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Francis Wlazinski

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Abstract

In some circumstances, if the image by a morphism of a $(k+1)$ -power-free word contains a $(k+1)$ -power, we can reduce this word to obtain a new word with a same scheme. Theses circumstances are verified in the case of uniform morphisms that allows us to state that a k -power-free uniform morphism is a $(k+1)$ -power-free morphism when $k \geq 4$.

Reduction in non- $(k + 1)$ -power-free morphisms

Francis Wlazinski

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1 Introduction

As many problems in combinatorics on words, the origin of which is our concern derives directly from the original problem that is to say, the construction of an infinite word without consecutive repetition of the same factor. Such a word is called a square-free word. Since [21] (see also [2]), we know how to build infinite square-free words on a three-letters alphabet but also infinite overlap-free words on a two-letters alphabet that is words that do not have any factor of the form $auaua$ with a a letter and u a word.

Most explicitly built infinite square-free words or infinite overlap-free words (for instance in [22, 17, 8]) are obtained by iterating a morphism. They are generated as fixed points of free monoid morphisms. Indeed, a non-erasing ($\forall x \in A, f(x) \neq \varepsilon$) endomorphism f on an alphabet A such that $f(a) = au$ with $u \neq \varepsilon$ verifies $f^{n+1}(a) = f^n(a)f^n(u)$ for any positive integer n . Consequently, $f^n(a)$ is a prefix of $f^{n+1}(a)$ and we can define the infinite word $\bar{a} = \lim_{n \rightarrow +\infty} f^n(a)$. We say that $\bar{a}$ is the (infinite) word generated by f .

The study of infinite of square-free words [1, 6] and overlap-free words [9] generated by morphisms was extended to words avoiding other repetitions: cube u^3 [10] and more generally k -power u^k [19].

Although other ways have therefore been adopted, the most commonly used method to produce square-free words remains to iterate a morphism from a letter of the alphabet. Some type of morphisms appears: the morphisms that preserve the absence of the repetition. Note that if the morphism preserves the absence of square, that is to say if the image of a square-free word by this morphism is also square-free, then the sequence that it will produce will be square-free. Two different definitions follows for the morphisms: those that generate square-free words and those that preserve the absence of square simply called square-free morphisms. The study of square-free morphisms is thus a more specific part of the previous problem. This definition can be extended to the other k -powers with $k \geq 3$ but also to sesqui-powers or to fractional powers.

Several methods exist to verify whether a morphism is square-free [5], overlap-free [3, 18], cube-free [20] or k -power-free [13, 12, 23]. In this search to verify whether a morphism preserves the absence of a repetition, the uniform morphisms that is whose where the images of the letters have the same length give specific results [7, 11].

In line with this approach, a natural question arises: is a k -power morphism also a $(k + 1)$ -power-free morphism? In other words, if the image of any k -power-free word by a morphism f

is k -power-free, is the image of a $(k+1)$ -power-free word also $(k+1)$ -power-free? The answer already exists for the Thue-Morse morphism [4]: it is k -power-free for every integer $k > 2$.

In the search for an answer, equations of words (Lemma 3.1) appears in the initial case of a non- $(k+1)$ -power-free morphism. We give some conditions (Lemma 3.9) where we can simplify the initial equations. We call this simplification a reduction of the initial word: we construct a new word whose image contains a $(k+1)$ -power but with a lower length. The fact that the powers are synchronised (Lemma 2.12) appears as a particular case and will allow us to conclude for uniform morphisms (Proposition 4.1).

2 Preliminaries

Let us recall some basic notions of Combinatorics of words.

2.1 Words

An *alphabet* A is a finite set of symbols called *letters*. A *word* over A is a finite sequence of letters from A . The *empty word* ε is the empty sequence of letters. Equipped with the concatenation operation, the set A^* of words over A is a free monoid with ε as neutral element and A as set of generators. Since an alphabet with one element is limited interest to us, we always assume the cardinal of considered alphabets is at least two. Given a non-empty word $u = a_1 \dots a_n$ with $a_i \in A$ for any integer i from 1 to n , the *length* of u denoted by $|u|$ is the integer n that is the number of letters of u . By convention, we have $|\varepsilon| = 0$. The *mirror image* of u , denoted by $\tilde{u}$, is the word $a_n \dots a_2 a_1$. In the particular case of the empty word, $\tilde{\varepsilon} = \varepsilon$.

A word u is a *factor* of a word v if there exist two (possibly empty) words p and s such that $v = pus$. We denote $\text{Fcts}(v)$ the set of all factors of v . If $u \in \text{Fcts}(v)$, we also say that v *contains* the word u (as a factor). If $p = \varepsilon$, u is a *prefix* of v . If $s = \varepsilon$, u is a *suffix* of v . If $u \neq v$, u is a *proper factor* of v . If u , p and s are non-empty, u is an *internal factor* of v .

Two words u and v are *conjugated* if $u = t_1 t_2$ and $v = t_2 t_1$ for two (possibly empty) words t_1 and t_2 .

Let w be a non-empty word and let i, j be two integers such that $0 \leq i - 1 \leq j \leq |w|$. We denote by $w[i..j]$ the factor of w such that $|w[i..j]| = j - i + 1$ and $w = pw[i..j]s$ for two words s and p verifying $|p| = i - 1$. Note that, when $j = i - 1$, we have $w[i..j] = \varepsilon$. When $i = j$, we also denote by $w[i]$ the factor $w[i..i]$ which is the i^{th} letter of w . In particular, $w[1]$ and $w[|w|]$ are respectively the first and the last letter of w .

Powers of a word are defined inductively by $u^0 = \varepsilon$, and for any integer $n \geq 1$, $u^n = uu^{n-1}$. Given an integer $k \geq 2$, since the case ε^k is of little interest, we called a k -power any word u^k with $u \neq \varepsilon$. Given an integer $k \geq 2$, a word is *k -power-free* if it does not contain any k -power as factor. A *primitive* word is a word which is not a k -power of an another word whatever the integer $k \geq 2$. A (non-empty) k -power v^k is called *pure* if any proper factor of v^k is k -power-free. In particular, we say that v^k is a pure k -power of a word w if $v \in \text{Fcts}(w)$ and v^k is pure. Repeating the fact that a non-pure k -power contains a k -power which is itself pure or not, we obtain that any k -power contains a pure k -power. Moreover, if v^k is a pure k -power then v is primitive but the converse does not hold. Let us also remark that a word can not start with two different pure k -powers.

The following proposition gives the well-known solutions (see [15]) to an elementary equation in words and will be widely used in the following sections:

Proposition 2.1 *Let A be an alphabet and u, v, w three words over A .*

1. *If $vu = uw$ and $v \neq \varepsilon$ then there exist two words r and s over A and an integer n such that $u = r(sr)^n$, $v = rs$ and $w = sr$.*
2. *If $vu = uv$, then there exist a word w over A and two integers n and p such that $u = w^n$ and $v = w^p$.*

We also need three other properties on words. The first one is an immediate consequence of Proposition 2.1(2).

Lemma 2.2 [12, 14] *If a non-empty word v is an internal factor of vv , i.e., if there exist two non-empty words x and y such that $vv = xvy$ then there exist a non-empty word t and two integers $i, j \geq 1$ such that $x = t^i$, $y = t^j$ and $v = t^{i+j}$.*

We also use a well-known result on combinatorics on words:

Proposition 2.3 (Fine and Wilf) [15, 16] *Let x and y be two words. If a power of x and a power of y have a common prefix of length at least equal to $|x| + |y| - \gcd(|x|, |y|)$ then x and y are powers of the same word.*

As a consequence of Proposition 2.3, we get:

Corollary 2.4 (Keränen) [12] *Let x and y be two words. If a power of x and a power of y have a common factor of length at least equal to $|x| + |y| - \gcd(|x|, |y|)$ then there exist two words t_1 and t_2 such that x is a power of $t_1 t_2$ and y is a power of $t_2 t_1$ with $t_1 t_2$ and $t_2 t_1$ primitive words. Furthermore, if $|x| > |y|$ then x is not primitive.*

2.2 Morphisms

Let A and B be two alphabets. A *morphism* f from A^* to B^* is a mapping from A^* to B^* such that $f(uv) = f(u)f(v)$ for all words u, v over A . When B has no importance, we will say that f is a morphism on A or that f is defined on A . Given an integer L , f is *L -uniform* if for each letter a in A we have $|f(a)| = L$. A morphism f is *uniform* if it is L -uniform for some integer $L \geq 0$. Given a set X of words over A , and given a morphism f on A , we denote by $f(X)$ the set $\{f(w) \mid w \in X\}$.

A morphism f on A is *k -power-free* if and only if $f(w)$ is k -power-free for all k -power-free words w over A . For instance, the *empty morphism* ϵ ($\forall a \in A, \epsilon(a) = \varepsilon$) or the *identity morphism* Id ($\forall a \in A, Id(a) = a$) are k -power-free.

A *non-erasing* morphism is a morphism for which $f(a) \neq \varepsilon$ for all letters $a \in A$. The empty morphism ϵ is the only morphism which is both erasing and k -power-free. Indeed, for any non-empty erasing morphism f , there exist two different letters a and b in A (remember $\text{Card}(A) \geq 2$) such that $f(a) \neq \varepsilon$, $f(b) = \varepsilon$, and so $f(aba^{k-1})$ contains a k -power.

A morphism on A is called *prefix* (resp. *suffix*) if, for all letters a and b in A , the word $f(a)$ is not a prefix (resp. not a suffix) of $f(b)$. A prefix (resp. suffix) morphism is non-erasing. A morphism is *bifix* if it is prefix and suffix.

Given a morphism f on A , the *mirror morphism* $\tilde{f}$ of f is defined for all words w over A , by $\tilde{f}(w) = \widetilde{f(w)}$. In particular, $\tilde{f}(a) = \widetilde{f(a)}$, $\forall a \in A$. Note that f is k -power-free if and only if $\tilde{f}$ is k -power-free.

Proofs of the three following lemmas are left to the reader.

Lemma 2.5 *Let f be a bifix morphism on an alphabet A and let u, v, w and t be words over A .*

The equality $f(u) = f(v)p$ with p be a prefix of $f(w)$ implies $u = vw'$ for a prefix w' of w such that $f(w') = p$.

And the equality $f(u) = sf(v)$ with s a suffix of $f(t)$ implies $u = t'v$ for a suffix t' of t such that $f(t') = s$.

Lemma 2.6 *Let f be a prefix morphism on an alphabet A , let u and v be words over A and let p_1 and p_2 be respectively prefixes of images by f of some letters a and b in A . If $(p_1; p_2) \neq (\varepsilon; f(b))$ and $(p_1; p_2) \neq (f(a); \varepsilon)$ then the equality $f(u)p_1 = f(v)p_2$ implies $u = v$ and $p_1 = p_2$.*

Lemma 2.7 *Let f be a suffix morphism on an alphabet A , let u and v be words over A and let s_1 and s_2 be respectively suffixes of images by f of some letters a and b in A . If $(s_1; s_2) \neq (\varepsilon; f(b))$ and $(s_1; s_2) \neq (f(a); \varepsilon)$ then the equality $s_1f(u) = s_2f(v)$ implies $u = v$ and $s_1 = s_2$.*

Definition 2.8 *A morphism f from A^* to B^* is a ps-morphism (Keränen [12] called f a ps-code) if and only if the equalities*

$$f(a) = ps, f(b) = ps' \text{ and } f(c) = p's$$

with $a, b, c \in A$ (possibly $c = b$), p, s, p', s' in B^ imply $b = a$ or $c = a$.*

Obviously, taking $c = b$, $s = \varepsilon$ in a first time and $p = \varepsilon$ in a second time, we get that a ps-morphism is a bifix morphism.

Lemma 2.9 [12, 14] *If f is not a ps-morphism then f is not a k -power-free morphism for any integer $k \geq 2$.*

Lemma 2.10 *Let f be a ps-morphism from A^* to B^* and let u, v and w be words over A such that $f(u) = \delta\beta$, $f(v) = \alpha\beta$ and $f(w) = \alpha\gamma$ for some non-empty words α, β, γ and δ over B . Then, it implies that $v = v_1av_2$, $u = u_1bv_2$ and $w = v_1cw_2$ for some words v_1, v_2, u_1, w_2 and some letters a, b, c . Moreover, we have either $b = a$ or $c = a$.*

In particular, if $|\delta| < |f(u[1])|$ then $u_1 = \varepsilon$ and if $|\gamma| < |f(w[|w|])|$ then $w_2 = \varepsilon$.

Proof.

Let us recall that, as any ps-morphism, f is bifix i.e. prefix and suffix.

Let $v[1..i]$ be the shortest prefix of v such that α is a prefix of $f(v[1..i])$. Since $\alpha \neq \varepsilon$, we have $v[1..i] \neq \varepsilon$ i.e. $i \geq 1$. We denote $v_1 = v[1..i-1]$, $v_2 = v[i+1..|v|]$ and $a = v[i]$. There exist two words $p \neq \varepsilon$ and $s (\neq f(a))$ such that $f(a) = ps$, $\alpha = f(v_1)p$ and $\beta = sf(v_2)$.

Let $u[j..|u|] (\neq \varepsilon)$ be the shortest suffix of u such that β is a suffix of $f(u[j..|u|])$. There exist two words $s_1 \neq \varepsilon$ and $p_1 (\neq f(u[j]))$ such that $f(u[j]) = p_1s_1$ and $\beta = s_1f(u[j+1..|u|])$. In particular, if $|\delta| < |f(u[1])|$ then $p_1 = \delta (\neq \varepsilon)$ and $j = 1$.

