On Controlling Unknown Scalar Systems with Low Gain Feedback

Mazen Alamir

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Mazen Alamir, *CNRS, University of Grenoble. (mazen.alamir@grenoble-inp.fr, http://www.mazenalamir.fr)*

In this paper, a new feedback law if proposed to force a highly uncertain system to track a desired set-point using saturated control. The proposed feedback is not based on high gain concept, rather, a saturated integrator is used. Stability analysis is performed under realistic feasibility assumption.

Problem Statement

Let us consider the following scalar system:

$$\dot{x} = \alpha\,[u - h] \quad (\alpha, h(\cdot)) \text{ unknown} \tag{1}$$

where $\alpha \in [\alpha_{min}, \alpha_{max}]$ is an unknown parameter with known bounds $\alpha_{min}, \alpha_{max}$, u is the control input, h is an unknown variable while x is the output of interest one would like to make it track some desired value x_d.

Regarding the unknown term h, the following assumption is assumed to hold:

Assumption 1 (Bounded unknown h)**.** *There is a known pair of lower and upper bounds h_{min} and h_{max} such that for all t the following inequalties hold:*

$$h(t) \in [h_{min}, h_{max}] \tag{2}$$

In order for the regulation problem to be feasible, it is assumed that the input control u can always *dominate* the term h had the latter be known. This is expressed through the following assumption which expresses also that u is constrained by lower and upper bounds:

Assumption 2 (u is authoritative enough)**.** *There exists lower and upper bound u_{min} and u_{max} such that for all t:*

$$u(t) \in [u_{min}, u_{max}] \tag{3}$$

Moreover, these bounds relate to the bound of h through the following inequalities:

$$u_{max} - h_{max} \geq \quad \varrho_+ > 0 \tag{4}$$
$$h_{min} - u_{min} \geq \quad \varrho_- > 0 \tag{5}$$

Note that if this assumption is not satisfied, the sign of x cannot be guaranteed to be imposed by the control.

The problem we are interested in here is the following:

Problem Statement

Define a dynamic state feedback law of the form:

$$\dot{z} = f(z, x, x_d) \tag{6}$$
$$u = K(z, x, x_d) \tag{7}$$

such that the tracking error $e = x - x_d$ is stabilized to a neighborhood of 0 that is as small as required.

Discussion Regarding Alternative Solutions

There are obviously two intuitive approaches to solve the tracking part of the problem. These are, high gain solution and observer-based solutions. These are sketched in the following sections and then compared to the solution proposed in this paper.

Elementary High-Gain Solution

The idea here is to *kill* the unknown term h by a high gain control. This is done by using VERY HIGH value of λ in the following feedback expression:

$$u = \lambda(x_d - x) \tag{8}$$

By doing so, the equation of the error $e = x - x_d$ becomes (assuming constant x_d):

$$\dot{e} = -\alpha\lambda e - \alpha h \tag{9}$$

which steers the errors towards the set:

$$\mathcal{E} := \left\{ e \quad \text{s.t.} \quad e \leq \frac{\max\{|h_{min}|, |h_{max}|\}}{\lambda} \right\} \tag{10}$$

which can be made as small as required by taking high values of λ. The problem with this simple solution is that the transient of the input variable u cannot be guaranteed to meet the saturation constraints (3).

Observer Based Control

In this options, the idea is not to *blindly kill* the unknown term but to estimate it through a dynamic observer using the measurement of both u and x. More precisely, the observer is given by the following dynamic system. Unfortunately, the fact that α is supposed to be unknown makes the quantity h non observable. Indeed, α and h have to be both reconstructed and a simple examination of equation (1) leads to the fact that each pair of possibilities:

$$(\alpha_1, h_1) \quad \text{and} \quad (\alpha_2, h_2) \tag{11}$$

are non distinguishable as soon as they are linked through the following relationships:

$$\alpha_1 = \beta\alpha_2 \quad \text{and} \quad h_2 = (1-\beta)u + \beta h_1 \tag{12}$$

which means that observability is rigorously lost as soon as u becomes almost constant. This is never a good news.

