon} + \frac{3}{2}N|\bar{x} - \bar{y}|^{2\epsilon-1}\omega'(|\bar{x} - \bar{y}|) \\ &\leq \frac{1}{2}\omega''(|\bar{x} - \bar{y}|) + \frac{N}{2}(\omega_o\tau(1 + \tau) + 3)|\bar{x} - \bar{y}|^{2\epsilon-1} \\ &\leq -\frac{1}{4}\omega_o(1 + \tau)\tau|\bar{x} - \bar{y}|^{-1+\tau} = \frac{\omega''(|\bar{x} - \bar{y}|)}{4}, \end{aligned}$$

and then (3.6) holds. This implies that the right hand side of (3.11) is for $x := \bar{x} - \bar{y}$ less than

$$cM^{p-1}\omega'(|\bar{x} - \bar{y}|)^{p-2}\omega''(|\bar{x} - \bar{y}|)|\bar{x} - \bar{y}|^{p-4} \leq -cM^{p-1}|\bar{x} - \bar{y}|^{-\hat{\tau}}$$

where by (3.27) $\tau - 1 + (p - 4)\epsilon := -\hat{\tau} < 0$, and using $4M^{p-1}|\Theta(\bar{x} - \bar{y})|^2 \leq cM^{p-1}$ one gets (3.15) for δ small enough.

Always by (3.9) and since $M^{p-1}|\Theta(\bar{x} - \bar{y}) \text{Id } \Theta(\bar{x} - \bar{y})| \leq M^{p-1}$, (3.16) is satisfied with $\tau_1 = 0$.

There remains to prove (3.17). For that aim observe that (3.24) still holds. Therefore, using (3.18), and (3.26), one gets

$$\begin{aligned} ||q^x|^{p-2} - |q|^{p-2}| |X| &\leq (p-2)|q^x - q|(|q^x| + |q|)^{p-3}|X| \\ &\leq c_2M^{p-1}|\bar{x} - \bar{y}|^{\frac{\gamma}{2}-1}. \end{aligned}$$

Hence, (3.17) holds with $\tau_2 = 1 - \frac{\gamma}{2} < 1 - \tau - (p - 4)\epsilon$ by (3.27).

Conclusion. From the previous proof one gets in the particular case where $u = v$ and $f = g$ that there exists some constant $C_{p,N,r}$ such that if u is a solution in B_1 , such that $|u|_\infty \leq 1$ and $|f|_\infty \leq 1$, $\text{lip}_{B_r} u \leq C_{p,N,r}$. Let now u be a bounded solution in the ball B_1 , then $v = \frac{u}{|u|_\infty + |f|_\infty^{\frac{1}{p-1}}}$ satisfies

the equation, with $|v|_\infty \leq 1$ and the right hand side $\tilde{f} = \frac{f}{(|u|_\infty + |f|_\infty^{\frac{1}{p-1}})^{p-1}}$.

Then $\text{lip}_{B_r} u \leq C_{p,N,r}(|u|_\infty + |f|_\infty^{\frac{1}{p-1}})$.

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REFERENCES

- [1] G. Barles, E. Chasseigne, and C. Imbert, Hölder continuity of solutions of second-order non-linear elliptic integro-differential equations, *J. Eur. Math. Soc.*, Vol 13, (2011), p. 1-26.
- [2] M. Belloni and B. Kawohl, The Pseudo p-Laplace eigenvalue problem and viscosity solutions, *ESAIM COCV*, Vol 10, (2004), p. 28-52.
- [3] I. Birindelli and F. Demengel, $C^{1,\beta}$ regularity for Dirichlet problems associated to fully nonlinear degenerate elliptic equations, *ESAIM COCV*, 20 (2014), p. 1009-1024.
- [4] I. Birindelli and F. Demengel, Existence and regularity result for sub and super-solutions of pseudo-Pucci's operators, work in preparation.
- [5] P. Bousquet and L. Brasco, Global Lipschitz Continuity for Minima of degenerate Problems, *Arxiv* : 1504.06101
- [6] P. Bousquet, L. Brasco, and V. Julin, Lipschitz regularity for local minimizers of some widely degenerate problem, *Calc. of Variations and Geometric Measure theory*, to appear.
- [7] L. Brasco and G. Carlier, On certain anisotropic elliptic equations arising in congestion optimal transport : Local gradient bounds, *Advances in Calculus of Variations*, Vol 7 (2014), p. 379-407.
- [8] X. Cabré, L. Caffarelli, "Fully-nonlinear equations", *Colloquium Publications* 43, American Mathematical Society, Providence, RI,1995.
- [9] M. Colombo, A. Figalli, Regularity Results for very degenerate elliptic equations, *J. Math. Pures Appl.* Vol 9, 101, (2014),p. 94-117.
- [10] M.G. Crandall, H. Ishii, P.L. Lions, User's guide to viscosity solutions of second order partial differential equations, *Bull. Amer. Math. Soc. (N.S.)* Vol 27,no. 1, (1992), p. 1-67.
- [11] M. G. Crandall, M. Kocan, P.L. Lions, A. Świech, On viscosity solutions of Fully Non-linear Equations with Measurable Ingredients, *Communications on pure and applied Mathematics*, Vol. XLIX, (1996), p 365-397.
- [12] E. Di Benedetto, $C^{1+\beta}$ local regularity of weak solutions of degenerate elliptic equations, *Nonlinear Analysis, Theory, Methods and Applications*, Vol. 7. No. 8, (1983), p. 827-850.
- [13] I. Fonseca, N. Fusco, P. Marcellini, An existence result for a non convex variational problem via regularity, *ESAIM: Control, Optimisation and Calculus of Variations*, Vol. 7, (2002), p. 69-95.
- [14] E. Giusti, "Direct methods in the calculus of variations", *World Scientific Publishing Co., Inc.*, River Edge, NJ, 2003.
- [15] C. Imbert, L. Silvestre, $C^{1,\alpha}$ regularity of solutions of degenerate fully non-linear elliptic equations , *Adv. Math.* Vol 233, (2013), p 196-206.
- [16] H. Ishii, Viscosity solutions of Nonlinear Partial Differential equations, *Sugaku Expositions* , Vol 9, no2, (1996), p . 135-152.
- [17] H. Ishii, P.L. Lions, Viscosity solutions of Fully-Nonlinear Second Order Elliptic Partial Differential Equations, *J. Differential Equations*, vol 83, (1990), p. 26-78.

- [18] P. Tolksdorff , Regularity for a more general class of quasilinear elliptic equations, J. Differential Equations, 51 (1984),no 1, p. 126-150.
- [19] N. Uraltseva, N. Urdaletova, The boundedness of the gradients of generalized solutions of degenerate quasilinear nonuniformly elliptic equations, Vest. Leningr. Univ. Math., 16 (1984), p. 263-270.