On Mittag-Leffler distributions and related stochastic processes

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THIERRY E. HUILLET

Abstract. Random variables with Mittag-Leffler distribution can take values either in the set of non-negative integers or in the positive real line. There can be of two different types, one (type-1) heavy-tailed with index $\alpha \in (0, 1)$, the other (type-2) possessing all its moments. We investigate various stochastic processes where they play a key role, among which: the discrete space/time Neveu branching process, the discrete-space continuous-time Neveu branching process, the continuous space/time Neveu branching process (CSBP) and renewal processes with rare events. Its relation to (discrete or continuous) self-decomposability and branching processes with immigration is emphasized. Special attention will be paid to the Neveu CSBP for its connection with the Bolthausen-Sznitman coalescent. In this context, and following a recent work of Möhle [49], a type-2 Mittag-Leffler process turns out to be the Siegmund dual to Neveu's CSBP block-counting process arising in sampling from $PD\left(e^{-t}, 0\right)$. Further combinatorial developments of this model are investigated.

Keywords: Mittag-Leffler random variables and processes, stochastic growth models, Neveu branching process with infinite mean, immigration and self-decomposability, renewal process, self-similarity, Bolthausen-Sznitman coalescent.

1. Sibuya random variables (rvs) and related branching processes

We first investigate a class of integral-valued rvs that will show important for our general purpose.

1.1. Sibuya rvs and related ones.

We start with their definition and main properties.

• **One parameter Sibuya**(α) **rv**. Let $X_\alpha \geq 1$ be an integer-valued random variable with support $\mathbb{N} = \{1, 2, ...\}$ defined as follows:

$$X_\alpha = \inf\left(l \geq 1 : \mathcal{B}_\alpha(l) = 1\right),$$

where $\left(\mathcal{B}_\alpha(l)\right)_{l\geq 1}$ is a sequence of independent Bernoulli rvs obeying $\mathbf{P}\left(\mathcal{B}_\alpha(l) = 1\right) = \alpha/l$ where $\alpha \in (0, 1)$. It is thus the first epoch of a success in a Bernoulli trial when the probability of success is inversely proportional to the number of the trial. X_α is called a Sibuya(α) rv. Then

$$\mathbf{P}\left(X_\alpha = k\right) = (-1)^{k-1}\binom{\alpha}{k}, \; k \geq 1,$$

with $\binom{\alpha}{k} = (\alpha)_k/k!$, $(\alpha)_k := \Gamma(\alpha + 1)/\Gamma(\alpha + 1 - k) = \alpha(\alpha - 1)\ldots(\alpha - k + 1)$, the Pochhammer's symbol (or decreasing factorial). Its probability generating function (pgf) is

$$\phi_{\alpha}(z) := \mathbf{E}\left(z^{X_{\alpha}}\right) = 1 - (1-z)^{\alpha}, \; z \leq 1.$$

We note that $\mathbf{P}(X_{\alpha} = k)$ is also $\mathbf{P}(X_{\alpha} = k) = \alpha \left[\overline{\alpha}\right]_{k-1} / k!$, where $\overline{\alpha} := 1 - \alpha$ and $[a]_k := a(a+1)\ldots(a+k-1)$, $k \geq 1$, are the rising factorials of a with $[a]_0 := 1$.

• **Discrete-stable**(μ, α) **rv** [66]**.** Consider the random variable $S_{\mu,\alpha}$ given by the random sum

$$S_{\mu,\alpha} = \sum_{l=0}^{P_{\mu}} X_{\alpha}(l),$$

where P_{μ} is Poisson distributed with mean $\mu > 0$ and $(X_{\alpha}(l))_{l \geq 0}$ is an iid sequence of Sibuya(α) rvs ($X_{\alpha}(l) \stackrel{d}{=} X_{\alpha}$), independent of P_{μ}. Then $\phi_{P_{\mu}}(z) = \mathbf{E}\left(z^{P_{\mu}}\right) = e^{-\mu(1-z)}$ and

$$\phi_{S_{\alpha,\mu}}(z) = \phi_{P_{\mu}}(\phi_{\alpha}(z)) = e^{-\mu(1-z)^{\alpha}}$$

the pgf of a discrete-stable(α, μ) rv, say $S_{\alpha,\mu}$. We will come back to this distribution below. Note that, with $S_{\alpha} := S_{\alpha,1}$, and in view of $S_{\alpha,\mu} \stackrel{d}{=} \mu^{1/\alpha} \circ S_{\alpha}$, μ is the scale parameter of $S_{\alpha,\mu}$.

• **Scaled Sibuya**(α, λ) **rv.** Let $c \in (0,1)$. Define the c-thinned version of the rv X_{α}, say $X_{\alpha,c} := c \circ X_{\alpha}$, as the random sum

$$X_{\alpha,c} = c \circ X_{\alpha} \stackrel{d}{=} \sum_{l=1}^{X_{\alpha}} B_c(l),$$

with $(B_c(l))_{l \geq 1}$ a sequence of independent and identically distributed (iid) Bernoulli variables such that $\mathbf{P}(B_c(1) = 1) = c$, independent of X_{α}. This binomial thinning operator, acting on discrete rvs, has been defined by [66]; it stands as the discrete version of the change of scale (note that if $X = n$ is a constant integral rv, $c \circ X$ is random with bin(n, c) distribution). The pgf of $X_{\alpha,c}$ is

$$\phi_{\alpha,c}(z) := \mathbf{E}\left(z^{X_{\alpha,c}}\right) = \phi_{X_{\alpha}}(1 - c(1-z)) = 1 - (c(1-z))^{\alpha}, \; z \leq 1.$$

With $\lambda = c^{\alpha} \in (0,1)$, we shall therefore call a rv $X_{\alpha,\lambda}$ with pgf $\phi_{\alpha,\lambda}(z) = 1 - \lambda(1-z)^{\alpha}$ a scaled Sibuya(α, λ) rv, with scale parameter λ, obeying $X_{\alpha,\lambda} \stackrel{d}{=} \lambda^{1/\alpha} \circ X_{\alpha}$. $X_{\alpha,\lambda} \geq 0$ is now an integer-valued random variable with support $\mathbb{N}_0 = \{0, 1, 2, \ldots.\}$, satisfying

$$\pi_{\alpha,\lambda}(0) := \mathbf{P}(X_{\alpha,\lambda} = 0) = 1 - \lambda \text{ and}$$

$$\pi_{\alpha,\lambda}(k) := \mathbf{P}(X_{\alpha,\lambda} = k) = \lambda(-1)^{k-1}\binom{\alpha}{k} = \alpha\lambda \frac{\left[\overline{\alpha}\right]_{k-1}}{k!}, \; k \geq 1. \tag{1}$$

Both $X_{\alpha,\lambda}$ and $X_{\alpha} = X_{\alpha,1}$ are heavy-tailed with exponent α : $\mathbf{P}(X > k) = L(k)\, k^{-\alpha}$ for some slowly-varying sequence $L(k)$.

• **Main properties** [9]**.** The rv $X_{\alpha,\lambda}$ is infinitely divisible (ID), or compound Poisson, iff $\lambda \leq 1 - \alpha$. This follows from the fact that, with $\mu = -\log(1-\lambda) \leq -\log \alpha$

$$\phi_{\alpha,\lambda}(z) = 1 - \lambda(1-z)^{\alpha} = e^{-\mu(1-h(z))}$$

for some absolutely monotone pgf $h(z)$ (the pgf of the sizes of the batches), obeying $h(0) = 0$.

It is even discrete self-decomposable (and thus unimodal) iff $\lambda \leq (1-\alpha)/(1+\alpha)$ with $X_{\alpha,\lambda}$ self-decomposable $\Rightarrow X_{\alpha,\lambda}$ ID, [66]. We will come back to this self-decomposability property below.

• **Three-parameters Sibuya**(α, β, λ) **rv.** Let $\beta > 0$. If $X_{\alpha,\lambda}$ is ID (else if $\lambda \leq 1-\alpha$), then for all $\beta > 0$

$$\phi_{\alpha,\beta,\lambda}(z) = (1 - \lambda(1-z)^{\alpha})^{\beta}$$

is the pgf of some rv $X_{\alpha,\beta,\lambda}$, called a generalized Sibuya(α, β, λ) rv. This is because, under our assumptions, $X_{\alpha,\lambda}$ is compound Poisson.

1.2. **Branching processes involving Sibuya rvs: discrete space-time Neveu process.** We describe here an integral-valued Bienaymé-Galton-Watson branching process in discrete time whose branching mechanism is a Sibuya(α, λ) rv. It turns out that the population size at generation n is itself again a Sibuya(α_n, λ_n) rv, so computable. We call it the discrete Neveu process. We investigate some of the consequences of this remarkable fact.

• **Branching process with Sibuya**(α, λ) **offspring distribution (discrete-time).** Let $\phi_{\alpha_1,\lambda_1}(z)$ and $\phi_{\alpha_2,\lambda_2}(z)$ be the pgfs of two independent scaled Sibuya rvs with parameters (α_1, λ_1) and (α_2, λ_2). We have the stability under composition property

$$\phi_{\alpha_2,\lambda_2}\left(\phi_{\alpha_1,\lambda_1}(z)\right) = \phi_{\alpha_2\alpha_1,\lambda_2\lambda_1^{\alpha_2}}(z).$$

In particular, with $\phi^{\circ n}(z) = \phi(\phi(...\phi(z)))$ (n times), the $n-$th composition of $\phi(z)$ with itself, then

$$\phi^{\circ n}_{\alpha,\lambda}(z) = \phi_{\alpha^n,\lambda^{(1-\alpha^n)/(1-\alpha)}}(z)$$

is the $n-$th composition of $\phi_{\alpha,\lambda}(z)$ with itself. It is thus itself a Sibuya(α_n, λ_n) rv with $\alpha_n := \alpha^n$ and $\lambda_n := \lambda^{(1-\alpha^n)/(1-\alpha)}$.

Consider a supercritical Galton-Watson process whose offspring number has distribution $X_{\alpha,\lambda}$ (so with infinite mean $1 < \mathbf{E}(X_{\alpha,\lambda}) = \infty$), defining the discrete Neveu model. Note $\mathbf{E}\left(X^q_{\alpha,\lambda}\right) < \infty$ if $q < \alpha$. Because $\mathbf{P}(X_{\alpha,\lambda} = 0) = 1 - \lambda > 0$, this process has a positive (non-zero) probability of extinction. Let N_n be the number of descendants of a unique common ancestor at generation n. Then $\phi_n(z) := \mathbf{E}_1\left(z^{N_n}\right) := \mathbf{E}\left(z^{N_n} \mid N_0 = 1\right)$, the pgf of N_n, obeys the recursion

$$\phi_{n+1}(z) = \phi_{\alpha,\lambda}(\phi_n(z)),\ n \geq 0,\ \phi_0(z) = z, \tag{2}$$

leading to

$$\phi_n(z) = \phi^{\circ n}_{\alpha,\lambda}(z) = 1 - \lambda_n (1-z)^{\alpha_n}. \tag{3}$$

Note that if $\lambda_1 = \lambda \leq 1 - \alpha_1$ ($X_{\alpha,\lambda}$ is ID), then N_n is ID for all $n \geq 1$ because $\lambda_n := \lambda^{(1-\alpha^n)/(1-\alpha)} \leq \lambda \leq 1 - \alpha \leq 1 - \alpha^n$ whereas if $\lambda > 1 - \alpha$, N_n becomes ID for all $n \geq n_0 = \inf(n : \lambda_n \leq 1 - \alpha_n)$. This is because $(\lambda_n)_{n\geq 1}$ is a decreasing sequence from λ to $\lambda^{1/(1-\alpha)}$ while $1-\alpha_n$ is increasing from $1-\alpha$ to 1. Therefore, only after time n_0,

$$\phi_n(z) = e^{-\mu_n(1-h_n(z))},\ n \geq n_0,$$

where $\mu_n = -\log(1-\lambda_n)$ and $h_n(z)$ a pgf (with $h_n(0)=0$) which can be identified from the latter equation and (3): the branching process N_n with $N_0 = 1$ has the same distribution as a (time-inhomogeneous rate) compound Poisson process with independent jumps but time dependent jumps $\Delta_k(n)$, $k \geq 1$:

$$N_n \overset{d}{=} \sum_{k=1}^{P(\mu_n)} \Delta_k(n). \tag{4}$$

It holds that the rate sequence μ_n is decreasing from $-\log(1-\lambda_{n_0})$ to $-\log\left(1-\lambda^{1/(1-\alpha)}\right)$ whereas $h_n(z) = \mu_n^{-1}\log\left(1+\frac{\lambda_n}{1-\lambda_n}\left(1-(1-z)^{\alpha_n}\right)\right)$ obeys $h_n(0)=0$ and $h_n(z) \to 0$ as $n \to \infty$, showing that $h_n(z)$ is the pgf of some jump rv $\Delta(n)$ taking values in $\{1,2,...\}$, whose size increases with n and with $\Delta(n) \overset{d}{\to} \infty$ as $n \to \infty$. As a compound Poisson process with jump sizes $\Delta(n)$ (only after time n_0), the law of N_n is the one of a process with non-decreasing sample paths.

Furthermore, from (3), with $\mathbf{P}_1(N_n = k) := \mathbf{P}(N_n = k \mid N_0 = 1)$

$$\begin{aligned} \pi_{\alpha_n,\lambda_n}(0) &:= \mathbf{P}_1(N_n = 0) = 1-\lambda_n \\ \pi_{\alpha_n,\lambda_n}(k) &:= \mathbf{P}_1(N_n = k) = \lambda_n(-1)^{k-1}\binom{\alpha_n}{k} = \alpha_n\lambda_n\frac{[\overline{\alpha}_n]_{k-1}}{k!},\ k \geq 1. \end{aligned}$$

Thus, $\mathbf{P}_1(N_\infty = 0) = \lim_{n\uparrow\infty}\mathbf{P}_1(N_n = 0) = 1-\lambda^{1/(1-\alpha)} =: \rho_e$, the probability of extinction. As required, the number ρ_e is the smallest solution to $\phi_{\alpha,\lambda}(z) = z$. If N_n does not go extinct, it explodes ($N_n \to \infty$ as $n \to \infty$) with probability $\lambda^{1/\alpha(1-\alpha)}$.

Remark: It can be checked that, with $\log_a b = \log b/\log a$ and $A(z) = 1 - \log_{1-\rho_e}(1-z)$, $z<1$,

$$\phi_{\alpha,\lambda}(z) = A^{-1}(\alpha A(z)),\ \text{so that}\ \phi_{\alpha,\lambda}^{\circ n}(z) = A^{-1}(\alpha^n A(z)). \tag{5}$$

This is an alternative way to see that the discrete Neveu branching model is 'integrable'.

Let τ_e be the extinction time (or the height of the Neveu branching process). The events $N_n > 0$ and $\tau_e > n$ coincide so

$$\mathbf{P}_1(\tau_e > n) = \mathbf{P}_1(N_n > 0) = \lambda_n,$$

with $\mathbf{P}_1(N_\infty = 0) = \mathbf{P}_1(\tau_e < \infty) = \rho_e = 1-\lambda^{1/(1-\alpha)} > 0$. One consequence of this fact is the following: if the population is observed to be alive after generation n, the probability that it will survive forever is

$$1-\mathbf{P}_1(\tau_e < \infty \mid \tau_e \geq n) = (1-\rho_e)/\lambda_{n-1} = \lambda^{\alpha^{n-1}/(1-\alpha)}.$$

This survival probability is doubly exponentially close to 1 as n grows (for instance, if $\alpha = \lambda = 1/2$, it is 0.9993233275 as soon as $n = 12$). So, if extinction is to occur, it occurs rapidly or nearly never. Indeed,

$$\mathbf{P}_1\left(\tau_e<\infty \mid \tau_e\geq n\right)=\frac{\mathbf{P}_1\left(\tau_e\geq n \mid \tau_e<\infty\right)\mathbf{P}_1\left(\tau_e<\infty\right)}{\mathbf{P}_1\left(\tau_e\geq n\right)}$$

$$=\frac{\left(1-\mathbf{P}_1\left(\tau_e<n \mid \tau_e<\infty\right)\right)\mathbf{P}_1\left(\tau_e<\infty\right)}{\mathbf{P}_1\left(\tau_e\geq n\right)}=\frac{\mathbf{P}_1\left(\tau_e<\infty\right)-\mathbf{P}_1\left(N_{n-1}=0\right)}{1-\mathbf{P}_1\left(N_{n-1}=0\right)}$$

$$=1-\frac{1-\mathbf{P}_1\left(\tau_e<\infty\right)}{1-\mathbf{P}_1\left(N_{n-1}=0\right)}=1-\frac{1-\rho_e}{\lambda_{n-1}}.$$

We also have [15]:

$$\mathbf{P}_1\left(\alpha^n\log\left(1+N_n\right)\leq x\right)\to F\left(x\right)\ \text{ as } n\to\infty,$$

with $x\geq 0$ and

$$F\left(x\right)=\lim_{n\to\infty}\mathbf{E}_1\left(\left(1-e^{-\alpha^{-n}x}\right)^{N_n}\right)=1-\lambda^{1/(1-\alpha)}e^{-x}=\left(1-\lambda^{1/(1-\alpha)}\right)+\lambda^{1/(1-\alpha)}\left(1-e^{-x}\right),$$

corresponding to a rv with law $\left(1-\lambda^{1/(1-\alpha)}\right)\delta_0+\lambda^{1/(1-\alpha)}e^{-x}$ (a mixture of a rv with Dirac mass at 0 and an exponentially distributed rv say E, with mean 1).

So if the population does not go extinct, N_n grows fast to infinity at a double-exponential speed: $\left(1+N_n\right)^{\alpha^n}\overset{d}{\to}Z=e^E>1$ as $n\to\infty$, with Z log-exponential (or standard Pareto) distributed: $\mathbf{P}\left(Z>x\right)=1/x,\ x>1$. Using martingale arguments, this convergence can be shown to be almost sure as well, ([25], [27], Proposition 3.8).

• **Randomizing the initial condition.** So far, we studied the number of descendants N_n of a single individual at generation 0. Suppose the initial number of particles is now random with geometric distribution $\mathbf{P}\left(N_0=n\right)=qp^n,\ n\geq 0$ and $q+p=1$ (the distribution of maximal entropy under the constraint that its mean $\mu=p/\left(1-p\right)$ is fixed). Then under the assumption of independence of all the progenies, with $\phi_{G_\mu}\left(z\right)=1/\left(1+\mu\left(1-z\right)\right)$ the pgf of N_0, the pgf of the full process reads

$$\mathbf{E}\left(z^{N_n}\right)=\phi_{G_\mu}\left(\phi_n\left(z\right)\right)=\frac{1}{1+\mu\lambda_n\left(1-z\right)^{\alpha_n}}. \tag{6}$$

We will identify later this pgf as the one of a (discrete-) Mittag-Leffler distribution with scale parameter $\mu_n=\mu\lambda_n$ and tail exponent $\alpha_n=\alpha^n$. Similarly, if the initial number of particles were random with Poisson(μ) distribution, the pgf of the full process would be a discrete stable law with scale parameter $\mu_n=\mu\lambda_n$ and tail exponent $\alpha_n=\alpha^n$:

$$\mathbf{E}\left(z^{N_n}\right)=\phi_{P_\mu}\left(\phi_n\left(z\right)\right)=e^{-\mu_n(1-z)^{\alpha_n}}=e^{-\mu_n\left[1-\left(1-(1-z)^{\alpha_n}\right)\right]}, \tag{7}$$

showing that $N_n\overset{d}{=}\sum_{k=0}^{P(\mu_n)}\Delta_k\left(n\right)$, a compound Poisson($\mu_n$) −sum of iid parts with distribution Sibuya(α_n). Poisson distributions are maximal entropy distributions in the class of ultra-log-concave distributions, [33]. We call such branching processes, with discrete-stable(μ_n,α^n) marginals, the discrete-Neveu branching process, in view of their continuous counterpart defined in [50]. Note finally that if $N_0\geq 1$ is Sibuya(α) distributed, then, by stability, N_n is again Sibuya distributed with parameters $\lambda_n = \lambda^{\alpha(1-\alpha^n)/(1-\alpha)}$ and $\alpha_n = \alpha^{n+1}$.