Let $w[1..\ell]$ be the shortest prefix of w such that α is a prefix of $f(w[1..\ell])$. We denote $w_2 = w[\ell+1..|w|]$ and $c = w[\ell]$. There exist two words $p_2 \neq \varepsilon$ and $s_2 (\neq f(c))$ such that $f(c) = p_2s_2$, $\alpha = f(w[1..\ell-1])p_2$ and $\gamma = s_2f(w_2)$. In particular, if $|\gamma| < |f(w[|w|])|$ then $\ell = |w|$, $s_2 = \gamma (\neq \varepsilon)$ and $w_2 = \varepsilon$.

If $s \neq \varepsilon$, we denote $u_1 = u[1..j-1]$ and $b = u[j]$. Let us note that, if $|\delta| < |f(u[1])|$, we get $u_1 = \varepsilon$. Since f is bifix and by Lemma 2.7, the equality $(\beta =)sf(v_2) = s_1f(u[j+1..|u|])$, with $(s; s_1) \neq (\varepsilon; f(b))$ implies $u[j+1..|u|] = v_2$ and $s = s_1$ that is $u = u_1bv_2$. Furthermore, since $p, p_2 \neq \varepsilon$, we get $(p; p_2) \neq (\varepsilon; f(c))$ and $(p; p_2) \neq (f(a); \varepsilon)$. By Lemma 2.6, the equality $(\alpha =)f(v_1)p = f(w[1..\ell-1])p_2$ implies $p = p_2$ and $v_1 = w[1..\ell-1]$ that is $w = v_1cw_2$. So, we have $f(a) = ps$, $f(b) = p_1s$ and $f(c) = ps_2$. Since f is a ps-morphism, then $b = a$ or $c = a$.

If $s = \varepsilon$ then $\beta = f(v_2) = s_1f(u[j+1..|u|])$ with $s_1 \neq \varepsilon$. By Lemma 2.5, we get $s_1 = f(u[j])$, $p_1 = \varepsilon$ and $v_2 = u[j..|u|]$. Since $\delta \neq \varepsilon$, it follows that $j \geq 2$ and so $|\delta| \geq |f(u[1])|$. We denote $u_1 = u[1..j-2]$ and $b = u[j-1]$. We have $u = u_1bv_2$ but also $p = f(a)$, $f(v_1a) = \alpha = f(w[1..\ell-1])p_2$ with $p_2 \neq \varepsilon$. Since f is bifix, by Lemma 2.5, we get $s_2 = \varepsilon$ and $w[1..\ell-1]c = v_1a$ i.e. $c = a$ and $w = v_1aw_2$.

□

Definition 2.11 Let $k \geq 2$ be an integer. Let f be a morphism from A^* to B^* , w be a word over A and u be a non-empty word over B such that $f(w)$ contains the k -power u^k . Let $\bar{w}$ be a shortest factor of w which image by f contains u^k i.e. $f(\bar{w}) = pu^ks$ with $|p| < |f(\bar{w}[1])|$ and $|s| < |f(\bar{w}[|\bar{w}|])|$.

We say that $f(w)$ and u^k are synchronised if there exist three words w_0 , w_1 and w_2 such that $|f(w_0)| = |u|$ and $\bar{w} = w_1w_0w_2$ with $p = \varepsilon$ if $w_1 = \varepsilon$ and $s = \varepsilon$ if $w_2 = \varepsilon$.

In this situation, we also simply say that $f(w)$ contains a synchronised k -power.

The following lemma and its proof are based on Reduction 2 of the proof of Theorem 5.1 in [23].

Lemma 2.12 Let $k \geq 2$ be an integer. If f is a ps-morphism and if $f(w)$ contains a synchronised k -power then w contains a k -power.

Remark 2.13 More precisely, we prove that $\bar{w}$ starts or ends with a k -power which image by f is a conjugated of the synchronised k -power.

Proof.

Let u be the word such that $f(w)$ and u^k are synchronised, let $\bar{w}$ be the shortest factor of w which image by f contains u^k and let w_0 be a factor of $\bar{w}$ such that $|f(w_0)| = |u|$.

There exist a proper prefix p of $f(\bar{w}[1])$ and a proper suffix s of $f(\bar{w}[|\bar{w}|])$ such that $f(\bar{w}) = pu^ks$. Moreover, there exist two integers $0 \leq \ell < m \leq |\bar{w}|$ such that $\bar{w}[\ell+1..m] = w_0$.

If $\ell = 0$ i.e. $\bar{w}$ starts with w_0 then $p = \varepsilon$ and $f(\bar{w})$ starts with u . By Lemma 2.5, we get that $u = f(w_0)$ and $\bar{w}$ starts with w_0^k : w contains a k -power.

If $m = |\bar{w}|$ i.e. $\bar{w}$ ends with w_0 then, in a similar way, w contains the k -power w_0^k .

From now, let us assume that $0 < \ell < m < |\bar{w}|$ i.e. w_0 is an internal factor of $\bar{w}$. It implies that $f(w_0)$ is an internal factor of u^k . In particular, it means that $f(w_0)$ and u are conjugated.

For any integer j in $[0, k]$, let i_j be the smallest integer such that pu^j is a prefix of $f(\bar{w}[1..i_j])$ that is $|f(\bar{w}[1..i_j - 1])| < pu^j \leq |f(\bar{w}[1..i_j])|$ (except the special case $j = 0$ and $p = \varepsilon$ where the first inequation is not strict). We have $i_0 = 1$ and $i_k = |\bar{w}|$. There exist words p_j ($\neq \varepsilon$ when $j \neq 0$) and s_j such that $f(\bar{w}[i_j]) = p_j s_j$ for any $j \in [0, k]$, $p = p_1$, $s = s_k$ and $u = s_j f(\bar{w}[i_j + 1..i_{j+1} - 1]) p_{j+1}$ for any $j \in [0, k - 1]$.

Let us first remark that $|s_q| = |s_n|$ for two integers $0 \leq q, n \leq k - 1$ implies that $|p_{q+1}| = |p_{n+1}|$ (the converse also holds using Lemma 2.7 and the fact that s_q and s_n are not images of a letter). Indeed, since $u = s_q f(\bar{w}[i_q + 1..i_{q+1} - 1]) p_{q+1} = s_n f(\bar{w}[i_n + 1..i_{n+1} - 1]) p_{n+1}$, we get $s_q = s_n$ and $f(\bar{w}[i_q + 1..i_{q+1} - 1]) p_{q+1} = f(\bar{w}[i_n + 1..i_{n+1} - 1]) p_{n+1}$ with $p_{q+1} \neq \varepsilon$ and $p_{n+1} \neq \varepsilon$. Since f is bifix and by Lemma 2.6, it follows that $\bar{w}[i_q + 1..i_{q+1} - 1] = \bar{w}[i_n + 1..i_{n+1} - 1]$ and $p_{q+1} = p_{n+1}$.

Let δ be the integer such that $\ell \in [i_\delta, i_{\delta+1}[$.

We have the equality $|s_\delta f(\bar{w}[i_\delta + 1..l])| = |u| - |f(\bar{w}[\ell + 1..i_{\delta+1} - 1]) p_{\delta+1}| = |f(\bar{w}[\ell + 1..m])| - |f(\bar{w}[\ell + 1..i_{\delta+1} - 1]) p_{\delta+1}| = |s_{\delta+1} f(\bar{w}[i_{\delta+1} + 1..m])| (\leq |u|)$. But the words $s_\delta f(\bar{w}[i_\delta + 1..l])$ and $s_{\delta+1} f(\bar{w}[i_{\delta+1} + 1..m])$ are both prefixes of u . Thus $s_\delta f(\bar{w}[i_\delta + 1..l]) = s_{\delta+1} f(\bar{w}[i_{\delta+1} + 1..m])$.

If $\delta = 0$ and $p_0 = p = \varepsilon$ then $s_\delta (= s_0) = f(\bar{w}[i_\delta])$ and $f(\bar{w}[i_\delta..l]) = s_{\delta+1} f(\bar{w}[i_{\delta+1} + 1..m])$ with $s_{\delta+1} \neq f(\bar{w}[i_{\delta+1}])$. By Lemma 2.5, we get that $s_1 (= s_{\delta+1}) = \varepsilon$, $p_1 (= p_{\delta+1}) = f(\bar{w}[i_{\delta+1}])$ and $u = f(\bar{w}[1..i_1])$. Again by Lemma 2.5 and by induction, it implies that $\bar{w}$ starts with $(\bar{w}[1..i_1])^k$ with $|f(\bar{w}[1..i_1])| = |u|$ i.e. $f(\bar{w}[1..i_1])$ is a conjugated of u .

From now let us assume $\delta \neq 0$ or $p \neq \varepsilon$. Since f is bifix, $s_\delta \neq f(\bar{w}[i_\delta])$ and $s_{\delta+1} \neq f(\bar{w}[i_{\delta+1}])$, by Lemma 2.7, we get $s_\delta = s_{\delta+1}$. Thus, we have $p_{\delta+1} = p_{\delta+2}$ for an integer δ such that $0 \leq \delta \leq k - 2$.

We will now show that for any integer r such that $1 \leq r \leq \delta + 1$, we necessarily have $p_r = p_{\delta+1}$. By contradiction, assume that there exists an integer r verifying $1 \leq r \leq \delta + 1$ and $p_r \neq p_{\delta+1}$ and let us choose the greatest one. By this way, $p_{r+1} = p_{r+2} (= p_{\delta+1})$. It follows that $s_r f(\bar{w}[i_r + 1..i_{r+1} - 1]) = s_{r+1} f(\bar{w}[i_{r+1} + 1..i_{r+2} - 1])$. Since $s_r \neq f(\bar{w}[i_r])$ and $s_{r+1} \neq f(\bar{w}[i_{r+1}])$, by Lemma 2.7, we get $s_r = s_{r+1}$. But p_r and p_{r-1} are both prefixes of u thus one of the two different words p_r or p_{r+1} is a suffix of the other. It means that one the two different words $f(\bar{w}[i_r])$ or $f(\bar{w}[i_{r+1}])$ is a suffix of the other: a contradiction with f bifix.

In a similar way, we prove that, for any integer r in $[\delta + 1, k - 1]$, we have $s_r = s_\delta$ with $s_r \neq f(\bar{w}[i_r])$. And it follows that $p_r = p_\delta$ for any integer r in $[\delta + 2, k]$.

Thus we have $p_q = p_\delta = p_1$ and $s_0 f(\bar{w}[2..i_1 - 1]) p_1 = u = s_{q-1} f(\bar{w}[i_{q-1} + 1..i_q - 1]) p_q$ for all integers q in $[1, k]$.

If $s_0 = f(\bar{w}[1])$, since f is bifix and by Lemma 2.5, we get that $\bar{w}[i_{q-1} + 1..i_q] = \bar{w}[1..i_1]$ and $s_{q-1} = \varepsilon$ for all $2 \leq q \leq k$ that is w starts with $(\bar{w}[1..i_1])^k$ with $f(\bar{w}[1..i_1])$ a conjugated of u .

If $s_0 \neq f(\bar{w}[1])$, since f is bifix and by Lemma 2.7, we get that $\bar{w}[i_{q-1} + 1..i_q - 1] = \bar{w}[2..i_1 - 1]$ and $s_{q-1} = s_0$ for all $2 \leq q \leq k$. In particular, it means that $s_0 = s_1$. It follows that $\bar{w} = \bar{w}[1](\bar{w}[2..i_1 - 1]\bar{w}[i_1])^{r-1}\bar{w}[2..i_1 - 1]\bar{w}[|\bar{w}|]$ with $f(\bar{w}[1]) = ps_1$, $f(\bar{w}[i_1]) =$

p_1s_1 and $f(\overline{w}[\overline{w}]) = p_1s$. Since f is a ps-morphism, it means that $\overline{w}[1] = \overline{w}[i_1]$ or $\overline{w}[\overline{w}] = \overline{w}[i_1]$ i.e. $\overline{w} = (\overline{w}[1..i_1 - 1])^r \overline{w}[\overline{w}]$ or $\overline{w} = \overline{w}[1](\overline{w}[2..i_1])^k$: the word $\overline{w}$ starts or ends with a k -power which image is a conjugated of u . $\square$

Lemma 2.14 *Let $k \geq 4$ be an integer.*

The image of a pure k -power by a k -power-free morphism is also a pure k -power.

Proof. Let f be a k -power-free morphism from A^* to B^* and let v^k be a pure k -power over A .

If $f(v)^k$ was not a pure k -power then there would exist a pure k -power $u^k \in \text{Fcts}(f(v)^k)$ such that $|u| < |f(v)|$.

Since f is k -power-free and since the three proper factors of v^k $v[2..|v|]v^{k-2}v[1..|v| - 1]$, $v^{k-1}v[1..|v| - 1]$ and $v[2..|v|]v^{k-1}$ are k -power-free, we get that $|u^k| > |f(v[2..|v|]v^{k-2}v[1..|v| - 1])| \geq 2|f(v)| > |u| + |f(v)|$. By Corollary 2.4, $f(v)$ and u are powers of conjugated words and $f(v)$ is not primitive: a contradiction with the hypotheses.

$\square$

3 Reduction of a power

3.1 About k -power-free morphisms

Even if it seems not obvious, hypotheses of Lemma 3.1 appear almost immediatly when the image of a word by f contains a $(k + 1)$ -power.

