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η and λ deformations as $\mathcal{E}$-models

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Abstract

We show that the so-called λ deformed σ-model as well as the η deformed one belong to a class of the $\mathcal{E}$-models introduced in the context of the Poisson–Lie-T-duality. The λ and η theories differ solely by the choice of the Drinfeld double; for the λ model the double is the direct product $G \times G$ while for the η model it is the complexified group $G^{\mathbb{C}}$. As a consequence of this picture, we prove for any G that the target space geometries of the λ-model and of the Poisson–Lie T-dual of the η-model are related by a simple analytic continuation.

1. Summary

Consider the actions $S_\eta(g)$ and $S_\lambda(g)$ of the so-called η and λ deformed σ-models on the target of a simple compact Lie group G:

$$S_\eta(g) = \frac{1}{2} \int d\xi^+ d\xi^- (g^{-1}\partial_+ g, (1-\eta R)^{-1} g^{-1}\partial_- g), \tag{1}$$

$$S_\lambda(g) = S_{WZW}(g) + \lambda \int d\xi^+ d\xi^- ((1-\lambda \mathrm{Ad}_g)^{-1}\partial_+ g g^{-1}, g^{-1}\partial_- g). \tag{2}$$

Here $g(\xi^+, \xi^-) \in G$, the derivatives $\partial_\pm$ are taken with respect to the light-cone variables $\xi^\pm$, $(.,.)$ is the Killing–Cartan form on the Lie algebra $\mathcal{G}^{\mathbb{C}}$ of $G^{\mathbb{C}}$, $R : \mathcal{G} \to \mathcal{G}$ is the so-called Yang–Baxter operator and $S_{WZW}(g)$ is the standard WZW action

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$$S_{WZW}(g) := \frac{1}{2}\int d\xi^+ d\xi^- (g^{-1}\partial_+ g, g^{-1}\partial_- g) + \frac{1}{12}\int d^{-1}(dgg^{-1}, [dgg^{-1}, dgg^{-1}]). \quad (3)$$

The models (1) and (2) were respectively introduced in [30,31] and [40], with the parameters η, λ real and $|\lambda| < 1$.

It may seem that the expression (2) defines a σ-model also on the complexified group $G^{\mathbb{C}}$, however, this is a false appearance. The reason is that the action S_λ evaluated on $G^{\mathbb{C}}$-valued configurations takes generically complex values. However, if we evaluate S_λ exclusively on configurations p with values in the space P of positive definite Hermitian elements of $G^{\mathbb{C}}$ and we take λ to be a complex number of modulus 1 then $-\mathrm{i}S_\lambda(p)$ is always real and defines some σ-model on P. Our **Result 1** (the principal one) then states:

The σ-model $-\mathrm{i}S_\lambda(p)$ on P for $\lambda = \frac{1-\mathrm{i}\eta}{1+\mathrm{i}\eta}$ is the Poisson–Lie T-dual of the η-model.

Few remarks are in order:

1) The replacing of the unitary argument g by the positive definite Hermitian one p in (2) can be interpreted as a simple analytic continuation of the coordinates parameterizing the Cartan torus; our result therefore generalizes to any G the $SU(2)$ result of Refs. [18,41] stating that the λ-model is related by analytical continuation to the Poisson–Lie T-dual of the η-model.
2) It is very probable that our purely bosonic result can be generalized to the supergroup context. This would mean that, up to the analytic continuation, the λ-deformed $(AdS_5 \times S^5)_\lambda$ superstring of Ref. [15] is the Poisson–Lie T-dual of the η-deformed $(AdS_5 \times S^5)_\eta$ superstring of Ref. [10].
3) For the group $G = SU(N)$, P coincides with the spaces of positive definite Hermitian $N \times N$ matrices.

The results of [18] and [41] on the analytic continuation were obtained by working in appropriate coordinates on the group $SU(2)$ and on its dual Borel group. It appears extremely difficult to generalize that method to higher dimensional groups because the action of the dual η-model (in its version known before the present paper; cf. Eq. (43)) becomes prohibitively complicated in any coordinate system. To move forward we have to find a completely coordinate-free framework to work with and this turns out to be possible thanks to our following **Result 2**:

The λ-model on any simple Lie group target G belongs to the class of the $\mathcal{E}$-models considered in [26,27] in the context of the Poisson–Lie T-duality.

The next **Result 3** is the consequence of the previous one:

For every simple compact Lie group G there exists a manifold $P_{\mathcal{D}}$, a distinguished function H on $P_{\mathcal{D}}$ and two compatible Poisson structures $\{.,.\}_0$, $\{.,.\}_1$ on $P_{\mathcal{D}}$ such that the dynamical system with the phase space $P_{\mathcal{D}}$, the Hamiltonian H and the Poisson structure $\{.,.\}_0 + \varepsilon\{.,.\}_1$ can be identified with

i) the principal chiral model on G, for $\varepsilon = 0$;
ii) the λ-model on G, for $\varepsilon > 0$, where $\lambda = (1 - \varepsilon^{\frac{1}{2}})(1 + \varepsilon^{\frac{1}{2}})$;
iii) the η-model on G, for $\varepsilon < 0$, where $\eta = (-\varepsilon)^{\frac{1}{2}}$.