• **Resolvent of the Neveu process.** It is also clear that N_n is a discrete-time Markov chain over the set of integers $\mathbb{N}_0 = \{0,1,2,...\}$, with transition probability matrix $P = [P_{i,j}]$ satisfying $P_{i,j} = \mathbf{P}(N_1 = j \mid N_0 = i) =: \mathbf{P}_i(N_1 = j) = [z^j]\,\phi_{\alpha,\lambda}(z)^i$, $i,j \in \mathbb{N}_0$, $P_{0,j} = \delta_{j,0}$. The pgf $\phi_{\alpha,\lambda}(z)^i$ is the one of a three-parameter Sibuya rv $\phi_{\alpha,i,\lambda}(z)$. $P_{i,j}$ can easily be computed by the Faa di Bruno formula for the composition of generating functions, involving ordinary Bell numbers $\widehat{B}_{j,i}$, ([14], Tome 1, p. 148). Specifically, with $\pi_{\alpha,\lambda}(k) = \mathbf{P}(X_{\alpha,\lambda} = k)$ the probability system given by (1)

$$P_{i,j} = \sum_{k=1}^{i} \binom{i}{k} \pi_{\alpha,\lambda}(0)^{i-k}\, \widehat{B}_{j,k}\left(\pi_{\alpha,\lambda}(1), \pi_{\alpha,\lambda}(2), ...\right),\ j,i \geq 1. \tag{8}$$

Let $x(\bullet)$ denote the sequence $x(1), x(2), x(3)$ ... Ordinary Bell numbers $\widehat{B}_{j,i}(x(\bullet))$ are related to standard Bell numbers $B_{j,i}(x(\bullet)) = B_{j,i}(x(1), x(1), ...)$ by

$$\widehat{B}_{j,i}(x(\bullet)) = B_{j,i}(\bullet! x(\bullet)),$$

so with [14]

$$B_{j,i}(x(\bullet)) = j! \sum^{*} \prod_{k\geq 1} \frac{1}{c_k!}\left(\frac{x(k)}{k!}\right)^{c_k} \text{ and } \widehat{B}_{j,i}(x(\bullet)) = j! \sum^{*} \prod_{k\geq 1} \frac{x(k)^{c_k}}{c_k!},$$

the latter star summations running over the integers c_k obeying $\sum_{k\geq 1} c_k = i$ and $\sum_{k\geq 1} k c_k = j$. Thus, with $\overline{\lambda} := 1-\lambda = \pi_{\alpha,\lambda}(0)$, $\pi_{\alpha,\lambda}(k) = \alpha\lambda\,[\overline{\alpha}]_{k-1}$, $k \geq 1$

$$P_{i,j} = \sum_{k=1}^{i} \binom{i}{k} \overline{\lambda}^{i-k} (\alpha\lambda)^k B_{j,k}\left([\overline{\alpha}]_{\bullet-1}\right) = \overline{\lambda}^{i} \sum_{k=1}^{i} \binom{i}{k} \left(\alpha\lambda/\overline{\lambda}\right)^k B_{j,k}\left([\overline{\alpha}]_{\bullet-1}\right),\ j,i \geq 1.$$

When $i = 1$, recalling that $B_{j,1}(x(\bullet)) = x(j)$, we recover that $P_{1,j} = \alpha\lambda\,[\overline{\alpha}]_{j-1}$, $j \geq 1$.

What we get here for free is that for all $j,i \geq 1$

$$P^n_{i,j} = [z^j]\left(\phi^{\circ n}_{\alpha,\lambda}(z)^i\right) = [z^j]\left(1-\lambda_n(1-z)^{\alpha_n}\right)^i$$

$$= \sum_{k=0}^{i} \binom{i}{k} \pi_{\alpha_n,\lambda_n}(0)^{i-k}\, \widehat{B}_{j,k}\left(\pi_{\alpha_n,\lambda_n}(\bullet)\right) = \overline{\lambda}_n^{i} \sum_{k=1}^{i} \binom{i}{k} \left(\alpha_n\lambda_n/\overline{\lambda}_n\right)^k B_{j,k}\left([\overline{\alpha}_n]_{\bullet-1}\right)$$

is obtained similarly as for $P_{i,j}$ simply while performing the substitution $(\alpha,\lambda) \to (\alpha_n,\lambda_n)$. For $j,i \geq 1$, we also obtain the resolvent of N_n as

$$g_{i,j}(\mathrm{z}) := \delta_{i,j} + \sum_{n\geq 1}^{n} \mathrm{z}^n P^n_{i,j} = \delta_{i,j} + \sum_{n\geq 1}^{n} \mathrm{z}^n \overline{\lambda}_n^{i} \sum_{k=1}^{i} \binom{i}{k} \left(\alpha_n\lambda_n/\overline{\lambda}_n\right)^k B_{j,k}\left([\overline{\alpha}_n]_{\bullet-1}\right).$$

In particular,

$$g_{i,i}(\mathrm{z}) = 1 + \sum_{n\geq 1} \mathrm{z}^n \overline{\lambda}_n^{i} \sum_{k=1}^{i} \binom{i}{k} \left(\alpha_n\lambda_n/\overline{\lambda}_n\right)^k B_{i,k}\left([\overline{\alpha}_n]_{\bullet-1}\right).$$

It holds that $\mathbf{E}\left(\mathrm{z}^{\tau_{i,j}}\right) = g_{i,j}(\mathrm{z})/g_{j,j}(\mathrm{z})$ where $\tau_{i,j}$ is the first passage time to state $j \neq i$ of N_n given $N_0 = i$, together with $\mathbf{E}\left(\mathrm{z}^{\tau_{i,i}^r}\right) = 1 - 1/g_{i,i}(\mathrm{z})$, where $\tau_{i,i}^r$ is the first return time to state i of N_n with $\mathbf{P}\left(\tau_{i,i}^r < \infty\right) = 1 - 1/g_{i,i}(1)$.

The Bell numbers $B_{j,i}\left([\overline{\alpha}]_{\bullet-1}\right)$ appearing in the above computations obey a simple 3−term recursion with appropriate boundary conditions

$$B_{j+1,i}\left([\overline{\alpha}]_{\bullet-1}\right) = B_{j,i-1}\left([\overline{\alpha}]_{\bullet-1}\right) + (j - i\alpha) B_{j,i}\left([\overline{\alpha}]_{\bullet-1}\right).$$

They constitute a class of generalized Stirling numbers studied in [8].

The Markov chain N_n is possibly absorbed at $\{0\}$ if extinction occurs. Let then $\tau_{i,e} = \inf\{n \geq 1 : N_n = 0 \mid N_0 = i\}$, $i \geq 1$ with $\tau_{i,e} := \infty$ in case of non-extinction (note $\tau_e = \tau_{1,e}$). $\tau_{i,e}$ is the extinction probability given i particles originally. Then,

$$\mathbf{P}(\tau_{i,e} \leq n) = \mathbf{P}_i(N_n = 0) = \left[z^0\right] \phi_n(z)^i = (1 - \lambda_n)^i \tag{9}$$

and $\mathbf{P}(\tau_{i,e} = \infty) = 1 - (1 - \lambda_\infty)^i = 1 - \rho_e^i$. Let $\overline{P}$ be obtained from P while removing its first row and column (corresponding to the absorbing state $\{0\}$). Let $\mathbf{h} \equiv (h(1), h(2), ...)'$ solve $\overline{P}\mathbf{h} = \mathbf{h}$ with boundary condition $h(1) = 1$. The column sequence $\mathbf{h}$ is called the scale (or harmonic) sequence of $\{N_n\}$. It is such that $h\left(N_{n\wedge\tau_{i,e}}\right)$ is a martingale and it holds that $h(i) = h(\infty)(1 - \mathbf{P}(\tau_{i,e} < \infty))$, [51]. It can easily be checked by hand to be

$$h(i) = \frac{1 - \rho_e^i}{1 - \rho_e}, \; i \geq 1. \tag{10}$$

• **Pgf versus log-Laplace transform (LLt) and associated random walk.** Let $M := X_{\alpha,\lambda}$ denote the offspring number per capita in the discrete Neveu branching process. Let $\psi_n(p) := -\log \phi_n(e^{-p})$ and $\psi(p) := -\log \mathbf{E}\left(e^{-p(M-1)}\right)$. The recursion (2) is also

$$\psi_{n+1}(p) - \psi_n(p) = \psi(\psi_n(p)), \; n \geq 0, \; \psi_0(p) = p.$$

Let $\widetilde{M} := M - 1$, taking values in $\{-1, 0, 1, 2, ...\}$. Discrete-time branching processes are intimately related to a random walk. Consider indeed the (skip-free to the left) random walk $S_{n+1} = S_n + \widetilde{M}_{n+1}$, $S_0 = 1$, with the sequence $\left\{\widetilde{M}_n\right\}$ iid all distributed like $\widetilde{M}$. Assume also that state $\{0\}$ is absorbing. Then, with $\overline{N}_n = N_1 + ... + N_n$ the cumulated number of offsprings up to time n, $S_{\overline{N}_{n+1}} - S_{\overline{N}_n} = \sum_{i=\overline{N}_n+1}^{\overline{N}_{n+1}} \widetilde{M}_i \overset{d}{=} \sum_{i=1}^{N_n} \widetilde{M}_i$, showing that

$$N_n \overset{d}{=} S_{\overline{N}_{n-1}}.$$

Therefore N_n is a time-changed version of S_n. The one-step transition matrix $P = [P_{i,j}]$ of the random walk $\{S_n\}$ started at $S_0 = i$ with $\{0\}$ absorbing, is $\mathbf{P}_0(S_1 = j) = P_{0,j} = \delta_{j,0}$ and

$$\mathbf{P}_i(S_1 = j) = P_{i,j} = \left[z^{j-i+1}\right] \phi_{\alpha,\lambda}(z) = \pi_{\alpha,\lambda}(j - i + 1), \; i \geq 1, \; j \geq i - 1.$$

It is an upper-Hessenberg type matrix. The scale or harmonic function of $\{S_n\}$, say $\mathbf{s} \equiv (s(1), s(2), ...)'$, therefore is the smallest solution to

$$\sum_{j \geq i-1} \pi_{\alpha,\lambda}(j-i+1) s(j) = s(i), \; i \geq 2,$$

with conventional boundary condition $s(1) = 1$. It is an increasing sequence. With $\rho_e = 1 - \lambda^{1/(1-\alpha)}$, the smallest solution to $\phi_{\alpha,\lambda}(z) = z$ and $b = 1/(1-\rho_e)$, we get

$$s(1) = 1, \; s(i) = b\left(1 - \rho_e^i\right), \; i \geq 2,$$

with $s(i) = h(i)$ given in (10). Indeed, if $i \geq 3$

$$\begin{aligned} \sum_{j \geq i-1} \pi_{\alpha,\lambda}(j-i+1) s(j) &= \sum_{k \geq 0} \pi_{\alpha,\lambda}(k) s(k+i-1) \\ &= b\left(1 - \sum_{k \geq 0} \pi_{\alpha,\lambda}(k) \rho_e^{k+i-1}\right) = b\left(1 - \rho_e^i\right). \end{aligned}$$

If $i = 2$,

$$\sum_{k \geq 0} \pi_{\alpha,\lambda}(k) s(k+1) = \pi_{\alpha,\lambda}(0) + b \sum_{k \geq 1} \pi_{\alpha,\lambda}(k)\left(1 - \rho_e^{k+1}\right)$$

$$= \pi_{\alpha,\lambda}(0) + b\left(1 - \pi_{\alpha,\lambda}(0) - \rho_e\left(\rho_e - \pi_{\alpha,\lambda}(0)\right)\right) = b\left(1 - \rho_e^2\right),$$

fixing the limiting constant $b = 1/(1-\rho_e) = s(\infty)$. This is consistent with the following result: let $\tau_{i,0} = \inf(n \geq 1 : S_n = 0 \mid S_0 = i)$. Then, by first-step analysis ([56], p. 92), $\mathbf{E}(z^{\tau_{i,0}}) = h(z)^i$, where $h(z)$ solves the functional equation

$$h(z) = z\phi_{\alpha,\lambda}(h(z)). \tag{11}$$

We therefore get $\mathbf{P}(\tau_{i,0} < \infty) = h(1)^i$ with $h(1) = \rho_e$. Note $\mathbf{E}(\tau_{i,0}) = \infty$ but $\mathbf{E}(z^{\tau_{i,0}} \mid \tau_{i,0} < \infty) = (h(z)/h(1))^i$, leading to $\mathbf{E}(\tau_{i,0} \mid \tau_{i,0} < \infty) = i/(1-\alpha)$. We also have [51],

$$s(i) = s(\infty)\left(1 - \mathbf{P}(\tau_{i,0} < \infty)\right),$$

together with $\tau_{i,j} = \inf(n \geq 1 : S_n = j \mid S_0 = i)$, $j > i$ and

$$\mathbf{P}(\tau_{i,j} < \tau_{i,0}) = \frac{s(i)}{s(j)} = \frac{1 - \rho_e^i}{1 - \rho_e^j}, \tag{12}$$

with $\mathbf{P}(\tau_{i,\infty} < \tau_{i,0}) = 1 - \rho_e^i$, the probability of non-extinction given $S_0 = i$, as required. Let $S_n^* = \max(S_m, m \leq n)$ be the ladder height process of $\{S_n\}$. It then holds from (12) that for $j \geq i$, $\mathbf{P}(S_\infty^* \geq j \mid S_0^* = i) = \left(1 - \rho_e^i\right)/\left(1 - \rho_e^j\right)$ and $\mathbf{P}(S_\infty^* = \infty \mid S_0^* = i) = 1 - \rho_e^i$. Thus

$$\mathbf{P}(S_\infty^* \geq j \mid S_0^* = i, S_\infty^* < \infty) = \frac{\rho_e^j}{1 - \rho_e^j}\left(\rho_e^{-i} - 1\right), \; j \geq i.$$

Because N_n is a time-changed version of S_n, with $N_\infty^* = \max(N_n, n \geq 1 \mid N_0 = i)$

$$\mathbf{P}(N_\infty^* \geq j \mid N_0^* = i) = \left(1 - \rho_e^i\right)/\left(1 - \rho_e^j\right) \text{ as well.}$$

The random variable N_∞^*, the maximal value which $\{N_n\}$ can take, is known as the width of $\{N_n\}$. Its distribution is given above.

We also observe that $h(z)^i$, as a solution to the functional equation (11), is the pgf of $\overline{N}_\infty$ started at i, the total limiting number of cumulated individuals which appeared over time in the population (possibly infinite on the set of explosion). It can be solved by the Lagrange inversion formula, [14]. Observing $\mathbf{E}\left(z^{S_n} \mid S_0 = i\right) = z^{i-n}\phi_{\alpha,\lambda}(z)^n$, by Lagrange formula, we get:

$$[z^n]\,h(z)^i = \frac{i}{n}\left[z^{n-i}\right]\phi_{\alpha,\lambda}(z)^n = \frac{i}{n}\left[z^0\right]\mathbf{E}\left(z^{S_n} \mid S_0 = i\right).$$

This yields the Kemperman formula ([56], p. 92) as

$$\mathbf{P}(\tau_{i,0} = n) = \frac{i}{n}\mathbf{P}_i(S_n = 0),\ n \geq i, \tag{13}$$

and more generally, while observing

$$\left[z^{n+j}\right]h(z)^i = \frac{i}{n+j}\left[z^{n+j-i}\right]\phi_{\alpha,\lambda}(z)^n = \frac{i}{n+j}\left[z^j\right]\mathbf{E}\left(z^{S_n} \mid S_0 = i\right),$$

$$\mathbf{P}(\tau_{i,0} = n+j) = \frac{i}{n+j}\mathbf{P}_i(S_n = j),\ n \geq i-j. \tag{14}$$

Finally, let $i, j \neq 1$ and $j \geq (i-n) \wedge 0$. Given $S_0 = i$, we have

$$S_n = 0 \cdot 1_{\{\tau_{i,0} \leq n\}} + \left(i + \sum_{k=1}^{n} \widetilde{M}_k\right) \cdot 1_{\{\tau_{i,0} > n\}}.$$

We therefore get, for $j \neq 0$

$$P_{i,j}^n := \mathbf{P}_i(S_n = j) = \left[z^{j-i+n}\right]\left(\phi_{\alpha,\lambda}(z)^n\right) = \frac{n+j}{i}\mathbf{P}(\tau_{i,0} = n+j),$$

so that the resolvent of $\{S_n\}$ reads ($j \neq 0$)

$$\begin{aligned} g_{i,j}(\mathrm{z}) \quad : &= \delta_{i,j} + \sum_{n \geq 1} \mathrm{z}^n P_{i,j}^n = \delta_{i,j} + \frac{1}{i}\sum_{n \geq i-j} \mathrm{z}^n (n+j)\,\mathbf{P}(\tau_{i,0} = n+j) \\ &= \delta_{i,j} + \frac{\mathrm{z}^{1-j}}{i}\sum_{k \geq i} \mathrm{z}^{k-1} k\mathbf{P}(\tau_{i,0} = k) = \delta_{i,j} + \frac{\mathrm{z}^{1-j}}{i}\frac{d}{d\mathrm{z}}\left(h(\mathrm{z})^i\right). \end{aligned}$$

In particular,

$$g_{i,i}(\mathrm{z}) = 1 + \frac{\mathrm{z}^{1-i}}{i}\frac{d}{d\mathrm{z}}\left(h(\mathrm{z})^i\right).$$

This leads to the first return time of $\{S_n\}$ to state i pgf: $\mathbf{E}\left(\mathrm{z}^{\tau_{i,i}^r}\right) = 1 - g_{i,i}(\mathrm{z})^{-1}$. In particular, $\mathbf{E}\left(\mathrm{z}^{\tau_{1,1}^r}\right) = 1 - 1/(1 + h'(\mathrm{z}))$ with, from (11), $\mathbf{P}\left(\tau_{1,1}^r < \infty\right) = 1 - 1/(1 + h'(1)) = \rho_e/(\rho_e + 1 - \alpha)$, in view of $h'(1) = \rho_e/\left(1 - \phi_{\alpha,\lambda}'(\rho_e)\right)$ and $\phi_{\alpha,\lambda}'(\rho_e) = \alpha \in (0,1)$.

• **Conditioned Neveu branching processes with Sibuya(α, λ) offspring distribution.** We know briefly address the problems of conditioning the latter process either on extinction or on explosion.

- When conditioning the supercritical branching process with branching mechanism $\phi_{\alpha,\lambda}(z)$ on extinction, one needs to consider a branching process with the modified branching mechanism

$$\phi_{\alpha,\lambda}(z) \to \widehat{\phi}_{\alpha,\lambda}(z) = \frac{\phi_{\alpha,\lambda}(z\rho_e)}{\rho_e} = \frac{1-\lambda(1-\rho_e z)^{\alpha}}{\rho_e}.$$

$\widehat{\phi}_{\alpha,\lambda}(z)$ is the branching mechanism of some rv say $\widehat{X}_{\alpha,\lambda}$ (the new modified offspring number), now with

$$\begin{aligned}\mathbf{P}\left(\widehat{X}_{\alpha,\lambda}=0\right) &= (1-\lambda)/\rho_e \\ \mathbf{P}\left(\widehat{X}_{\alpha,\lambda}=k\right) &= \frac{\lambda}{\rho_e}(-1)^{k-1}\binom{\alpha}{k}\rho_e^k,\ k\geq 1.\end{aligned}$$

Remark: In view of (5), $\widehat{\phi}_{\alpha,\lambda}(z) = \widehat{A}^{-1}\left(\alpha\widehat{A}(z)\right)$ where $\widehat{A}(z) = A(A_e(z))$, $A_e(z) = z\rho_e$ so that $\widehat{A}(z) = 1-\log_{1-\rho_e}(1-\rho_e z)$, $z<1/\rho_e$.

We have

$$\mathbf{E}\left(\widehat{X}_{\alpha,\lambda}\right) = \phi'_{\alpha,\lambda}(z=\rho_e) = \lambda\alpha(1-\rho_e)^{\alpha-1} = \alpha < 1.$$

The new process is therefore subcritical. When the process starts from one particle, it holds that

$$\widehat{\phi}^{\circ n}_{\alpha,\lambda}(z) = \widehat{\phi}_{\alpha^n,\lambda^{(1-\alpha^n)/(1-\alpha)}}(z) = \frac{1-\lambda_n(1-\rho_e z)^{\alpha_n}}{\rho_e} = \widehat{A}^{-1}\left(\alpha^n\widehat{A}(z)\right).$$

The $n-$th composition of $\widehat{\phi}_{\alpha,\lambda}(z)$ with itself is again of the same form $\widehat{\phi}_{\alpha_n,\lambda_n}(z)$, with $\alpha_n=\alpha^n$ and $\lambda_n=\lambda^{(1-\alpha^n)/(1-\alpha)}$. Thus

$$\begin{aligned}\mathbf{P}_1\left(\widehat{N}_n=0\right) &= (1-\lambda_n)/\rho_e \\ \mathbf{P}_1\left(\widehat{N}_n=k\right) &= \frac{\lambda_n}{\rho_e}(-1)^{k-1}\binom{\alpha_n}{k}\rho_e^k = \frac{\alpha_n\lambda_n}{\rho_e}\frac{[\overline{\alpha}_n]_{k-1}}{k!}\rho_e^k,\ k\geq 1,\end{aligned}$$

now with geometric tails. As required, we have

$$\mathbf{P}_1\left(\widehat{N}_n=0\right) \underset{n\to\infty}{\to} \left(1-\lambda^{1/(1-\alpha)}\right)/\rho_e = 1,\ \text{(almost sure extinction)}.$$

The Yaglom limit exists, with explicit logarithmic limiting distribution. Indeed, from the large n inspection of the probabilities $\mathbf{P}_1\left(\widehat{N}_n=k\right)$, $k\geq 0$,

$$\lim_{n\to\infty}\mathbf{P}_1\left(\widehat{N}_n=k\mid\widehat{N}_n>0\right) = -\frac{1}{\log(1-\rho_e)}\frac{\rho_e^k}{k},\ k\geq 1.$$

The mean value of the Yaglom distribution is: $\rho_e/(-(1-\rho_e)\log(1-\rho_e))$. The corresponding pgf

$$g(z) = \frac{-\log(1-\rho_e z)}{-\log(1-\rho_e)} =: \log_{1-\rho_e}(1-\rho_e z),\ z<1/\rho_e$$

solves the associated Schröder functional equation $1-g\left(\widehat{\phi}_{\alpha,\lambda}(z)\right) = \alpha(1-g(z))$, known to characterize the Yaglom limit, [28]. Observe that, with $\overline{g}(z) := 1-g(z) = \widehat{A}(z)$

$$\widehat{\phi}_{\alpha,\lambda}(z) = \overline{g}^{-1}(\alpha\overline{g}(z)),\ \text{so that } \widehat{\phi}^{\circ n}_{\alpha,\lambda}(z) = \overline{g}^{-1}(\alpha^n\overline{g}(z)).$$

The time to extinction, say $\widehat{\tau}_e$, of this new branching process is now finite with probability 1 and

$$\mathbf{P}_1\left(\widehat{\tau}_e \leq n\right) = \mathbf{P}_1\left(\tau_e \leq n \mid \tau_e < \infty\right) = \frac{1-\lambda_n}{1-\lambda^{1/(1-\alpha)}},$$

with $\alpha^{-n}\mathbf{P}_1\left(\widehat{\tau}_e > n\right) \underset{n\to\infty}{\to} -\log\left(\lambda\right)\lambda^{1/(1-\alpha)}/\left(1-\lambda^{1/(1-\alpha)}\right)$ translating the fact that $\widehat{\tau}_e$ is tail equivalent to a rv with geometrically decaying tails α^n.