In what follows, a solution that avoid the explicit use of both explicit observer and high gain principle is proposed while α is still completely unknown.

The Proposed Solution

Define the saturation map according to:

$$S(v) := \begin{cases} u_{max} & \text{if } v \geq u_{max} \\ u_{min} & \text{if } v \leq u_{min} \\ v & \text{otherwise} \end{cases} \tag{13}$$

The proposed feedback is defined by:

$$\dot{z} = \lambda_f \left[S(\lambda(x_d - x) + z) - z\right] \tag{14}$$

$$u = S(\lambda(x_d - x) + z) \tag{15}$$

where $\lambda > 0$ and $\lambda_f > 0$ are two design parameters to be appropriately chosen.

This control law if obviously of the form (6)-(7). In the following section, the properties of the resulting closed-loop are analyzed.

Analysis of The Closed-Loop Properties

The closed-loop system equations can be written as follows:

$$\dot{x} = \alpha\left[S(\lambda(x_d - x) + z) - h\right] \tag{16}$$

$$\dot{z} = \lambda_f \left[S(\lambda(x_d - x) + z) - z\right] \tag{17}$$

Let us write these equations in the three different configuration given by the definition (13), namely:

Case ($u = u_{max}$)

$$\dot{x} = \alpha\left[u_{max} - h\right] \tag{18}$$

$$\dot{z} = \lambda_f \left[u_{max} - z\right] \tag{19}$$

Case ($u = u_{min}$)

$$\dot{x} = \alpha\left[u_{min} - h\right] \tag{20}$$

$$\dot{z} = \lambda_f \left[u_{min} - z\right] \tag{21}$$

Case (no saturation)

$$\begin{bmatrix}\dot{x}\\ \dot{z}\end{bmatrix} = \begin{bmatrix}-\alpha\lambda & \alpha \\ -\lambda_f\lambda & 0\end{bmatrix}\begin{bmatrix}x\\ z\end{bmatrix} + \begin{bmatrix}\alpha\lambda & -\alpha \\ \lambda\lambda_f & 0\end{bmatrix}\begin{bmatrix}x_d\\ h\end{bmatrix} \tag{22}$$

or equivalently:

$$\begin{bmatrix}\dot{x}\\ \dot{z}\end{bmatrix} = \begin{bmatrix}-\alpha\lambda & \alpha \\ -\lambda_f\lambda & 0\end{bmatrix}\begin{bmatrix}x - x_d\\ z - h\end{bmatrix} \tag{23}$$

Note that the above three cases defines three regions in the 2D plan defined by the coordinates $(x - x_d, z)$ as it is shown in Figure 1. More precisely, these three regions are defined by:

$$\mathcal{A}_+ := \{(\xi, z) \mid z - \lambda\xi \geq u_{max}\} \tag{24}$$

$$\mathcal{A}_- := \{(\xi, z) \mid z - \lambda\xi \leq u_{min}\} \tag{25}$$

$$\mathcal{A}_0 := \{(\xi, z) \mid z - \lambda\xi \in (u_{min}, u_{max})\} \tag{26}$$

Figure 1 shows also some information regarding the inclinations of the vector fields in the different regions that enable the following results to be proved.

Lemma 1. *For any $\lambda > 0$ used in (14)-(15), if λ_f satisfies the following inequality:*

$$\lambda_f < \left[\frac{\varrho_+}{u_{max} - u_{min}}\right]\lambda \tag{27}$$

then the set $\mathcal{A}_0$ is attractive for all initial state such that $(x - x_d, z) \in \mathcal{A}_+$. Δ

Proof. This can be proved if one can prove that the angle θ depicted in Figure 1 is lower than $\arctan(\lambda)$. But Assumptions (4)-(5) together with the fact that z necessarily belongs to $[u_{min}, u_{max}]$ enable to write (see Figure 1):

$$\tan\theta \leq \frac{\lambda_f(u_{max} - u_{min})}{\varrho_+} \tag{28}$$

which obviously gives the result. $\square$

Using the same arguments, the following Lemma follows:

Lemma 2. *For any $\lambda > 0$ used in (14)-(15), if λ_f satisfies the following inequality:*

$$\lambda_f < \left[\frac{\varrho_-}{u_{max} - u_{min}}\right]\lambda \tag{29}$$

then the set $\mathcal{A}_0$ is attractive for all initial state such that $(x - x_d, z) \in \mathcal{A}_-$. Δ

The previous two Lemmas obviously lead to the following corollary:

Corollary 1. *For any $\lambda > 0$ used in (14)-(15), if λ_f satisfies the following inequality:*

$$\lambda_f < \left[\frac{\min\{\varrho_+, \varrho_-\}}{u_{max} - u_{min}}\right]\lambda \tag{30}$$

then $\mathcal{A}_0$ is globally attractive and invariant. Δ

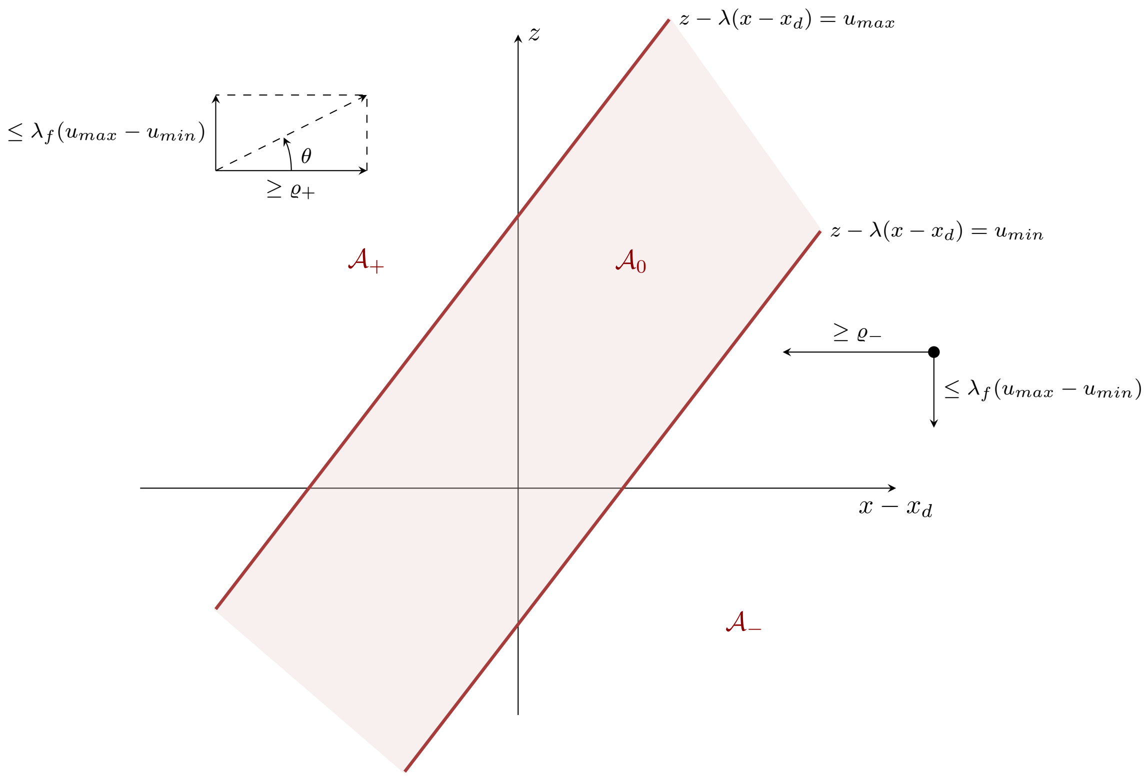

Figure 1: *Definition of the three regions $\mathcal{A}_0$, $\mathcal{A}_+$ and $\mathcal{A}_-$ used in the analysis of the stability of the closed-loop system with the proposed feedback law.*