Lemma 3.1 *Let $k \geq 4$ be an integer. Let f be a ps-morphism from A^* to B^* . Let v and T be non-empty words over A such that v^k is a pure k -power. Let us assume that $f(T) = \pi_1 f(v)^k \sigma_2$ with $|\pi_1| < |f(T[1])|$ and $|\sigma_2| < |f(T[|T|])|$. Then, we are in one of the following situations:*

- (P.1) : *There exist a pure k -power x^k , a word y over A and a word Z over B such that*
 - (P.1.1) : $T = x^k y$, $|y| \leq 1$, $f(y) = \pi_1 \sigma_2$, $f(x) = \pi_1 Z$ and $f(v) = Z \pi_1$
 - (P.1.2) : *or* $T = y x^k$, $|y| = 1$, $f(y) = \pi_1 \sigma_2$, $f(x) = Z \sigma_2$ and $f(v) = \sigma_2 Z$.
- (P.2) : *There exist a pure k -power x^k and a non-empty word y over A such that*
 - (P.2.1) : $T = x^k y$ with $|f(x^{k-1})| < |\pi_1 f(v)|$
 - (P.2.2) : *or* $T = y x^k$ with $|f(x^{k-1})| < |f(v) \sigma_2|$.
- (P.3) : *f is not k -power-free.*

Proof.

If T is k -power-free, it ends the proof: f is not k -power-free.

So T contains at least one k -power. Among the k -powers of T , we choose one which image by f is a shortest: we can write $T = y_1 x^k y_2$ where $|f(x)| = \min\{|f(x')| \text{ where } x'^k \in \text{Fcts}(T)\}$. By this definition, since f is bifix (as any ps-morphism) and so non-erasing, x^k is a pure k -power. Otherwise, x^k (and T) would contain a proper factor $\check{x}^k$ with $f(\check{x})^k$ proper factor of $f(x)^k$ i.e. $|f(\check{x})| < |f(x)|$: a contradiction with the definition of x .

Case 1: A power of $f(x)$ and a power of $f(v)$ have a common factor of length at least $|f(x)| + |f(v)|$

In a first time, we are going to list two cases where this situation necessarily happens.

If $y_1 \neq \varepsilon$ and $y_2 \neq \varepsilon$ since $|\pi_1| < |f(T[1])| \leq |f(y_1)|$ and $|\sigma_2| < |f(T[|T|])| \leq |f(y_2)|$, we get that $f(x)^k$ is an internal factor of $f(v)^k$. It follows that $|f(x)| < |f(v)|$. If $|f(x)^k| < |f(v)^{k-2}|$, by a length criterion, we get that $f(x)^k$ is an internal factor of $f(v)^{k-1}$ with v^{k-1} k -power-free: f is not k -power-free. Thus it is bound to $|f(x)^k| \geq |f(v)^{k-2}| \geq |f(v^2)| \geq |f(v)| + |f(x)|$.

If $y_1 = \varepsilon$ and $y_2 = \varepsilon$ then $T = x^k$, $f(x)^k = \pi_1 f(v)^k \sigma_2$, π_1 is a prefix of $f(x[1])$, σ_2 a suffix of $f(x[|x|])$ and $|f(x)| \geq |f(v)|$. The word $f(x)^{k-2} \in \text{Fcts}(f(v)^k)$ is a common factor of powers of the two words $f(x)$ and $f(v)$. Furthermore, we have $|f(x)|^{k-2} \geq |f(x)|^2 \geq |f(x)| + |f(v)|$.

Let us now really deal with this case 1. By Corollary 2.4, there exist two words t_1 and t_2 and two integers p and r such that $f(v) = (t_1 t_2)^p$ and $f(x) = (t_2 t_1)^r$ with $t_1 t_2$ and $t_2 t_1$ primitive words.

If $p \geq 2$ then $f(v^{\lceil k/2 \rceil})$ contains a k -power. Indeed, $f(v^{\lceil k/2 \rceil}) = (t_1 t_2)^{p \times \lceil k/2 \rceil}$ with $p \times \lceil k/2 \rceil \geq k$. In the same way, if $r \geq 2$ then $f(x^{\lceil k/2 \rceil})$ contains a k -power. But $v^{\lceil k/2 \rceil}$ (proper factor of v^k) and $x^{\lceil k/2 \rceil}$ (proper factor of x^k) are both k -power-free: f is not k -power-free.

So we can assume that $p = r = 1$. We have $f(T) = f(y_1)(t_2 t_1)^k f(y_2) = \pi_1 (t_1 t_2)^k \sigma_2$ with $|f(y_1 y_2)| = |\pi_1 \sigma_2|$, $|\pi_1| < |f(T[1])| = |f((y_1 x)[1])|$ and $|\sigma_2| < |f(T[|T|])| = |f((x y_2)[|x y_2|])|$.

If $y_2 \neq \varepsilon$, we get $|f(y_2)| > |\sigma_2|$ and thus $|f(y_1)| < |\pi_1|$. It means that $y_1 = \varepsilon$. Furthermore $|\pi_1| < |f(x)| = |t_2 t_1|$ and $f(T) = f(x^k y_2)$ starts with $(t_2 t_1)^k$ and $\pi_1 (t_1 t_2)^k$. Since $t_1 t_2$ is not an internal factor of $(t_1 t_2)^2$ (we know that $f(v) = t_1 t_2$ is primitive) and by a length criterion, we necessarily obtain that $t_2 = \pi_1$ and $f(y_2) = \pi_1 \sigma_2$. It follows that $f(v) = t_1 \pi_1$ and $f(x) = \pi_1 t_1$. Since $|t_2| = |\pi_1| < |f(T[1])| = |f(x[1])|$ and $|\sigma_2| < |f(y_2[|y_2|])|$, if $|y_2| \geq 2$, we get that $y_2[1..|y_2| - 1] \neq \varepsilon$ and $f(y_2[1..|y_2| - 1])$ is a prefix of $t_2 = \pi_1$ itself a prefix $f(x[1])$: a contradiction with f bifix. So $|y_2| = 1$ and $|f(y_2)| - |\sigma_2| = |t_2| \leq |f(v)|$.

In the same way, when $y_1 \neq \varepsilon$ we successively get $y_2 = \varepsilon$, $\sigma_2 = t_1$, $f(y_1) = \pi_1 \sigma_2$, $f(v) = \sigma_2 t_2$, $f(x) = t_2 \sigma_2$, $|y_1| = 1$ and $|f(y_1)| - |\pi_1| = |t_1| \leq |f(v)|$.

If $y_1 = y_2 = \varepsilon$ then $\pi_1 = \sigma_2 = \varepsilon$, $t_1 t_2 = t_2 t_1$ i.e. $x = v$ and $T = x^k$.

Case 2: Any power of $f(x)$ and any power of $f(v)$ do not have common factor of length at least $|f(x)| + |f(v)|$.

If $y_1 = \varepsilon$ and $y_2 \neq \varepsilon$, we have $T = x^k y_2$ and $f(x)f(x)^{k-1}f(y_2) = \pi_1 f(v)^k \sigma_2$. But π_1 is a prefix of $f(T[1]) = f(x[1])$: there exist a word σ_1 such that $f(x[1]) = \pi_1 \sigma_1$. Furthermore, $\sigma_1 f(x[2..|x|])f(x^{k-1}) \in \text{Fcts}(f(x^k)) \cap \text{Fcts}(f(v^k))$. Thus $|\sigma_1 f(x[2..|x|])f(x^{k-2})| < |f(v)|$ and $|f(x^{k-1})| < |\pi_1 f(v)|$.

In a same way, if $y_1 \neq \varepsilon$ and $y_2 = \varepsilon$, we have $T = y_1 x^k$ and $|f(x^{k-1})| < |f(v)\sigma_2|$. $\square$

By Lemma 2.12 and Remark 2.13, we immediatly get:

Corollary 3.2 *With hypotheses and notations of Lemma 3.1, if $f(T)$ and $f(v)^k$ are synchronised (this is obviously the case when f is a uniform ps-morphism) then either f is not k -power-free or T verifies (P.1).*

Corollary 3.3 *Let $k \geq 4$ be an integer. Let f be a ps-morphism from A^* to B^* . Let v^k and t^k be two pure k -powers over A . Let us assume that $f(t^k) = \pi_1 f(v)^k \sigma_2$ with $|\pi_1| < |f(t[1])|$ and $|\sigma_2| < |f(t[|t|])|$. If $\pi_1 \neq \varepsilon$ or if $\sigma_2 \neq \varepsilon$ then f is not k -power-free.*

Proof.

By Lemma 3.1, if f is k -power-free then t^k verifies (P.1) or (P.2). Since t^k is a pure k -power, it follows that t^k can only verify (P.1.1) with $|y| = 0$. But that contradicts the fact that $|f(y)| = |\pi_1\sigma_2| > 0$.

□

Corollary 3.4 *Let $k \geq 4$ be an integer. Let f be a ps-morphism from A^* to B^* . Let v and T be non-empty words over A such that v^k is a pure k -power. Let us assume that $f(T) = \pi_1 f(v)^{k+1} \sigma_2$ with $|\pi_1| < |f(T[1])|$ and $|\sigma_2| < |f(T[|T|])|$. Then, either f is not k -power-free or there exist a pure k -power x^k , a word Y over A and a word Z over B such that*

$$(P.1.1)' : T = x^{k+1}Y, |Y| \leq 1, f(Y) = \pi_1\sigma_2, f(x) = \pi_1Z \text{ and } f(v) = Z\pi_1$$

$$(P.1.2)' : \text{or } T = Yx^{k+1}, |Y| = 1, f(Y) = \pi_1\sigma_2, f(x) = Z\sigma_2 \text{ and } f(v) = \sigma_2Z.$$

Proof.

Let T_1 be the shortest prefix of T such that $f(T_1)$ starts with $\pi_1 f(v)^k$ i.e. $f(T_1) = \pi_1 f(v)^k \sigma'_2$ with $|\sigma'_2| < |f(T_1[|T_1|])|$ and let $\bar{T}_1$ be the word such that $T = T_1 \bar{T}_1$.

Let T_2 be the shortest suffix of T such that $f(T_2)$ ends with $f(v)^k \sigma_2$ i.e. $f(T_2) = \pi'_1 f(v)^k \sigma_2$ with $|\pi'_1| < |f(T_2[1])|$ and let $\bar{T}_2$ be the word such that $T = \bar{T}_2 T_2$.

By Lemma 3.1, either f is not k -power-free or each of the words T_1 and T_2 verify one of the condition (P.1.1), (P.1.2), (P.2.1) or (P.2.2).

If T_1 verifies (P.1.1) that is there exist a pure k -power x^k , a word y over A and a word Z over B such that $T_1 = x^k y$, $|y| \leq 1$, $f(y) = \pi_1 \sigma'_2$, $f(x) = \pi_1 Z$ and $f(v) = Z \pi_1$ then we get that $f(T) = f(x^k y \bar{T}_1) = (\pi_1 Z)^k \pi_1 \sigma'_2 f(\bar{T}_1) = \pi_1 (Z \pi_1)^{k+1} \sigma_2$. It means that $\sigma'_2 f(\bar{T}_1)$ starts with Z and $f(y \bar{T}_1)$ starts with $\pi_1 Z = f(x)$. Since f is injective, we get that $y \bar{T}_1$ starts with x : there exist a word Y such that $y \bar{T}_1 = xY$ with $f(Y) = \pi_1 \sigma_2$. If $|Y| \geq 2$, since $|\sigma_2| < |f(T[|T|])| = |f(Y[|Y|])|$, then $f(Y[1..|Y| - 1])$ is a prefix of π_1 itself a prefix of $f(x[1])$: a contradiction with f bifix. It follows that T verifies (P.1.1)'.

If T_1 verifies (P.1.2) then $f(v) = \sigma'_2 Z$ and $f(T) = f(y x^k \bar{T}_1) = \pi_1 \sigma'_2 (Z \sigma'_2)^k f(\bar{T}_1) = \pi_1 (\sigma'_2 Z)^{k+1} \sigma_2$. That is $f(\bar{T}_1) = Z \sigma_2$ with $|\sigma_2| < |f(T[|T|])| = |f(\bar{T}_1[|\bar{T}_1|])|$. Moreover, $f(y) = \pi_1 \sigma'_2$ with $|\pi_1| < |f(T[1])| = |f(y[1])| = |f(y)|$ and $f(x) = Z \sigma'_2$. By Lemma 2.10 and since $|\sigma'_2| < |f(T_1[|T_1|])| = |f(x[|x|])|$, we get that $x = x_1 a$ and $\bar{T}_1 = x_1 c$ for some word x_1 and some letters a and c with either $y = a$ or $c = a$. If $c = a$ then $\bar{T}_1 = x$, $\sigma'_2 = \sigma_2$ and T verifies (P.1.2)'. If $y = a$ then it means that $f(x)$ ends with $f(y) = f(a) = \pi_1 \sigma'_2$. It implies that Z ends with π_1 : there exist a word Z_1 such that $Z = Z_1 \pi_1$ and $f(x_1) = Z_1$. Since $f(\bar{T}_1) = Z \sigma_2 = Z_1 \pi_1 \sigma_2 = f(x_1) f(c)$, we get that $f(c) = \pi_1 \sigma_2$. Taking $a x_1$ for x , c for Y and $\sigma'_2 Z_1$ for Z , we get that T verifies (P.1.1)'.

In the same way, if T_2 verifies (P.1.2) then T verifies (P.1.2)' and if T_2 verifies (P.1.1) then T verifies (P.1.1)' or (P.1.2)'.

If T_1 verifies (P.2.2) that is $T_1 = y x^k$ with $|f(x^{k-1})| < |f(v) \sigma'_2|$, then, by definition of T_1 , we get $|\sigma'_2| < |f(x)| \leq |f(x)^{k-2}| < |f(v)|$ and $|f(y x^{k-1} x[1..|x| - 1])| < |\pi_1 f(v)^k|$. It follows $|f(x \bar{T}_1)| \geq |f(x[|x|] \bar{T}_1)| > |f(v) \sigma_2|$ and it implies that $\bar{T}_1 \neq \varepsilon$. Thus $f(x)^k$ is an internal factor of $f(v^{k-1})$: f is not k -power-free.