We finish by two more remarks:

4) The statement of Result 3 could be in principle reconstructed by composing together several facts established already in [9,40,46], however, we consider as an independent result the way how we obtain it directly and naturally from the formalism of the $\mathcal{E}$-models.
5) The Poisson structure $\{.,.\}_0 + \varepsilon\{.,.\}_1$ is the (symplectic version of) the current algebra built on a one-parameter family $\mathcal{D}_\varepsilon$ of the Drinfeld doubles of the Lie algebra $\mathcal{G} \equiv \mathrm{Lie}(G)$. The Hamiltonian H is a quadratic expression in the currents and it is *completely determined* by the Hamiltonian of the principal chiral model because it does not depend on ε.

2. Introduction

A problem how to deform an integrable non-linear σ-model on group manifold in a way preserving the integrability was formulated some forty years ago and it turned out to be a difficult one. Several integrable deformations of the principal chiral model have been found in the eighties and the nineties for the simplest case of the group $SU(2)$ [4,7,13,14] but for long decades no examples were constructed for higher dimensional groups. Some effort (see e.g. [37]) has been made to determine a complete system of conditions which a target geometry on a general Lie group must fulfill in order to guarantee integrability, however, attempts to find solutions of this complicated highly overdetermined system of conditions essentially failed for other groups than $SU(2)$. This situation lasted until 2008 when, in [31], the present author established the integrability of the so-called η-deformed (or, equivalently, Yang–Baxter) σ-model [30] for any simple compact Lie group target G.

The integrable η-deformation of the principal chiral model described in [31] was generalized to the context of integrable coset and supercoset targets in [9] and [10], respectively. In particular, the result [10] has triggered an important activity in the field because of its relevance in the AdS/CFT story [1–3,5,8,12,17,19,22,20,23,33,35,36,42,44,45]. In a short period of few years, several new integrable deformations of the integrable nonlinear σ-models were obtained, some of them multi-parametric [6,11,16,21,32,34,40]. In the present paper, we shall concentrate mainly on the integrable deformation of the WZW model proposed in [40]. It is now called the "λ-deformation", it belongs to a class of σ-models introduced in [43] and, similarly as in the η case, it was later generalized to the integrable supercoset targets [15].

Three papers [46,18] and [41] have recently discussed the issue of possible structural relations between the integrable η- and λ-deformations and all of them emphasized the relevance of the concept of the Poisson–Lie T-duality [24–26,39] in this context. In particular, Vicedo [46] studied extensively the case of the λ-model on a non-compact simple Lie group admitting the so-called split Yang–Baxter operator on its Lie algebra and pointed out the existence of the Poisson–Lie T-dual theory[1] resembling the variant of the η-model with real poles of the so-called twist functions (the poles of the twist function of the original η-model [30,31] are complex conjugated). On the other hand, Hoare and Tseytlin [18] and Sfetsos, Siampos and Thompson [41] have sticked to the compact case and showed that the λ-deformation on the $SU(2)$ target is related by an appropriate analytic continuation to the Poisson–Lie T-dual of the η-deformation. The principal goal of the present paper is to generalize this result of [18,41] to any target G.

[1] A second-order action of this dual theory is not explicitly given in [46] because of the problems with the factorizability of the underlying Drinfeld double. In this respect, the formula (14) of the present paper includes also the case of the non-factorizable doubles and its usefulness for the further development of the results of [46] looks very probable.

The other goal of the present article is to point out that the structural relation between the η- and λ-deformations is particularly explicit, obvious and neat in the framework of the theory of the $\mathcal{E}$-models developed in the context of the Poisson–Lie T-duality in [26,27]. In this regard, we wish to stress the conceptual and technical utility of several papers on the Poisson–Lie-T-duality like [27,28] which so far remain somewhat in the shadow of the initial works [24–26,39]. Indeed, as we shall show, the results of the paper [27] permit to establish that not only the η-deformation but also their λ-counterpart belongs to the class of the $\mathcal{E}$-models introduced in [26,27]. In fact, the difference between η- and λ-deformations turns out to be given *solely* by the choice of the Drinfeld double encoding the Hamiltonian structure of the integrable σ-model in question. The choice of the complexified group $G^{\mathbb{C}}$ yields the η-deformation while the double $G \times G$ corresponds to the λ-deformation.

The paper is organized as follows: In Section 3, we review the notion of the Drinfeld double current algebra as well as that of the $\mathcal{E}$-model [26,27]. In Section 4, we show that the λ-model on arbitrary compact simple group target G is a particular case of the $\mathcal{E}$-model and, for completeness, we review also the result of [30] establishing the same thing for the η-model. In Section 5, we establish the result concerning the analytical continuation relation between the λ and the dual η target geometries for any G and, finally, we devote Section 6 to a discussion of the results and to an outlook.

3. $\mathcal{E}$-models

Let $\mathcal{D}$ denote a real finite dimensional Lie algebra and let $(.,.)_\mathcal{D}$ be an ad-invariant non-degenerate symmetric bilinear form on $\mathcal{D}$. We then construct an infinite-dimensional Poisson manifold $P_\mathcal{D}$ the coordinates $j^A(\sigma)$ of which are labeled by one discret parameter $A = 1,\dots,\dim\mathcal{D}$ and one continuous (loop) parameter σ, with the defining Poisson brackets given by

$$\{j^A(\sigma), j^B(\sigma')\} = F^{AB}{}_C j^C(\sigma)\delta(\sigma-\sigma') + D^{AB}\partial_\sigma\delta(\sigma-\sigma'). \tag{4}$$

Here $F^{AB}{}_C$ are the structure constants of $\mathcal{D}$ in some basis $T^A \in \mathcal{D}$ and

$$D^{AB} := (T^A, T^B)_\mathcal{D}. \tag{5}$$

The Poisson manifold $P_\mathcal{D}$ is referred to as the (symplectic[2] version of the) current algebra associated to $\mathcal{D}$.