- When conditioning the supercritical branching process with branching mechanism $\phi_{\alpha,\lambda}\left(z\right)$ now on explosion, one needs to consider a branching process with the modified branching mechanism (the Harris-Sevastyanov transform ([26], [61]))

$$\begin{aligned}\phi_{\alpha,\lambda}\left(z\right) \quad \rightarrow \quad \widetilde{\phi}_{\alpha,\lambda}\left(z\right) &= \frac{\phi_{\alpha,\lambda}\left(\rho_e + z\left(1-\rho_e\right)\right)-\rho_e}{1-\rho_e} \\ &= 1-\frac{\lambda\left(1-\rho_e\right)^{\alpha}\left(1-z\right)^{\alpha}}{1-\rho_e} = 1-\left(1-z\right)^{\alpha} =: \widetilde{\phi}_{\alpha}\left(z\right),\end{aligned}$$

independent of λ. The transformed pgf $\widetilde{\phi}_{\alpha,\lambda}\left(z\right) = \widetilde{\phi}_{\alpha}\left(z\right)$ is the branching mechanism of some rv say $\widetilde{X}_{\alpha}$, now with $\mathbf{P}\left(\widetilde{X}_{\alpha}=0\right)=0$. $\widetilde{X}_{\alpha}$ is an (unscaled) Sibuya(α) rv. It is the generating function of the reproduction law of some Galton-Watson process $\left\{\widetilde{N}_n\right\}$ which is obtained by the restriction of $\left\{N_n\right\}$ to prolific individuals (disregarding doomed particles of $\left\{N_n\right\}$). If $\widetilde{N}_n$ is the number of offspring at generation n of this conditioned process, we have

$$\mathbf{E}_1\left(z^{\widetilde{N}_n}\right) = \widetilde{\phi}_{\alpha}^{\circ n}\left(z\right) = 1-\left(1-z\right)^{\alpha_n}.$$

Remark: In view of (5), $\widetilde{\phi}_{\alpha,\lambda}\left(z\right) = \widetilde{A}^{-1}\left(\alpha\widetilde{A}\left(z\right)\right)$ where $\widetilde{A}\left(z\right) = A\left(A_{\overline{e}}\left(z\right)\right)$, $A_{\overline{e}}\left(z\right) = \rho_e + z\left(1-\rho_e\right)$. Thus, with $\widetilde{A}\left(z\right) = -\log_{1-\rho_e}\left(1-z\right)$, $\widetilde{\phi}_{\alpha}^{\circ n}\left(z\right) = \widetilde{A}^{-1}\left(\alpha^n\widetilde{A}\left(z\right)\right)$.

We note that $\mathbf{E}_1\left(z^{\widetilde{N}_n}\right) = \frac{\phi_n(z)-\phi_n(0)}{1-\phi_n(0)}$, emphasizing that $\widetilde{N}_n$ is N_n conditionally given $N_n \geq 1$. Equivalently,

$$\begin{aligned}\mathbf{P}_1\left(\widetilde{N}_n = 0\right) &= 0, \\ \mathbf{P}_1\left(\widetilde{N}_n = k\right) &= \left(-1\right)^{k-1}\binom{\alpha_n}{k} = \alpha_n\frac{\left[\overline{\alpha}_n\right]_{k-1}}{k!},\ k \geq 1.\end{aligned}$$

The time to extinction of this $^{\sim}-$process, say $\widetilde{\tau}_e$, is $\widetilde{\tau}_e = \infty$ with probability 1 and $\mathbf{P}_1\left(\widetilde{N}_n \to \infty\right) = 1$. And $\widetilde{N}_n$ has non-decreasing sample-paths.

Remark: With $\phi_n\left(z\right) := \mathbf{E}_1\left(z^{N_n}\right)$, consider a (time-inhomogeneous) branching process whose pgf obeys $\phi_{n+1}\left(z\right) = f_{n+1}\left(\phi_n\left(z\right)\right)$ where now $f_n\left(z\right) = \phi_{\alpha^n,\lambda}\left(z\right)$. Then

$$\phi_n\left(z\right) = \phi_{\alpha^n,\lambda}\left(...\left(\phi_{\alpha^2,\lambda}\left(\phi_{\alpha,\lambda}\left(z\right)\right)\right)\right) = 1-\lambda_n\left(1-z\right)^{\alpha_n},$$

with now $\lambda_n = \lambda^{1+\sum_{m=1}^{n-1}\alpha^{m(m+1)/2}}$ and $\alpha_n = \prod_{m=1}^{n}\alpha^m = \alpha^{n(n+1)/2}$. By so doing, the tail (but not the scale) parameter α^n of the branching mechanism f_n now depends on the number of the generation, with $\alpha^n \to 0$ as $n \to \infty$. When conditioning on explosion ($\lambda = 1$), $\phi_n\left(z\right) = 1-\left(1-z\right)^{\alpha_n}$ and N_n is distributed like an (unscaled) Sibuya$\left(\alpha^{n(n+1)/2}\right)$ and one expects N_n to grow fast to infinity at a rate still faster than double-exponential: $\left(1+N_n\right)^{\alpha^{n^2}/2} \overset{d}{\to} Z = e^W > 1$ as $n \to \infty$.

1.3. **Branching processes involving Sibuya rvs: discrete-space, continuous–time Neveu process.** We now turn to discrete-space and continuous-time branching processes involving Sibuya rvs. A central and critical one will be the Neveu process.

• **Branching process with Sibuya(α, λ) offspring distribution in continuous time.** We consider a continuous-time version of the latter model in discrete-time ([26], [61]), also with independent branching particles. For continuous-time branching processes, the rates $\lambda_k\left(dt\right)$ at which one individual gives birth to k individuals in the time interval dt are given by ($\lambda_k > 0$)

$$\lambda_1\left(dt\right) = 1 - \left(\sum_{k\neq 1} \lambda_k\right) dt + o\left(dt\right) \text{ and } \lambda_k\left(dt\right) = \lambda_k dt + o\left(dt\right) \text{ if } k \neq 1.$$

Each particle lives a random time with exponential distribution with frequency parameter $\mu = \sum_{k\neq 1} \lambda_k$. At the time of its death, it gives birth to a random number $M \neq 1$ of particles of the same type with law

$$\mathbf{P}\left(M = k\right) = \lambda_k/\mu,\ k \neq 1.$$

The induced branching mechanism to consider is thus $f\left(z\right) = \mu\left(\phi\left(z\right) - z\right)$ where $\phi\left(z\right) = \sum_{k\neq 1} \mathbf{P}\left(M = k\right) z^k$ is the pgf of M. With $\lambda_1 > 0$ arbitrary, we also have

$$f\left(z\right) = \sum_{k\neq 1} \lambda_k z^k - \left(\sum_{k\neq 1} \lambda_k\right) z = \sum_{k} \lambda_k z^k - \left(\sum_{k} \lambda_k\right) z = \mu\left(\phi\left(z\right) - z\right),$$

now with $\mu = \sum_k \lambda_k$, $\phi\left(z\right) = \sum_k \mathbf{P}\left(M = k\right) z^k$ and $\mathbf{P}\left(M = k\right) = \lambda_k/\mu$, $k \in \{0, 1, 2, ...\}$. Thus, $\phi\left(z\right)$ is now the pgf of a rv M taking values in $\{0, 1, 2, ...\}$, including $\{1\}$. And the two models are equivalent.

Consider therefore a continuous-time branching process with Sibuya(α, λ) branching mechanism, so with $\phi\left(z\right) = \phi_{\alpha,\lambda}\left(z\right)$. Then:

$$f\left(z\right) = \mu\left(\left(1 - z\right) - \lambda\left(1 - z\right)^{\alpha}\right) = \mu\left(\phi_{\alpha,\lambda}\left(z\right) - z\right).$$

The rates $\lambda_k > 0$ at which one individual gives birth to k individuals are $\lambda_k = \left[z^k\right] f\left(z\right)$. We get here

$$\begin{aligned} \lambda_0 &= \mu\left(1 - \lambda\right), \\ \lambda_k &= \mu\lambda\left(-1\right)^{k-1}\binom{\alpha}{k},\ k \geq 1. \end{aligned}$$

Then, $\phi_t\left(z\right) = \mathbf{E}_1\left(z^{N(t)}\right)$, $t \geq 0$, the pgf of the population size $N\left(t\right)$ at time t started with a single individual, obeys [26]

$$\partial_t \phi_t\left(z\right) = f\left(\phi_t\left(z\right)\right),\ \phi_0\left(z\right) = z, \tag{15}$$

which can be solved explicitly to give

$$\phi_t\left(z\right) = 1 - \left(\lambda\left(1 - e^{-\mu(1-\alpha)t}\right) + e^{-\mu(1-\alpha)t}\left(1 - z\right)^{1-\alpha}\right)^{1/(1-\alpha)}.$$

We have $\phi_t(1) = 1 - \left(\lambda\left(1 - e^{-\mu(1-\alpha)t}\right)\right)^{1/(1-\alpha)} = \mathbf{P}_1(N_t < \infty) < 1$ and the process is not regular, or defective (for each $t > 0$, there is a positive probability that $N(t) = \infty$, translating the opportunity of explosion in finite time; the process $N(t)$ loses mass at infinity). For small values of $t > 0$ already, we have

$$\mathbf{P}_1(N(t) = \infty) \propto t^{1/(1-\alpha)}.$$

We also have

$$\mathbf{P}_1(N(t) = 0) = \phi_t(0) = 1 - \left(\lambda + (1-\lambda)e^{-\mu(1-\alpha)t}\right)^{1/(1-\alpha)} \underset{t\to\infty}{\to} \rho_e := 1 - \lambda^{1/(1-\alpha)},$$

the smallest solution in $[0,1]$ to $f(z) = 0$. ρ_e is the probability of extinction of $N(t)$; it coincides with the one obtained in the discrete-time setup. Also,

$$\mathbf{P}_1(\tau_e > t) = \mathbf{P}_1(N(t) > 0) = \left(\lambda + (1-\lambda)e^{-\mu(1-\alpha)t}\right)^{1/(1-\alpha)},$$

giving the tail probability distribution function of the extinction time, with exponential tails conditionally given $\tau_e < \infty$

$$e^{\mu(1-\alpha)t}\mathbf{P}_1(\tau_e > t \mid \tau_e < \infty) \underset{t\to\infty}{\to} C = \frac{(1-\lambda)\lambda^{\alpha/(1-\alpha)}}{\left(1 - \lambda^{1/(1-\alpha)}\right)(1-\alpha)} > 0.$$

Note that from (15)

$$\phi_t(z) = A^{-1}\left(e^{-\mu t}A(z)\right)$$

where

$$A(z) = e^{-\mu\int_0^z \frac{1}{f(x)}dx} = \left(\frac{(1-z)^{1-\alpha} - \lambda}{1-\lambda}\right)^{1/(1-\alpha)}, \quad 0 \leq z < \rho_e$$

and $A^{-1}(\cdot)$ is the inverse of $A(\cdot)$. We have that

$$\widetilde{\phi}_t(z) := \mathbf{E}_1\left(z^{\widetilde{N}(t)}\right) = \frac{\phi_t(z) - \phi_t(0)}{1 - \phi_t(0)}$$

is the pgf of $N(t)$ conditionally given $N(t) \geq 1$, with $\mathbf{P}_1\left(\widetilde{N}(t) = 0\right) = 0$. Again, for all $t > 0$, $\widetilde{\phi}_t(1) = \frac{\phi_t(1) - \phi_t(0)}{1 - \phi_t(0)} = \mathbf{P}_1\left(\widetilde{N}(t) < \infty\right) < \mathbf{P}_1(N(t) < \infty) < 1$.

Remark:

So far, we studied the number of descendants $N(t)$ of a single individual at generation 0. Suppose the initial number of particles is now random with Sibuya$(1-\alpha)$ distribution. Then under the assumption of independence of all the progenies, with $\phi_{1-\alpha}(z) = 1 - (1-z)^{1-\alpha}$ the pgf of $N(0) \geq 1$, the pgf of the full process reads

$$\mathbf{E}_1\left(z^{N(t)}\right) = \phi_{1-\alpha}(\phi_t(z)) = 1 - \left[\lambda\left(1 - e^{-\mu(1-\alpha)t}\right) + e^{-\mu(1-\alpha)t}(1-z)^{1-\alpha}\right],$$

which is a defective Sibuya$\left(1-\alpha, e^{-\mu(1-\alpha)t}\right)$ rv in that $\mathbf{P}_1(N(t) = \infty) = \lambda\left(1 - e^{-\mu(1-\alpha)t}\right) > 0$.

• **Neveu branching process in continuous-time.** Let $\lambda \in (0,1]$ and let $\mu > 0$. One can check that

$$\phi_\lambda(z) = (1-\lambda) + \lambda\left[(1-z)\log(1-z) + z\right]$$

is a pgf with $[z]\,\phi_\lambda(z)=0$. Consider a continuous-time branching process with branching mechanism

$$f(z)=\mu\left((1-\lambda)(1-z)+\lambda(1-z)\log(1-z)\right)=\mu\left(\phi_\lambda(z)-z\right). \tag{16}$$

The rates $\lambda_k>0$ at which one individual gives birth to $k\neq 1$ individuals are $\lambda_k=\left[z^k\right]f(z)$. We get

$$\lambda_0=\mu(1-\lambda)\ \text{ and } \lambda_k=\frac{\mu\lambda}{k(k-1)},\ k\geq 2.$$

Each particle lives a random time with exponential distribution with frequency parameter $\sum_{k\neq 1}\lambda_k=\mu$. At the time of its death, it gives birth to a random number $M\neq 1$ of particles of the same type with law

$$\mathbf{P}(M=k)=\lambda_k/\mu,\ k\neq 1.$$

As before, $\phi_t(z)=\mathbf{E}_1\left(z^{N(t)}\right),\ t\geq 0$, the pgf of the population size $N(t)$ at time t started with a single individual, obeys

$$\partial_t\phi_t(z)=f(\phi_t(z)),\ \phi_0(z)=z,$$

which can be solved explicitly to give

$$\phi_t(z)=1-e^{-\frac{(1-\lambda)}{\lambda}\left(1-e^{-\lambda\mu t}\right)}(1-z)^{e^{-\lambda\mu t}}. \tag{17}$$

It can easily be checked that

$$\phi_t(z)=A^{-1}\left(e^{-\mu t}A(z)\right),$$

where in the Neveu case

$$A(z)=e^{-\mu\int_0^z\frac{1}{f(x)}dx}=\left(1+\frac{\lambda}{1-\lambda}\log(1-z)\right)^{1/\lambda},\ 0\leq z<1-e^{-(1-\lambda)/\lambda}.$$

Therefore $N(t)$ has a Sibuya$\left(\alpha_t:=e^{-\lambda\mu t},\lambda_t:=e^{-\frac{(1-\lambda)}{\lambda}\left(1-e^{-\lambda\mu t}\right)}\right)$ distribution with

$$\begin{aligned}\mathbf{P}_1(N(t)=0)&=1-\lambda_t=:\overline{\lambda}_t\\ \mathbf{P}_1(N(t)=k)&=(-1)^{k-1}\lambda_t\binom{\alpha_t}{k}=\alpha_t\lambda_t\frac{[\overline{\alpha}_t]_{k-1}}{k!},\ k\geq 1.\end{aligned}$$

We conclude that $N(t)$ is ID iff $\lambda_t\leq 1-\alpha_t$ (else $e^{-\frac{(1-\lambda)}{\lambda}\left(1-e^{-\lambda\mu t}\right)}\leq 1-e^{-\lambda\mu t}$) or equivalently if $t>t_0:=-\frac{1}{\lambda\mu}\log(1-x_0)$ where $x_0\in(0,1)$ is defined implicitly by $e^{-\frac{(1-\lambda)}{\lambda}x_0}=x_0$. Therefore, only after time t_0,

$$\phi_t(z)=e^{-\mu_t(1-h_t(z))},\ t>t_0$$

where $\mu_t=-\log(1-\lambda_t)$ and $h_t(z)$ a pgf (with $h_t(0)=0$) which can be identified from the latter equation and (17). The Neveu branching process $N(t)$ with $N(0)=1$ has the same distribution as a (time-inhomogeneous) compound Poisson process:

$$N(t)\overset{d}{=}\sum_{k=1}^{P(\mu_t)}\Delta_k(t). \tag{18}$$

The right-hand-side of (18) is indeed a Poisson-$P(\mu_t)$ sum (with intensity μ_t) of independent jumps $\Delta_\cdot(t)$, each with time-dependent pgf $h_t(z)$.

Note that if $\lambda=1$, $N(t)\geq 1$ and absorption is not possible (explosion has probability 1) and $N(t)$ cannot be ID (although $N(t)-1$ is, [37]). If $\lambda=1$, we have

$\phi_t(z) = 1 - (1-z)^{\alpha_t}$ and, conditionally on non-extinction, $N(t)$ has a Sibuya(α_t) distribution:

$$\mathbf{P}_1(N(t) = k) = (-1)^{k-1} \binom{\alpha_t}{k} = \alpha_t \frac{[\overline{\alpha}_t]_{k-1}}{k!}, k \geq 1.$$

In any case, we have $\phi_t(1) = \mathbf{P}_1(N(t) < \infty) = 1$ and the discrete Neveu process in continuous-time is regular or conservative (finite-time explosion is impossible). When $\lambda \in (0,1)$, we also have

$$\mathbf{P}_1(N(t) = 0) = \phi_t(0) = 1 - e^{-\frac{(1-\lambda)}{\lambda}\left(1-e^{-\lambda\mu t}\right)} \underset{t\to\infty}{\to} \rho_e := 1 - e^{-\frac{(1-\lambda)}{\lambda}},$$

the smallest solution in $[0,1]$ to $f(z) = 0$. ρ_e is the probability of extinction of $N(t)$. Also,

$$\mathbf{P}_1(\tau_e > t) = \mathbf{P}_1(N(t) > 0) = e^{-\frac{(1-\lambda)}{\lambda}\left(1-e^{-\lambda\mu t}\right)} \to 1 - \rho_e,$$

the explosion probability at $\tau_e = \infty$. This gives the tail probability distribution function of the extinction time as one with exponential tails

$$e^{\lambda\mu t}\mathbf{P}_1(\tau_e > t \mid \tau_e < \infty) \underset{t\to\infty}{\to} C = \frac{\lambda}{1-\lambda}\frac{1-\rho_e}{\rho_e} > 0,$$

conditionally given $\tau_e < \infty$. Here again, see [27], Proposition 3.8,

$$e^{-\mu t}\log(1 + N(t)) \underset{t\to\infty}{\overset{a.s.}{\to}} \rho_e \cdot 0 + (1-\rho_e) E, \text{ where } E \overset{d}{\sim} \exp(1).$$

If the initial number of particles is i, we also note from (8) that, with $i, j \geq 1$

$$\mathbf{P}_i(N(t) = j) = \overline{\lambda}_t^i \sum_{k=1}^{i} \binom{i}{k} \left(\alpha_t\lambda_t/\overline{\lambda}_t\right)^k B_{j,k}\left([\overline{\alpha}_t]_{\bullet-1}\right),$$

showing that the full transition probability system of the Neveu model is integrable by quadrature. Furthermore,

$$g_{i,j}(q) := \int_0^\infty e^{-qt}\mathbf{P}_i(N(t) = j)\, dt$$

is the resolvent of $N(t)$. If $j = 0$, $B_{j,k} \neq 0$ only if $k = 0$, so that

$$\mathbf{P}_i(N(t) = 0) = \overline{\lambda}_t^i = \left(1 - e^{-\frac{(1-\lambda)}{\lambda}\left(1-e^{-\lambda\mu t}\right)}\right)^i = \mathbf{P}_i(\tau_e \leq t) =: \mathbf{P}(\tau_{i,e} \leq t).$$

$\tau_{i,e}$ is the extinction time given i particles at the origin of times. This is the continuous-time version of the discrete-time hitting time result (9).

If now the initial number of particles is random and Poisson(1) distributed, the pgf of $N(t)$ becomes

$$\mathbf{E}\left(z^{N(t)}\right) = \mathbf{E}\mathbf{E}_{P(1)}\left(z^{N(t)}\right) = e^{-(1-\phi_t(z))} = e^{-\lambda_t(1-z)^{\alpha_t}} = e^{-\lambda_t(1-[1-(1-z)^{\alpha_t}])}.$$

Alternatively, $N(t)$ is discrete-stable(α_t, λ_t) distributed. Therefore,

$$N(t) \overset{d}{=} \sum_{k=1}^{P(\lambda_t)} \Delta_k(t),$$

a Poisson-$P(\lambda_t)$ sum (with intensity $\lambda_t = e^{-\frac{(1-\lambda)}{\lambda}\left(1-e^{-\lambda\mu t}\right)}$) of independent jumps $\Delta_{\cdot}(t)$ each with unscaled Sibuya($\alpha_t = e^{-\mu t}$) distribution.