In the same way, if T_2 verifies (P.2.1) then f is not k -power-free.

Let us now assume that T_1 verifies (P.2.1) and T_2 verifies (P.2.2) i.e. there exist two pure k -powers x^k and x'^k and two non-empty words y and y' over A such that $T_1 = x^k y$

with $|f(x^{k-1})| < |\pi_1 f(v)|$ and $T_2 = y'x'^k$ with $|f(x'^{k-1})| < |f(v)\sigma_2|$. In particular, $|f(x)| < \frac{1}{2}|\pi_1 f(v)| < |\pi_1| + \frac{1}{2}|f(v)|$ and $|f(x')| < |\sigma_2| + \frac{1}{2}|f(v)|$. It follows that $|f(T[2..|T| - 1])| \geq |f(T)| - |f(x)| - |f(x')| > |f(v)^k|$: there exist a word V' conjugated of $f(v)$ such that $f(T[2..|T| - 1])$ contains V'^k . If $T[2..|T| - 1]$ is k -power-free then f is not. Thus $T[2..|T| - 1]$ contains a pure k -power t^k . But $f(t)^k$ is an internal factor of $f(v)^{k+1}$. So, if $|f(t)^k| \leq |f(t)| + |f(v)|$ then $f(t)^k$ is factor of $f(v^3)$ with v^3 k -power-free: f is not k -power-free. If $|f(t)^k| > |f(t)| + |f(v)|$, by Corollary 2.4, then there exist two words t_1, t_2 and two integers p, q such that $f(t) = (t_1 t_2)^p$ and $f(v) = (t_2 t_1)^q$ with $t_1 t_2$ and $t_2 t_1$ primitive words. Moreover, since v^k and t^k are pure k -powers, we get $p = q = 1$. Let T' and T'' be the non-empty words such that $T = T' t^k T''$. We have $f(T) = f(T')(t_1 t_2)^k f(T'') = \pi_1 (t_2 t_1)^{k+1} \sigma_2$ with $|\pi_1| < |f(T'[1])|$ and $|\sigma_2| < |f(T''[|T''|])|$. Since $t_1 t_2$ is a primitive word, $t_1 t_2$ is not an internal factor of $(t_1 t_2)^2$. So $f(T') = \pi_1 t_2$ and $f(T'') = t_1 \sigma_2$. By Lemma 2.10, it implies that $t = v_1 a v_2$, $T' = b v_2$ and $T'' = v_1 c$ for some words v_1, v_2, u_1, w_2 and some letters a, b, c . with either $b = a$ or $c = a$. That is $T = (a v_2 v_1)^{k+1} c$ or $T = b (v_2 v_1 a)^{k+1}$: T verifies $(P.1.1)'$ or $(P.1.2)'$. $\square$

3.2 Equations of reduction

When $f(w) = pu^\kappa s$, the different occurrences of u give us equations on the images of the factors of w . Some equations can be reduce:

Lemma 3.5 *Let $\alpha_1, \alpha_2, \beta_1, \beta'_1, \beta_2, \gamma_1, \gamma_2$ be words over an alphabet B such that $|\beta_1| = |\beta_2| \neq 0$, β'_1 is a proper suffix of β_1 and $0 \leq |\alpha_2| - |\alpha_1| \leq |\beta'_1|$. In these hypotheses, the equality $\alpha_2 \beta_2 \gamma_2 = \alpha_1 \beta'_1 \beta_1 \gamma_1$ implies $\alpha_2 \gamma_2 = \alpha_1 \beta'_1 \gamma_1$.*

Proof.

Let us denote $w = \alpha_1 \beta'_1 \beta_1 \gamma_1 = \alpha_2 \beta_2 \gamma_2$.

The words α_1, α_2 are both prefixes of w . Since $|\alpha_2| \geq |\alpha_1|$, the word α_1 is a prefix of α_2 : there exists a word α'_2 such that $\alpha_2 = \alpha_1 \alpha'_2$ with $|\alpha'_2| = |\alpha_2| - |\alpha_1| \leq |\beta'_1|$.

We have $|\gamma_2| - |\gamma_1| = |w| - |\alpha_2 \beta_2| - |\gamma_1| = |\beta'_1| - (|\alpha_2| - |\alpha_1|)$ so $0 \leq |\gamma_2| - |\gamma_1| \leq |\beta'_1|$. The words γ_1, γ_2 are both suffixes of w : there exists a word γ'_2 such that $\gamma_2 = \gamma'_2 \gamma_1$ with $0 \leq |\gamma'_2| \leq |\beta'_1|$.

The equality $\alpha_2 \beta_2 \gamma_2 = \alpha_1 \beta'_1 \beta_1 \gamma_1$ becomes $\alpha_1 \alpha'_2 \beta_2 \gamma'_2 \gamma_1 = \alpha_1 \beta'_1 \beta_1 \gamma_1$ that is $\alpha'_2 \beta_2 \gamma'_2 = \beta'_1 \beta_1$. But $|\alpha'_2| + |\gamma'_2| = |\alpha'_2 \beta_2 \gamma'_2| - |\beta_2| = |\beta'_1 \beta_1| - |\beta_1| = |\beta'_1|$. Thus α'_2 is a prefix of β'_1 and γ'_2 is a suffix of β_1 so of β'_1 with $|\alpha'_2| + |\gamma'_2| = |\beta'_1|$ that is $\alpha'_2 \gamma'_2 = \beta'_1$.

It follows that $\alpha_2 \gamma_2 = \alpha_1 \alpha'_2 \gamma'_2 \gamma_1 = \alpha_1 \beta'_1 \gamma_1$. $\square$

The situation described in Figure 1 is an example of a case where the hypotheses of the following lemma are verified. Figure 2 deals with Point 4 of Remark 3.7.

Lemma 3.6 *Let $\kappa \geq 3$ be an integer.*

Let f be a morphism from A^ to B^* and let $(w_i)_{i=1.. \kappa+1}, (x_i)_{i=1.. \kappa}$ be words over A such that $|f(x_i)| = |f(x_j)| \neq 0$ for all integers i, j in $[1, \kappa]$.*

We denote by w the word $w_1 x_1 \dots w_\kappa x_\kappa w_{\kappa+1}$.

We assume that there exist $u, p, s, (X_i)_{i=1..κ}$ and $(Y_i)_{i=1..κ}$ words over B such that $f(w_1) = pX_1$, $f(w_{κ+1}) = Y_κs$ and $f(w_i) = Y_{i-1}X_i$ for all $2 \leq i \leq κ$. Furthermore, we assume that, for all integers i in $[1, κ]$, we have $u = X_i f(x_i) Y_i$: it means that $f(w) = pu^κs$.

Let us also assume that there exists an integer q such that, for any integer i in $[1, κ]$, $0 \leq |X_q| - |X_i| \leq |X_q''|$ where X_q'' is a common suffix of X_q and $f(x_q)$. Then the word $\tilde{w} = w_1 w_2 \dots w_κ w_{κ+1}$ verifies $f(\tilde{w}) = p\tilde{u}^κs$ with $\tilde{u} = X_i Y_i$ for any integer i in $[1, κ]$.

In particular, $f(\tilde{w})$ and $\tilde{u}^κ$ are synchronised only if $f(w)$ and $u^κ$ are synchronised.

Figure 1: Reduction of a power

Figure 2: Case 4 of Remark 3.7

We say that we have reduced w .

Point 4 of Remark 3.7 will be treated in the proof of Lemma 3.6. Point 5 is the mirror image of point 4. And point 6 is a combination of point 4 and point 5.

Remark 3.7 1. Using the mirror image and exchanging $|X_q|$ the maximum of $|X_i|$ by the maximum $|Y_q|$ of $|Y_i|$ (i.e. $|X_q|$ is the minimum of $|X_i|$), the condition " $0 \leq |X_q| - |X_i| \leq |X_q''|$ where X_q'' is a common suffix of X_q and $f(x_q)$ " of Lemma 3.6 can be replaced by " $0 \leq |Y_q| - |Y_i| \leq |Y_q'|$ where Y_q' is a common prefix of Y_q and $f(x_q)$ ".

2. A prefix u_1 of u is also a prefix of $\tilde{u}$ if $|u_1| < |X_q|$ and a suffix u_2 of u is also a prefix of $\tilde{u}$ if $|u_2| < \max |Y_j|$.

3. If, instead of $u = X_\kappa f(x_\kappa) Y_\kappa$, we only have that $X_\kappa f(x_\kappa) Y_\kappa$ is a prefix of u then $f(\tilde{w}) = p\tilde{u}^{\kappa-1} X_\kappa Y_\kappa s$ with $X_\kappa Y_\kappa$ prefix of $\tilde{u}$.
4. If $q \neq 1$ and X_q is a suffix of $f(x_q)$ i.e. $X'_q = \varepsilon$ (see Figure 2) then we do not need x_1 and optionally not w_1 in the hypotheses of Lemma 3.6. Conclusion remains true with $u = X_1 Y_1$, $w'_2 = w_1 w_2$ or w_2 , $f(w'_2) = p X_1 Y_1 X_2$, $w = w'_2 x_2 w_3 \dots w_\kappa x_\kappa w_{\kappa+1}$ and $\tilde{w}$ a (not necessarily proper) suffix of $w'_2 w_3 \dots w_\kappa w_{\kappa+1}$.
5. If $q \neq \kappa$ and Y_q is a prefix of $f(x_q)$ then we do not need x_κ and optionally not $w_{\kappa+1}$ in the hypotheses of Lemma 3.6. Conclusion remains true with $u = X_\kappa Y_\kappa$, $w'_\kappa = w_\kappa w_{\kappa+1}$ or w_κ , $f(w'_\kappa) = Y_{\kappa-1} X_\kappa Y_\kappa s$, $w = w_1 x_1 w_2 \dots w_{\kappa-1} x_{\kappa-1} w'_\kappa$ and $\tilde{w}$ a (not necessarily proper) prefix of $w_1 w_2 \dots w_{\kappa-1} w'_\kappa$.
6. If $q \neq 1$, $q \neq \kappa$, X_q is a suffix of $f(x_q)$ and Y_q is a prefix of $f(x_q)$ then we do not need neither x_1 nor x_κ in the hypotheses of Lemma 3.6. Conclusion remains true with $u = X_1 Y_1 = X_\kappa Y_\kappa$, $w'_2 = w_1 w_2$ or w_2 , $w'_\kappa = w_\kappa w_{\kappa+1}$ or w_κ , $f(w'_2) = p X_1 Y_1 X_2$, $f(w'_\kappa) = Y_{\kappa-1} X_\kappa Y_\kappa s$, $w = w'_2 x_2 w_3 \dots w_{\kappa-1} x_{\kappa-1} w'_\kappa$ and $\tilde{w}$ a (not necessarily proper) factor of $w'_2 w_3 \dots w_{\kappa-1} w'_\kappa$.

For any positive integer ℓ , since $|f(x_i)| = |f(x_j)|$ is equivalent to $|f(x_i^\ell)| = |f(x_j^\ell)|$ and since a prefix (resp. a suffix) of $f(x_i)$ is a prefix (resp. a suffix) of $f(x_i^\ell)$, we immediately obtain the following Corollary that will be the central point of the proof of Proposition 4.1.

Corollary 3.8 (method of reduction)

Let $\kappa \geq 3$ and $\ell \geq 1$ be two integers, let α be an integer in $\{1, 2\}$ and let β be an integer in $\{\kappa - 1, \kappa\}$

Let f be a morphism from A^* to B^* and let $(w_i)_{i=\alpha.. \beta+1}$, $(x_i)_{i=\alpha.. \beta}$ be words over A such that $|f(x_i)| = |f(x_j)| \neq 0$ for all integers i, j in $[\alpha, \beta]$.

We denote by w the word $w_\alpha x_\alpha^\ell \dots w_\beta x_\beta^\ell w_{\beta+1}$.

We assume that there exist u, p, s , $(X_i)_{i=\alpha.. \beta}$ and $(Y_i)_{i=\alpha.. \beta}$ words over B such that $f(w_i) = Y_{i-1} X_i$ for all integers i in $[1 + \alpha; \beta]$. Furthermore, we also assume that $f(w_\alpha) = p u^{\alpha-1} X_1$ and $f(w_{\beta+1}) = Y_\kappa u^{\kappa-\beta} s$ where $u = X_i f(x_i^\ell) Y_i (\neq \varepsilon)$ for all integers i in $[\alpha, \beta]$: it means that $f(w) = p u^\kappa s$.

Finally, we assume that there exists an integer q such that, for any integer i in $[\alpha, \beta]$, $0 \leq |X_q| - |X_i| \leq |X_q''|$ where X_q'' is a common suffix of X_q and $f(x_q)$, $0 \leq |X_q| - |X_i| \leq |f(x_q)|$ when $\alpha = 2$, or $0 \leq |Y_i| - |Y_q| \leq |f(x_q)|$ when $\beta = \kappa - 1$.

Then, for any integer $0 \leq \phi < \ell$, the word $\tilde{w} = w_\alpha x_\alpha^\phi \dots w_\beta x_\beta^\phi w_{\beta+1}$ verifies $f(\tilde{w}) = p \tilde{u}^\kappa s$ with $\tilde{u} = X_i f(x_i^\phi) Y_i$ for any integer i in $[1; \kappa]$.

In particular, $f(\tilde{w})$ and $\tilde{u}^\kappa$ are synchronised only if $f(w)$ and u^κ are synchronised.