In what follows, we shall study only quadratic Hamiltonians in $j^A(\sigma)$ based on a choice of an $\mathbb{R}$-linear self-adjoint idempotent operator $\mathcal{E}: \mathcal{D} \to \mathcal{D}$ and given by the following formula

$$H_\mathcal{E} := \frac{1}{2}\int d\sigma (j(\sigma), \mathcal{E}j(\sigma))_\mathcal{D}. \tag{6}$$

Here we have used a $\mathcal{D}$-valued coordinates $j(\sigma)$ on $P_\mathcal{D}$ defined by

$$(j(\sigma), T^A)_\mathcal{D} := j^A(\sigma). \tag{7}$$

We also state, for the completeness, that the self-adjointness and the idempotency of $\mathcal{E}$ (which are essential for the world-sheet Lorentz invariance of the Hamiltonian) mean, respectively

[2] The invertibility of the Poisson tensor may fail only in a finite-dimensional zero mode sector in the Fourier-transformed current components $j^A(\sigma)$ which is determined by boundary conditions imposed on the currents.

$$(\mathcal{E}x, y)_{\mathcal{D}} = (x, \mathcal{E}y)_{\mathcal{D}}, \quad \forall x, y \in \mathcal{D}; \qquad \mathcal{E}^2 x = x, \quad \forall x \in \mathcal{D}. \tag{8}$$

The dynamical system on the phase space $P_{\mathcal{D}}$ defined by the current algebra Poisson brackets (4) and by the quadratic Hamiltonian (6) is referred to as an $\mathcal{E}$-model. It was originally defined in [26,27] and its equations of motion have the zero-curvature form valued in $\mathcal{D}$, that is

$$\partial_\tau j = \partial_\sigma(\mathcal{E}j) + [\mathcal{E}j, j]. \tag{9}$$

Here τ stands for the time.

Remark 1. In [26,27], we have been using a parametrization of the phase space $P_{\mathcal{D}}$ in terms of a group-like variable $l(\sigma)$ taking values in the loop group of the Drinfeld double D. (D is a Lie group the Lie algebra of which is $\mathcal{D}$.) The relation with the current algebra description $j(\sigma)$ reads

$$j(\sigma) = \partial_\sigma l(\sigma) l(\sigma)^{-1} \tag{10}$$

and the equation of motion (9) takes form

$$\partial_\tau l l^{-1} = \mathcal{E}\partial_\sigma l l^{-1}. \tag{11}$$

The Poisson brackets expressed in terms of the variables $l(\sigma)$ are more cumbersome than the elegant current algebra formula (4), nevertheless, the expression for the symplectic form on $P_{\mathcal{D}}$ is simpler in the $l(\sigma)$ language (see [26,27] for details).

Remark 2. Suppose that there is a linear one-parameter family of the Lie algebra structures on the vector space $\mathcal{D}$, which means that the structure constants $F^{AB}{}_C$ can be written as

$$F^{AB}{}_C = F_0^{AB}{}_C + \varepsilon F_1^{AB}{}_C, \quad \varepsilon \in \mathbb{R}. \tag{12}$$

Then the current algebra Poisson structure (4) can be represented accordingly as

$$\{j^A(\sigma), j^B(\sigma')\} = \{j^A(\sigma), j^B(\sigma')\}_0 + \varepsilon\{j^A(\sigma), j^B(\sigma')\}_1. \tag{13}$$

The Poisson structures $\{.,.\}_0$ and $\{.,.\}_1$ appearing in this relation can be readily read off from Eq. (4) and they are automatically compatible because the structure constants $F^{AB}{}_C$ verify the Lie algebra Jacobi identity for every ε.

Suppose now that there is a Lie subalgebra $\mathcal{G} \subset \mathcal{D}$ isotropic with respect to the bilinear form $(.,.)_{\mathcal{D}}$ and such that $\dim \mathcal{G} = \frac{1}{2}\dim D$ (the isotropy means $(x, x)_{\mathcal{D}} = 0, \forall x \in \mathcal{G}$). Then it was shown in [27] that there is a non-linear σ-model on the target D/G which can be identified with the $\mathcal{E}$-model $(P_{\mathcal{D}}, H_{\mathcal{E}})$. Here G is the subgroup of D corresponding to the subalgebra $\mathcal{G}$ and "can be identified" means the existence of a symplectomorphism (i.e. a canonical transformation) taking the phase space and the Hamiltonian of the D/G σ-model onto $P_{\mathcal{D}}$ and $H_{\mathcal{E}}$, respectively. The target space geometry of the D/G model was worked out in detail in [27,28,30] and it is encoded in the following action:

$$S_{\mathcal{E}}(f) = S_{WZW,\mathcal{D}}(f) - \int d\xi^+ d\xi^- (P_f(\mathcal{E}) f^{-1}\partial_+ f, f^{-1}\partial_- f)_{\mathcal{D}}. \tag{14}$$