Proof. The attractivity is a direct consequence of Lemmas 1 and 2. The invariance results from the simple facts that when the state approaches the boundaries of $\mathcal{A}_0$ with any of $\mathcal{A}_+$ or $\mathcal{A}_-$, it is repulsed back before reaching the boundary whenever the requirements of these Lemmas hold. This is precisely implied by the condition (30). □

The direct consequence of Corollary 1 is that the only thing that remains to be analyzed regarding the stability of the closed-loop behavior is related to the behavior of the closed-loop system inside the region $\mathcal{A}_0$. But the behavior inside $\mathcal{A}_0$ is determined by the dynamic equation (23). This enables the following result to be established:

Lemma 3. *For any λ, if λ_f satisfies:*

$$\lambda_f = r \times \left[\frac{\alpha_{min}\lambda}{4}\right] \quad ; \quad r \in (0,1) \tag{31}$$

then the matrix:

$$A_0 := \begin{bmatrix} -\alpha\lambda & \alpha \\ -\lambda_f\lambda & 0 \end{bmatrix} \tag{32}$$

possesses two real and strictly negative eigenvalues $p_{1,2}$. More precisely:

$$p_{1,2} = -\alpha\lambda\left[1 \pm \sqrt{1-r}\right] \tag{33}$$

Proof. Straightforward since the discriminant of the characteristic equation is given by:

$$\Delta = \alpha\lambda\left[\alpha\lambda - 4\lambda_f\right] \tag{34}$$

the remaining facts directly follows. Δ

The previous results lead to the following Proposition:

Proposition 1. *Take some $\lambda > 0$.* **If** *the following conditions holds:*

1. *λ_f is such that*

$$\lambda_f < \left[\min\left\{\frac{\min\{\varrho_+, \varrho_-\}}{u_{max} - u_{min}}, \frac{\alpha_{min}}{4}\right\}\right] \times \lambda \tag{35}$$

2. *x_d is constant and*
3. *the dynamic of the unknown term h satisfies:*

$$\left|\frac{dh}{dt}\right| \leq \delta \tag{36}$$

then *the tracking error $e_x = x - x_d$ and the estimation error $e_h = z - h$ asymptotically satisfy the following inequalities:*

$$\lim_{t\to\infty} |x(t) - x_d| \leq \frac{\delta}{\lambda\lambda_f} \;;\; \lim_{t\to\infty} |z(t) - h(t)| \leq \frac{\delta}{\lambda_f} \tag{37}$$

Proof. This is because condition (35) implies that the set $\mathcal{A}_0$ is globally attractive. Therefore, the dynamic defined by (23) prevails after a finite time t_0. Therefore, one has for all $t \geq t_0$:

$$\begin{bmatrix} e_x(t) \\ e_z(t) \end{bmatrix} = e^{A_0(t-t_0)}\begin{bmatrix} e_x(t_0) \\ e_z(t_0) \end{bmatrix} - \int_{t_0}^{t} e^{A_0(t-\tau)}\begin{bmatrix} 0 \\ \dot{h}(\tau) \end{bmatrix} d\tau \tag{38}$$

and since (35) makes A_0 hurwitz invertible, the last expression asymptotically behaves like:

$$A_0^{-1}\begin{bmatrix}0\\ \dot h(t)\end{bmatrix} = \frac{1}{\lambda_f}\begin{bmatrix}0 & -\frac{1}{\lambda}\\ \frac{\lambda_f}{\alpha} & -1\end{bmatrix}\begin{bmatrix}0\\ \dot h(t)\end{bmatrix} = \begin{bmatrix}-\frac{1}{\lambda\lambda_f}\\ -\frac{1}{\lambda_f}\end{bmatrix}\dot h(t) \quad (39)$$