Proof of Lemma 3.6.

Without loss of generality, we can assume that $|p| < |f(w_1[1])|$ and $|s| < |f(w_{\kappa+1}[[w_{\kappa+1}]]|)$.

Let X'_q be the word such that $X_q = X'_q X_q''$. For any integer $i \in [1, \kappa]$, we have $X_i f(x_i) Y_i = X'_q X_q'' f(x_q) Y_q (= u)$. Since $0 \leq |X_i| - |X'_q| \leq |X_q''|$, by lemma 3.5, we get $X_i Y_i = X_q Y_q$ (it is

$\check{u}$).

That is $f(\check{w}) = pX_1Y_1X_2..Y_{\kappa-1}X_\kappa Y_\kappa s = p\check{u}^\kappa s$ and $|\check{u}| = |u| - |f(x_q)|$.

Let us treat Point 4 of Remark 3.7 that is $q \neq 1$, X_q is a suffix of $f(x_q)$ (i.e. $X'_q = \varepsilon$), $u = X_1Y_1$, $w'_2 = w_1w_2$ or w_2 , $f(w'_2) = pX_1Y_1X_2$ and $w = w'_2x_2..w_\kappa x_\kappa w_{\kappa+1}$. Let $\bar{X}_q$ be the word such that $f(x_q) = \bar{X}_q X_q$. We have $|X_i| \leq |X_q| \leq |f(x_q)|$ for any integer i in $[2, \kappa]$. Since $X_i f(x_i)$ and $X_q f(x_q)$ are both prefixes of u , let z_i be the prefix of X_q such that $X_q \bar{X}_q z_i = X_i f(x_i)$. We have $|z_i| = |X_i f(x_i)| - |X_q \bar{X}_q| = |X_i|$. Thus $X_i = z_i$ is a suffix of $f(x_i)$. It follows that, for all integers i in $[2, \kappa]$, $X_i Y_i = \check{u}$ (as $X_i f(x_i) Y_i$) is a suffix of $pu = pX_1Y_1$: $f(\check{w})$ ends with $\check{u}^\kappa s$.

Even if it seems elementary, the delicate point of this proof is the property of synchronization. In the next part, we are interested in it. This will also give the basic ideas of the proof for the specific assumptions of Remark 3.7. A re-reading of the general case adjusting the conditions (most frequently considering a suffix of w_1 or w_1w_2 in case 4 and a prefix of $w_{\kappa+1}$ or $w_\kappa w_{\kappa+1}$ in case 5) gives the solution to these specific cases.

If $f(\check{w})$ and $\check{u}^\kappa$ are synchronised, there exist two integers $0 \leq \ell < m \leq |\check{w}|$ such that $|\check{u}| = |f(\check{w}[\ell+1..m])| = |f(\check{w}[1..m])| - |f(\check{w}[1..\ell])|$ and, specifically, $p = \varepsilon$ when $\ell = 0$ and $s = \varepsilon$ when $m = |\check{w}|$.

If $\ell = 0$ then $\check{u} = f(\check{w}[1..m]) = X_1Y_1$. Since f is injective, there exists a prefix w'_2 of w_2 such that $f(w_1) = X_1$ and $f(w'_2) = Y_1$. It follows that w starts with $w_1x_1w'_2$ and $f(w)$ starts with $f(w_1x_1w'_2) = X_1f(x_1)Y_1 = u$: $f(w)$ and u^κ are synchronised.

In a similar way, if $m = |\check{w}|$ then $f(w)$ and u^κ are synchronised.

From now, we assume that $0 < \ell < m < |\check{w}|$ and let $r \geq 1$ be the integer such that $|p\check{u}^{r-1}| \leq |f(\check{w}[1..\ell])| < |p\check{u}^r|$. Let us recall that the words $\check{w}[1..\ell]$ and $w_1..w_r$ are both prefixes of $\check{w}$ and that $f(w_1..w_r) = p\check{u}^{r-1}X_r$.

Case 1: $|f(\check{w}[1..\ell])| \leq |p\check{u}^{r-1}| + \min\{|X_r|; |X_{r+1}|\}$.

We have $|f(\check{w}[1..\ell])| \leq |p\check{u}^{r-1}X_r| = |f(w_1..w_r)|$ and so $\check{w}[1..\ell]$ is a prefix of $w_1..w_r$. More precisely, since $|f(w_1..w_{r-1})| \leq |p\check{u}^{r-1}| \leq |f(\check{w}[1..\ell])|$, there exists a suffix y_r of w_r such that $\check{w}[1..\ell]y_r = w_1..w_r$. Since $|f(\check{w}[1..m])| = |\check{u}| + |f(\check{w}[1..\ell])|$, we get $|p\check{u}^r| \leq |f(\check{w}[1..m])| \leq |p\check{u}^r| + |X_{r+1}|$. There exists a suffix y_{r+1} of w_{r+1} such that $\check{w}[1..m]y_{r+1} = w_1..w_r w_{r+1}$. In particular, we have $|f(w_{r+1})| = |f(\check{w}[1..m])| + |f(y_{r+1})| - |f(\check{w}[1..\ell])| - |f(y_r)| = |\check{u}| + |f(y_{r+1})| - |f(y_r)|$.

Since y_r is a suffix of w_r and y_{r+1} is a suffix of w_{r+1} , let i be the integer such that $w[1..i]y_r = w_1x_1w_2..x_{r-1}w_r$ and let j be the integer such that $w[1..j]y_{r+1} = w_1x_1w_2..w_r x_r w_{r+1}$. Since $i = 0$ implies $\ell = 0$ and $j = |w|$ implies $m = |w|$, we have $0 < i < j < |w|$. Furthermore, $|f(w[i+1..j])| = |f(w[1..j])| - |f(w[1..i])| = |f(x_r w_{r+1})| - |f(y_{r+1})| + |f(y_r)| = |\check{u}| + |f(x_r)| = |\check{u}| + |f(x_q)| = |u|$. That is $f(w)$ and u^κ are synchronised.

Case 2: $|f(\check{w}[1..\ell])| \geq |p\check{u}^{r-1}| + \max\{|X_r|; |X_{r+1}|\}$.

The inequalities $|p\check{u}^r X_r| \leq |f(\check{w}[1..\ell])| < |p\check{u}^r|$ mean that $|f(w_1..w_r)| \leq |f(\check{w}[1..\ell])| < |f(w_1..w_{r+1})|$: there exists a prefix z_{r+1} of w_{r+1} such that $\check{w}[1..\ell] = w_1..w_r z_{r+1}$. Since $|p\check{u}^r X_{r+1}| \leq |f(\check{w}[1..m])| < |p\check{u}^{r+1}|$, there exists a prefix z_{r+2} of w_{r+2} such that $\check{w}[1..m] = w_1..w_{r+1} z_{r+2}$. We have $|\check{u}| = |f(\check{w}[1..m])| - |f(\check{w}[1..\ell])| = |f(w_{r+1} z_{r+2})| - |f(z_{r+1})|$.

Let i be the integer such that $w[1..i] = w_1x_1..w_r x_r w_{r+1} x_{r+1} z_{r+1}$ and let j be the integer such that $w[1..j] = w_1x_1..w_r x_r w_{r+1} x_{r+1} z_{r+2}$. Since $0 < \ell < m < |\check{w}|$, we have $0 < i < j < |w|$. Furthermore, $|f(w[i+1..j])| = |f(w[1..j])| - |f(w[1..i])| = |f(w_{r+1} x_{r+1})| + |f(z_{r+2})| -$

$|f(z_{r+1})| = |\tilde{u}| + |f(x_{r+1})| = |\tilde{u}| + |f(x_q)| = |u|$. That is $f(w)$ and u^κ are synchronised.

Case 3: $|p\tilde{u}^{r-1}| + \min\{|X_r|; |X_{r+1}|\} < |f(\tilde{w}[1..\ell])| < |p\tilde{u}^{r-1}| + \max\{|X_r|; |X_{r+1}|\}$.

In particular, it means that $|X_r| \neq |X_{r+1}|$.

If $|X_r| < |X_{r+1}|$ then $|f(w_1..w_r)| = |p\tilde{u}^{r-1}X_r| < |f(\tilde{w}[1..\ell])| < |p\tilde{u}^r| \leq |f(w_1..w_{r+1})|$ and $|f(w_1..w_r)| \leq |p\tilde{u}^r| < |f(\tilde{w}[1..m])| < |p\tilde{u}^rX_{r+1}| = |f(w_1..w_{r+1})|$: There exists a prefix z_{r+1} of w_{r+1} such that $\tilde{w}[1..\ell] = w_1..w_rz_{r+1}$ and there exists a suffix y_{r+1} of w_{r+1} such that $\tilde{w}[1..m]y_{r+1} = w_1..w_rw_{r+1}$.

We have $|\tilde{u}| = |f(\tilde{w}[1..m])| - |f(\tilde{w}[1..\ell])| = |f(w_{r+1})| - |f(z_{r+1})| - |f(y_{r+1})|$.

Since $0 < |f(\tilde{w}[1..\ell])| - |f(w_1..w_r)| = |f(\tilde{w}[1..\ell])| - |p\tilde{u}^{r-1}X_r| < |X_{r+1}| - |X_r|$, we get $|X_rf(z_{r+1})| < |X_{r+1}|$ and so the word $X_rf(z_{r+1})$ is a prefix of X_{r+1} . Since $||X_r| - |X_{r+1}|| \leq |X_q''| \leq |f(x_q)|$, we get that $|X_rf(x_r)| \geq |X_{r+1}|$. But X_{r+1} is a prefix of $X_rf(x_r)$ (they are both prefixes of u) so $f(z_{r+1})$ is a prefix of $f(x_r)$. By Lemma 2.6, it implies that z_{r+1} is a prefix of x_r .

Let i be the integer such that $w[1..i] = w_1x_1..w_rz_{r+1}$ and let j be the integer such that $w[1..j]y_{r+1} = w_1x_1w_2..w_rx_rw_{r+1}$. As previously, we have $0 < i < j < |w|$ and $|f(w[i+1..j])| = |f(w[1..j])| - |f(w[1..i])| = |f(x_rw_{r+1})| - |f(y_{r+1})| - |f(z_{r+1})| = |\tilde{u}| + |f(x_r)| = |\tilde{u}| + |f(x_q)| = |u|$. That is $f(w)$ and u^κ are synchronised.

The case $|X_r| > |X_{r+1}|$ is solved in the same way using the fact that z_{r+2} is a prefix of $f(x_{r+1})$. $\square$

3.3 Situations of reduction

Let $k \geq 3$ be an integer and let $\kappa \in \{k; k+1\}$. Let f be a morphism from A^* to B^* and let ω be a word over A such that $f(\omega) = pU^\kappa S$ for some words p, S and $U \neq \varepsilon$ over B such that $|p| < |f(\omega[1])|$. Moreover, we assume $|S| < |f(\omega[|\omega|])|$ when $\kappa = k+1$. It is important to note that in this section the word S is not necessary a proper suffix of $f(\omega[|\omega|])$ when $\kappa = k$.

For any integer j in $[1, \kappa+1]$, let i_j the smallest integer such that pU^{j-1} is a prefix of $f(\omega[1..i_j])$. We have $i_1 = 1$ and there exist words p_j and s_j such that $f(\omega[i_j]) = p_js_j$, $p_1 = p$, $s_{\kappa+1}$ is a prefix of S ($s_{\kappa+1} = S$ when $\kappa = k+1$), $p_j \neq \varepsilon$ if $j \neq 1$ and $s_1 \neq \varepsilon$. Furthermore, we have $f(\omega[1..i_j]) = pU^{j-1}s_j$ for any integer j in $[1, \kappa+1]$, $U = s_jf(\omega[i_j+1..i_{j+1}-1])p_{j+1}$ for any integer j in $[1, \kappa]$.

Since a factor of ω can appear many times in ω , it is necessary to indicate which exact factor we are going to work with: if $\omega[n..m] = z$, we denote $n = n_z$ and $m = m_z$ fixing by this way the considered occurrence of z in ω . For any positive integer α , if $\omega[n..m] = z^\alpha$, we also denote $n = n_z$ and $m = m_z$ without specifying α . It is the same notation as the case $\alpha = 1$: we will precise only if necessary.

To simplify notations, let us recall that, given two integers $1 \leq n_z \leq m_z \leq |\omega|$, the word $\omega[n_z..m_z] = z^\alpha$ define two words we denote z_p and z_s such that $\omega = z_pz^\alpha z_s$, with $n_z = |z_p| + 1$ and $m_z = |z_pz^\alpha|$. This means that $z_p = \omega[1..n_z-1]$ and $z_s = \omega[m_z+1..|\omega|]$.

Given two integers $1 \leq n_z \leq m_z \leq i_{\kappa+1}$, we also define a word D_z and three integers λ_z , d_z and c_z (even if c_z is not used in this section). Eventually, we will precise $D_{z,\omega}$, $\lambda_{z,\omega}$, $d_{z,\omega}$ and $c_{z,\omega}$ if a doubt may occur. Briefly, λ_z is the integer such that $f(\omega[n_z..m_z]) = f(z^\alpha)$ starts in the λ_z^{th} occurrence of U ; d_z indicates if the first occurrence of $f(z)$ in $f(\omega[n_z..m_z])$

covers or not two consecutive occurrences of U ; c_z is the number of occurrences of U covers by $f(\omega[n_z..m_z])$ and D_z is a prefix of U such that $f(z_p z)$ ends with D_z or $D_z f(z)$.