Here the action $S_{WZW,\mathcal{D}}(f)$ is given by

$$S_{WZW,\mathcal{D}}(f) :=$$
$$:= \frac{1}{2}\int d\xi^+ d\xi^- (f^{-1}\partial_+ f, f^{-1}\partial_- f)_{\mathcal{D}} + \frac{1}{12}\int d^{-1}(dff^{-1}, [dff^{-1}, dff^{-1}])_{\mathcal{D}}, \tag{15}$$

the usual light-cone variables $\xi^\pm$ and derivatives $\partial_\pm$ read

$$\xi^\pm := \frac{1}{2}(\tau \pm \sigma), \qquad \partial_\pm := \partial_\tau \pm \partial_\sigma, \tag{16}$$

f stands for the parametrization of the right coset D/G by elements f of D (if there exists no global section of this fibration, we can choose several local sections covering the whole base space D/G) and, finally, $P_f(\mathcal{E})$ is a projection from $\mathcal{D}$ into $\mathcal{D}$ defined by the relations

$$\mathrm{Im} P_f(\mathcal{E}) = \mathcal{G}, \qquad \mathrm{Ker} P_f(\mathcal{E}) = (\mathbf{1} + \mathrm{Ad}_{f^{-1}}\mathcal{E}\mathrm{Ad}_f)\mathcal{G}.$$

Remark 3. The use of the projection $P_f(\mathcal{E})$ in the formula (14) is a new result (a by-line one) of the present paper which encompasses the results of [27,28,30] (e.g. the formula (12) of [27]) in a basis independent way.

We do not repeat here the derivation of the formula (14) for the σ-model action from the $\mathcal{E}$-model data $(P_\mathcal{D}, H_\mathcal{E})$ as it is presented in [27,28,30] but we do write down the symplectomorphism associating to every solution of the equation of motion of the σ-model (14) the solution of the equation of motion (9) because this result is not contained in [27,28,30]:

$$j = \partial_\sigma f f^{-1} - \frac{1}{2} f \left(P_f(\mathcal{E}) f^{-1}\partial_+ f - P_f(-\mathcal{E}) f^{-1}\partial_- f \right) f^{-1}. \tag{17}$$

4. Current algebras of η and λ deformations

Consider a simple compact real Lie algebra $\mathcal{G}$ equipped with its standard Killing–Cartan form $(.,.)$. We introduce one-parameter family of real Lie-algebras $\mathcal{D}_\varepsilon$ which all have the property of being the Drinfeld doubles of $\mathcal{G}$. As the vector space, $\mathcal{D}_\varepsilon$ is just the direct sum of the vector space $\mathcal{G}$ with itself:

$$\mathcal{D}_\varepsilon := \mathcal{G} \dot{+} \mathcal{G}, \tag{18}$$

the Lie algebra bracket $[.,.]_\varepsilon$ on $\mathcal{D}_\varepsilon$ is defined in terms of the commutator $[.,.]$ in $\mathcal{G}$ as follows

$$[x_1 \dot{+} x_2, y_1 \dot{+} y_2]_\varepsilon := ([x_1, y_1] + \varepsilon[x_2, y_2]) \dot{+} ([x_1, y_2] + [x_2, y_1]), \qquad x_i, y_i \in \mathcal{G}, \tag{19}$$

and, finally, the ad-invariant non-degenerate symmetric bilinear form $(.,.)_\mathcal{D}$ does not depend on ε and it is given by

$$(x_1 \dot{+} x_2, y_1 \dot{+} y_2)_\mathcal{D} := (x_2, y_1) + (x_1, y_2). \tag{20}$$

Note that $\mathcal{G}$ is embedded in $\mathcal{D}_\varepsilon$ as $\mathcal{G} \dot{+} 0$, or, said in other words, $\mathcal{D}_\varepsilon$ is the Drinfeld double of its subalgebra $\mathcal{G} \dot{+} 0 \simeq \mathcal{G}$.

We now introduce a one-parameter family of $\mathcal{E}$-models $(P_{\mathcal{D}_\varepsilon}, H)$ based on the current algebra (4) for the Drinfeld double $\mathcal{D}_\varepsilon$ and equipped with the quadratic Hamiltonian (6) given by the following choice of the self-adjoint idempotent operator $\mathcal{E}$:

$$\mathcal{E}(x_1 \dot{+} x_2) := (x_2 \dot{+} x_1). \tag{21}$$

Because here we speak about the particular operator $\mathcal{E}$ given by Eq. (21), we denote just by H the Hamiltonian associated to it via (6), reserving the notation $H(\mathcal{E})$ to situations when a generic operator $\mathcal{E}$ occurs.

Remark 4. We note that the structure constants of the Lie algebra $\mathcal{D}_\varepsilon$ have precisely the structure (12) of Remark 2 which means that the symplectic structure of the $\mathcal{E}$-model $(P_{\mathcal{D}_\varepsilon}, H)$ has the form of the linear combination $\{.,.\}_0 + \varepsilon\{.,.\}_1$ of two compatible Poisson structures as mentioned in the Result 3 of Section 1.

We now evaluate, for every ε, the second order σ-model action (14) of the $\mathcal{E}$-model $(P_{\mathcal{D}_\varepsilon}, H)$. We start with the simplest case $\varepsilon = 0$ where it turns out to hold:

The $\mathcal{E}$-model $(P_{\mathcal{D}_0}, H)$ can be identified with the principal chiral model on G.

Let us demonstrate this statement:

We first remark, that the Drinfeld double D_0 is the semi-direct product of manifolds G and $\mathcal{G}$, i.e. the group law reads

$$(g_1, x_1)(g_2, x_2) = (g_1 g_2, x_1 + g_1 x_2 g_1^{-1}), \qquad g_1, g_2 \in G, \quad x_1, x_2 \in \mathcal{G}. \tag{22}$$

It can be easily checked that, indeed, the law (22) gives rise to the Lie algebra commutator (19) for $\varepsilon = 0$. Now note that the commutation relation (19) implies

$$[0 \dot{+} x_2, 0 \dot{+} y_2]_0 = 0. \tag{23}$$

Denote the Abelian Lie algebra $0 \dot{+} \mathcal{G}$ by the symbol $\tilde{\mathcal{G}}$ and the corresponding Lie group by $\tilde{G}$. (The elements of $\tilde{G}$ are therefore $(e, x) \in D_0$, e being the unit element of G.)