Remark: Recall that a Bienaymé-Galton-Watson processes with generating functions $\phi_n(z)$ is called embeddable, if there is a continuous-time $t > 0$ semi-group of probability generating functions $\phi_t(z)$ obeying $\phi_{t+s}(z) = \phi_t(\phi_s(z))$ such that $\phi_n(z) = \phi_{t=n}(z)$, $n = 1, 2, ...$ Not every Bienyamé-Galton-Watson process is embeddable. However, the discrete Neveu process clearly is in the continuous-time Neveu process. Calling λ_c and λ_d the $(0,1)-$valued λs appearing respectively in the continuous and discrete-time Neveu model, the embedding goes through the mapping σ in parameter space and its inverse σ^{-1}

$$\begin{pmatrix}\alpha\\ \lambda_d\end{pmatrix} \overset{\sigma}{\to} \begin{pmatrix}\mu = \frac{-\log\alpha}{1-\alpha}\left(1-\alpha-\log\lambda_d\right)\\ \lambda_c = \frac{1-\alpha}{1-\alpha-\log\lambda_d}\end{pmatrix} \overset{\sigma^{-1}}{\to} \begin{pmatrix}\alpha = e^{-\mu\lambda_c}\\ \lambda_d = e^{-\left(\frac{1-\lambda_c}{\lambda_c}\right)\left(1-e^{-\mu\lambda_c}\right)}\end{pmatrix}.$$

• $\alpha-$stable continuous-time branching process with branching mechanism having finite mean but infinite variance: $\boldsymbol{\alpha} \in (1,2)$.

Let $m, \mu > 0$. Consider a continuous-time Galton-Watson process whose branching mechanism is $f(z)$ with $f(z)/\mu =: \phi(z) - z = (m-1)(z-1) + C(1-z)^{\alpha}$ with $\alpha \in (1,2)$ and $m/\alpha \geq C > m-1$. This model with three parameters (m, C, α) was considered in [1] and [3]. We have $\phi(z) = \mathbf{E}\left(z^M\right)$ with $\phi'(1) = \mathbf{E}(M) = m > 0$ and $\phi''(1) = \infty$ (finite mean number m of offspring per individual at birth but infinite variance). It can be checked that, with $N(0) = 1$ and $\phi_t(z) := \mathbf{E}_1\left(z^{N(t)}\right)$,

$$\phi_t(z) = \begin{cases} 1 - \left[\frac{C}{m-1}\left(1-e^{-\mu(\alpha-1)(m-1)t}\right) + e^{-\mu(\alpha-1)(m-1)t}(1-z)^{-(\alpha-1)}\right]^{-1/(\alpha-1)}, & m \neq 1\\ 1 - \left[(\alpha-1)C\mu t + (1-z)^{-(\alpha-1)}\right]^{-1/(\alpha-1)}, & \text{if } m = 1 \text{ (critical case).}\end{cases}$$

This follows from the fact that $\phi_t(z) = A^{-1}\left(e^{-\mu t}A(z)\right)$ with

$$\begin{aligned} A(z) &= e^{-\mu\int_0^z \frac{dz'}{f(z')}} = e^{-\frac{1}{1-m}I(z)} = \left(\frac{1-m+C(1-z)^{\alpha-1}}{(1-m+C)(1-z)^{\alpha-1}}\right)^{1/[(\alpha-1)(m-1)]},\\ I(z) &= \int_0^z \frac{dz'}{(1-z') + \frac{C}{1-m}(1-z')^{\alpha}} = \log\left(\frac{1-m+C(1-z)^{\alpha-1}}{(1-m+C)(1-z)^{\alpha-1}}\right)^{1/(\alpha-1)}. \end{aligned}$$

In particular therefore,

$$1 - \phi_t(0) = \mathbf{P}_1(N(t) > 0) = \mathbf{P}_1(\tau_e > t), \text{ with}$$

$$\mathbf{P}_1(\tau_e > t) = \begin{cases} \left[\frac{C}{m-1}\left(1-e^{-\mu(\alpha-1)(m-1)t}\right) + e^{-\mu(\alpha-1)(m-1)t}\right]^{-1/(\alpha-1)}, & \text{if } m \neq 1\\ \left(1+(\alpha-1)C\mu t\right)^{-1/(\alpha-1)}, & \text{if } m = 1 \text{ (critical case),}\end{cases}$$

with tail behaviors (for some computable constants $C_1, C_2 > 0$)

$$\begin{aligned} e^{\mu(1-m)t}\mathbf{P}_1(\tau_e > t) \underset{t\to\infty}{\to} C_1, \text{ if } m &< 1\\ t^{1/(\alpha-1)}\mathbf{P}_1(\tau_e > t) \underset{t\to\infty}{\to} C_2, \text{ if } m &= 1. \end{aligned}$$

If $m>1$ (supercritical case), $\rho_e=1-((m-1)/C)^{1/(\alpha-1)}$ whereas $\rho_e=1$ if $m\leq 1$ (sub- and critical cases). We also have:

$$\lim_{z\to 1}\phi_t(z)=\mathbf{P}_1(N(t)<\infty)=1 \text{ for all } m>0,$$

and the process is always regular (no explosion in finite time). Furthermore, for large t,

$$\mathbf{E}_1(N(t))\sim\begin{cases} e^{\mu(m-1)t} \text{ if } m\neq 1\\ 1 \text{ if } m=1\end{cases},$$

and the mean value of $N(t)$ exists.

- In the subcritical case with $m<1$, $\mathbf{P}_1(N(t)=k\mid N(t)>0)\underset{t\to\infty}{\to}\pi_k$ where the π_ks, $k\geq 1$, constitute a limiting system of probabilities of some discrete rv W, with proper probability generating function $\phi_W(z)=\mathbf{E}\left(z^W\right)=\sum_{k\geq 1}\pi_k z^k$, here explicit with, [26],

$$\phi_W(z)=1-\exp\left\{-\mu(1-m)\int_0^z\frac{dz'}{f(z')}\right\}=1-\left(\frac{(1-m+C)(1-z)^{\alpha-1}}{1-m+C(1-z)^{\alpha-1}}\right)^{\frac{1}{\alpha-1}}.$$

In other words, $N(t)\mid N(t)>0\underset{t\to\infty}{\overset{d}{\to}}W$. The mean of the limiting rv W exists and takes the value $(1+C/(1-m))^{1/(\alpha-1)}>1$.

- In the supercritical case with $m>1$, $e^{-\mu(m-1)t}N(t)$ is a martingale, with $e^{-\mu(m-1)t}N(t)\underset{t\to\infty}{\overset{d}{\to}}W$, where the LSt of W, $\varphi(s):=\mathbf{E}\left(e^{-sW}\right)$, is the solution of the differential equation $\varphi'(s)=f(\varphi(s))/(\mu(m-1)s)$, $\varphi(0)=1$, [26]. Using the above expression of $I(z)$, we find explicitly

$$\varphi(s)=1-\left(\frac{s^{\alpha-1}}{1-\frac{C}{m-1}(1-s^{\alpha-1})}\right)^{1/(\alpha-1)}.$$

Note $\varphi(s)\underset{s\to\infty}{\to}\rho_e=1-((m-1)/C)^{1/(\alpha-1)}$, expressing that W has an atom at 0 with mass ρ_e.

- In the critical case with $m=1$, it also holds that given $N(0)=1$,

$$\mathbf{E}_1(N(t)\mid N(t)>0)=\partial_z\left(\frac{\phi_t(z)-\phi_t(0)}{1-\phi_t(0)}\right)\Big|_{z=1}=\frac{1}{1-\phi_t(0)}=(1+(\alpha-1)C\mu t)^{1/(\alpha-1)},$$

displaying algebraic superlinear growth in time, with exponent $1/(\alpha-1)>1$ and,

$$\frac{N(t)\mid(N(t)>0,N(0)=1)}{\mathbf{E}_1(N(t)\mid N(t)>0)}\underset{t\to\infty}{\overset{d}{\to}}E\overset{d}{\sim}\exp(1).$$

In the critical case with $m=1$, the process $N(t,N_0=1)$ started at $N_0=1$ is discrete-self-similar with Hurst index $H=-1/(\alpha-1)<0$. For all $c\in(0,1)$ indeed, the one-dimensional marginals obey

$$N\left(ct,c^H\circ N_0\right)\overset{d}{=}c^H\circ N(t,N_0).$$

The continuous-time Neveu model with parameters (α_t,λ) is obtained from the $(m,C,\alpha)-$branching model while putting $C=\frac{\lambda}{\alpha-1}$, $m=C\alpha$, $\lambda\in(0,1)$ and taking the limit $\alpha \to 1^+$ (in which $m \to \infty$, $C/(m-1) \sim 1 + \frac{1-\lambda}{\lambda}(\alpha-1) \to 1$ and $(\alpha-1)(m-1) \to \lambda$). Indeed, the limiting branching mechanism in this case is the one of Neveu (16)

$$\begin{aligned} \lim_{\alpha \downarrow 1} f(z)/\mu &= \lim_{\alpha \downarrow 1} \frac{1}{\alpha-1}\left[(\lambda-(1-\lambda)(\alpha-1))(z-1)+\lambda(1-z)^{\alpha}\right] \\ &= (1-\lambda)(1-z)+\lambda(1-z)\log(1-z). \end{aligned}$$

The limiting Neveu model with $\alpha=1$ is still regular. It is critical in the sense that if the branching mechanism has tail index $\alpha<1$ (as in the branching process with Sibuya(α,λ) offspring distribution), then the process no longer is regular, rather it is defective. And it is regular if $\alpha \geq 1$.

• **Pgf versus log-Laplace transform (LLt) and associated random walk (continuous-time).** Consider any of the three continuous-time branching processes just considered and let M be their offspring number per capita at birth. Let $\psi_t(p) := -\log \phi_t(e^{-p})$ and $\psi(p) := 1 - \mathbf{E}\left(e^{-p(M-1)}\right) = 1 - e^p \phi(e^{-p})$, with $\phi(z) = \mathbf{E}\left(z^M\right)$. The differential equation (15) also reads in terms of LLts

$$\partial_t \psi_t(p) = \psi(\psi_t(p)),\ n \geq 0,\ \psi_0(p) = p.$$

Let $\widetilde{M} := M-1$ taking values in $\{-1,0,1,2,...\}$. With $P(t)$ the number of events of a standard Poisson point process over $[0,t]$, consider the (skip-free to the left) compound Poisson process $S(t) = \sum_{n=1}^{P(t)} \widetilde{M}_n$, $S_0=1$, with the sequence $\left\{\widetilde{M}_n\right\}$ iid all distributed like $\widetilde{M}$. Suppose $\{S(t)\}$ is stopped when it first hits 0 if ever. Then, as a process with independent and stationary increments, $\mathbf{E}\left(e^{-p(S(t)-1)}\right) = e^{-t\psi(p)}$. With $\int_0^t N(s)\,ds$ the integrated area under the profile of $\{N(.)\}$ started at 1,

$$N(t) \overset{d}{=} S\left(\overline{N}(t)\right). \tag{19}$$

Therefore $N(t)$ (also stopped when it first hits 0 if ever) is a time-changed version of $S(t)$. This constitutes the continuous-time analog to the similar result discussed above in discrete-time, [38]. Given $S_0=i$, defining $\tau_{i,0} = \inf(t>0 : S(t)=0 \mid S(0)=i)$, it holds that $\tau_{i,0} \overset{d}{=} \int_0^\infty N(s)\,ds \mid N(0)=i$ and

$$\mathbf{E}\left(e^{-q\tau_{i,0}}\right) = e^{-i\psi^{-1}(-q)} =: e^{-ia(q)},\ q \geq 0.$$

Introducing $h(q) = e^{-a(q)}$, $h(q)$ obeys the functional equation

$$h(q) = \frac{1}{1+q}\phi(h(q)),$$

which is the continuous-time version of (11).

Any of the three continuous-time branching processes just considered $N(t)$ are continuous-time Markov chains over the set of integers $\mathbb{N}_0 = \{0,1,2,...\}$, whose transition rate matrix is $Q = [Q_{i,j}]$ with $Q_{i,j} = \lambda_{i,j} = \left[z^j\right] f(z)^i$, $i,j \in \mathbb{N}_0$, $i \neq j$ and $Q_{i,i} = \lambda_{i,i} = -\sum_{j \neq i} \lambda_{i,j}$. This Markov chain is possibly absorbed in finite time at $\{0\}$ and possibly also at $\{\infty\}$ for the (not regular) $\alpha-$stable branching model. When $\{N(t)\}$ is regular as in the last two examples, let $\tau_{i,e} = \inf\{t>0 : N(t)=0 \mid N(0)=i\}$, $i \geq 1$ and $\tau_{i,\overline{e}} = \inf\{t>0 : N(t)=\infty \mid N(0)=i\}$, $i \geq 1$, with $\tau_{i,\overline{e}} = \infty$ in case of explosion in infinite time. Let $\overline{Q}$ be obtained from

Q while removing its first row and column (corresponding to the absorbing state $\{0\}$). Let $\mathbf{h} \equiv (h(1), h(2), ...)'$ solve $\overline{Q}\mathbf{h} = \mathbf{0}$ with boundary condition $h(1) = 1$. The column sequence $\mathbf{h}$ is called the scale sequence of $\{N(t)\}$ and again

$$h(i) = \frac{1 - \rho_e^i}{1 - \rho_e}, \; i \geq 1.$$

It is such that $h(N(t \wedge \tau_{i,e}))$ is a martingale, [51]. We have

$$\mathbf{P}(\tau_{i,e} < \tau_{i,\overline{e}}) = \frac{h(\infty) - h(i)}{h(\infty)} = \rho_e^i.$$

When dealing with the $\alpha-$stable branching model, there is a positive probability that $N(t)$ hits ∞ in finite time. So $\tau_{i,\overline{e}} < \infty$ with positive probability. We have

$$\mathbf{P}(\tau_{i,\overline{e}} \leq t) = \mathbf{P}_i(N_t = \infty) = \left(\lambda\left(1 - e^{-\mu(1-\alpha)t}\right)\right)^{i/(1-\alpha)} > 0.$$

With $\tau_i := \tau_{i,e} \wedge \tau_{i,\overline{e}}$, $h(N(t \wedge \tau_i))$ is a martingale and $h(\infty) = (1 - \rho_e)^{-1} < \infty$ with again

$$\mathbf{P}(\tau_{i,e} < \tau_{i,\overline{e}}) = \frac{h(\infty) - h(i)}{h(\infty)} = \rho_e^i \text{ and } \rho_e = 1 - e^{-\frac{(1-\lambda)}{\lambda}}.$$

1.4. **Mittag-Leffler rvs and related processes.** We describe here Mittag-Leffler rvs which can be of two different types and we also discuss some related processes. These were briefly encountered previously.

• **Type-1 Mittag-Leffler, [53].** Consider the pgf of a discrete Mittag-Leffler(α, μ) rv obtained as the random sum

$$M_{\alpha,\mu} = \sum_{l=0}^{G_\mu} X_\alpha(l)$$

where $(X_\alpha(l))_{l\geq 0}$ is an iid sequence of unscaled Sibuya(α) rvs and G_μ is geometric with mean μ, independent of the latter sequence. Thus, $\phi_{G_\mu}(z) = 1/(1 + \mu(1-z))$, and

$$\phi_{M_{\alpha,\mu}}(z) = \mathbf{E}\left(z^{M_{\alpha,\mu}}\right) = \phi_{G_\mu}\left(\phi_{X_\alpha}(z)\right) = \left(1 + \mu(1-z)^\alpha\right)^{-1}.$$

$M_{\alpha,\mu}$ also has infinite mean. Clearly, $M_{\alpha,\mu}$, as a geometric-stable rv, is ID (compound Poisson) and it is even discrete self-decomposable because for all $c \in (0,1)$,

$$\frac{\phi_{M_{\alpha,\mu}}(z)}{\phi_{M_{\alpha,\mu}}(1 - c(1-z))} = c^\alpha + (1 - c^\alpha)\left(1 + \mu(1-z)^\alpha\right)^{-1}$$

is the pgf of a mixture of discrete rvs, one of which is degenerate at 0, [54]. We have

$$\begin{aligned}
\left(1 - \lambda(1-z)^\alpha\right)\left(1 + \lambda(1-z)^\alpha\right) &= \left(1 - \lambda^2(1-z)^{2\alpha}\right) \text{ or} \\
\left(1 - \lambda^2(1-z)^{2\alpha}\right)\left(1 + \lambda(1-z)^\alpha\right)^{-1} &= 1 - \lambda(1-z)^\alpha.
\end{aligned}$$

If $\alpha < 1/2$, $\phi_{2\alpha,\lambda^2}(z) = 1 - \lambda^2 (1-z)^{2\alpha}$ is the pgf of a scaled Sibuya$\left(2\alpha, \lambda^2\right)$ rv and $\left(1 + \lambda (1-z)^{\alpha}\right)^{-1}$ the pgf of discrete Mittag-Leffler rv $M_{\alpha,\mu}$ with $\mu = \lambda < 1$. Thus if $\alpha < 1/2$,

$$\phi_{\alpha,\lambda}(z) = \phi_{2\alpha,\lambda^2}(z)\, \phi_{M_{\alpha,\lambda}}(z)$$

and $\phi_{\alpha,\lambda}(z)$ factorizes.

• **A (discrete-) self-similar Type-1 Mittag-Leffler process.** Consider a counting process $\overline{N}(t)$, $t \geq 0$, so taking integral values, whose marginal pgf is

$$\phi_t(z) := \mathbf{E}_1\left(z^{\overline{N}(t)}\right) = \left(1 + t^{\alpha H} (1-z)^{\alpha}\right)^{-1}.$$

This is the pgf of a type-1 discrete Mittag-Leffler distribution with scale parameter $\mu = t^{\alpha H}$ and tail exponent α ($\alpha \in (0,1)$). For each $t > 0$ and each $c \in (0,1)$, we have

$$\left\{\overline{N}(ct)\right\} \overset{d}{=} \left\{c^H \circ \overline{N}(t)\right\},$$

so that $\overline{N}(t)$ is discrete-self-similar (in the latter sense), with Hurst exponent H. If $H = \alpha$, this process (call it $\overline{N}_{\alpha}(t)$) is self-similar with Hurst exponent α. Its pgf is thus $\phi_t(z) = \left(1 + t^{\alpha^2} (1-z)^{\alpha}\right)^{-1}$, with $\overline{N}(t) \to \infty$ as $t \to \infty$ with probability 1.

The process $\overline{N}_{\alpha}(t)$ is (time-inhomogeneous) Markovian. Indeed, observing $\phi_1(z) = \left(1 + (1-z)^{\alpha}\right)^{-1}$, we have

$$\partial_t \phi_t(z) = \frac{\alpha^2}{t}\left(\phi_t^2(z) - \phi_t(z)\right),\ t \geq 1.$$

Define the (deterministic) time-change $\tau_t = \alpha^2 \log t$ and its inverse $t_\tau = \exp\left(\tau/\alpha^2\right)$. Introducing $\widetilde{\phi}_\tau(z) = \phi_{t_\tau}(z)$, with $\widetilde{\phi}_0(z) = \phi_1(z)$, we thus obtain

$$\partial_\tau \widetilde{\phi}_\tau(z) = \widetilde{\phi}_\tau^2(z) - \widetilde{\phi}_\tau(z),\ \tau \geq 0 \text{ and } \phi_t(z) = \widetilde{\phi}_{\tau_t}(z).$$

$\overline{N}_{\alpha}(t) = \widetilde{N}_{\alpha}\left(\alpha^2 \log t\right)$ is thus an appropriate time-changed version of the Markov process $\widetilde{N}_{\alpha}(\tau)$ with $\mathbf{E}\left(z^{\widetilde{N}_{\alpha}(\tau)}\right) = \widetilde{\phi}_\tau(z) = \left(1 + e^{\tau}(1-z)^{\alpha}\right)^{-1}$. The process $\widetilde{N}_{\alpha}(\cdot)$ can be interpreted as some variation of a binary branching process: start with $\widetilde{N}_{\alpha}(0)$ particles, with type-1 Mittag-Leffler(α) distribution; after some random time with exponential(1) distribution, split the initial population into two parts, each of independent size drawn again from Mittag-Leffler(α) distribution. Iterate the process for the two independent subfamilies and then for the subsequent families. $\widetilde{N}_{\alpha}(\tau)$ is the total population size at time $\tau \geq 0$.

Applying the (discrete-) Lamperti transformation [39] L to $\left\{\overline{N}(t)\right\}$, $t \geq 0$, namely: $\left\{\overline{N}(t)\right\} \overset{L}{\to} \left\{e^{-\alpha t} \circ \overline{N}\left(e^t\right)\right\} =: \left\{\overline{\mathfrak{N}}(t)\right\}$, we end up with a (strictly) stationary process $\left\{\overline{\mathfrak{N}}(t)\right\}$, with $\overline{\mathfrak{N}}(t) \overset{d}{=} \overline{\mathfrak{N}}(0)$ for all $t \geq 0$, and

$$\mathbf{E}\left(z^{\overline{\mathfrak{N}}(0)}\right) = \frac{1}{1 + (1-z)^{\alpha}},$$

emphasizing the link between self-similar and stationary processes.

Remark: The case $H = 1/\alpha$ is interesting as well, thus with pgf for $\left\{\overline{N}(t)\right\}$: $\phi_t(z) = \left(1 + t(1-z)^{\alpha}\right)^{-1}$. Everything works the same except that the time changes involved here are just $\tau_t = \log t$ and its inverse $t_\tau = \exp \tau$, independent of α.

• **Type-2 Mittag-Leffler rv.** There exists a type-2 (discrete-) Mittag-Leffler distribution, defined as follows. Let $\alpha \in (0,1)$ and

$$E_\alpha(x) := \sum_{m\geq 0} \frac{x^m}{\Gamma(\alpha m+1)},$$

the standard Mittag-Leffler function, extending the exponential function obtained as $\alpha \to 1$, [46]. With $\mu > 0$, define $N_{\alpha,\mu}$ to be a discrete type-2 Mittag-Leffler rv if it has the pgf

$$\phi_{N_{\alpha,\mu}}(z) := \mathbf{E}\left(z^{N_{\alpha,\mu}}\right) = E_\alpha\left(-\mu\left(1-z\right)\right).$$

The falling factorial moments of $N_{\alpha,\mu}$ are

$$\mathbf{E}\left(N_{\alpha,\mu}\right)_m = m!\left[(z-1)^m\right]\phi_{N_{\alpha,\mu}}(z) = \mu^m \frac{\Gamma(m+1)}{\Gamma(\alpha m+1)},\ m \geq 1$$

so $N_{\alpha,\mu}$ has all its moments finite. In particular, $\mathbf{E}\left(N_{\alpha,\mu}\right) = \mu/\Gamma(\alpha+1) > \mu$, $\mathbf{E}\left(N_{\alpha,\mu}\right)_2 = \mathbf{E}\left[\left(N_{\alpha,\mu}\right)\left(N_{\alpha,\mu}-1\right)\right] = 2\mu^2/\Gamma(2\alpha+1)$,....