More precisely, if $n_z = 1$ i.e. z is a prefix of ω then $\lambda_z = 0$, $d_z = 1$ and D_z is the word such that $f(z) = pD_z$. When $n_z \geq 2$, let λ_z be the integer such that $n_z \in]i_{\lambda_z}; i_{\lambda_z+1}]$ i.e. $|pU^{\lambda_z-1}| \leq |f(\omega[1..n_z-1])| = |f(z_p)| < |pU^{\lambda_z}|$. If $|f(z_p z)| \leq |pU^{\lambda_z}|$ then let $d_z = 0$ otherwise let $d_z = 1$. Let D_z be the word such that $f(z_p z^{d_z}) = pU^{\lambda_z-1+d_z} D_z$. It means that $D_z = s_{\lambda_z} f(\omega[i_{\lambda_z} + 1..n_z - 1])$ when $d_z = 0$ and $s_{\lambda_z} f(\omega[i_{\lambda_z} + 1..n_z - 1]) f(z) = U D_z$ when $d_z = 1$. In particular, D_z is a proper suffix of $f(z)$ when $d_z = 1$. Finally, c_z is the lowest integer such that $|f(\omega[1..m_z])| \leq |pU^{\lambda_z+c_z-1}|$.

It is important to remark that, if $\omega[n_z..m_z] = z^\alpha$, the integers n_z and m_z define z^α and z . But, since we may have several occurrences of z^α in ω , we do not have the contrary. In other words, the equality $z = z'$ not necessarily implies $n_z = n_{z'}$ or $m_z = m_{z'}$. In the same vein, λ_z , d_z , c_z and D_z depend on n_z and m_z but not directly of z . But if no question exists over the considered factor of ω or if the choice of the considered factor does not matter, we will write z^α instead of $\omega[n_z..m_z]$.

For any integer $\alpha \geq 2$ and for any word $\omega[n_z..m_z] = z^\alpha$ with $n_z, m_z \in [1, i_{\kappa+1}]$, the word $f(\omega[n_y..m_y]) = f(y^\alpha) = f(y)^\alpha$ with $n_y, m_y \in [1, i_{\kappa+1}]$ is a *conjugated shift to the left* of $f(\omega[n_z..m_z]) = f(z^\alpha) = f(z)^\alpha$ (in $f(\omega)$) if there exist two words $t_1 \neq \varepsilon$ and t_2 such that $f(y) = t_2 t_1$, $f(z) = t_1 t_2$ and if we have one of the following conditions:

1. $D_z = D_y t_2$ when $d_y = d_z$
2. $D_y = D_z t_1$ when $d_y = 1$ and $d_z = 0$
3. $D_y f(y) t_2 = U D_z$ when $d_y = 0$ and $d_z = 1$

Let us remark that conditions (2) and (3) imply $|D_z| < |t_2|$. Moreover, Taking $t_2 = \varepsilon$, let us also note that $f(z^\alpha)$ is a conjugated shift to the left of itself.

We say that $f(y)^\alpha$ is a *conjugated shift to the right* of $f(z)^\alpha$ if $f(z)^\alpha$ is a conjugated shift to the left of $f(y)^\alpha$. We simply say that $f(y)^\alpha$ is a *conjugated shift* of $f(z)^\alpha$ if it is a conjugated shift to the left or to the right of $f(z)^\alpha$.

For a general use of conjugated shifts of $f(z)^\alpha$, we will switch t_1 and t_2 roles in definition and conditions (1) to (3) for a conjugated shift to the right.

For any pure k -power $\omega[n_v..m_v] = v^k$ of ω , there are $k-2$ choices for the factor v^3 in v^k . We denote $v_{(\beta)}^3$ the β^{th} factor of v^3 in v^k that is $\omega[n_v..m_v] = v^{\beta-1} v_{(\beta)}^3 v^{k-\beta-2}$ with $1 \leq \beta \leq k-2$.

We will focus on these different cubes v^3 but without specifying β in this section.

For any factor $\omega[n_v..m_v] = v^3$ of $\omega[1..i_{\kappa+1}]$ and for any integer $j \in [1; \kappa]$, let $L_{j,v}$ be the set of the words $\omega[n_x..m_x] = x^3$ such that $f(\omega[n_x..m_x]) = f(x)^3$ is a conjugated shift to the left of $f(\omega[n_v..m_v]) = f(v)^3$ with $\lambda_x = j$ if $d_x = d_v = 0$ and $\lambda_x = j-1$ otherwise.

We also denote $R_{j,v}$ the set of the words $\omega[n_x..m_x] = x^3$ such that $f(\omega[n_x..m_x]) = f(x)^3$ is a conjugated shift to the right of $f(v)^3$ with $\lambda_x = j - d_v \times d_x$.

If $\omega[n_{x_j}..m_{x_j}] = x_j^3$ is a word in $L_{j,v} \cup R_{j,v}$, we denote $t_{1,j}, t_{2,j}$ the words such that $f(v) = t_{1,j} t_{2,j}$ and $f(x_j) = t_{2,j} t_{1,j}$.

More specifically, let j_0 be an integer such that $\omega[n_v..m_v] = v^3 \in L_{j_0,v} \cup R_{j_0,v}$. We will always assume that $n_{x_{j_0}} = n_v$ and $m_{x_{j_0}} = m_v$ that is $x_{j_0} = v$.

Lemma 3.9 *We use all previous definitions and notations of this section. In particular, v^3 is a chosen factor of a pure k -power v^k . When one of the four following situations happen, there exist a word $\tilde{\omega}$ such that $f(\tilde{\omega}) = p'(U')^\kappa S'$ for some words p' , S' and $U' \neq \varepsilon$ over B verifying $|p'| < |f(\tilde{\omega}[1])|$, $0 < |U'| < |U|$ and $f(\tilde{\omega})$ and $(U')^\kappa$ are synchronised if $f(\omega)$ and U^κ are synchronised.*

1. $d_v = 1$, $|D_v f(v)^2| < |U|$ and $L_{j,v} \cup R_{j,v} \neq \emptyset$ for any integer $j \in [2, \kappa]$.
2. $d_v = 1$, $L_{j,v} \cup R_{j,v} \neq \emptyset$ for any integer $j \in [2, \kappa - 1]$ and there exists a positive integer ϕ such that $\omega[n_{x_j}..|\omega|]$ starts with $v^{\phi+2}$ and $\sup\{2|f(v)|; |D_v f(v)^\phi|\} \leq |U| < |D_v f(v)^{\phi+1}|$.
3. $d_v = 0$, $|D_v f(v)^2| \leq |U|$ and $L_{j,v} \cup R_{j,v} \neq \emptyset$ for any integer $j \in [1, \kappa]$.
4. $d_v = 0$, $|U| < |D_v f(v)^2| < |D_v U|$ and $L_{j,v} \cup R_{j,v} \neq \emptyset$ for any integer $j \in [1, \kappa - 1]$.

Proof.

For any integer j , let $\omega[n_{x_j}..m_{x_j}] = x_j^3$ be a word in $L_{j,v} \cup R_{j,v}$.

1. $d_v = 1$, $|D_v f(v)^2| < |U|$ and $L_{j,v} \cup R_{j,v} \neq \emptyset$ for any integer $j \in [2, \kappa]$.

If $x_j^3 \in L_{j,v}$ and $d_{x_j} = 1$ (including $x_{j_0} = v$) or if $x_j^3 \in R_{j,v}$, let X_j be the word D_{x_j} and let e_j be the integer d_{x_j} . If $x_j^3 \in L_{j,v}$ and $d_{x_j} = 0$, let X_j be the suffix of $f(x_j)$ such that $D_{x_j} f(x_j)^2 = U X_j$ and let $e_j = 2$. Let q be an integer such that $|X_q| = \max\{|X_j|; j \in [2, \kappa]\}$. For all integers $j \in [2, \kappa]$, if $d_{x_j} = 0$ with $x_j^3 \in L_{j,v}$ or if $d_{x_j} = 1$, by definitions, we have that X_j is a suffix of $f(x_j)$. If $d_{x_j} = 0$ with $x_j^3 \in R_{j,v}$ then it means that $D_v = D_{x_j} t_{2,j}$. But D_v is a suffix of $f(v) = t_{1,j} t_{2,j}$. So it implies that $X_j = D_{x_j}$ is a suffix of $t_{1,j}$ and of $f(x_j) = t_{2,j} t_{1,j}$.

In particular, X_q is a suffix of $f(x_q)$ and we get that $0 \leq |X_q| - |X_j| \leq |X_q| \leq |f(x_q)|$ for all integers $j \in [2, \kappa]$. Furthermore, if $d_{x_j} = 0$ with $x_j^3 \in R_{j,v}$ then $\lambda_x = j$ and $\lambda_x = j - 1$ otherwise. It follows that $f(\omega[1..n_{x_j} - 1])f(x_j^{e_j}) = pU^{j-1}X_j$.

Since $|X_j f(x_j)| \leq 2|f(x_j)| = |f(v)^2| \leq |U|$, $X_j f(x_j)$ is a prefix of U : there exists a word Y_j such that $U = X_j f(x_j) Y_j$ for all integers $j \in [2, \kappa]$. Let w_2 be the prefix of ω such that $f(w_2) = pU X_2$ i.e. $w_2 = \omega[1..n_{x_2} - 1]x_2^{e_2}$ and let $w_{\kappa+1}$ be the suffix of ω such that $f(w_{\kappa+1}) = Y_\kappa S$ i.e. $\omega = \omega[1..n_{x_\kappa} - 1]x_\kappa^{1+e_\kappa} w_{\kappa+1}$. In particular, we have $f(\omega[n_{x_j}..n_{x_{j+1}} - 1])f(x_{j+1}^{e_{j+1}}) = f(x_j^{1+e_j})Y_j X_{j+1}$ for all integers $j \in [2, \kappa - 1]$. Since f is bifix, it implies that there exists a word w_j such that $f(w_j) = Y_{j-1} X_j$ for all integers $j \in [3, \kappa]$.

In summary, $\omega = w_2 x_2 w_3 x_3 .. w_\kappa x_\kappa w_{\kappa+1}$, $f(\omega) = pU^\kappa S$ with $U = X_j f(x_j) Y_j$ for all integers $j \in [2, \kappa]$, there exists an integer $q \in [2, \kappa]$ such that $0 \leq |X_q| - |X_j| \leq |X_q| \leq |f(x_q)|$ and X_q is a suffix of $f(x_q)$.

By Corollary 3.8 (or Lemma 3.6 and using Remark 3.7(4)), in particular the property of synchronised words, we can reduce $f(\omega)$. More precisely, let U' be the non-empty word $X_q Y_q$ and let $\bar{w}_2$ be the shortest suffix of w_2 such that $f(\bar{w}_2)$ ends with $U' X_2$ and let

$\tilde{\omega}$ be the word $\bar{w}_2 w_3 \dots w_\kappa w_{\kappa+1}$. We get that $f(\tilde{\omega}) = p'(U')^\kappa S$ with $|p'| < |f(\tilde{\omega}[1])|$ and $|U'| = |U| - |f(x_q)| < |U|$.

Fact 1 : Let us note that U' is a suffix of U and, if U starts with a word z prefix of $D_v f(v)$ (for instance X_q) then z is also a prefix of U' .

Fact 2 : For all integers $j \in [1, \kappa]$, if $x_j^3 \in L_{j,v}$ then w_j ends with x_j and besides if $d_{x_j} = 1$ then w_{j+1} starts with x_j . If $x_j^3 \in R_{j,v}$ then w_{j+1} starts with x_j and besides if $d_{x_j} = 1$ then w_j ends with x_j .

Fact 3 : If there exists an integer j_1 such that $x_{j_1}^3 \in L_{j_1,v}$ with $d_{x_{j_1}} = 0$ and if there exists an integer j_2 such that $x_{j_2}^3 \in R_{j_2,v}$ with $d_{x_{j_2}} = 0$ then w_{j_1+1} starts with x_{j_1} and w_{j_2} ends with x_{j_2} .

2. $d_v = 1$, $L_{j,v} \cup R_{j,v} \neq \emptyset$ for any integer $j \in [2, \kappa - 1]$ and there exists a positive integer ϕ such that $\omega[n_v \dots |\omega|]$ starts with $v^{\phi+2}$ and $\sup \{2|f(v)|; |D_v f(v)^\phi|\} \leq |U| < |D_v f(v)^{\phi+1}|$.

In this case, U is a prefix of $D_v f(v)^{\phi+1}$.

For any integer $j \in [2, \kappa - 1]$, we define X_j and e_j as Case 1 and we find that X_j is also a suffix of $f(x_j)$ (thus of $f(x_j)^{\phi+1}$).

If $x_j^3 \in L_{j,v}$ with $d_{x_j} = 1$, then U is a prefix of $D_v f(v)^{\phi+1} = X_j t_{2,j}(t_{1,j} t_{2,j})^{\phi+1}$ thus U is a prefix of $X_j f(x_j)^{\phi+2}$. If $x_j^3 \in L_{j,v}$ with $d_{x_j} = 0$, since U^2 is a prefix of $U D_v f(v)^{\phi+1} = U D_v(t_{1,j} t_{2,j})^{\phi+1} = D_{x_j}(t_{2,j} t_{1,j})^{\phi+2} t_{2,j} = U X_j f(x_j)^{\phi+2}$, it follows that U is a prefix of $X_j f(x_j)^{\phi+1}$.

In the same way, we show that U is a prefix of $X_j f(x_j)^{\phi+1}$ when $x_j^3 \in R_{j,v}$.