Consider now the σ-model (14) on the target $D_0/\tilde{G}$. This coset can be obviously identified with the subgroup G of D_0, the elements of which are $f = (g, 0) \in \mathcal{D}_0$. Thus the field f featuring in (14) can be chosen to take values $(g, 0) \in D_0$. In this case the part $S_{WZW,\mathcal{D}}(f)$ of the action (14) vanishes because the Lie algebra $\mathcal{G}$ of G is maximally isotropic (i.e. $(\mathcal{G} \dot{+} 0, \mathcal{G} \dot{+} 0)_{\mathcal{D}} = 0$). Since the operator $\mathcal{E}$ given by (21) evidently commutes with $\mathrm{Ad}_{(g,0)}$, the projection $P_{(g,0)}(\mathcal{E})$ does not depend on g and it is easily found to be given by

$$P_{(g,0)}(\mathcal{E})(x_1 \dot{+} x_2) = (0 \dot{+} (x_2 - x_1)), \tag{24}$$

hence

$$P_{(g,0)} f^{-1}\partial_+ f = P_{(g,0)}(g^{-1}\partial_+ g \dot{+} 0) = (0 \dot{+} -g^{-1}\partial_+ g). \tag{25}$$

Combining (14), (20) and (25) we find the following action of the σ-model on $D_0/\tilde{G}$:

$$S_{\mathcal{E},0}(g) = \int d\xi^+ d\xi^- (g^{-1}\partial_+ g, g^{-1}\partial_- g). \tag{26}$$

This is indeed the action of the principal chiral model on the group G [47].

Now we show that the evaluation of the second order σ-model action (14) of the $\mathcal{E}$-models $(P_{\mathcal{D}_\varepsilon}, H)$ for $\varepsilon > 0$ gives the λ-model of [40]. More precisely, it holds

For $\varepsilon > 0$, the $\mathcal{E}$-model $(P_{\mathcal{D}_\varepsilon}, H)$ can be identified with the λ-model on G characterized by the action

$$\begin{aligned} S_\lambda(g) = {}& \frac{1}{2}\int d\xi^+ d\xi^- (g^{-1}\partial_+ g, g^{-1}\partial_- g) + \frac{1}{12}\int d^{-1}(dgg^{-1}, [dgg^{-1}, dgg^{-1}]) \\ & + \lambda \int d\xi^+ d\xi^- ((1 - \lambda \mathrm{Ad}_g)^{-1}\partial_+ g g^{-1}, g^{-1}\partial_- g), \end{aligned} \tag{27}$$

where

$$\lambda = \frac{1-\varepsilon^{\frac{1}{2}}}{1+\varepsilon^{\frac{1}{2}}}. \tag{28}$$

We start the argument by considering the Lie algebra $\mathcal{G} \oplus \mathcal{G}$ (i.e. the direct sum of the Lie algebra $\mathcal{G}$ with itself), the elements of which will be typically denoted (α_1, α_2). There is an ad-invariant non-degenerate symmetric bilinear form on $\mathcal{G} \oplus \mathcal{G}$ given by the formula

$$((\alpha_1, \alpha_2), \beta_1, \beta_2))_{\mathcal{G}\oplus\mathcal{G}} := (\alpha_1, \beta_1) - (\alpha_2, \beta_2). \tag{29}$$

For each ε positive there is an isomorphism of Lie algebras $\Phi_\varepsilon : \mathcal{D}_\varepsilon \to \mathcal{G} \oplus \mathcal{G}$ given by

$$\Phi_\varepsilon(x_1 \dot{+} x_2) = (x_1 + \varepsilon^{\frac{1}{2}} x_2, x_1 - \varepsilon^{\frac{1}{2}} x_2). \tag{30}$$

This isomorphism preserves the bilinear forms (20) and (29) up to normalization, that is

$$(\Phi_\varepsilon(x), \Phi_\varepsilon(y))_{\mathcal{G}\oplus\mathcal{G}} = 2\varepsilon^{\frac{1}{2}}(x, y)_{\mathcal{D}}, \qquad x, y \in \mathcal{D}_\varepsilon. \tag{31}$$

The existence of the isomorphism Φ_ε means that we can work with the double $\mathcal{G} \oplus \mathcal{G}$ instead of $\mathcal{D}_\varepsilon$, if we translate by Φ_ε to the $\mathcal{G} \oplus \mathcal{G}$ context also the operator $\mathcal{E} : \mathcal{D}_\varepsilon \to \mathcal{D}_\varepsilon$ given by (21). The translated operator $\mathcal{E}_\varepsilon : \mathcal{G} \oplus \mathcal{G} \to \mathcal{G} \oplus \mathcal{G}$ is defined by the requirement

$$\mathcal{E}_\varepsilon \circ \Phi_\varepsilon = \Phi_\varepsilon \circ \mathcal{E}, \tag{32}$$

which gives

$$\mathcal{E}_\varepsilon(\alpha, \beta) = \frac{1}{2}(\varepsilon^{\frac{1}{2}} + \varepsilon^{-\frac{1}{2}})(\alpha, -\beta) + \frac{1}{2}(\varepsilon^{\frac{1}{2}} - \varepsilon^{-\frac{1}{2}})(\beta, -\alpha). \tag{33}$$