• **Linnik rv** [29]**.** Let $\alpha \in (0,1)$ and $\beta, \mu > 0$. Consider the discrete Linnik(α,β,μ) random variable, say $L_{\alpha,\beta,\mu}$, with pgf

$$\phi_{L_{\alpha,\beta,\mu}}(z) = \mathbf{E}\left(z^{L_{\alpha,\beta,\mu}}\right) = \left(1+\mu\left(1-z\right)^\alpha/\beta\right)^{-\beta}.$$

We have $M_{\alpha,\mu} = L_{\alpha,1,\mu}$ and $L_{\alpha,\beta,\mu} \stackrel{d}{\to} S_{\mu,\alpha}$ when $\beta \to \infty$. Such discrete Linnik distributions are obtained as gamma$(\beta, \mu/\beta)$ mixtures of the scale parameter of discrete stable(μ,α) rvs $S_{\mu,\alpha}$, namely

$$\phi_{L_{\alpha,\beta,\mu}}(z) = \mathbf{E}\left(z^{L_{\alpha,\beta,\mu}}\right) = \mathbf{E}\left(e^{-\Gamma_{\beta,\mu/\beta}(1-z)^\alpha}\right),$$

where $\Gamma_{\beta,\mu/\beta}$ is a positive Gamma$(\beta,\mu/\beta)$ rv with density

$$f(x) = \frac{(\mu/\beta)^{-\beta}}{\Gamma(\beta)} x^{\beta-1} e^{-\beta x/\mu}.$$

So $\left(1+\mu\left(1-z\right)^\alpha/\beta\right)^{-\beta}$ is indeed a pgf. We note that $L_{\alpha,\beta,\mu}$ with $\beta = 1$ (which is a Mittag-Leffler rv $M_{\alpha,\mu}$) is also obtained as an exponential(μ) mixture of the scale parameter of a discrete stable(μ,α) rv $S_{\mu,\alpha}$.

Suppose $\lambda \leq 1-\alpha < 1$. Then $\phi_{\alpha,\beta,\lambda}(z) = \left(1-\lambda\left(1-z\right)^\alpha\right)^\beta$ is a pgf for all $\beta > 0$. We have the identity

$$\begin{aligned} \left(1-\lambda(1-z)^\alpha\right)^\beta \left(1+\lambda(1-z)^\alpha\right)^\beta &= \left(1-\lambda^2(1-z)^{2\alpha}\right)^\beta \text{ or} \\ \left(1-\lambda^2(1-z)^{2\alpha}\right)^\beta \left(1+\lambda(1-z)^\alpha\right)^{-\beta} &= \left(1-\lambda(1-z)^\alpha\right)^\beta. \end{aligned}$$

If $\alpha < 1/2$ and $\lambda^2 \leq 1-2\alpha$, $\phi_{2\alpha,\beta,\lambda^2}(z) = \left(1-\lambda^2\left(1-z\right)^{2\alpha}\right)^\beta$ is the pgf of a scaled Sibuya$\left(2\alpha,\beta,\lambda^2\right)$ rv and $\phi_{L_{\alpha,\beta,\lambda\beta}}(z) := \left(1+\lambda\left(1-z\right)^\alpha\right)^{-\beta}$ is the pgf of a Linnik rv with parameters $\alpha, \beta, \mu = \lambda\beta$.

Thus if $\alpha < 1/2$ and $\lambda^2 \leq 1 - 2\alpha$, $\phi_{\alpha,\beta,\lambda}(z)$ is also a pgf (because $\lambda \leq \sqrt{1-2\alpha} < 1 - \alpha$) and

$$\phi_{\alpha,\beta,\lambda}(z) = \phi_{2\alpha,\beta,\lambda^2}(z)\,\phi_{L_{\alpha,\beta,\lambda/\beta}}(z).$$

Thus $\phi_{\alpha,\beta,\lambda}(z)$ factorizes.

1.5. **Discrete self-decomposability and stability and related stochastic processes.** We here briefly address the notions of discrete self-decomposability and stability which have been met previously.

• **Discrete additive self-decomposability.** Let now $X \geq 0$ be an integer-valued random variable. There exists a discrete version of the notion of self-decomposability [66]. Some accounts on the sub-class of discrete-stable random variables may also be found here and in [12].

The pgf $\phi(z) := \mathbf{E}z^X$ is the one of a discrete self-decomposable variable X if for any $c \in (0,1)$, there is a pgf $\phi_c(z)$ (depending on c) such that

$$\phi(z) = \phi(1 - c(1-z)) \cdot \phi_c(z)$$

This is the standard (discrete) version of self-decomposability of probability distributions on the integers, through a functional equation. We then have the characterization property:

It follows from the definition of self-decomposable distributions that if $\phi(z)$ is the pgf of the random variable X, then X can be additively decomposed as

$$X \overset{d}{=} c \circ X' + X_c$$

where the c-thinned random variable $c \circ X$, for $c \in (0,1]$, is defined above. X and X' have the same distribution and $c \circ X'$ is independent of the remaining random variable X_c whose pgf is $\phi_c(z)$.

Observing that for any two real numbers c_1 and c_2 of $(0,1]$, $c_1 \circ (c_2 \circ X) = (c_1 \cdot c_2) \circ X$, and that the following convergence in law to zero holds

$$c^n \circ X \underset{n\uparrow+\infty}{\overset{d}{\to}} 0,$$

we get, iterating the above decomposition

$$\sum_{m=0}^{n} X_m \underset{n\uparrow+\infty}{\overset{d}{\to}} X,$$

where $X_m \overset{d}{=} c^m \circ X_c$ are independent random variables with pgfs: $\phi_{X_m}(z) = \phi_{X_c}(1 - c^m(1-z))$. $\diamond$

Discrete self-decomposable random variables are thus obtained as limits in law for sums of independent scaled discrete random variables.

A slightly different way to see this is as follows. Consider the discrete-time integral-valued Ornstein-Uhlenbeck process

$$X(n+1) = c \circ X(n) + X_c(n+1),\ X(0) \text{ random},$$

where $(X_c(n); n \geq 1)$ is an iid driving sequence, each distributed like X_c. Then $X(n)$ is a discrete perpetuity [70], clearly with

$$X(n) = c^n \circ X(0) + \sum_{m=1}^{n} c^{n-m} \circ X_c(m) \overset{d}{\to} X \overset{d}{=} \sum_{m=0}^{\infty} X_m \text{ as } n \to \infty.$$

In such models typically, a population whose fate is to die out predictably at shrinking rate c is regenerated by the incoming of immigrants in random number.
The following representation result is also known to hold true [66]:

The random variable X is discrete self-decomposable iff, with $R(z)$ the canonical measure, defined through

$$\phi(z) = e^{\int_1^z R(u)du},$$

the function $h(z) := 1 - (1-z) R(z)$ defines a pgf such that $h(0) = 0$. As a consequence, X is discrete SD iff its pgf is of the form

$$\phi(z) = e^{\int_1^z \frac{1-h(u)}{1-u} du}.$$

This means that the Taylor coefficients of $R(z)$, say $(r_k,\, k \geq 1)$, constitute a non-increasing sequence of k. As a result, the associated probability system $\mathbf{P}(X = k) := p_k,\ k \geq 0$ is unimodal, with mode at the origin iff $r_0 = \frac{p_1}{p_0} \leq 1$, [66]. The self-decomposable (SD) subclass of infinitely divisible (ID) distributions therefore focuses on unimodal distributions, with mode possibly at the origin.

• **Self-decomposable rvs and pure-death branching processes with immigration,** [69]**.** Consider a continuous-time homogeneous compound Poisson process $P_\gamma(t),\ t \geq 0,\ P_\gamma(0) = 0$, so with pgf

$$\mathbf{E}_0\left(z^{P_\gamma(t)}\right) = \exp -\gamma t\left(1 - h(z)\right),$$

where $h(z)$ (with $h(0) = 0$) is the pgf of the sizes of the batches arriving at the jump times of $P_\gamma(t)$ having rate $\gamma > 0$. Let now

$$\phi_t(z) = 1 - e^{-t}(1-z),$$

be the pgf of a pure-death branching process started with one particle at $t = 0$ (more general subcritical branching processes could be considered as well). The lifetime of the initial particle is thus larger (smaller) than t with probability e^{-t} (respectively $1-e^{-t}$). Let X_t with $X_0 = 0$ be a random process counting the current size of some population for which a random number of individuals (determined by $h(z)$) immigrate at the jump times of $P_\gamma(t)$, each of which being independently and immediately subject to the latter pure death process. We thus have

$$\Phi_t(z) := \mathbf{E}\left(z^{X_t}\right) = \exp -\gamma \int_0^t \left(1 - h\left(\phi_s(z)\right)\right) ds,\ \Phi_0(z) = 1,$$

with $\Phi_t(0) = \mathbf{P}(X_t = 0) = \exp -\gamma \int_0^t \left(1 - h\left(1 - e^{-s}\right)\right) ds$, the probability that the population is extinct at t. As $t \to \infty$,

$$\Phi_t(z) \to \Phi_\infty(z) = e^{-\gamma \int_0^\infty \left(1 - h\left(1 - e^{-s}(1-z)\right)\right) ds} = e^{-\gamma \int_z^1 \frac{1-h(u)}{1-u} du}.$$

So, $X := X_\infty$, as the limiting population size of this pure-death process with immigration, is a self-decomposable rv, [69]. With $X \overset{d}{=} c \circ X' + X_c$ defining the rv X_c, we have that

$$\phi_{X_c}(z) = \frac{\Phi_\infty(z)}{\Phi_\infty(1 - c(1 - z))} = e^{-\gamma \int_z^{1-c(1-z)} \frac{1-h(u)}{1-u} du} \text{ is a pgf.}$$

In such models typically, a subcritical population whose fate is to die out (along the stochastic pure-death process striking each individual alive) is regenerated by the incoming of immigrants at random times and in random number.
Example: Let us now identify the pgf $h(z)$ (with $h(0) = 0$) of the number of immigrants leading to a type-1 Mittag-Leffler(α, μ) distribution for X_∞, known to be discrete self-decomposable. For some $\gamma > 0$, we must solve

$$\Phi_\infty(z) = e^{\gamma \int_1^z \frac{1-h(u)}{1-u} du} = \left(1 + \mu(1 - z)^\alpha\right)^{-1} = \phi_{M_{\alpha,\mu}}(z),$$

and we find $\gamma = \alpha\mu/(1 + \mu)$ and

$$h(z) = \frac{1 - (1 - z)^\alpha}{1 + \mu(1 - z)^\alpha}.$$

The function $h(z)$ is thus the product of $1 - (1 - z)^\alpha$ (the pgf of an unscaled Sibuya(α) rv) times $\left(1 + \mu(1 - z)^\alpha\right)^{-1}$ (the pgf of a type-1 Mittag-Leffler(α, μ) rv). It is indeed a pgf, the one of the sizes of the immigrant incoming population.

Both in discrete or continuous time, the occurrence of a self-decomposable limit law is related to a competition between a mechanism which tends to shrink the population size at a constant rate, against another input mechanism which tends to have it increased.

• **Discrete stability,** [66]**.** Let X be a self-decomposable rv so that X can be additively decomposed as $X \overset{d}{=} c_1 \circ X' + R$. Suppose the remaining term R itself is obtained as $R \overset{d}{=} c_2 \circ X''$ where X'' is independent of X', both distributed like X $(c_1, c_2 \in (0, 1))$. Then, $\phi(z) = \mathbf{E}\left(z^X\right)$ obeys,

$$\phi(z) = \phi(1 - c_1(1 - z)) \cdot \phi(1 - c_2(1 - z)),$$

whose solution is $\phi(z) = e^{-\mu(1-z)^\alpha}$, $(\mu > 0)$ iff the following structural equation holds: $c_1^\alpha + c_2^\alpha = 1$, $\alpha \in (0, 1)$. This is the pgf of a discrete-stable(α, μ) rv. The process $X(t)$ with pgf

$$\phi_t(z) = \mathbf{E}\left(z^{X(t)}\right) = e^{-\mu t(1-z)^\alpha},$$

is a compound Poisson process which is discrete-self-similar with Hurst index $H = 1/\alpha$.

Remark: Discrete-stable rvs are in particular self-decomposable. The pgf $h(z)$ (with $h(0) = 0$) of the number of immigrants leading to a discrete-stable(α, μ) distribution for X_∞ in the latter branching with immigration construction is: $h(z) = 1 - (1 - z)^\alpha$, $\gamma = \alpha\mu$. It is an unscaled Sibuya(α) rv.

2. Mittag-Leffler distributions in the continuum

Like in the discrete setting, there are two types of Mittag-Leffler rvs taking values now in $\mathbb{R}_{+}$. They are related to other famous rvs such as stable or Weibull rvs.

• **Type 1 Mittag-Leffler rv in the continuum**. Let $T>0$ be a positive rv and $\alpha \in (0,1)$. It is a type-1 Mittag-Leffler rv if it has the cumulative probability distribution function, say cpdf,

$$\mathbf{P}\left(T>t\right)=E_{\alpha}\left(-t^{\alpha}\right).$$

We have

$$\phi_{T}\left(p\right):=\mathbf{E}\exp\left(-pT\right)=\frac{1}{1+p^{\alpha}}\text{ and }\mathbf{E}\left(T^{q}\right)=\frac{\Gamma\left(1+q/\alpha\right)\Gamma\left(1-q/\alpha\right)}{\Gamma\left(1-q\right)},$$

respectively the Laplace-Stieltjes transform (LSt) and the moment generating-function (with $q<\alpha$) of T. Type-1 Mittag-Leffler rvs are heavy-tailed with tail exponent α. Just like their discrete counterparts, type-1 Mittag-Leffler rvs are ID and even self-decomposable.

- Weibull distribution: This is the law of $W_{\alpha}:=E^{1/\alpha}$ where E has standard exponential distribution. Then $\mathbf{P}\left(W_{\alpha}>t\right)=e^{-t^{\alpha}}$ with

$$\mathbf{E}\left(W_{\alpha}^{q}\right)=\Gamma\left(1+q/\alpha\right),\ q>-\alpha.$$

W_{α} is not heavy-tailed. As we will see just below, S_{α} as a one-sided stable rv with index α has moment function $\mathbf{E}\left(S_{\alpha}^{q}\right)=\frac{\Gamma(1-q/\alpha)}{\Gamma(1-q)}$, showing that the type-1 Mittag-Leffler rv T admits the multiplicative factorization into two independent factors:

$$T\overset{d}{=}W_{\alpha}\cdot S_{\alpha}.$$

- Fréchet distribution: This is the law of $F_{\alpha}:=E^{-1/\alpha}$, with $\mathbf{P}\left(F_{\alpha}\leq t\right)=e^{-t^{-\alpha}}$ and $\mathbf{E}\left(F_{\alpha}^{q}\right)=\Gamma\left(1-q/\alpha\right)$, $q<\alpha$. F_{α} is heavy-tailed with tail index α. We have

$$T^{-1}\overset{d}{=}F_{\alpha}\cdot S_{\alpha}^{-1},$$

as a multiplicative decomposition of the reciprocal T^{-1} of a type-1 Mittag-Leffler rv.

• **Type 2 Mittag-Leffler rv** ([20], p. 453, [59]). Let $T>0$ be a positive rv. It is a type-2 Mittag-Leffler(α) rv if it has the LSt and moment function ($\alpha\in(0,1)$)

$$\phi_{T}\left(p\right):=\mathbf{E}\exp\left(-pT\right)=E_{\alpha}\left(-p\right)\text{ and }\mathbf{E}\left(T^{q}\right)=\frac{\Gamma\left(q+1\right)}{\Gamma\left(q\alpha+1\right)}\ (q>-1).$$

The type-2 Mittag-Leffler(α) rv has all its integral moments finite which can be read from $\mathbf{E}\left(T^{q}\right)$ with q integer. The pdf of a type 2 rv is

$$\mathbf{P}\left(T\leq t\right)=\frac{1}{\pi}\sum_{k\geq 1}\frac{\left(-1\right)^{k-1}\Gamma\left(k\alpha\right)}{k!}\sin\left(k\pi\alpha\right)t^{k}. \tag{20}$$

We recall that a rv, say S_{α}, is one-sided stable with index α if

$$\phi_{S_{\alpha}}\left(p\right):=\mathbf{E}\exp\left(-pS_{\alpha}\right)=e^{-p^{\alpha}},\ \mathbf{E}\left(S_{\alpha}^{q}\right)=\frac{\Gamma\left(1-q/\alpha\right)}{\Gamma\left(1-q\right)},\ q<\alpha.$$

One-sided stable rvs are heavy-tailed with tail exponent α. Note that if S_α is one-sided stable with index α, $S_\alpha^{-\alpha}$ is a type-2 Mittag-Leffler(α) rv. Thus, for a type-2 Mittag-Leffler(α) rv T,

$$T \stackrel{d}{=} S_\alpha^{-\alpha}.$$

The pdf of a type 2 Mittag-Leffler rv (20) can be obtained using this information and the known expression

$$f_{S_\alpha}(s) = \frac{1}{\pi} \sum_{k \geq 1} \frac{(-1)^{k-1} \Gamma(k\alpha + 1)}{k!} \sin(k\pi\alpha) \, s^{-(k\alpha+1)}$$

of the density of a one-sided stable rv, [68]. Type-2 Mittag-Leffler(α) rvs are neither heavy-tailed nor infinitely divisible.

• From continuous ML rvs to discrete ML rvs.

Recall that if $P_\mu(t)$ is a Poisson process with intensity μt, $\mu > 0$, and T a positive random variable with LSt $\phi_T(p)$, independent of $P_\mu(t)$, then the random value at $t = T$ of $P_\mu(t)$, namely $P_\mu(T)$, has pgf

$$\mathbf{E}\left(z^{P_\mu(T)}\right) = \int_0^\infty F_T(dt) \, e^{-\mu t(1-z)} = \phi_T(\mu(1-z)).$$

In particular, if $T > 0$ is a type-1 (respectively type-2) Mittag-Leffler positive rv with LSt $\phi_T(p) = \frac{1}{1+p^\alpha}$ (respectively $E_\alpha(-p)$), then $P_{\mu^{1/\alpha}}(T)$ (respectively $P_\mu(T)$) is a discrete type-1 (respectively type-2) Mittag-Leffler rv with pgf $\left(1 + \mu(1-z)^\alpha\right)^{-1}$ (respectively $E_\alpha(-\mu(1-z))$). This shows that discrete type-2 Mittag-Leffler rvs are not compound Poisson (or ID).

• Although the type-2 Mittag-Leffler(α) rv T is not infinitely divisible and thus not self-decomposable, there is some relation of such a rv, obeying $\mathbf{E}(T^q) = \frac{\Gamma(q+1)}{\Gamma(q\alpha+1)}$, with self-decomposability. Let indeed E be a standard exponential(1) rv so that $G = -\log E$ has standard Gumbel distribution. Gumbel rvs are self-decomposable on $\mathbb{R}$, [66]. Thus, with G_α independent of $G' \stackrel{d}{=} G$, for all $\alpha \in (0,1)$

$$G \stackrel{d}{=} \alpha G' + G_\alpha.$$

Taking the LSt on both sides and observing $\mathbf{E}\left(e^{-qG}\right) = \mathbf{E}(E^q) = \Gamma(q+1)$, we can identify T as $\exp(-G_\alpha)$ with $\mathbf{E}\left(e^{-qG_\alpha}\right) = \mathbf{E}\left(e^{-qG}\right) / \mathbf{E}\left(e^{-q\alpha G}\right)$. So $-\log T \stackrel{d}{=} G_\alpha$, the independent factor appearing in the additive decomposition of G. Stated equivalently and multiplicatively, with $W_{1/\alpha} = E^\alpha$ a Weibull rv with parameter $1/\alpha$

$$E \stackrel{d}{=} W_{1/\alpha} \cdot T,$$

translating that E is log-self-decomposable. The type-2 Mittag-Leffler(α) rv T therefore appears as an independent multiplicative factor in the multiplicative decomposition of the standard exponential rv.

3. Mittag-Leffler distributions and renewal processes

We first recall salient facts arising from the modelling of events occurring randomly in time [20], [13]. Then we observe that a renewal process whose interarrival times are continuous type-1 Mittag-Leffler(α) distributed generates a counting process with discrete type-2 Mittag-Leffler distribution with parameter $\mu = t^{\alpha}$. The latter process, as a fractional Poisson process, is also a subordinated version of the traditional Poisson process, with time replaced by an independent inverse stable subordinator.