This together with (36) obviously gives the result. Δ

Illustrative Simulations

Consider the unknown signal h given by:

$$h(t) = 5 - 1.3\cos(1.2t) - 1.6\sin(2t+\pi/6); \quad (40)$$

The time evolution of h on the time interval $[0,10]$ is shown in Figure 2. Straightforward computation shows that $\delta = 4.76$ satisfies the inequality (36). Moreover, h_{min} and h_{max} are given by:

$$h_{min} = 2 \quad ; \quad h_{max} = 8 \quad (41)$$

Se take the unknown $\alpha = 1$ which is not used in the feedback design although it changes the dynamics of the closed-loop system. Let assume however that one knows that $\alpha_{min} = 0.2$ and that the control bounds are given by:

$$u_{min} = 1 \quad ; \quad u_{max} = 10 \quad (42)$$

so that $\varrho_- = 1$ and $\varrho_+ = 2$ can be guaranteed in (4)-(5). It results that the condition (35) on λ and λ_f becomes:

$$\lambda_f < \left[\min\left\{\frac{\min\{1,2\}}{10-1}, \frac{0.2}{4}\right\}\right] \times \lambda$$
$$< 0.05 \times \lambda \quad (43)$$

In the simulation, $\lambda_f = \{0.04, 0.001\} \times \lambda$ are usedfor several values of λ are used to illustrate the properties of the closed-loop systems.

The results are shown on Figure 3, 4 and 5. More precisely:

Figure 3 uses $\lambda = 400$ and $\lambda_f = 0.04\lambda = 16$.

Figure 4 uses $\lambda = 4000$ and $\lambda_f = 0.04\lambda = 160$.

Figure 5 uses $\lambda = 4000$ and $\lambda_f = 0.001\lambda = 4$.

The message from these simulations are the following:

1. The theoretical predicted bounds given by (37) of Proposition 1 seems not only correct but rather tight (no over conservatism) as the effectively encountered levels get very close to the bound regularly.
2. Very small error can be achieved without uncontrollable transient on the control as the latter respects the saturation constraints anyway.
3. Low time derivative of u can be obtained low values of λ_f as it can be observed by comparing Figures 4 and Figure 5
4. The better estimation of h is obtained through u and not necessarily z. This is especially true for low values of λ_f. This can be confirmed by observing Figures 6, 7 and 8 that show the evolution of u and h as well as their difference $|u-h|$ for the same scenarios.

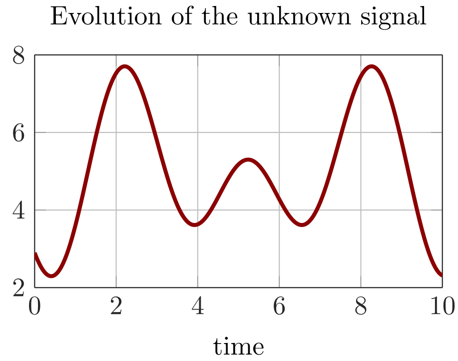

Figure 2: *Illustrative simulation: Evolution of the unknown signal $h(\cdot)$ used in the simulation of $\dot x = \alpha(u-h)$ under the proposed feedback which ignores α and h.*