The group Drinfeld double of the Lie algebra $\mathcal{G} \oplus \mathcal{G}$ is evidently $G \times G$ (i.e. the direct product of G with itself) and its elements will be typically denoted as (a_1, a_2). The diagonal subgroup of $G \times G$ generated by the elements of the form (a, a) will be denoted as G^δ. The corresponding Lie algebra $\mathcal{G}^\delta$ is maximally isotropic (it is the image of the subalgebra $\mathcal{G} \dot{+} 0 \subset \mathcal{D}_\varepsilon$ under the isomorphism Φ_ε) and its elements are (α, α). In order to apply to the present situation the general formula (14), there remains to parametrize the cosets D/G^δ by the elements of D and to identify the projection $P_f(\mathcal{E}_\varepsilon)$. Obviously, the coset D/G^δ can be identified with the first copy G in the direct product $G \times G$ which gives the parametrization $f = (g, e)$. $P_{(g,e)}(\mathcal{E}_\varepsilon)$ is then straightforwardly found to be equal to

$$P_{(g,e)}(\mathcal{E}_\varepsilon)(\alpha, \beta) = (\frac{\lambda}{\lambda - \mathrm{Ad}_{g^{-1}}}\alpha + \frac{1}{1 - \lambda \mathrm{Ad}_g}\beta, \frac{\lambda}{\lambda - \mathrm{Ad}_{g^{-1}}}\alpha + \frac{1}{1 - \lambda \mathrm{Ad}_g}\beta), \tag{34}$$

where λ is given by the formula (28).

Finally, taking into account that $f^{-1}\partial_+ f = (g^{-1}\partial_+ g, 0)$, the wanted formula (27) follows directly (up to an overall normalization) from Eqs. (14), (29) and (34).

Remark 5. Note that when the parameter ε ranges from 0 to $+\infty$, the parameter λ given by (28) ranges from -1 to 1. This is to be compared with the original paper [40] where the way of obtaining the action (27) (by a gauging procedure) leads to the interval of the values of λ between 0 and 1. Thus the vantage point based on the $\mathcal{E}$-models "sees" more possible values of λ.

The fact that for $\varepsilon < 0$ the evaluation of the second order σ-model action (14) of the $\mathcal{E}$-models $(P_{\mathcal{D}_\mathcal{E}}, H)$ gives the η-model of [30] was proven already in [30]. However, to keep the exposition self-contained we outline here the argument:

Consider the Lie algebra $\mathcal{G}^{\mathbb{C}}$ (i.e. the complexification of $\mathcal{G}$) the elements of which will be typically denoted as z. There is an ad-invariant non-degenerate symmetric bilinear form on $\mathcal{G}^{\mathbb{C}}$ given by the formula

$$(z_1, z_2)_{\mathcal{G}^{\mathbb{C}}} := -\mathrm{i}(z_1, z_2) + \mathrm{i}\overline{(z_1, z_2)}, \tag{35}$$

where $(.,.)$ is the Killing–Cartan form on $\mathcal{G}^{\mathbb{C}}$ and $\overline{number}$ stands for the complex conjugation of the *number*.

For each ε negative, there is an isomorphism of Lie algebras $\Psi_\varepsilon : \mathcal{D}_\varepsilon \to \mathcal{G}^{\mathbb{C}}$ given by

$$\Psi_\varepsilon(x_1 \dot{+} x_2) = x_1 + |\varepsilon|^{\frac{1}{2}} \mathrm{i} x_2. \tag{36}$$

This isomorphism relates the bilinear forms (20) and (35) up to normalization, that is

$$(\Psi_\varepsilon(x), \Psi_\varepsilon(y))_{\mathcal{G}^{\mathbb{C}}} = 2|\varepsilon|^{\frac{1}{2}}(x, y)_{\mathcal{D}}, \qquad x, y \in \mathcal{D}_\varepsilon. \tag{37}$$

The existence of the isomorphisme Ψ_ε means that we can work with the double $\mathcal{G}^{\mathbb{C}}$ instead of $\mathcal{D}_\varepsilon$, if we translate to the $\mathcal{G}^{\mathbb{C}}$ context also the operator $\mathcal{E} : \mathcal{D}_\varepsilon \to \mathcal{D}_\varepsilon$ given by (21). The translated operator $\mathcal{E}_\varepsilon : \mathcal{G}^{\mathbb{C}} \to \mathcal{G}^{\mathbb{C}}$ is defined by the requirement

$$\mathcal{E}_\varepsilon \circ \Psi_\varepsilon = \Psi_\varepsilon \circ \mathcal{E}, \tag{38}$$

which gives

$$\mathcal{E}_\varepsilon z = \frac{\mathrm{i}}{2}(|\varepsilon|^{\frac{1}{2}} - |\varepsilon|^{-\frac{1}{2}})z - \frac{\mathrm{i}}{2}(|\varepsilon|^{\frac{1}{2}} + |\varepsilon|^{-\frac{1}{2}})z^*. \tag{39}$$

Here z^* stands for the Hermitian conjugation.