3.1. **Counting events.** Suppose at time $t = 0$, some event occurs for the first time. Suppose successive events occur in the future in such a way that the inter-arrival times between consecutive events form an iid sequence $(T_m,\, m \geq 1)$ with common distribution $T_m \overset{d}{=} T, m \geq 1$. Inter-arrival time T is assumed to have a density function (df), say $f_T(t)$. (We shall also need $F_T(t) := \mathbf{P}(T \leq t)$ and $\overline{F}_T(t) := 1 - F_T(t)$ i.e. the probability distribution function (pdf) of T and its complement to one (cpdf)). We are then left with a sequence of events occurring at times

$$\overline{T}_0 = 0,\, \overline{T}_n := \sum_{m=1}^{n} T_m,\, n \geq 1. \tag{21}$$

Let $N(t)$, $t > 0$, count the random number of events which occurred in the time interval $[0, t)$. Clearly,

$$N(t) = \sum_{n \geq 0} \mathbf{1}_{\left\{\overline{T}_n \leq t\right\}} \tag{22}$$

with $\mathbf{1}_{\{.\}}$ the set indicator function which takes the value one if the event is realized, zero, otherwise. As a result, an essential feature of such processes is that the events "$N(t) > n$" and "$\overline{T}_n \leq t$" coincide. Such random processes are called pure counting renewal processes (the adjective pure is relative to the hypothesis which has been made that the origin of time is an instant at which some event occurred; if this not the case, the adjective delayed is currently employed and the first event occurs at time $\overline{T}_0 := T_0 > 0$, independent of $(T_m,\, m \geq 1)$ but not necessarily with the same distribution). If in addition $\int_0^{+\infty} f_T(s)\, ds = 1$ (T is "proper") such renewal processes are said to be recurrent; this has to be opposed to transient renewal processes for which $\int_0^{+\infty} f_T(t)\, dt < 1$, corresponding to "defective" T, allowing for a finite probability that the first event never occurs, i.e. occurs at time $t = +\infty$. We shall avoid transient processes in the sequel and limit ourselves to recurrent ones. However, among recurrent processes, we shall distinguish between positive recurrent processes for which the average renewal time $\mathbf{E}T := \theta < +\infty$ and null recurrent for which $\mathbf{E}T = +\infty$.

If $\mathbf{E}T = +\infty$, we shall limit ourselves to situations where this occurs as a result of "heavy-tailedness" of the inter-arrival time: $\overline{F}_T(t) \sim c_\alpha t^{-\alpha}$, as $t \to +\infty$, with $\alpha \in (0, 1)$. Here, $c_\alpha > 0$ is a scale factor for T. In other words $c_\alpha = t_0^{\alpha}$ for some $t_0 > 0$ fixing the time-scale itself.

We note that if the inter-arrival times $(T_m,\, m \geq 1)$ are exponentially distributed, this counting process boils down into the familiar (shifted) Poisson process.

We shall call $\Lambda(t) := \mathbf{E}N(t)$ the intensity of the pure renewal process and $\lambda(t) := d\Lambda(t)/dt$ its rate (i.e. the instantaneous frequency at which events occur at time t); the function $\Lambda(t)/t$ is called the frequency of the phenomenon.

If follows from (22) that

$$\Lambda(t) = 1 + \sum_{n\geq 1} \int_0^t f_T^{*n}(s)\, ds \tag{23}$$

where f_T^{*n} stands for the n-fold convolution of f_T with itself (the df of $\overline{T}_n$).

If we let $\Lambda(p) := \int_0^{+\infty} e^{-ps}\Lambda(t)\, dt$, $\lambda(p) := \int_0^{+\infty} e^{-ps}\lambda(t)\, dt$ and $\phi_T(p) := \int_0^{+\infty} e^{-pt} f_T(t)\, dt$ stand for the Laplace transforms of $\Lambda(t)$, $\lambda(t)$ and $f_T(t)$, respectively, (23) yields

$$\Lambda(p) = \frac{1}{p(1-\phi_T(p))}, \text{ and } \lambda(p) = \frac{1}{1-\phi_T(p)}, \tag{24}$$

underlining the connection between probability theory and physical rate processes.

As p tends to zero, we get some informations on the way the intensity and rate functions behave for large times. These are strongly connected to the tail distribution of the variable T and we have to distinguish between two cases.

1./ $\theta < +\infty$. In this case, $\phi_T(p) \sim 1 - \theta p$, as $p \to 0^+$. Hence, $\Lambda(p) \sim 1/\left(\theta p^2\right)$ and $\lambda(p) \sim 1/(\theta p)$, as $p \to 0^+$. This means $\Lambda(t) \sim t/\theta$ and $\lambda(t) \sim 1/\theta$ as $t \to +\infty$. For recurrent positive processes, the rate function tends to $1/\theta$ as time drifts to infinity.

2./ $\theta = +\infty$. In this case, i.e. for recurrent null processes, the rate function tends to zero: this is a "rare event" hypothesis, as the expected time between consecutive events is infinite.

For example, if time T has a heavy-tailed cpdf such that $\overline{F}_T(t) \sim c_\alpha t^{-\alpha}$, as $t \to +\infty$, with $\alpha \in (0,1)$, $c_\alpha > 0$, in such a way that $\theta = +\infty$, then $\phi_T(p) \sim 1 - c_\alpha p^\alpha$, as $p \to 0^+$. Hence, $\Lambda(p) \sim 1/\left(c_\alpha p^{\alpha+1}\right)$ and $\lambda(p) \sim 1/(c_\alpha p^\alpha)$, as $p \to 0^+$. This means $\Lambda(t) \sim t^\alpha/c_\alpha$ and $\lambda(t) \sim t^{\alpha-1}/c_\alpha$ as $t \to +\infty$: the intensity goes to infinity slower than t and the rate function tends to zero algebraically. As time goes to infinity, the events get sparser and sparser, owing to the infinite average hypothesis of the inter-arrival times.

The process $N(t)$ counts the number of events which occurred before time t: each time an event occurs, the counter is incremented by one unity. We have

$$N(t) \overset{d}{=} N_0 \cdot \mathbf{1}_{\{T>t\}} + (1 + N'(t-T)) \cdot \mathbf{1}_{\{T\leq t\}}, \tag{25}$$

where $T > 0$ is a proper positive random variable known as the first renewal time of $N(t)$. Let us comment this identity: fix a time t at which $N(t)$ is to be evaluated. If the realization of time T exceeds the time t of interest, the process $N(t)$ is in its initial state, say $N(0) = N_0$. If $T = s \leq t$, the value of $N(t)$ is the independent sum of the first jump of amplitude 1 plus a statistical copy $N'(.)$ of the process $N(.)$ in the remaining time $t - s$, conditionally to the event $T = s$. Let us now translate the definition (25) in terms of the evolution of the pgf of $N(t)$. Let

$$\Phi_N(t, \lambda) := \mathbf{E}z^{N(t)} \tag{26}$$

be the pgf of the cumulative process $N(t)$. Then

$$\Phi_N(t,z) = \phi_{N_0}(z)\,\mathbf{P}(T>t) + z\int_0^t \Phi_{\overline{X}}(t-s,\lambda)\,f_T(s)\,ds \tag{27}$$

where $\phi_{N_0}(e) := \mathbf{E}z^{N_0}$. Introducing the Laplace transforms

$$\Phi_N(p,z) := \int_0^{+\infty} e^{-pt}\Phi_N(t,z)\,dt \text{ and } \phi_T(p) := \int_0^{+\infty} e^{-ps} f_T(s)\,ds \tag{28}$$

respectively of $\Phi_N(.,z)$ and f_T, yields

$$\Phi_N(p,z) = \frac{(1-\phi_T(p))\,\phi_{N_0}(z)}{p\,(1-\phi_T(p)\,z)}, \tag{29}$$

provided $\phi_T(p)\,z<1$. We shall call $\Phi_N(p,z)$ the Laplace functional of the counting process $N(t)$, $t\geq 0$. The two quantities $\left(\phi_{N_0}(z), \phi_T(p)\right)$ completely determine the law of the process $N(t)$, $t\geq 0$.

The pure elementary counting renewal process may be recovered from $N_0=1$ which states that the initial condition of the counting process is one. Equation (29) in this case reads

$$\Phi_N(p,z) = \frac{(1-\phi_T(p))\,z}{p\,(1-\phi_T(p)\,z)} \tag{30}$$

provided $\phi_T(p)\,z<1$. Thus the equation $\phi_T(p)=z^{-1}$ is the location of the poles of $\Phi_N(p,z)$.

If $N_0=0$, we get a counting process $N(t)$ with $N(0)=N_0=0$ and

$$\Phi_N(p,z) = \frac{1-\phi_T(p)}{p\,(1-\phi_T(p)\,z)}$$

to be compared with (30) for which $N_0=1$.

3.2. **Applications to the Mittag-Leffler distributions.** Suppose T is a type-1 Mittag-Leffler rv so with $\phi_T(\lambda) := \mathbf{E}\exp(-pT) = \frac{1}{1+p^\alpha}$. It is heavy-tailed with index α indicating that the times between consecutive events are long (a rare event hypothesis). If $N_0=0$, we get a counting process $N(t)$ with Laplace functional

$$\Phi_N(p,z) = \frac{p^\alpha}{p\,(1+p^\alpha-z)},$$

leading to

$$\Phi_N(t,z) = E_\alpha(-t^\alpha(1-z)).$$

Indeed, with $\int_0^\infty e^{-pt}t^{\alpha m}dt = \Gamma(\alpha m+1)/p^{\alpha m+1}$

$$\begin{aligned}
\Phi_N(p,z) &= \int_0^\infty e^{-pt}E_\alpha(-t^\alpha(1-z))\,dt \\
&= \sum_{m\geq 0}\frac{(-1)^m(1-z)^m}{\Gamma(\alpha m+1)}\int_0^\infty e^{-pt}t^{\alpha m}dt \\
&= \frac{1}{p}\sum_{m\geq 0}(-1)^m\left(\frac{1-z}{p^\alpha}\right)^m = \frac{p^\alpha}{p\,(1+p^\alpha-z)}.
\end{aligned}$$

Thus the $t-$marginal of the corresponding count process $N(t)$ has a type-2 discrete Mittag-Leffler marginal distribution with parameter $\mu=t^\alpha$. $N(t)$ is often called the fractional Poisson process, [40], [2], [43]. Clearly $\{N(t)\}$ is not Markov and in particular not a process with independent increments. The process $\{N(t)\}$ has a slow algebraic growth. We have

$$\begin{aligned}
\mathbf{E}(N(t)) &= \frac{t^{\alpha}}{\Gamma(\alpha+1)}, \\
\mathrm{Var}(N(t)) &= \frac{t^{\alpha}}{\Gamma(\alpha+1)} + 2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \left(\frac{t^{\alpha}}{\Gamma(\alpha+1)}\right)^2 \\
&\underset{\text{large } t}{\sim} \left(\frac{2}{\Gamma(2\alpha+1)} - \frac{1}{\Gamma(\alpha+1)^2}\right) t^{2\alpha} \\
\mathbf{E}(N(t), N(t+\tau)) &= \frac{t^{\alpha}}{\Gamma(\alpha+1)} + 2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{(t\tau)^{\alpha}}{\Gamma(\alpha+1)^2},
\end{aligned}$$

so that $N(t)$ exhibits long-range negative correlation

$$\mathrm{Covar}(N(t), N(t+\tau)) \underset{\text{large } \tau,\ t \text{ fixed}}{\sim} -\frac{\alpha t^{\alpha+1}}{\Gamma(\alpha+1)^2} \tau^{-(1-\alpha)}.$$

The algebraic decay parameter of the correlation function is $1-\alpha$. We note that, for each $t>0$ and each $c \in (0,1)$

$$N(ct) \overset{d}{=} c^{\alpha} \circ N(t),$$

suggesting that $N(t)$ is discrete-self-similar with Hurst exponent $\alpha \in (0,1)$.

There exists a $\mathbb{R}_+-$valued mass process $X(t)$ such that $N(t) = P(X(t))$, a Poisson rv with random intensity $X(t)$. From this construction, $N(t)$ is the number of individuals carrying the mass $X(t)$. Clearly, $\mathbf{E}e^{-pX(t)} = E_{\alpha}(-pt^{\alpha})$, so that

$$\mathbf{E}z^{N(t)} = \mathbf{E}e^{-X(t)(1-z)} = E_{\alpha}(-pt^{\alpha}) \mid_{p=1-z} = E_{\alpha}(-t^{\alpha}(1-z)).$$

The marginal distribution of the process $\{X(t)\}$ is thus the one of a continuous type-2 discrete Mittag-Leffler(α) rv with parameter $\mu = t^{\alpha}$. It is now known (see Example 2 in [29], p. $2639-2641$, [23] Chapter 9, [44] and [47]), that the subordinating process $\{X(t)\}$ can be obtained as a (non-Markovian) inverse Lévy-stable(α) compound renewal process, namely: $X(t) = \inf(s>0 : S(s)>t)$, with $S(0)=0$ and $\mathbf{E}\left(e^{-pS(s)}\right) = e^{-sp^{\alpha}}$.

The two Mittag-Leffler(α) distributions are thus intimately related through this renewal process structure.

4. Möhle's construction [49] and further developments

There is a continuous-state, continuous-time version of the Neveu process. We briefly report here on a recent construction [49] relating type-2 Mittag-Leffler Markov processes to Neveu continuous-state branching process. In this construction, the Mittag-Leffler parameter α is allowed to vary with time: $\alpha = \alpha_t = e^{-t}$. Additional details are given.

4.1. **The Siegmund dual of the Neveu continuous-state branching process.** For each $t \geq 0$, let $\widehat{X}_{e^{-t}}$ be a type-2 Mittag-Leffler rv with parameter $\alpha = \alpha_t = e^{-t}$. With $x > 0$, let $\widehat{X}(t) = x^{\alpha_t} \widehat{X}_{\alpha_t} > 0$, so with $\widehat{X}(0) = x$ and $\widehat{X}(\infty) \overset{d}{\sim} \exp(1)$. The marginal law at t of $\widehat{X}(t)$ started at x is characterized by its LSt

$$\phi_{\widehat{X}(t)}(p) = \mathbf{E}_x \left(e^{-p\widehat{X}(t)} \right) = E_{\alpha_t}(-px^{\alpha_t}).$$

Equivalently, $\widehat{X}(t)$ has marginal transition pdf

$$\mathbf{P}_x \left(\widehat{X}(t) \leq y \right) = \frac{1}{\pi} \sum_{k \geq 1} \frac{(-1)^{k-1} \Gamma(k\alpha_t)}{k!} \sin(k\pi\alpha_t) x^{-k\alpha_t} y^k. \tag{31}$$

Then $T_t \phi(x) := \mathbf{E}_x \phi \left(\widehat{X}(t) \right)$ defines a Markov jump semigroup with infinitesimal generator L acting on $\phi \in C^\infty$ as [49]

$$L\phi(x) = \lim_{t \to 0^+} \frac{T_t \phi(x) - \phi(x)}{t} = f(x) \phi'(x) + \int_0^\infty \left(\phi(x-h) - \phi(x) + h\phi'(x) \right) \nu_x(dh),$$

with drift $f(x) = x(\psi(2) - \log x)$ ($\psi(2) = 1 - \gamma$, the value at point 2 of the digamma function, γ the Euler constant) and unbounded local jump measure at state x

$$\nu_x(dh) = xh^{-2} \mathbf{1}_{\{h \in (0,x)\}} dh.$$

The Markov process $\left\{ \widehat{X}(t) \right\}$ is continuous in probability and stochastically monotone [49] in the initial condition x. Its invariant measure is standard exponentially distributed so that if $\widehat{X}(0)$ is made random with standard (mean 1) exponentially distribution, $\left\{ \widehat{X}(t) \right\}$ is stationary, observing $\int_0^\infty dx e^{-x} E_{\alpha_t}(-px^{\alpha_t}) = (1+p)^{-1}$, the LSt of a standard exponential distribution.

As a stochastically monotone process, a Siegmund dual of $\left\{ \widehat{X}(t) \right\}$, say $\{X(t)\}$, can be defined from [64], [49] as from

$$\mathbf{P}_x \left(\widehat{X}(t) \leq y \right) = \mathbf{P}_y (X(t) > x).$$

From (31), $\mathbf{P}_y(X(t) > x)$ is thus known, with $\mathbf{P}_y(X(t) > x) \sim \Gamma(\alpha_t) \sin(\pi\alpha_t) x^{-\alpha_t}$ for large x. It turns out that $\{X(t)\}$ is a continuous-state (supercritical) Neveu branching process with log-Laplace exponent [50], [5]

$$\psi_{t,x}(p) := -\log \mathbf{E}_x e^{-pX(t)} = xp^{\alpha_t},$$

where $x = X(0)$ (to be compared with its discrete version as from 7). We have $\psi_{t,x}(p) = x\psi_t(p)$ and the quantity $\psi_t(p) := \psi_{t,1}(p)$ obeys the Markov property $\psi_{t+s}(p) = \psi_t(\psi_s(p))$ with

$$\partial_t \psi_t(p) = \psi(\psi_t(p)), \ \psi_0(p) = p \tag{32}$$

and it has branching mechanism [4], [5], [31]

$$\psi(p) = -p\log p = cp + \int_0^\infty \left(1 - px\mathbf{1}_{\{x \leq 1\}} - e^{-px} \right) \pi(dx),$$

with unbounded Lévy measure for jumps $\pi(dx) = x^{-2}dx$ obeying $\int \left(1 \wedge x^2\right) \pi(dx) < \infty$ and c a drift constant. We have

$$\psi_t(p) = B^{-1}(t + B(p)), \text{ where } B(p) = \int^p \frac{dq}{\psi(q)} = -\log\log p.$$

Note that $\psi'(0^+) = +\infty$, so that $\mathbf{E}_x(X(t)) = +\infty$, and it may be shown, again using martingale arguments [50], that, conditionally given $X(t)$ drifts to ∞, it does so at double exponential speed a.s., [25]. Note also that, as a result of $\int_{0+} dp/|\psi(p)| = \infty$, $\psi_t(0) = 0$ for all $t \geq 0$: the continuous-state Neveu branching process is regular or conservative and there cannot be explosion in finite time. The Neveu process is also supercritical with extinction probability $\rho_e = e^{-p_0}$, with $p_0 = 1$ solving $\psi(p_0) = 0$, [31]. Because $\psi_t(p)$ is the log-Laplace exponent of a stable(α_t) random variable, we have the Mellin transform formula

$$\mathbf{E}_x\left(X(t)^q\right) = x^{q/\alpha_t}\frac{\Gamma(1 - q/\alpha_t)}{\Gamma(1-q)},\ q < \alpha_t.$$

It can also easily be shown, following the argument in [31] stating that $\int^{\infty} dp/\psi(p) = -\log\log(p)\mid_{p=\infty} = -\infty$, that if the supercritical Neveu process started at x goes extinct (an event with probability $\rho_{x,e} = \rho_e^x = e^{-x}$), this can only happen at time $\tau_{x,e} = \infty$, with probability one. So both extinction and explosion times are infinite with probability one for this model.

• The Lévy process associated to the Neveu branching process.

Let $\{S(t)\}$ with $S(0) = x$ be the Lévy process with Laplace exponent $\psi(p)$ so with $\mathbf{E}\left(e^{-p(S(t)-S(0))}\right) = \exp(-t\psi(p))$. Suppose this process is stopped when it first hits 0 if ever. This process has unbounded variations; it is spectrally positive (it displays non-negative jumps) although not a subordinator and it is the continuous-state branching process analog to the skip-free to the left compound Poisson process discussed in the discrete-space, continuous-time version of the Neveu process. The process $\{S(t)\}$ has the scale function characterized by its LSt [4]

$$\int_0^{\infty} e^{-px}s(x)\,dx = -\frac{1}{\psi(p)} = \frac{1}{p\log p},\ p > p_0 = 1.$$

We have $s(x) = e^x\overline{s}(x)$ where $\overline{s}(x)$ has Laplace transform

$$\frac{1}{(p+1)\log(p+1)} = \frac{1}{p}\frac{1}{2-\phi(p)}.\text{ Here } \phi(p) = 1 + \sum_{n\geq 1}\frac{(-1)^n}{n(n+1)}p^n$$

is the LSt of some positive random variable Z with log-convex Stieltjes moment sequence $\mathbf{E}(Z^n) = (n-1)!/(n+1)$, $n \geq 1$. Thus $\overline{s}(x) = \int_0^x \rho(y)\,dy$ where $\rho(y)$is the density of a geometric(1/2) sum of such iid Zs, [31]. The factor $\overline{s}(x)$ is thus a probability distribution function of some infinitely divisible random variable, and $s(x)$ an exponentially modulated version of $\overline{s}(x)$.

And as in the discrete set-up, if $\tau_{x,y} = \inf(t > 0 : S(t) = y \mid S(0) = x)$, $y > x > 0$:

$$\mathbf{P}(\tau_{x,y} < \tau_{x,0}) = \frac{s(x)}{s(y)} = e^{-(y-x)}\frac{\overline{s}(x)}{\overline{s}(y)}.$$

More generally, [67], with $s_q(x)$, $q \geq 0$, the function whose Laplace transform is $\int_0^\infty e^{-px} s_q(x)\,dx = -(\psi(p)+q)^{-1}$, $p > p_0 = 1$,

$$\mathbf{E}\left(e^{-q\tau_{x,y}} \mathbf{1}_{\{\tau_{x,y}<\tau_{x,0}\}}\right) = \frac{s_q(x)}{s_q(y)}$$

is a classical identity giving the law of $\tau_{x,y}$ on the event $\{\tau_{x,y} < \tau_{x,0}\}$.

It is also known [4] from the theory of spectrally positive processes that $\tau_{x,0}$ is a subordinator process with independent increments and log-Laplace exponent

$$-\log \mathbf{E}\left(e^{-q\tau_{x,0}}\right) = x\psi^{-1}(-q). \tag{33}$$

In the Neveu process case under study, $h(q) := \psi^{-1}(-q) = e^{W(q)}$,where $W(q)$ is the Lambert function solving the functional equation $W(q)\,e^{W(q)} = q$. This equation can be solved by the Lagrange inversion formula, leading to its principal branch $W(q) = \sum_{n\geq 1}(-1)^{n-1}\frac{n^{n-1}}{n!}q^n$, $q > -p_0$ and $h(q) = 1+\sum_{n\geq 1}(-1)^{n-1}\frac{(n-1)^{n-1}}{n!}q^n$. $\overline{h}(q) := h(q) - 1$ is a Bernstein function (i.e. $\overline{h} \geq 0$ and $\overline{h}'$ is completely monotone: for all $q > 0 : \overline{h}' \geq 0$, $\overline{h}'' \leq 0$, $\overline{h}''' \geq 0$,...). Note $\mathbf{P}(\tau_{x,0} < \infty) = e^{-xh(0)} = e^{-x}$ and $\mathbf{E}\left(e^{-q\tau_{x,0}} \mid \tau_{x,0} < \infty\right) = e^{-x\overline{h}(q)}$. The corresponding Lévy measure of the jumps has $n-$moments: $(n-1)^{n-1}$, $n \geq 1$.