Let q be an integer such that $|X_q| = \max\{|X_j|; j \in [2, \kappa - 1]\}$. If $x_j^3 \in L_{j,v}$ with $d_{x_j} = 1$ or $x_j^3 \in R_{j,v}$ with $d_{x_j} = 0$ then $|X_j| \leq |X_{j_0}|$. Thus, if $q \neq j_0$, either $x_j^3 \in L_{j,v}$ with $d_{x_j} = 0$ or $x_j^3 \in R_{j,v}$ with $d_{x_j} = 1$. Let δ be the greatest integer such that $|X_q f(x_q)^\delta| \leq |U| < |X_q f(x_q)^{\delta+1}|$.

For any integer $j \in [2, \kappa - 1]$, since $|X_j f(x_j)^\delta| \leq |X_q f(x_q)^\delta| \leq |U|$, there exists a word Y_j such that $U = X_j f(x_j)^\delta Y_j$. Since U is a prefix of $X_q f(x_q)^{\delta+2}$, we get that $U = X_q f(x_q)^\delta Y_q$ with Y_q a prefix of $f(x_q)$.

Let w_2 be the prefix of ω such that $f(w_2) = p U X_2$, let w_κ be the suffix of ω such that $f(w_\kappa) = Y_{\kappa-1} U S$ and, for all integers $j \in [3, \kappa - 1]$, let w_j be the word such that $f(w_j) = Y_{j-1} X_j$.

By Corollary 3.8 (or Lemma 3.6 and using Remark 3.7(6)), we can reduce $f(\omega)$. More precisely, let U' be non-empty the word $X_q Y_q$: U' is both a prefix and a suffix of U . Let $\bar{w}_2$ be the shortest suffix of w_2 such that $f(\bar{w}_2)$ ends with $U' X_2$ and let $\tilde{\omega}$ be the word $\bar{w}_2 w_3 \dots w_{\kappa-1} w_\kappa$. We get that $f(\tilde{\omega}) = p(U')^{\kappa-1} U S$ and so starts with $p(U')^\kappa$ where $|U'| = |U| - |f(x_q)| < |U|$.

3. $d_v = 0$, $|D_v f(v)^2| \leq |U|$ and $L_{j,v} \cup R_{j,v} \neq \emptyset$ for any integer $j \in [1, \kappa]$.

For any integer $j \in [1, \kappa]$, let X_j be the word $D_{x_j} f(x_j)$ if $x_j \in L_{j,v}$ with $d_{x_j} = 0$ (including $x_{j_0} = v$), or the word D_{x_j} if $x_j \in L_{j,v}$ with $d_{x_j} = 1$ or if $x_j \in R_{j,v}$.

If $x_j \in L_{j,v}$, let $e_j = 1$ and if $x_j \in R_{j,v}$, let $e_j = 0$.

For any word $x_j^3 \in R_{j,v}$, since $|D_v f(v)^2| \leq |U|$, we necessarily have $d_{x_j} = 0$. Furthermore, $0 \leq |X_{j_0}| - |X_j| = |t_{2,j}| < |f(x_{j_0})| = |f(v)|$.

If $x_j^3 \in L_{j,v}$ and $d_{x_j} = 0$, we have $X_{j_0} = D_v f(v) = D_{x_j} t_{2,j} t_{1,j} t_{2,j} = X_j t_{2,j}$ and so $0 \leq |X_{j_0}| - |X_j| = |t_{2,j}| < |f(x_{j_0})|$. If $x_j^3 \in L_{j,v}$ and $d_{x_j} = 1$, we have $X_j t_{2,j} = D_{x_j} t_{2,j} = D_v t_{1,j} t_{2,j} = X_{j_0}$ and so $0 \leq |X_{j_0}| - |X_j| = |t_{2,j}| < |f(x_{j_0})| = |f(v)|$.

We have $|X_{j_0}| = \max\{|X_j|; j \in [1; \kappa]\}$ and $f(\omega[1..n_{x_j} - 1])f(x_j^{e_j}) = pU^{j-1}X_j$ for all integers $j \in [1, \kappa]$.

Since $|X_j f(x_j)| \leq |D_v f(v)^2| \leq |U|$, the word $X_j f(x_j)$ is a prefix of U : there exist words Y_j such that $U = X_j f(x_j) Y_j$ for all integers j in $[1, \kappa]$. Let w_1 be the word $\omega[1..n_{x_1} - 1]x_1^{e_{x_1}}$ and $w_{\kappa+1}$ be the word such that $\omega[n_{x_\kappa}..|\omega|] = x_\kappa^{1+e_\kappa} w_{\kappa+1}$. In particular, we have $f(w_1) = pX_1$, $f(w_{\kappa+1}) = Y_\kappa S$ and, for any integer $j \in [1, \kappa - 1]$, $f(\omega[n_{x_j}..n_{x_{j+1}} - 1])f(x_{j+1}^{e_{j+1}}) = f(x_j^{1+e_j})Y_j X_{j+1}$. Since f is bifix, it implies that there exists a word w_j such that $f(w_j) = Y_{j-1}X_j$ for all integers $j \in [2, \kappa]$.

By Corollary 3.8 (or Lemma 3.6), we can reduce $f(\omega)$. More precisely, $\tilde{\omega} = w_1 w_2 .. w_\kappa w_{\kappa+1}$ and $U' = X_i Y_i (\neq \varepsilon)$ for all integers $i \in [1, \kappa]$. We get that $f(\tilde{\omega}) = p'(U')^\kappa S$ with $|p'| = |p| < |f(W[1])| = |f(w_1[1])| = |f(\tilde{\omega}[1])|$ and $|U'| = |U| - |f(x_q)| < |U|$.

Fact 1 : A prefix of U of length at most $\max\{|X_i|\} = |X_{j_0}| = |D_v f(v)|$ is also a prefix of U' and a suffix of U of length at most $\max\{|Y_i|\}$ is also a suffix of U' .

Fact 2 : If $|D_v f(v^3)| \leq |U|$ (i.e. Y_q starts with $f(v)$), we can work with $e_j + 1$ instead of e_j and we find that a prefix of U of length at most $\max\{|X_j|\} = |X_{j_0}| = |D_v f(v^2)|$ is also a prefix of U' .

Fact 3 : If $L_{j,v} \neq \emptyset$ for any integer $j \in [1, \kappa]$ then w_j ends with x_j and w_{j+1} starts with x_j .

4. $d_v = 0$, $|U| < |D_v f(v)^2| < |D_v U|$ and $L_{j,v} \cup R_{j,v} \neq \emptyset$ for any integer $j \in [1, \kappa - 1]$.

Let us recall that, by definition, $|D_v f(v)| \leq |U|$.

Let S_2 be the set of integers j such that there exists a word x_j^3 in $R_{j,v}$ with $d_{x_j} = 1$ but no word in $R_{j,v}$ with $d_{x_j} = 0$ and no word in $L_{j,v}$.

Case 4.1: $d_v = 0$, $|U| < |D_v f(v)^2| < |D_v U|$, $S_2 = \emptyset$ and $L_{j,v} \cup R_{j,v} \neq \emptyset$ for any integer $j \in [1, \kappa - 1]$.

If $x_j^3 \in R_{j,v}$, if $x_j^3 \in L_{j,v}$ with $d_{x_j} = 1$ or if $x_j^3 \in L_{j,v}$ with $d_{x_j} = 0$ and $|D_{x_j} f(x_j)^2| \geq |U|$ then let X_j be the word D_{x_j} and let $e_j = d_{x_j}$. If $x_j^3 \in L_{j,v}$ with $d_{x_j} = 0$ and $|D_{x_j} f(x_j)^2| < |U|$ then let X_j be the word $D_{x_j} f(x_j)$ and let $e_j = 1$. For all integers $j \in [1, \kappa - 1]$, we have $f(\omega[1..n_{x_j} - 1])f(x_j^{e_j}) = pU^{j-1}X_j$.

For all integers $j \in [1, \kappa - 1]$, $X_j f(x_j)$ is a prefix of U : there exists a word Y_j such that $U = X_j f(x_j) Y_j$. Since $|U f(x_j)| > |X_j f(x_j)^2| \geq |U|$, we get that $X_j f(x_j)^2$ is a prefix of U^2 and it follows that Y_j is a prefix of $f(x_j)$.

Let q be an integer such that $|X_q| = \max\{|X_j|; j \in [1; \kappa - 1]\}$. In particular, we have $|Y_q| \leq |f(x_q)|$ and $0 \leq |X_q| - |X_j| = |Y_j| - |Y_q| \leq |f(x_j)| = |f(x_q)|$ for any integer j in $[1; \kappa - 1]$.

Let w_1 be the word $\omega[1..n_{x_1} - 1]x_1^{e_{x_1}}$ and w_κ be the word such that $\omega[n_{x_{\kappa-1}}..|\omega|] = x_{\kappa-1}^{1+e_{\kappa-1}}w_\kappa$. We have $f(w_1) = pX_1$, $f(w_\kappa) = Y_{\kappa-1}US$. Furthermore, for all integers $j \in [1, \kappa - 2]$, we have $f(\omega[n_{x_j}..n_{x_{j+1}} - 1])f(x_{j+1}^{e_{j+1}}) = f(x_j^{1+e_j})Y_jX_{j+1}$. Since f is bifix, it implies that there exists a word w_j such that $f(w_j) = Y_{j-1}X_j$ for all integers $j \in [2, \kappa - 1]$.

By Lemma 3.6 and using Remark 3.7(5), we can reduce $f(\omega)$.

The non-empty word $U' = X_\kappa Y_\kappa$ is a prefix of U and any suffix of U of length at most $\max\{|Y_i|\}$ is also a prefix of U' . We take $\tilde{\omega} = w_1w_2..w_\kappa$. We get that $f(\tilde{\omega}) = p(U')^{\kappa-1}US$ starts with $p'U'^\kappa$ with $|p'| = |p| < |f(W[1])| = |f(w_1[1])| = |f(\tilde{\omega}[1])|$ and $|U'| < |U|$.

Case 4.2: $d_v = 0$, $|U| < |D_v f(v)^2| < |D_v U|$, $S_2 \neq \emptyset$ and $L_{j,v} \cup R_{j,v} \neq \emptyset$ for any integer $j \in [1, \kappa - 1]$.

If $j \in S_2$, let X_j be the word D_{x_j} and let $e_j = 1$.

If $j \notin S_2$, we assume that if $x_j^3 \in R_{j,v}$ then $d_{x_j} = 0$ else we take $x_j^3 \in L_{j,v}$. If $x_j^3 \in R_{j,v}$ (with $d_{x_j} = 0$) or if $x_j^3 \in L_{j,v}$ with $d_{x_j} = 0$ and $|D_{x_j}f(x_j)^2| > |U|$ (for instance x_{j_0}), let X_j be the word such that $D_{x_j}f(x_j)^2 = UX_j$ and let $e_j = 2$. If $x_j^3 \in L_{j,v}$ with $d_{x_j} = 1$ or $d_{x_j} = 0$ and $|D_{x_j}f(x_j)^2| \leq |U|$, let X_j be the word such that $D_{x_j}f(x_j)^3 = UX_j$ and let $e_j = 3$. For all integers $j \in [1, \kappa - 1]$, we have $f(\omega[1..n_{x_j} - 1])f(x_j^{e_j}) = pU^jX_j$. Especially, the word X_j is a suffix of $f(x_j)$ for any integer $j \in [1, \kappa - 1]$.

Let j_1 be an integer in S_2 i.e. $x_{j_1} \in R_{j,v}$ and $d_{x_{j_1}} = 1$. We get that U^2 starts with $X_{j_1}f(x_{j_1}^2)$. By definition, we have $UX_{j_0} = D_v f(v)^2 = UX_{j_1}t_{2,j}$. For any word $x_j^3 \in L_{j,v}$ with $d_{x_j} = 1$ or with $d_{x_j} = 0$ and $|D_{x_j}f(x_j)^2| \leq |U|$, always by definitions, we obtain $UX_j = UX_{j_0}t_{1,j}$. It follows that $|f(x_{j_1})| = |f(x_j)| \geq |X_j| > |X_{j_0}| \geq |X_{j_1}|$. Furthermore, the words $f(x_j)$ and $f(x_{j_1})$ are conjugated.

Let $\tau_{2,j}$ be the non-empty suffix of X_j (and of $f(x_j)$) such that $X_j = X_{j_1}\tau_{2,j}$ and let $\tau_{1,j}$ be the word such that $f(x_j) = \tau_{1,j}\tau_{2,j}$. Since UX_{j_1} ends with $\tau_{1,j}$, we get $f(x_{j_1}) = \tau_{2,j}\tau_{1,j}$. Thus U^2 starts with $X_{j_1}(\tau_{2,j}\tau_{1,j})^2 = X_jf(x_j)\tau_{1,j}$. Since f is bifix, it implies that $\omega[m_{x_j} + 1..|\omega|]$ also starts with x_j . In other words, x_j^3 is followed by x_j in ω .

Let q be an integer such that $|X_q| = \max\{|X_j|; j \in [1; \kappa - 1]\}$. In particular, $0 \leq |X_q| - |X_j| \leq |f(x_q)|$.

For all integers $j \in [1, \kappa - 1]$, $X_jf(x_j)$ is a prefix of U : there exists a word Y_j such that $U = X_jf(x_j)Y_j$.

Let w_2 be the prefix of ω such that $f(w_2) = pUX_1$ i.e. $w_2 = \omega[1..n_{x_1} - 1]x_1^{e_1}$ and let $w_{\kappa+1}$ be the suffix of ω such that $f(w_{\kappa+1}) = Y_\kappa S$ i.e. $\omega = \omega[1..n_{x_\kappa} - 1]x_\kappa^{1+e_\kappa}w_{\kappa+1}$. Furthermore, we have $f(\omega[n_{x_j}..n_{x_{j+1}} - 1])f(x_{j+1}^{e_{j+1}}) = f(x_j^{1+e_j})Y_jX_{j+1}$ for all integers $j \in [1, \kappa - 2]$. Since f is bifix, it implies that there exists a word w_j such that $f(w_j) = Y_{j-1}X_j$ for all integers $j \in [3, \kappa]$.