The group Drinfeld double of the Lie algebra $\mathcal{G}^{\mathbb{C}}$ is evidently the complexified group $G^{\mathbb{C}}$ viewed as the real group. We shall evaluate the σ-model action (14) on the target $G^{\mathbb{C}}/\tilde{G}$ where for the $\tilde{G}$ we take the isotropic AN subgroup of $G^{\mathbb{C}}$ featuring in the standard Iwasawa decomposition $G^{\mathbb{C}} \simeq GAN$ [48]. It then follows that the space of cosets $G^{\mathbb{C}}/\tilde{G}$ can be identified with the group G thus the field f in (14) can be chosen G-valued: $f = g$. However, the operator $\mathcal{E}_\varepsilon$ as given by (39) obviously commutes with Ad_g therefore the projection $\tilde{P}_{f=g}(\mathcal{E}_\varepsilon)$ does not depend on f (we put tilde over $P(\mathcal{E}_\varepsilon)$ in order to indicate that the image of this projection is $\tilde{\mathcal{G}}$ and, in what follows, we suppress the subscript f). In order to find $\tilde{P}(\mathcal{E}_\varepsilon)$ explicitly, we note that the elements of $\tilde{\mathcal{G}}$ can be parametrized by the elements of $\mathcal{G}$ by using the so-called Yang–Baxter operator $R : \mathcal{G} \to \mathcal{G}$ (the explicit formula for R can be found in [30,31]). Explicitly, every $\zeta \in \tilde{\mathcal{G}}$ can be uniquely written as

$$\zeta = (R - \mathrm{i})u \tag{40}$$

for some $u \in \mathcal{G}$. With this insight, we find straightforwardly

$$\tilde{P}(\mathcal{E}_\varepsilon)z = \frac{1}{2}\frac{R - \mathrm{i}}{1 + \sqrt{|\varepsilon|}R}\left((\mathrm{i} + \sqrt{|\varepsilon|})z + (\mathrm{i} - \sqrt{|\varepsilon|})z^*\right). \tag{41}$$

Taking into account the isotropy of the group G (which eliminates the $S_{WZW,\mathcal{D}}(f)$ term from the action (14)), applying $\tilde{P}(\mathcal{E}_\varepsilon)$ on $g^{-1}\partial_+ g$ and inserting the result in the general formula (14) we find

$$S_\eta(g) = \frac{1}{2}\int d\xi^+ d\xi^- (g^{-1}\partial_+ g, (1 - \eta R)^{-1} g^{-1}\partial_- g), \tag{42}$$

where $\eta = \sqrt{|\varepsilon|}$. This coincides with the action of the η-model of Ref. [30,31].

We note finally, that in the present Section 4 we have established the Results 2 and 3 as stated in Section 1.

5. T-duality and analytic continuation

By the Poisson–Lie T-dual of the η-model (42) we shall mean the model (14) based on the same $\mathcal{E}_\varepsilon$ operator (39) as the original model (42) but with the target space being D/G instead of $D/\tilde{G}$. As in [30,31], we can identify the coset D/G with the group $\tilde{G} = AN$ and, by setting $f = b \in AN$ and realizing that $S_{WZW,\mathcal{D}}(b) = 0$, we trivially obtain from the basic formula (14) the action of the dual model in the following form

$$\tilde{S}_\eta(b) = \frac{1}{2}\int d\xi^+ d\xi^- (\partial_+ bb^{-1}, \tilde{O}(b)^{-1}\partial_- bb^{-1})_{\mathcal{D}}. \tag{43}$$

We do not specify further[3] the b-dependent linear operator $\tilde{O} : \mathcal{G} \to \tilde{\mathcal{G}}$ because it is not the form (43) of the dual action that we are going to compare with the λ-model action (27). Indeed, in trying to do so we would hurt on a very complicated dependence of $\tilde{O}(b)$ on b. Fortunately, we find in this paper a way out of these technical difficulties by identifying the coset D/G not with the group AN but with the space P of all positive definite Hermitian elements of the group $G^{\mathbb{C}}$. This new identification is based on the well-known fact that every element of $D = G^{\mathbb{C}}$ admits a unique polar decomposition as the product of a positive definite Hermitian element with a unitary element. From this statement it can be easily derived that the AN-parametrization and the P-parametrization of the coset D/G is related by the diffeomorphism $\Upsilon : AN \to P$:

$$\Upsilon(b) = \sqrt{bb^*}. \tag{44}$$

To obtain the action of the dual model in the P-parametrization, it is now sufficient to set $f = \Upsilon(b)$ and to identify the projection $P_{\Upsilon(b)}(\mathcal{E}_\varepsilon)$:

$$P_{\Upsilon(b)}(\mathcal{E}_\varepsilon)z = \left(\sqrt{|\varepsilon|} - \mathrm{i} + (\sqrt{|\varepsilon|} + \mathrm{i})\mathrm{Ad}_{bb^*}\right)^{-1}\left((\sqrt{|\varepsilon|} + \mathrm{i})Ad_{bb^*}z - (\sqrt{|\varepsilon|} - \mathrm{i})z^*\right). \tag{45}$$

Here z^* means the Hermitian conjugation of the element z.