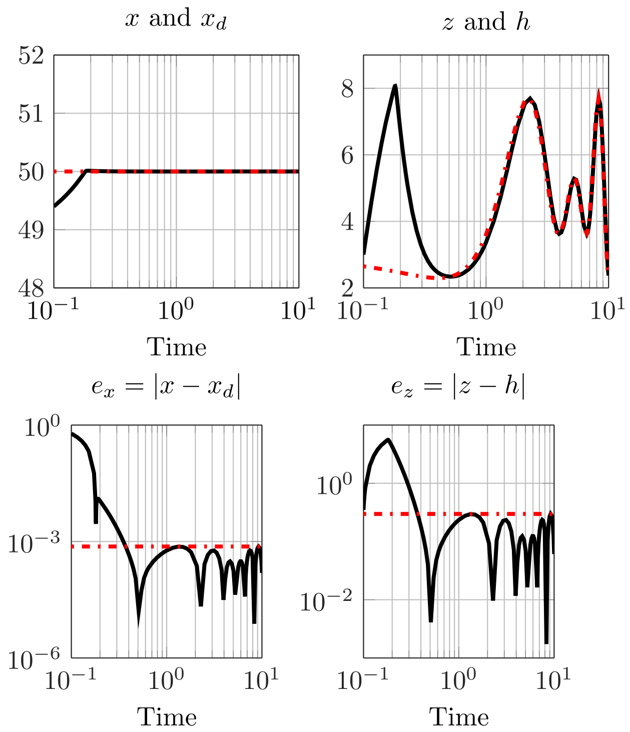

Figure 3: *Illustrative simulation: Evolution of the closed-loop system with the feedback defined by $\lambda = 400$ and $\lambda_f = 0.04\lambda$. The dashed red lines in the bottom plots represents the asymptotic bounds predicted by the theory through (37) of Proposition 1.*

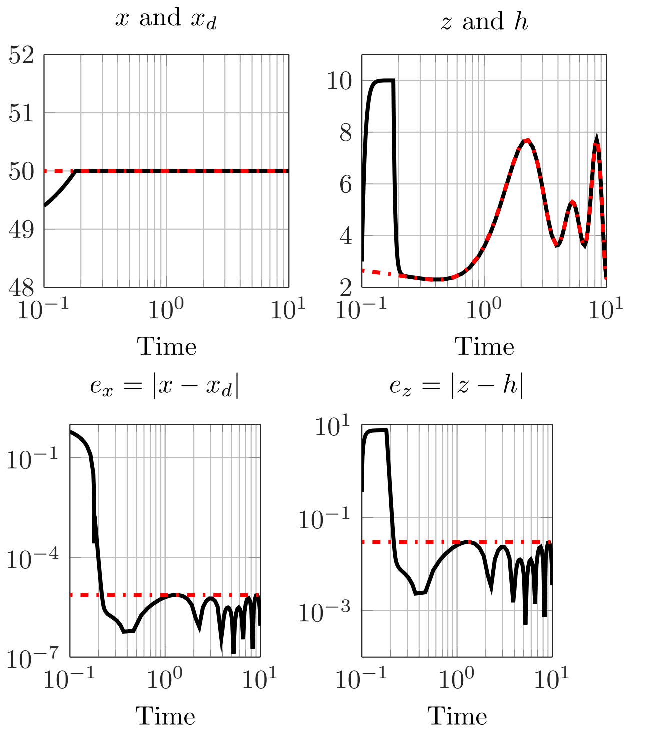

Figure 4: *Illustrative simulation: Evolution of the closed-loop system with the feedback defined by $\lambda = 4000$ and $\lambda_f = 0.04\lambda$. The dashed red lines in the bottom plots represents the asymptotic bounds predicted by the theory through (37) of Proposition 1.*