Let finally $\{S^*(t)\}$, with $S^*(0) = x$ be the supremum (or ladder height) process of $\{S(t)\}$, viz

$$S^*(t) = \sup_{0\leq s\leq t} S(s).$$

As in the discrete set-up, from the previous arguments, with $y > x$, we have

$$\mathbf{P}(S^*(\infty) \leq y \mid S^*(0) = x) = \frac{s(y-x)}{s(y)},$$

together with [6]

$$\mathbf{P}(X^*(\infty) \leq y \mid X^*(0) = x) = \frac{s(y-x)}{s(y)} = e^{-x}\frac{\overline{s}(y-x)}{\overline{s}(y)},$$

where $X^*(t) = \sup_{0\leq s\leq t} X(s)$ is the supremum process of the Neveu branching process $X(t)$. This is because, [38], $X(t)$ is a time-changed version of $S(t)$, a fact generalizing (19):

$$X(t) \overset{d}{=} S\left(\int_0^t X(s)\,ds\right),$$

with, from (33), and as in the discrete Neveu process case,

$$\tau_{x,0} \overset{d}{=} \int_0^\infty (X(s) \mid X(0) = x)\,ds,$$

recalling $\{X(s)\}$ does not go extinct in finite times.

The process $\{S^*(t)\}$ enjoys some nice properties. It is known, as a result of the Wiener-Hopf factorization for the spectrally positive process $\{S(t)\}$, that this new process is a subordinator (with non-decreasing sample paths with bounded variations), with log-Laplace exponent $\psi^*(p) = \psi(p)/(p_0 - p)$, [36], [65]. $\psi^*(p)$ is smooth at $p = p_0 = 1$. Therefore,

$$\mathbf{E}e^{-p(S^*(t)-S^*(0))} = e^{-t\psi^*(p)} = e^{-t(p\log p)/(p-1)}.$$

It can be checked that $\psi^*(p) \geq 0$, $p \geq 0$, is indeed a Bernstein function, with $\psi^*(0^+) = 0$ and $\psi^*(0^+)' = +\infty$. As a result,

$$\psi^*(p) = \frac{p \log p}{p-1} = \int_0^\infty \left(1 - e^{-px}\right) \pi^*(dx) = p \int_0^\infty e^{-px} \overline{\pi}^*(x)\, dx,$$

for some new unbounded Lévy measure π^* of its jumps with tails $\overline{\pi}^*(x) = \int_x^\infty \pi^*(dz)$ (Note $\psi^*(p)$ has no killing component). Because $\pi^*(dx)$ integrates $1 \wedge x$, $\{S^*(t)\}$ has bounded variations.

• Randomizing the initial condition of the Neveu process.

Assuming $X(0)$ random and standard exponentially distributed

$$\mathbf{E}e^{-pX(t)} = \mathbf{E}e^{-X(0)p^{\alpha_t}} = \int_0^\infty dx e^{-x} e^{-xp^{\alpha_t}} = \frac{1}{1 + p^{\alpha_t}},$$

which is a type-1 Mittag-Leffler rv with variable exponent e^{-t} (to be compared with its discrete version as from (6)). In this latter sense, $\left\{\widehat{X}(t)\right\}$ as a (stationary) type-2 Mittag-Leffler Markov process and $\{X(t)\}$ as a type-1 Mittag-Leffler branching process are Siegmund duals of one another.

This also suggests the following discrete-space version of Möhle's construction:

Let $\{P(x)\}$ be a Poisson process with intensity $x > 0$ with: $\mathbf{E}\left(z^{P(x)}\right) = e^{-x(1-z)}$, counting the number of individuals with mass belonging to $[0, x]$. Consider the Markov process $\widehat{N}(t, x) := P\left(\widehat{X}(t)\right)$ where $\left\{\widehat{X}(t)\right\}$ is the type-2 Möhle-Mittag-Leffler process started at $\widehat{X}(0) = x$, giving the temporal evolution of the mass process. The new process $\widehat{N}(t, x)$ is thus a subordinated version of the traditional Poisson process, with time replaced by an independent type-2 Möhle-Mittag-Leffler process started at $\widehat{X}(0) = x$. We have

$$\mathbf{E}\left(z^{\widehat{N}(t,x)}\right) = \mathbf{E}_x\left(z^{P\left(\widehat{X}(t)\right)}\right) = \mathbf{E}_x\left(e^{-\widehat{X}(t)(1-z)}\right) = E_{\alpha_t}\left(-x^{\alpha_t}(1-z)\right),$$

which is a discrete type-2 Mittag-Leffler process with parameters $\mu_t(x) = x^{\alpha_t}$ and $\alpha_t = e^{-t}$. Initially, $\widehat{N}(0, x) \overset{d}{=} P(x)$. Assuming $X(0) = x$ random and standard exponentially distributed, $\left\{\widehat{N}(t)\right\}$ is strictly stationary with

$$\mathbf{E}\left(z^{\widehat{N}(t)}\right) = \int_0^\infty dx e^{-x} \mathbf{E}\left(z^{\widehat{N}(t,x)}\right) = \int_0^\infty dx e^{-x} E_{\alpha_t}\left(-x^{\alpha_t}(1-z)\right) = \frac{1}{2-z},$$

the pgf of a geometric$(1/2)$ random variable.

Consider also the Markov process $N(t, x) := P(X(t))$ with $\{X(t)\}$ the Neveu branching process started at $X(0) = x$ (Siegmund dual to $\left\{\widehat{X}(t)\right\}$). We have

$$\mathbf{E}\left(z^{N(t,x)}\right) = \mathbf{E}_x\left(z^{P(X(t))}\right) = \mathbf{E}_x\left(e^{-X(t)(1-z)}\right) = e^{-x(1-z)^{\alpha_t}}$$

which is a discrete-stable Neveu process with parameters $\mu(x) = x$ and $\alpha_t = e^{-t}$. Assuming $X(0) = x$ random and standard exponentially distributed

$$\mathbf{E}\left(z^{N(t)}\right) = \int_0^\infty dx e^{-x} e^{-x(1-z)^{\alpha_t}} = \frac{1}{1 + (1-z)^{\alpha_t}},$$

which is a discrete type-1 Mittag-Leffler process with variable tail exponent $\alpha_t = e^{-t}$. Therefore, if one assumes that the continuous space-time mass process is carried by a random Poisson number of individuals, then we are back to the continuous-time discrete-space setup.

4.2. **Poisson-Dirichlet $PD(\alpha_t, 0)$ partition and the block-counting process $\{I(t)\}$.** We consider here some combinatorics related to $PD(\alpha_t, 0)$ thereby completing some further properties of the Siegmund duality.

• **Neveu process and Poisson-Dirichlet $PD(\alpha_t, 0)$ partition.** For the Neveu process $\{X(t)\}$ started at x, the log-Laplace exponent $\psi_{t,x}(p) = xp^{\alpha_t}$ satisfies

$$\psi_{t,x}(p) = x\int_0^\infty \left(1 - e^{-ps}\right)\Pi_t(ds) = xp\int_0^\infty e^{-ps}\overline{\Pi}_t(s)\,ds = x\psi_t(p)$$

where $\Pi_t(ds) = \frac{\alpha_t}{\Gamma(1-\alpha_t)}s^{-(\alpha_t+1)}ds$ and $\overline{\Pi}_t(s) = \frac{1}{\Gamma(1-\alpha_t)}s^{-\alpha_t}$, [31]. Therefore $\{X(t)\}$ is distributed like a subordinator whose unbounded Lévy measure of its jumps $\Pi_t(ds)$ is time-inhomogeneous. Because $\Pi_t(ds)$ integrates $1 \wedge x$, this subordinator has bounded variations. With $B(p) = \int_p^{p_0} dq/\psi(q)$,

$$\psi_t(p) = B^{-1}(B(p) - t)$$

is indeed the integrated version of (32) observing $-\log(\log(p))$ is the primitive of $1/\psi$.

With $\beta_t = \alpha_t^{-1} = e^t$, the Poisson point-process decomposition of $\{X(t)\}$ is thus (as a generalization of (4), (18))

$$X(t) \overset{d}{=} \sum_{k\geq 1}\overline{\Pi}_t^{-1}(S_k/x) = \sum_{k\geq 1}\left(x^{-1}\Gamma(1-\alpha_t)S_k\right)^{-\beta_t} =: \sum_{k\geq 1}\Delta_{(k)}(t),$$

where the $(S_k)_{k\geq 1}$ are points of a standard Poisson point process on $(0,\infty)$ with Lebesgue measure intensity: $0 < S_1 < S_2 < ....$ Note $\Delta_{(1)}(t) > \Delta_{(2)}(t) > ....$ with the law of $\Delta_{(k)}(t)$ given by

$$\mathbf{P}\left(\Delta_{(k)}(t) \leq z\right) = e^{-s}\sum_{l=0}^{k-1}\frac{s^k}{k!}\mid_{s=z^{-\alpha_t}/(x\Gamma(1-\alpha_t))}.$$

Thus, upon normalizing the jumps' sizes, for each $t > 0$, $\delta_{(k)}(t) := \Delta_{(k)}(t)/X(t)$, $k \geq 1$, constitute a random partition of unity into an infinite sequence of ranked pieces, known as the Poisson-Dirichlet $PD(\alpha_t, 0)$ partition of the unit interval, [57]. It can also be understood as follows: consider a system $(\pi_k,\, k \geq 1)$ of independent random variables each with distribution beta$(1-\alpha_t, k\alpha_t)$. Set $\chi_1 = \pi_1$; $\chi_k = \prod_{j=1}^{k-1}(1-\pi_j)\pi_k$, $k \geq 2$, so with Griffith-Engen-McCloskey distribution. The ranked values of the χ_ks are the $\delta_{(k)}(t)$s, with $PD(\alpha_t, 0)$-distribution.

By Campbell formula [35], [34], for all non-negative measurable function g for which the involved integrals converge:

$$\begin{aligned}
\mathbf{E}e^{-p\sum_{k\geq 1} g\left(\Delta_{(k)}(t)\right)} &= \exp -x\int_0^\infty \left(1-e^{-pg(s)}\right)\Pi_t\left(ds\right) \\
&= \exp -\frac{x\alpha_t}{\Gamma\left(1-\alpha_t\right)}\int_0^\infty \left(1-e^{-pg(s)}\right)s^{-(\alpha_t+1)}ds \\
\mathbf{E}\sum_{k\geq 1} g\left(\Delta_{(k)}\left(t\right)\right) &= x\int_0^\infty g\left(s\right)\Pi_t\left(ds\right) = \frac{x\alpha_t}{\Gamma\left(1-\alpha_t\right)}\int_0^\infty g\left(s\right)s^{-(\alpha_t+1)}ds
\end{aligned}$$

and (see [58], Proposition 2.1), for the ranked segments,

$$\mathbf{E}\sum_{k\geq 1} g\left(\delta_{(k)}\left(t\right)\right) = \int_0^\infty g\left(s\right)\sigma_t\left(ds\right),\ \text{where} \tag{34}$$

$$\sigma_t\left(ds\right) := \mathbf{E}\sum_{k\geq 1}\mathbf{1}_{\left\{\delta_{(k)}(t)\in ds\right\}} = \frac{1}{\Gamma\left(1-\alpha_t\right)\Gamma\left(\alpha_t\right)}s^{-(\alpha_t+1)}\left(1-s\right)^{\alpha_t-1}\mathbf{1}_{\{s\in(0,1)\}}ds,$$

is the structural (intensity) measure of $PD\left(\alpha_t,0\right)$.

In particular, if $g\left(s\right)=s\mathbf{1}_{\{s\leq\varepsilon\}}$, the average contribution to $X\left(t\right)=\sum_{k\geq 1}\Delta_{(k)}\left(t\right)$ of the $\Delta_{(k)}\left(t\right)$ which are smaller than some $\varepsilon>0$ is

$$\mathbf{E}\sum_{k\geq 1}\Delta_{(k)}\left(t\right)\mathbf{1}_{\left\{\Delta_{(k)}(t)\leq\varepsilon\right\}} = \frac{x\alpha_t}{\Gamma\left(1-\alpha_t\right)}\int_0^\varepsilon s^{(1-\alpha_t)-1}ds = \frac{x\alpha_t}{\Gamma\left(2-\alpha_t\right)}\varepsilon^{1-\alpha_t}.$$

For $q>\alpha_t$, taking $g\left(s\right)=s^q$ in (34)

$$\mathbf{E}\left(\sum_{k\geq 1}\delta_{(k)}\left(t\right)^q\right) = \frac{1}{\Gamma\left(1-\alpha_t\right)\Gamma\left(\alpha_t\right)}\int_0^1 s^{(q-\alpha_t)-1}\left(1-s\right)^{\alpha_t-1}ds = \frac{\Gamma\left(q-\alpha_t\right)}{\Gamma\left(1-\alpha_t\right)\Gamma\left(q\right)}$$

whereas taking $g\left(s\right)=s\mathbf{1}_{\{s>\varepsilon\}}$

$$\begin{aligned}
\mathbf{E}\left(\sum_{k\geq 1}\delta_{(k)}\left(t\right)\mathbf{1}_{\left\{\delta_{(k)}(t)>\varepsilon\right\}}\right) &= \frac{1}{\Gamma\left(1-\alpha_t\right)\Gamma\left(\alpha_t\right)}\int_\varepsilon^1 s^{(1-\alpha_t)-1}\left(1-s\right)^{\alpha_t-1}ds \\
&\underset{\varepsilon\ \text{small}}{\sim} 1-\frac{\varepsilon^{1-\alpha_t}}{\Gamma\left(2-\alpha_t\right)\Gamma\left(\alpha_t\right)}
\end{aligned}$$

gives the average contribution of the segments in $PD\left(\alpha_t,0\right)$ whose sizes are larger than $\varepsilon\in\left(0,1\right)$ in terms of an incomplete beta function.

• **Mean Shannon entropy of** $PD\left(\alpha_t,0\right)$**.** Letting $z_t\left(q\right)=\mathbf{E}\left(\sum_{k\geq 1}\delta_{(k)}\left(t\right)^q\right)$, with $\psi\left(x\right)$ the digamma function ($\psi\left(1\right)=-\gamma$, γ the Euler constant), we obtain the following expression of the mean Shannon entropy of $PD\left(\alpha_t,0\right)$ as

$$s_t := \mathbf{E}\left(-\sum_{k\geq 1}\delta_{(k)}\left(t\right)\log\delta_{(k)}\left(t\right)\right) = -\partial_q z_t\left(q+1\right)|_{q=0} = \psi\left(1\right)-\psi\left(1-\alpha_t\right). \tag{35}$$

As t varies from 0^+ to $+\infty$, s_t decreases from $+\infty$ to 0 : as $t\rightarrow 0^+$, $PD\left(\alpha_t,0\right)$ is made of infinitely many segments all of very small sizes (disorder is maximal), whereas as $t\rightarrow\infty$, as shown just below, the largest segment $\delta_{(1)}\left(t\right)$ will dominate the other ones with small room left for the other segments.

• **The size of the largest segment in** $PD(\alpha_t, 0)$**.** We will consider now the problem of computing the law of $\delta_{(1)}(t)$. For the functions g for which it converges, consider the integral

$$H_{t,g}(p) := \int_0^\infty \left(1 - e^{-ps} g(s)\right) s^{-(\alpha_t+1)} ds. \tag{36}$$

In particular, $H_{t,g\equiv 1}(p) = -\Gamma(-\alpha_t) p^{\alpha_t}$. Then ([57], Corollary 47)

$$\int_0^\infty e^{-px} \mathbf{E} \prod_{k\geq 1} g\left(x\delta_{(k)}(t)\right) dx = \frac{1}{\alpha_t} \partial_p K_{t,g}(p) \text{ with } K_{t,g}(p) = \log H_{t,g}(p). \tag{37}$$

With $b \leq 0$, take $g(s) = \mathbf{1}_{\{s\leq b\}}$ in (36, 37). Then, with

$$I_t(p) = \alpha_t \int_0^1 \left(1 - e^{-ps}\right) s^{-(\alpha_t+1)} ds,$$

$$\begin{aligned}
\int_0^\infty e^{-px} \mathbf{E} \prod_{k\geq 1} g\left(x\delta_{(k)}(t)\right) dx &= \int_0^\infty e^{-px} \mathbf{P}\left(\delta_{(1)}(t)^{-1} > \frac{x}{b}\right) dx \\
&= b \int_0^\infty e^{-pby} \mathbf{P}\left(\delta_{(1)}(t)^{-1} > y\right) dy \\
\frac{1}{\alpha_t} \partial_p \log H_{t,g}(p) &= \frac{1}{\alpha_t} \frac{bI_t'(pb)}{1 + I_t(pb)}.
\end{aligned}$$

Thus, observing $I_t'(p) = \frac{\alpha_t}{p}\left(I_t(p) + 1 - e^{-p}\right)$

$$\mathbf{E}\left(e^{-q/\delta_{(1)}(t)}\right) = 1 - \frac{q}{\alpha_t} \frac{I_t'(q)}{1 + I_t(q)} = \frac{e^{-q}}{1 + I_t(q)}$$

is the LSt of $1/\delta_{(1)}(t)$. This shows that $\delta_{(1)}(t) \overset{d}{=} 1/(1 + \Delta(t))$ where

$$\mathbf{E}\left(e^{-q\Delta(t)}\right) = 1/\left(1 + I_t(q)\right) = 1/\left(1 + \alpha_t \int_0^1 \left(1 - e^{-qs}\right) s^{-(\alpha_t+1)} ds\right).$$

Next, $xI_t(q)$ is the log-Laplace exponent of a truncated Lévy subordinator $x \to L_t(x)$ with time-inhomogeneous jumps law supported by $(0,1)$ and $\Delta(t) = L_t(X)$, where $X \sim$exp(1) is an independent (subordination) random variable:

$$\mathbf{E}\left(e^{-q\Delta(t)}\right) = \int_0^\infty dx e^{-x} e^{-xI_t(q)} = 1/\left(1 + I_t(q)\right).$$

When $t \to \infty$, $\alpha_t \to 0$ and $\Delta(t) \overset{d}{\to} 0$ or $\delta_{(1)}(t) \overset{d}{\to} 1$: in the long-time run, as already observed in the context of the Shannon entropy, the largest segment $\delta_{(1)}(t)$ tends to dominate the other ones.

• **Moments of** $Z_t(q) := \sum_{k\geq 1} \delta_{(k)}(t)^q$**.** Taking $g(s) = 1 + s^q$ in (36, 37), with $a := \frac{\alpha_t \Gamma(q-\alpha_t)}{\Gamma(1-\alpha_t)}$,

$$K_{t,g}(p) = \frac{1}{p}\left(1 + \frac{q}{\alpha_t} \frac{ap^{-q}}{1 - ap^{-q}}\right) = \frac{1}{p}\left(1 + \frac{q}{\alpha_t} \sum_{j\geq 1} a^j p^{-qj}\right)$$

and

$$\int_0^\infty e^{-px}\mathbf{E}\prod_{k\geq 1} g\left(x\delta_{(k)}(t)\right)dx = \int_0^\infty e^{-px}\left(1+\sum_{j\geq 1} x^{qj}\mathbf{E}\sum_{k_1<...<k_j}\prod_{l=1}^{j}\delta_{(k_l)}(t)^q\right)dx$$

$$= \frac{1}{p}\left(1+\sum_{j\geq 1} p^{-qj}\cdot\Gamma(qj+1)\cdot\mathbf{E}\sum_{k_1<...<k_j}\prod_{l=1}^{j}\delta_{(k_l)}(t)^q\right)$$

Therefore,

$$\mathbf{E}\sum_{k_1<...<k_j}\prod_{l=1}^{j}\delta_{(k_l)}(t)^q = \frac{1}{\alpha_t j\Gamma(qj)}\left(\frac{\alpha_t\Gamma(q-\alpha_t)}{\Gamma(1-\alpha_t)}\right)^j, \, q>\alpha_t.$$

More generally (proceeding as in [30] p. 740), with $q_l > \alpha_t$, $l = 1, ..., j$

$$\mathbf{E}\sum_{k_1<...<k_j}\prod_{l=1}^{j}\delta_{(k_l)}(t)^{q_l} = \frac{1}{\alpha_t j\Gamma\left(\sum_{l=1}^{j} q_l\right)}\prod_{l=1}^{j}\frac{\alpha_t\Gamma(q_l-\alpha_t)}{\Gamma(1-\alpha_t)} =: \frac{1}{j!}\phi_{j,\alpha_t}(q_1,...,q_j),$$

where

$$\phi_{j,\alpha_t}(q_1,...,q_j) := \alpha_t^{j-1}\frac{\Gamma(j)}{\Gamma\left(\sum_{l=1}^{j} q_l\right)}\prod_{l=1}^{j}\frac{\Gamma(q_l-\alpha_t)}{\Gamma(1-\alpha_t)}.$$

We conclude that, with the star sum a sum running over the positive integers i_l summing to i, the integral moments of $Z_t(q) = \sum_{k\geq 1}\delta_{(k_l)}(t)^q$, $q > \alpha_t$ are given by

$$\mathbf{E}\left[Z_t(q)^i\right] = \sum_{j=1}^{i}\frac{1}{j!}\sum_{i_1+...+i_j=i}^{*}\binom{i}{i_1...i_j}\phi_{j,\alpha_t}(qi_1,...,qi_j).$$

In particular, consistently with ([31] p. 490) and (34), taking $i = 1, 2$

$$\begin{aligned}\mathbf{E}[Z_t(q)] &= \phi_{1,\alpha_t}(q) = \frac{\Gamma(q-\alpha_t)}{\Gamma(q)\Gamma(1-\alpha_t)},\\ \mathbf{E}\left[Z_t(q)^2\right] &= \phi_{1,\alpha_t}(2q)+\phi_{2,\alpha_t}(q,q) = \frac{\Gamma(2q-\alpha_t)}{\Gamma(2q)\Gamma(1-\alpha_t)}+\frac{\alpha_t}{\Gamma(2q)}\left[\frac{\Gamma(q-\alpha_t)}{\Gamma(1-\alpha_t)}\right]^2.\end{aligned}$$

Note the expression of the complete LSt of $Z_t(q)$:

$$\mathbf{E}\left(e^{-\lambda Z_t(q)}\right) = \sum_{i\geq 0}\frac{(-\lambda)^i}{i!}\sum_{j=1}^{i}\frac{1}{j!}\sum_{i_1+...+i_j=i}^{*}\binom{i}{i_1...i_j}\phi_{j,\alpha_t}(qi_1,...,qi_j) \tag{38}$$

and observe that

$$\mathbf{E}\left(Z_t(q)^\beta\right) = \frac{1}{\Gamma(-\beta)}\int_0^\infty d\lambda\cdot\lambda^{-\beta-1}\mathbf{E}\left(e^{-\lambda Z_t(q)}\right).$$

With $R_t(q) = \frac{1}{1-q}\log Z_t(q)$, the Rényi entropy of the random partition $PD(\alpha_t, 0)$, from the series expansion (38) in powers of λ, we obtain

$$\mathbf{E}\left(e^{-\beta R_t(q)}\right) = \frac{1}{\Gamma(-\beta/(1-q))}\int_0^\infty d\lambda\cdot\lambda^{-\frac{\beta}{1-q}-1}\mathbf{E}\left(e^{-\lambda Z_t(q)}\right),$$

as a formal expression of the LSt of $R_t(q)$. Recall $R_t(q) \underset{q\to 1}{\to} S_t$, the random Shannon entropy of $PD(\alpha_t, 0)$, with mean s_t given in (35).