Inserting (45) and (35) into the basic formula (14) and taking into account that $\Upsilon(b)$ is Hermitian (this gives e.g. $(\Upsilon(b)^{-1}\partial_+\Upsilon(b), \Upsilon(b)^{-1}\partial_-\Upsilon(b))_{\mathcal{G}^{\mathbb{C}}} = 0$) we obtain for the action of the dual η-model

$$\begin{aligned}\tilde{S}_\eta(b) &= -2\mathrm{i}S_{WZW}(\Upsilon(b)) + \\ &+ 2\mathrm{i}\int d\xi^+ d\xi^- \left(\frac{\mathrm{i} + (\eta + \mathrm{i})\mathrm{Ad}_{\Upsilon(b)}}{(\eta + \mathrm{i})\mathrm{Ad}_{\Upsilon(b)} + (\eta - \mathrm{i})\mathrm{Ad}_{\Upsilon(b)^{-1}}}\Upsilon(b)^{-1}\partial_+\Upsilon(b), \Upsilon(b)^{-1}\partial_-\Upsilon(b)\right).\end{aligned} \tag{46}$$

Here $\eta = \sqrt{|\varepsilon|}$ and the action $S_{WZW}(\Upsilon(b))$ appearing in (46) is based on the ordinary Killing–Cartan form $(.,.)$ and not on $(.,.)_{\mathcal{G}^{\mathbb{C}}}$. Explicitly,

$$\begin{aligned}S_{WZW}(\Upsilon(b)) :=& \frac{1}{2}\int d\xi^+ d\xi^- (\Upsilon(b)^{-1}\partial_+\Upsilon(b), \Upsilon(b)^{-1}\partial_-\Upsilon(b)) + \\ &+ \frac{1}{12}\int d^{-1}(d\Upsilon(b)\Upsilon(b)^{-1}, [d\Upsilon(b)\Upsilon(b)^{-1}, d\Upsilon(b)\Upsilon(b)^{-1}]).\end{aligned} \tag{47}$$

[3] The interested reader can find the explicit expression for $\tilde{O}(b)$ in [30] where $\tilde{O}(b)$ is related to the well-known Poisson–Lie structure $\tilde{\Pi}(b)$ on the group AN via the formula $\tilde{\Pi}(b) = \tilde{O}(b) - \tilde{O}(\mathbf{1})$.

Note that the hermiticity of $\Upsilon(b)$ implies that the dual action $\tilde{S}_\eta(b)$ is *real* inspite of the factor i standing in front of the r.h.s. of (46). In particular, the WZW term in the r.h.s. of (47) is purely imaginary. Finally, we use the Polyakov–Wiegmann formula [38]

$$S_{WZW}(bb^*) = 2S_{WZW}(\Upsilon(b)) + \int d\xi^+ d\xi^- (\Upsilon(b)^{-1}\partial_-\Upsilon(b), \partial_+\Upsilon(b)\Upsilon(b)^{-1}), \tag{48}$$

and the identity

$$(bb^*)^{-1}\partial_\pm(bb^*) = \Upsilon(b)^{-1}(\Upsilon(b)^{-1}\partial_\pm\Upsilon(b))\Upsilon(b) + \Upsilon(b)^{-1}\partial_\pm\Upsilon(b), \tag{49}$$

which gives together

$$\tilde{S}_\eta(b) = -\mathrm{i}S_{WZW}(bb^*) - \mathrm{i}\lambda\int d\xi^+ d\xi^- ((1-\lambda\mathrm{Ad}_{bb^*})^{-1}\partial_+(bb^*)(bb^*)^{-1}, (bb^*)^{-1}\partial_-(bb^*)) \tag{50}$$

with

$$\lambda = \frac{1-\mathrm{i}\eta}{1+\mathrm{i}\eta}. \tag{51}$$

Comparing the resulting expression (50) with the λ-model action S_λ given by the formula (2) or (27), we conclude

$$\tilde{S}_\eta(b) = -\mathrm{i}S_\lambda(bb^*), \quad \lambda = \frac{1-\mathrm{i}\eta}{1+\mathrm{i}\eta}. \tag{52}$$

Of course, the replacing the unitary argument g by the positive definite Hermitian argument bb^* in the λ-model action (2) can be interpreted as a simple analytic continuation of the coordinates parameterizing the Cartan torus. This is because both g and bb^* can be parametrized in the Cartan way:

$$g = hth^{-1}, \quad bb^* = hah^{-1}, \tag{53}$$

where h is in G, t is in the compact Cartan torus T of G and a is in the noncompact part A of the complex Cartan torus $T^{\mathbb{C}}$ of $G^{\mathbb{C}}$. Note in this respect that here A is the same A which appears in the Iwasawa decomposition $G^{\mathbb{C}} = GAN$.

As an example, let us explicitly describe the analytic continuation from the non-compact to the compact Cartan torus in the case of the group $SU(N)$ in which A is formed by the real diagonal matrices of the form

$$a_{ij} = e^{\psi_i}\delta_{ij}, \quad \sum_j \psi_j = 0, \quad i, j = 1, \dots N. \tag{54}$$

The analytic continuation of the real Cartan coordinates ψ_j to the strictly imaginary values $\mathrm{i}\psi_j$ obviously transforms a_{ij} into an element of the compact Cartan torus T hence it switches from the positive definite Hermitian bb^* to the unitary g.

We note finally, that in the present Section 5 we have established the Result 1 as stated in Section 1 with the notation $p = bb^*$.

6. Conclusions and outlook

We have identified the λ-model on a simple compact Lie group G as a particular case of the $\mathcal{E}$-model and we have used this result to relate the λ-model to the Poisson–Lie T-dual of the η-model by the analytic continuation for any simple compact Lie group G. We have also interpreted the λ-model and the η-model as two branches of a single one-parameter family of dynamical systems characterized by the same Hamiltonian but by the varying Poisson brackets.

It is probable that the framework of the $\mathcal{E}$-models will be useful to establish, for general G, the analytic continuation relating the two-parametric λ models of Ref. [41] with the duals of the bi-Yang–Baxter models of Ref. [32]. It is also plausible that the dressing cosets generalization of the $\mathcal{E}$-models of Ref. [29] will represent a suitable framework for establishing the analytic continuation relation between the η and the λ deformations of the σ-models living on cosets of G.

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