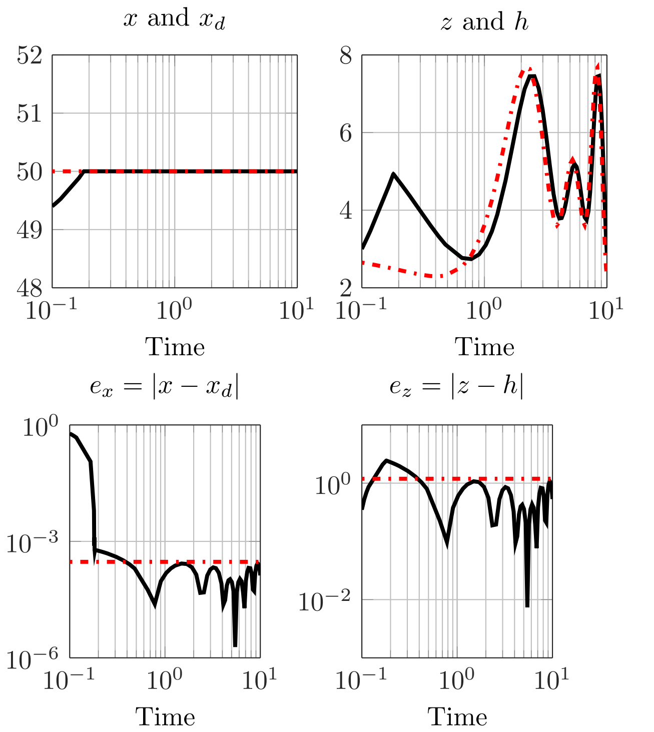

Figure 5: *Illustrative simulation: Evolution of u and h and their difference when $\lambda = 4000$ and $\lambda_f = 0.001\lambda$ is used. in the bottom plots represents the asymptotic bounds predicted by the theory through (37) of Proposition 1.*

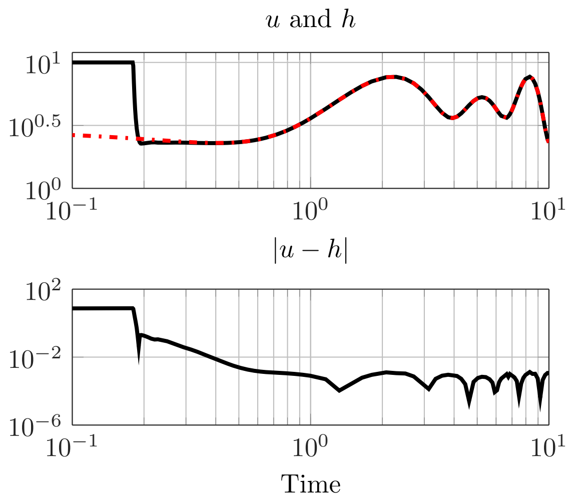

Figure 6: *Illustrative simulation: Comparison between u and h when $\lambda = 400$ and $\lambda_f = 0.04\lambda$ are used.*

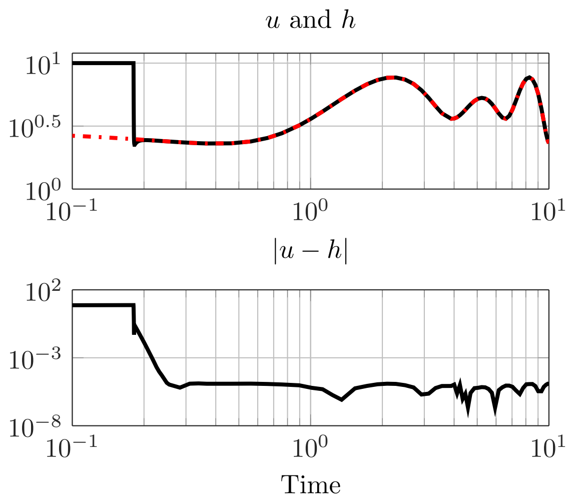

Figure 7: *Illustrative simulation: Comparison between u and h when $\lambda = 4000$ and $\lambda_f = 0.04\lambda$ are used.*

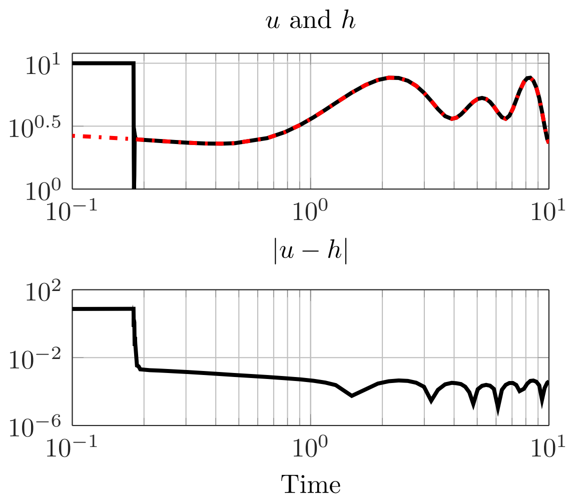

Figure 8: *Illustrative simulation: Comparison between u and h when $\lambda = 4000$ and $\lambda_f = 0.001\lambda$ are used.*