By Lemma 3.6 and using Remark 3.7(4), we can reduce $f(\omega)$. Reduction is almost the same that case where $d_v = 1$, $|D_v f(v)^2| < |U|$ and $L_{j,v} \cup R_{j,v} \neq \emptyset$ for any integer $j \in [2, \kappa]$. Let us note that U' is a suffix of U and that any prefix of U of length at most $\max\{|X_j|\}$ is also a prefix of U' .

□

4 Special case of uniform morphisms

As a consequence of Corollary 3.2 and using Lemma 3.9, we will be able to reduce a word whose image by a k -power-free uniform morphism contains a $(k+1)$ -power. It follows:

Proposition 4.1 *Let A and B be two alphabets and let $k \geq 4$ be an integer. A k -power-free uniform morphism is a $(k+1)$ -power-free morphism.*

Proof.

Let f be a uniform morphism from A^* to B^* . We assume that f is not $(k+1)$ -power-free and we want to show that f is not k -power-free.

The morphism f must be a ps-morphism. Otherwise, f is not k -power-free: it ends the proof.

Let w a shortest $(k+1)$ -power-free word whose image by f contains a $(k+1)$ -power. That is $f(w) = pu^{k+1}s$ for two words p and s and a non-empty word u over B .

If $f(w)$ and u^{k+1} are synchronised, by Lemma 2.12, w contains a $k+1$ -power: a contradiction with the definition of w .

Now, let us assume that f is a ps-morphism and that $f(w)$ and u^{k+1} are not synchronised. In particular, it implies that f is bifix and injective.

The central point of this proof is that, starting with w and u , we use iteratively reduction of Lemma 3.8 (that is of Lemma 3.6 and including the special cases of remark 3.7) on the word whose image contains a $(k+1)$ -power in such a way that there is no reduction left. That is, no situation of the hypotheses of Lemma 3.8 can be founded after this procedure: we obtain new words W et U such that $f(W) = pU^{k+1}s$ with p a proper prefix of $W[1]$, s a proper suffix of $W[|W|]$ and $f(W)$ and U^{k+1} are not synchronised.

We will show that either f is not k -power-free or $f(W)$ and U can again be reduced using Lemma 3.8: a contradiction with their definitions.

We focus on the fact that W necessarily contains a k -power. Indeed, since whatever the conjugated U_c of U , $f(W)$ contains U_c^k , the contrary ends the proof: f is not k -power-free. Moreover, if $W \neq w$ i.e. $|W| < |w|$ then, by definition of w , it means that W contains a $(k+1)$ -power.

Step 1: For any pure k -power v^k of W , the words U^{k+1} and $f(v)^k$ do not have any common factor of length at least $|U| + |f(v)|$.

By contradiction, let us assume that U^{k+1} and $f(v)^k$ have a common factor of length at least $|U| + |f(v)|$. By Corollary 2.4, there exist two words t_1 and t_2 and two integers r and q such that $f(v) = (t_1 t_2)^r$ and $U = (t_2 t_1)^q$ with $t_1 t_2$ and $t_2 t_1$ primitive words.

If $r \geq 2$ then $f(v^{k-1}) = (t_1 t_2)^{(k-1) \times r}$ with $(k-1) \times r \geq 2k-2$. And, since $k \geq 3$, we have $2k-2 \geq k$ that is $f(v^{k-1})$ contains a k -power with v^{k-1} k -power-free by definition of v : f is not k -power-free.

If $r = 1$ then it implies $q \geq k-1$. Otherwise, v^q would be an internal factor of v^k that is of W with $|f(v)^q| = |U|$: $f(W)$ and U^k would be synchronised. Thus, if $W = v_1 v^k v_2$ for some words v_1 and v_2 then $f(W) = f(v_1)(t_1 t_2)^k f(v_2) = p U^{k+1} s = p(t_2 t_1)^{q \times (k+1)} s$ with $q \geq k-1$.

Let x be the greatest integer such that $p(t_2 t_1)^x$ is a prefix of $f(v_1 v)$ and let y be the greatest integer such that $(t_2 t_1)^y s$ is a suffix of $f(v v_2)$. There exist four words t'_p , t''_p , t'_s and t''_s such that $t_2 t_1 = t'_p t''_p = t'_s t''_s$, $f(v_1 v) = p(t_2 t_1)^x t'_p$, $f(v v_2) = t''_s (t_2 t_1)^y s$ and $f(v^{k-2}) = t''_p (t_2 t_1)^{q(k+1)-x-y-2} t'_s$.

If $x = 0$ then $|f(v_1)| < |p|$ which implies that $v_1 = \varepsilon$. We get that $f(v_1 v^2) = p t'_p f(v) = (t_1 t_2)^2$ starts with a prefix of $p(t_2 t_1)^2$. Since $t_2 t_1$ is not a primitive word, by Lemma 2.2, $(t_2 t_1)$ is not an internal factor of $(t_2 t_1)^2$. It implies that $p = t_1$ and $t'_p = t_2$. In the same way, if $y = 0$, we get $s = t_2$ and $t''_s = t_1$.

Since $f(v_1 v)$ ends with $t_1 t_2$, since $f(v v_2)$ starts with $t_1 t_2$, if $x \geq 1$ and $t'_p \neq t_2$ or if $y \geq 1$ and $t''_s \neq t_1$ then $(t_2 t_1)$ is an internal factor of $(t_2 t_1)^2$. By Lemma 2.2, we get that $t_2 t_1$ is not a primitive word: a contradiction with the definition of $t_2 t_1$.

Thus $t'_p = t_2 = t'_s$, $t''_p = t_1 = t''_s$, $f(v_1 v) = p t_2 f(v)^x$, $f(v v_2) = f(v)^y t_1 s$ and $x + y + k - 2 = q \times (k+1) - 1$. Since f is bifix, it follows that $f(v_1 v)$ ends with $f(v)^x$ and $f(v v_2)$ starts with $f(v)^y$. We get that $v^{q \times (k+1) - 1}$ is an internal factor W with $q \times (k+1) - 1 \geq q$. It implies that v^q is an internal factor W with $|f(v)^q| = |U|$ i.e. $f(W)$ and U^k are synchronised: a contradiction with the hypotheses.

Step 2: $W[2..|W| - 1]$ contains a k -power and so a pure- k -power.

By contradiction, let us assume that $W[2..|W| - 1]$ is k -power-free. It implies that W starts or ends with a pure k -power. Let s_1 and p_{k+2} be the words such that $f(W[1]) = p s_1$ and $f(W[|W|]) = p_{k+2} s$ that is $U^{k+1} = s_1 f(W[2..|W| - 1]) p_{k+2}$.

If $|s_1| \leq |U^k|$ then there exists a word U_c such that $s_1 U_c$ is the prefix of $s_1 f(W[2..|W| - 1]) p_{k+2} = U^{k+1}$ of length $|s_1 U|$. Trivially, the word U_c is a conjugated of U (and $|U_c| = |U|$).

If $|s_1| + |p_{k+2}| \leq |U|$, we naturally have $|s_1| \leq |U^k|$. Moreover $|s_1| + |U_c^k| + |p_{k+2}| \leq |U^{k+1}|$. It means that $f(W[2..|W| - 1])$ starts with U_c^k . Since $W[2..|W| - 1]$ is a k -power-free word, it ends the proof : f is not k -power-free.

Let us now study the case where $|s_1| + |p_{k+2}| > |U|$.

Let us recall that, since we assume that $W[2..|W| - 1]$ is k -power-free, any pure k -power of $W = W[1..|W|]$ is necessarily a prefix or a suffix of it.

If W starts with a pure k -power v^k , let W_{com} be the greatest prefix of $s_1 f(v[2..|v|]) f(v^{k-1})$ which is factor of U^{k+1} so a common factor of a power of $f(v)$ and a power of U . Let us note that if $W = v^k$ then $W_{com} = U^{k+1}$ otherwise $W_{com} = s_1 f(v[2..|v|]) f(v^{k-1})$.

If $|W_{com}| \geq |U| + |f(v)|$, by Corollary 2.4, there exist two words t_1 and t_2 and two integers r and q such that $f(v) = (t_1 t_2)^r$ and $U = (t_2 t_1)^q$ with $t_1 t_2$ and $t_2 t_1$ primitive words. Since v^k is a pure k -power, we get that $r = 1$. Otherwise, f is not k -power-free. Since $f(W[2..|W| - 1])$ contains $U^{\lceil k/2 \rceil} = (t_2 t_1)^{q \times \lceil k/2 \rceil}$ if $q \geq 2$ then f is not k -power-free. Thus $r = q = 1$ and $|f(v)| = |U|$: a contradiction with the assumption that $f(W)$ and U are not synchronised.

So, we have $|W_{com}| < |U| + |f(v)|$. By definition of W_{com} , if $W = v^k$ then $W_{com} = U^{k+1} = s_1 f(v[2..|v|]) f(v^{k-2}) f(v[1..|v| - 1]) p_{k+2}$ would be a common factor of $f(v)^k$ and U^k with $|W_{com}| \geq |s_1| + |f(v)| + |p_{k+2}| > |f(v)| + |U|$: a contradiction. Thus $W \neq v^k$ and $|W_{com}| = |f(v)| + |s_1 f(v[2..|v|]) f(v^{k-2})| > |f(v)| + 2|s_1|$ which implies that $|s_1| < |U|/2$.

In the case where W ends with a k -power v^k , we similarly get $|p_{k+2}| < |U|/2$.

If W starts with a k -power then $|s_1| < |U|/2$ and, since $|s_1| + |p_{k+2}| > |U|$, it implies that $|p_{k+2}| > |U|/2$ and so $W[2..|W|]$ is k -power-free. But $f(W[1..|W|])$ starts with $ps_1 U_c^k$ i.e. $f(W[2..|W|])$ contains the k -power U_c^k : f is not k -power-free.

In the same way, if W ends with a k -power, we obtain either a contradiction with the assumptions or that f is not k -power-free.

Step 3: For any pure k -power $v^k \in \text{Fcts}(W[2..|W| - 1])$, the word $f(v)^k$ is an internal factor of U^3 and $|f(v^{k-1})| < |U|$.

For any pure k -power $v^k \in \text{Fcts}(W[2..|W| - 1])$, the word $f(v)^k$ is an internal factor of U^{k+1} . So $|f(v)^k| < |U| + |f(v)|$ i.e. $|f(v)^{k-1}| < |U|$. In particular, we get $|f(v)| < \frac{1}{2}|U|$ and $|f(v)^k| < \frac{3}{2}|U|$. That is $f(v)^k$ is an internal factor of U^3 . It implies that $c_v = 1, 2$ or 3 .

Let us recall that, for any integer $j \in [1; k+2-c_v]$, $f(v)^k$ is an internal factor of $p_j U^{c_v} s_{j+c_v}$. Thus if $\hat{v}_j$ is the shortest factor of $W[i_j..i_{j+c_v}]$ such that $f(\hat{v}_j)$ contains $f(v)^k$ then, by Corollary 3.2, we get that $\hat{v}_j$ verifies Property (P.1) for all integers $j \in [1; k+2-c_v]$.

We are going to see that it implies that W can be reduced: a final contradiction.

Case 3.1: $c_v = 3$

We necessarily have $d_v = 1$ and $|D_v f(v^{k-2})| (\leq |f(v^{k-1})|) < |U| \leq |D_v f(v^{k-1})|$. For any integer $j \in [1; k-1]$, if $\hat{v}_j$ verifies (P.1.1) then $x_{j(1)}^3 \in L_{j+1, v(1)}$ and if $\hat{v}_j$ verifies (P.1.2) then $x_{j(1)}^3 \in R_{j+1, v(1)}$. In other words, we have $L_{j+1, v(1)} \cup R_{j+1, v(1)} \neq \emptyset$ with $j+1 \in [2; k]$: by Lemma 3.9(2), we can reduce W .

Case 3.2: $c_v \neq 3$ and there exists a positive integer $\beta (\leq k-2)$ such that $d_{v(\beta)} = 1$

We necessarily have $c_v = 2$ thus $k+2-c_v = k$. For any integer $j \in [1; k]$, if $\hat{v}_j$ verifies (P.1.1) then $x_{j(\beta)}^3 \in L_{j+1, v(\beta)}$ and if $\hat{v}_j$ verifies (P.1.2) then $x_{j(\beta)}^3 \in R_{j+1, v(\beta)}$. That is $L_{j, v(\beta)} \cup R_{j, v(\beta)} \neq \emptyset$ for any integer $j \in [2; k+1]$: by Lemma 3.9(1), a reduction can be done.

Case 3.3: $c_v \neq 3$ and, for any positive integer $\beta (\leq k-2)$, we have $d_{v(\beta)} = 0$

If $c_{v(1)} = 1$ then $|D_{v(1)} f(v_{(1)})^2| \leq |U|$ and $L_{j, v(1)} \cup R_{j, v(1)} \neq \emptyset$ for any integer $j \in [1; k+1]$. By Lemma 3.9(3), a reduction can be done.

If $c_{v(1)} = 2$, there exists an integer ϕ such that $|U| < |D_{v(\phi)} f(v_\phi)^2|$ and $L_{j, v(\phi)} \cup R_{j, v(\phi)} \neq \emptyset$ for any integer $j \in [1; k]$: by Lemma 3.9(4), a reduction can be done.

□

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