• **The number of segments in $PD(\alpha_t, 0)$ with size larger than some threshold $1 > \varepsilon > 0$.** This is $I_+(t,\varepsilon) := \sum_{k\geq 1} \mathbf{1}_{\left\{\delta_{(k)}(t) > \varepsilon\right\}}$. Taking $g(s) = \mathbf{1}_{\{s>\varepsilon\}}$ in (34), we already know that

$$\mathbf{E}\sum_{k\geq 1} g\left(\delta_{(k)}(t)\right) = \mathbf{E}I_+(t,\varepsilon) = \frac{1}{\Gamma(1-\alpha_t)\Gamma(\alpha_t)}\int_\varepsilon^1 s^{-(\alpha_t+1)}(1-s)^{\alpha_t-1}\,ds.$$

Taking into account that

$$\varepsilon^{\alpha_t}\int_\varepsilon^1 s^{-(\alpha_t+1)}(1-s)^{\alpha_t-1}\,ds = \int_1^{1/\varepsilon} u^{-(\alpha_t+1)}(1-\varepsilon u)^{\alpha_t-1}\,du \underset{\varepsilon \text{ small}}{\sim} \frac{1}{\alpha_t},$$

it holds that $\mathbf{E}I_+(t,\varepsilon) \sim \varepsilon^{-\alpha_t}/\left(\Gamma(1-\alpha_t)\Gamma(1+\alpha_t)\right) = \varepsilon^{-\alpha_t}\text{sinc}(\pi\alpha_t)$, as ε goes small. When t is close to 0 (α_t close to 1 and $\text{sinc}(\pi\alpha_t)$ close to 0), all segments are very small; in the opposite direction, when t gets large (α_t approaches 0 and $\text{sinc}(\pi\alpha_t)$ approaches 1), only one segment (the one of biggest size) will prevail and $I_+(t,\varepsilon)$ tends to 1 in average.

Take now $g(s) = e^{-q\mathbf{1}_{\{s>\varepsilon\}}} = 1 + \left(e^{-q}-1\right)\mathbf{1}_{\{s>\varepsilon\}} =: 1 + \left(e^{-q}-1\right)g_\varepsilon$ in (36, 37). Then, with $J_t(p) = \int_1^\infty e^{-pu}u^{-(\alpha_t+1)}du$,

$$\begin{aligned}
\int_0^\infty e^{-px}\mathbf{E}\prod_{k\geq 1} g\left(x\delta_{(k)}(t)\right)dx &= \int_0^\infty e^{-px}\mathbf{E}\left(e^{-qI_+(t,\varepsilon/x)}\right)dx \\
H_{t,g}(p) \quad : &= \int_0^\infty \left(1 - e^{-ps}\left(1+\left(e^{-q}-1\right)\mathbf{1}_{\{s>\varepsilon\}}\right)\right)s^{-(\alpha_t+1)}ds \\
&= H_{t,1}(p) - \varepsilon^{-\alpha_t}\left(e^{-q}-1\right)J_t(p\varepsilon).
\end{aligned}$$

Thus, recalling $H_{t,1}(p) = -\Gamma(-\alpha_t)p^{\alpha_t}$ and observing $J_t'(p) = \frac{1}{p}\left(\alpha_t J_t(p) - e^{-p}\right)$

$$\begin{aligned}
\frac{1}{\alpha_t}\partial_p K_{t,g}(p) &= \frac{1}{p}\frac{-\Gamma(-\alpha_t)p^{\alpha_t} - \varepsilon^{-\alpha_t}\left(e^{-q}-1\right)\left(J_t(p\varepsilon) - e^{-p\varepsilon}/\alpha_t\right)}{-\Gamma(-\alpha_t)p^{\alpha_t} - \varepsilon^{-\alpha_t}\left(e^{-q}-1\right)J_t(p\varepsilon)} \\
&= \frac{1}{p}\left(1 + \frac{\left(e^{-q}-1\right)e^{-p\varepsilon}}{\Gamma(1-\alpha_t)(p\varepsilon)^{\alpha_t} - \left(e^{-q}-1\right)\alpha_t J_t(p\varepsilon)}\right).
\end{aligned}$$

Now, observing $I_+(t,u) = 0$ unless $u \in (0,1)$,

$$\int_0^\infty e^{-px}\mathbf{E}\left(e^{-qI_+(t,\varepsilon/x)}\right)dx = \varepsilon\int_0^1 u^{-2}e^{-p\varepsilon/u}\mathbf{E}\left(e^{-qI_+(t,u)}\right)du + \frac{1}{p}\left(1-e^{-p\varepsilon}\right).$$

Putting $\lambda = p\varepsilon$, we therefore obtain

$$\begin{aligned}
\int_0^1 u^{-2}e^{-\lambda/u}\mathbf{E}\left(e^{-qI_+(t,u)}\right)du &= \int_1^\infty e^{-\lambda v}\mathbf{E}\left(e^{-qI_+(t,1/v)}\right)dv \\
&= \frac{e^{-\lambda}}{\lambda}\left(1 + \frac{e^{-q}-1}{\Gamma(1-\alpha_t)\lambda^{\alpha_t} - \left(e^{-q}-1\right)\alpha_t J_t(\lambda)}\right).
\end{aligned}$$

Using $\alpha_t J_t(\lambda) \sim 1 - \Gamma(1-\alpha_t)\lambda^{\alpha_t}$ when λ is small, a small λ estimate of the right-hand-side gives $\Gamma(1-\alpha_t)\lambda^{-(1-\alpha_t)}\left(1/\left(1-e^{-q}\right)-1\right)$. This gives a large v estimate of $\mathbf{E}\left(e^{-qI_+(t,1/v)}\right)$ as

$$\mathbf{E}\left(e^{-qI_+(t,1/v)}\right) \sim v^{-\alpha_t}\frac{1}{e^q-1}.$$

which means $\mathbf{E}\left(e^{-qv^{-\alpha_t}I_+(t,1/v)}\right) \sim q^{-1}$ as $v \to \infty$ or $\varepsilon^{\alpha_t} I_+(t,\varepsilon) \overset{d}{\to} \mathbf{1}_{\{\varepsilon>0\}}$ as $\varepsilon \to 0$.

• **Sampling from $PD(\alpha_t, 0)$ and the block-counting process $\{I(t)\}$.** Throw uniformly at random a sample of size $i \geq 1$ over the unit interval partitioned according to $PD(\alpha_t, 0)$ and evaluate the probability of an i to $(i_1, ..., i_j)-$ merger with $i_l \geq 1$ and $\sum_{l=1}^{j} i_l = i$, which is the probability that the i sampling particles hit any size-j subset of segments (labeled in an arbitrary way) from $PD(\alpha_t, 0)$, $I_l(t) = i_l \geq 1$ times, $l = 1, ..., j \leq i$. With $[\alpha]_k := \Gamma(\alpha + k)/\Gamma(\alpha) = \alpha(\alpha + 1)...(\alpha + k - 1)$ defining the rising factorials, it is known by Pitman sampling formula (see eg [22] p. 58) that this probability is equal to

$$\mathbf{P}\left(I_l(t) = i_l, l = 1, ..., j; \sum_{l=1}^{j} I_l(t) = i\right) = \frac{1}{j!}\binom{i}{i_1 ... i_j}\phi_{j,\alpha_t}(i_1, ..., i_j)$$

where

$$\phi_{j,\alpha_t}(i_1, ..., i_j) = \alpha_t^{j-1}\frac{\Gamma(j)}{\Gamma(i)}\prod_{l=1}^{j}\frac{\Gamma(i_l - \alpha_t)}{\Gamma(1 - \alpha_t)} = \alpha_t^{j-1}\frac{(j-1)!}{(i-1)!}\prod_{l=1}^{j}[1 - \alpha_t]_{i_l - 1},$$

and this defines the continuous-time-t partition-valued Bolthausen-Sznitman coalescent, so with $PD(\alpha_t, 0)$ distribution.

Let $I(t)$ count the number of distinct pieces of the partition $PD(\alpha_t, 0)$ which are being visited in the sampling process, starting from $I(0) = i$. Then, with $c_{j,\alpha_t} := \prod_{l=1}^{j}\frac{\Gamma((l-1)\alpha_t + 1)}{\Gamma(1-\alpha_t)\Gamma(l\alpha_t)}$,

$$\begin{aligned}\mathbf{P}(I(t) = j \mid I(0) = i) &= \frac{1}{j!}\sum_{i_1 + ... + i_j = i}^{*}\binom{i}{i_1 ... i_j}\phi_{j,\alpha_t}(i_1, ..., i_j)\\ &= c_{j,\alpha_t}\frac{i!}{j!}\frac{\Gamma(\alpha_t j)}{\Gamma(i)}\sum_{i_1 + ... + i_j = i}^{*}\prod_{l=1}^{j}\frac{\Gamma(i_l - \alpha_t)}{i_l!}.\end{aligned}$$

When t gets large, using $\Gamma(\alpha_t) \sim \alpha_t^{-1}$, $c_{j,\alpha_t} \sim j!\alpha_t^j$ and

$$\mathbf{P}_i(I(t) = j) \sim \alpha_t^{j-1}r_{i,j} \text{ with } r_{i,j} = \frac{i}{j}\sum_{i_1 + ... + i_j = i}^{*}\frac{1}{\prod_{l=1}^{j} i_l} = \frac{(j-1)!}{(i-1)!}|s_{i,j}|,$$

where $s_{i,j}$ are the Stirling numbers of the first kind. It is well-known [14] that if $R = [r_{i,j}]$, then $L = R^{-1} = [l_{i,j}]$ with $l_{i,j} = (-1)^{i-j}\frac{(j-1)!}{(i-1)!}S_{i,j}$ and $S_{i,j}$ are the Stirling numbers of the second kind. Furthermore [48], the following spectral representation holds

$$\mathbf{P}_i(I(t) = j) = \sum_{k=j}^{i} e^{-(k-1)t}r_{i,k}l_{k,j}. \tag{39}$$

In particular,

$$\mathbf{P}_i(I(t) = 1) = \frac{\Gamma(i - \alpha_t)}{\Gamma(i)\Gamma(1 - \alpha_t)} = \mathbf{P}(\tau_i \leq t),$$

where $\tau_i = \inf(t > 0 : I(t) = 1 \mid I(0) = i)$ is the time to absorption (time to most recent common ancestor). With ψ the digamma function again, we have

$$\mathbf{P}(\tau_i > t) = \sum_{2 \leq j \leq i} \mathbf{P}(I(t) = j \mid I(0) = i) \underset{\text{large } t}{\sim} (\psi(i) - \psi(1)) e^{-t}$$

and τ_i is tail-equivalent to an exp(1) random variable. It follows as well from (39) (see Lemma 3.1 of [49]) that the rising factorial moments of $I(t)$ are given by

$$\mathbf{E}_i\left([I(t)]_j\right) = \frac{\Gamma(1+j)}{\Gamma(1+j\alpha_t)} \frac{\Gamma(i+j\alpha_t)}{\Gamma(i)},\ i \geq j \geq 1. \tag{40}$$

We have:

$$\mathbf{E}\left[(1-u)^{-I(t)} \mid I(0) = i\right] = \frac{1}{\Gamma(i)} \sum_{j \geq 0} \frac{\Gamma(i+j\alpha_t)}{\Gamma(1+j\alpha_t)} u^j,$$

so that the pgf of $I(t)$ given $I(0) = i$ is given by

$$\mathbf{E}\left(x^{I(t)} \mid I(0) = i\right) = \frac{1}{\Gamma(i)} \sum_{j \geq 0} (-1)^j \frac{\Gamma(i+j\alpha_t)}{\Gamma(1+j\alpha_t)} \left(x^{-1} - 1\right)^j,\ x \in [0,1]. \tag{41}$$

In fact, $\phi_i(x) := \mathbf{E}\left(x^{I(t)} \mid I(0) = i\right)$ is a degree$-i$ polynomial pgf in x (so absolutely monotone with $\phi_i(1) = 1$), obeying the recurrence

$$\phi_{i+1}(x) = \phi_i(x) - \frac{\alpha_t}{i} x(1-x)\phi_i'(x),\ i \geq 1,\ \phi_1(x) = x. \tag{42}$$

As t varies, we are then left with a Markov dynamics for $I(t) \in \{1, 2, ...\}$ which is the block-counting process of the Bolthausen-Sznitman coalescent, so with explicit global transition probabilities $\mathbf{P}_i(I(t) = j)$ and moments. Introducing the infinitesimal transition rates $q_{i,j}$ of $I(t)$ from state i to state $j \in \{1, ..., i-1\}$ by $\mathbf{P}(I(t+dt) = j \mid I(t) = i) = q_{i,j}dt + o(dt)$, these are known [55] to be given by $\Lambda(du) = du$ and (with $B(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$ the beta functions),

$$\begin{aligned} q_{i,j} &= \binom{i}{j-1} \int_0^1 u^{i-j-1}(1-u)^{j-1} \Lambda(du) = \binom{i}{j-1} B(i-j, j) \text{ if } 1 \leq j < i \\ q_{i,i} &= -\sum_{j \neq i} q_{i,j} \text{ if } j = i. \end{aligned}$$

The pure-death process $I(t)$ on the positive integers with such transition rates is known as the Bolthausen-Sznitman (block-counting) coalescent process with $\mathbf{P}_i(I(t) = j) = \left(e^{tQ}\right)_{i,j}$ and Q the lower-triangular matrix with entries

$$\begin{aligned} q_{i,j} &= \frac{i}{(i-j)(i-j+1)} \text{ if } 1 \leq j < i \\ q_{i,i} &= -(i-1) \text{ if } j = i. \end{aligned}$$

It has $\{1\}$ as an absorbing state where coalescence stops at time τ_i. The Bolthausen-Sznitman coalescent is a fundamental but particular case of $\Lambda-$coalescents with multiple (but not simultaneous) collisions when the collision measure $\Lambda(du)$ is uniform on the unit interval, [55].

Considering the restriction of Q to its i first rows and columns yields the Bolthausen-Sznitman $i-$coalescent $I_i(t) := I(t) \mid I(0) = i$ for which Möhle [49] shows, upon scaling, that, with $I_i(0)=i=[x]$,

$$\left\{\frac{I_i(t)}{i^{\alpha_t}}\right\}\underset{i\to\infty}{\overset{d}{\to}}\left\{\widehat{X}(t)\right\}.$$

Note from (40) that $\mathbf{E}_i(I(t))=\frac{1}{\Gamma(1+\alpha_t)}\frac{\Gamma(i+\alpha_t)}{\Gamma(i)}\underset{i\text{ large}}{\sim}\frac{i^{\alpha_t}}{\Gamma(1+\alpha_t)}$, explaining the power-law scaling.

• **A related occurrence of the Mittag-Leffler process.** The process $\widehat{X}_{\alpha_t}$ appearing in the construction of the Siegmund dual $\widehat{X}(t)=x^{\alpha_t}\widehat{X}_{\alpha_t}$ to $X(t)$ also appears in the following context. In view of $S_k/k\to 1$ a.s. as $k\to\infty$ by the strong law of large numbers, we conclude following ([57], Prop. 9) that, with $R(t):=xX(t)^{-\alpha_t}/\Gamma(1-\alpha_t)$,

$$R(t):=\lim_{k\to\infty}k\cdot\delta_{(k)}(t)^{\alpha_t}$$

exists both a.s. and in $q-$mean, $q>-1$. This is an expression of the algebraic rate of decrease of the $k-$th contribution $\Delta_{(k)}(t)$ to the total current state of $X(t)$, which is $\delta_{(k)}(t)$. As time passes by, the contribution $\delta_{(k)}(t)$ becomes smaller and smaller, starting from $\delta_{(k)}(t)\sim k^{-1}$ (as $t\to 0$), till $\delta_{(1)}(t)$ approaches 1 as $t\to\infty$ with no room left for other $\delta_{(k)}(t)$s.

Taking into account that $X(t)\overset{d}{=}x^{\beta_t}S_{\alpha_t}$ where S_{α_t} has stable(α_t) distribution and recalling $\widehat{X}_{\alpha_t}\overset{d}{=}S_{\alpha_t}^{-\alpha_t}$, we get $R(t)\overset{d}{=}\widehat{X}_{\alpha_t}/\Gamma(1-\alpha_t)$ with $\widehat{X}_{\alpha_t}$ defining a type-2 Mittag-Leffler process with moment generating function $\mathbb{E}\left[\widehat{X}_{\alpha_t}^q\right]=\frac{\Gamma(q+1)}{\Gamma(q\alpha_t+1)}$ (or equivalently with LSt the Mittag-Leffler function $E_{\alpha_t}(-p)$).

4.3. **Moment dual process $x(t)$ to the Bolthausen-Sznitman block-counting process $I(t)$.** So far, we saw that the Siegmund dual to the Neveu process (intimately related to $PD(\alpha_t,0)$) was a Mittag-Leffler process, useful in the asymptotic description of the block-counting process $\{I(t)\}$ arising from sampling from $PD(\alpha_t,0)$. Define now a $[0,1]-$valued Markov process $\{x(t)\}$ by the moment duality relation, [52]:

$$\mathbf{E}\left(x(t)^i\mid x(0)=x\right)=\mathbf{E}\left(x^{I(t)}\mid I(0)=i\right).$$

$\{x(t)\}$ is thus now the moment dual to $\{I(t)\}$. From (41) for instance, the mean, variance and skewness read

$$\begin{aligned}
\mathbf{E}(x(t)\mid x(0)=x) &= \phi_1(x)=x,\ \text{if } i=1\\
\mathbf{E}\left(x(t)^2\mid x(0)=x\right) &= \phi_2(x)=x-\alpha_t x(1-x),\ \text{if } i=2\\
\operatorname{Var}(x(t)\mid x(0)=x) &= \phi_2(x)-\phi_1(x)^2=(1-\alpha_t)x(1-x),\\
\mathbf{E}_x\left((x(t)-x(0))^3\right) &= \frac{-x(1-x)}{2}\left[1-\alpha_t^2+2x(1-\alpha_t)(2-\alpha_t)\right],
\end{aligned}$$

and the full moment sequence of $\{x(t)\}$ can be computed recursively using (42). With $\Lambda(du)=du$ (uniform), the dual process $\{x(t)\}$ is a well-defined two-types $\Lambda-$Fleming-Viot (Λ uniform) Markov jump process in continuous-time. It describes the forward in time (neutral) evolution of the fraction of type-1 alleles whose genealogical process backward in time is precisely $I(t)$. The states $\{0,1\}$ are absorbing for $\{x(t)\}$. More precisely, $\{x(t)\}$, as a martingale, has backward infinitesimal generator G (see [7], [19], [32])

$$\phi \in C^2([0,1]) \to G\phi(x) =$$
$$\int_{[0,1]\setminus\{0\}} [x\phi(x+(1-x)u) + (1-x)\phi(x(1-u)) - \phi(x)] \frac{1}{u^2}\Lambda(du),$$

which is the one of a pure jump process because Λ has no atom at $\{0\}$, so with

$$u(x,t) = \mathbf{E}_x\phi(x(t)) \text{ obeying } \partial_t u = G(u); \ u(x,0) = \phi(x).$$

Equivalently, the sample-paths of $x(t)$ obey the stochastic evolution

$$x(t) - x(0) = \int_{(0,t]\times(0,1]\times[0,1]} \left(1_{v\leq x_{s_-}} u\left(1-x_{s_-}\right) - 1_{v>x_{s_-}} u x_{s_-}\right) \mathcal{N}(ds \times du \times dv),$$

where $\mathcal{N}$ is a random Poisson measure on $[0,\infty) \times (0,1] \times [0,1]$ with intensity $ds \times \frac{1}{u^2}\Lambda(du) \times dv = ds \times \frac{1}{u^2} du \times dv$.

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