Absence of quantum Yang-Mills theory with mass gap in the strict sense of Wightman’s axioms

Emmanuel Kanambaye

To cite this version:

Emmanuel Kanambaye, Absence of quantum Yang-Mills theory with mass gap in the strict sense of Wightman’s axioms. 2016. <hal-01214159v9>

HAL Id: hal-01214159
https://hal.archives-ouvertes.fr/hal-01214159v9
Submitted on 12 Nov 2016

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
Absence of quantum Yang-Mills theory with mass gap in the strict sense of Wightman’s axioms

1EMMANUEL KANAMBAYE.
1wadouba@gmail.com
2Institut Nationale de Formation en Sciences de la Santé de Bamako (INFSS Bamako), Mali.

PACS 03.70.+k Theory of quantized fields.
PACS 03.00. Quantum mechanics, field theories, and special relativity.

Abstract

We prove that on four-dimensional Minkowski space ($\mathbb{R}^4$); quantum Yang-Mills theory with mass gap satisfying strictly all the axioms of Wightman cannot exist.

Keywords: quantum Yang-Mills existence; mass gap; color confinement; classical lumps.

1. Introduction:


Since, Yang-Mills theory has been a lot of surveyed; this permitted for example to define Standard Model of Particles Physics [4], which describes very well the three following fundamental interactions: electromagnetism, weak interactions and strong interaction.

Nevertheless, in spite of its experimental and computational success; there is no rigorous mathematical proof that Standard Model of Particles Physics is Quantum Field Theory [5] in the sense of Wightman’s axioms [6].

The challenge is therefore, to prove that for any compact simple gauge group $G$, quantum Yang-Mills theory (in the sense of Wightman axioms) on $\mathbb{R}^4$ exists and has a mass gap $\Delta > 0$ ($7^{th}$ Millennium Problems [7] of Clay Mathematics Institute [8]).

In this paper, we prove that this challenge cannot be completely overcome without modifying at least slightly the W1 Wightman axiom i.e. the assumption on the domain and continuity of the fields.

Indeed, we will show that, Minkowski four-dimensional quantum Yang-Mills theory with mass gap $\Delta > 0$, in the strict sense of Wightman’s axioms is disqualified by a theorem that we will demonstrate.

2. Quantum Yang-Mills with mass gap:

The Lagrangian of Yang-Mills theory [9] is the following:

$$ L = -\frac{1}{4g^2} G^a_{\mu\nu} G^{a}_{\mu\nu} $$

(1)
Where \( g \) represents the coupling constant of the theory; \( G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g f^{abc} A_\mu A^b A^c \) is the fields’ strength tensor; \( A_\mu^a \) the vectors gauge fields and \( f^{abc} \) the structure constant of the gauge group; \( \mu \) is space-time index and \( a \) an internal index.

To prove that on \( \mathbb{R}^4 \), quantum version of this theory exists and has mass gap would consist to establishing on the one hand that the theory is well defined on four-dimensional Minkowski space following the axioms of Wightman and on the other hand that there exists a certain constant \( \Delta > 0 \), such that any free physical state (particle) allowed by the theory has an energy \( E \) satisfying:

\[
E \geq \Delta
\]  

(2)

In other words, the theory would not contain free particles having energy inferior to \( \Delta \).

The vectors fields \( A_\mu^a \) being massless fields; it results therefore that, only their massive charge-singlet bounded states will be allowed.

As the energy of these charge-singlets bounded states should obey to the above equation (2); we conclude that, charge-singlet bounded states of energy inferior to \( \Delta \) would be forbidden.

Yet, it is possible to demonstrate the following Theorem:

**On four-dimensional Minkowski space; any forbidden charge-singlet bounded states of quantum Yang-Mills fields is equivalent to the classical Yang-Mills lumps.**

Indeed, as Coleman made it brilliantly [10]; we can prove that, on four-dimensional Minkowski space; any finite-energy (non-singular) solutions of classical Yang-Mills theory that do not radiate energy out to spatial infinity are forbidden while interpreting them as “lumps” of energy held together through their own self-interactions.

To convince us, let’s begin by giving the motion equation of \( G_{\mu\nu}^a \):

\[
\partial^\mu G_{\mu\nu}^a + f^{abc} A^b G_{\mu\nu}^c = 0
\]  

(3)

While remembering that in classical physics; the field strength \( G_{\mu\nu}^a \) has the statute of real-valued functions; we can define the \( \theta^{\mu\nu} \) corresponding energy-momentum tensor:

\[
\theta^{\mu\nu} = F^{\mu\nu}_a F^a_{\nu\alpha} - \frac{1}{2} s^{\mu\nu} F^a_{\mu\nu} F^{\alpha\gamma}_a
\]  

(4)

Let’s pose:

\[
G_{\mu}^a = E_i^a
\]  

(5)

\[
\frac{1}{2} \varepsilon_{ijk} G_{jk}^a = B_i^a
\]  

(6)

From there, we can deduce the \( \theta^{00} \) total energy and the \( \theta^{0i} \) total momentum of the fields:

\[
\theta^{00} = \frac{1}{2} (E^a \cdot E^a + B^a \cdot B^a)
\]  

(7)

\[
\theta^{0i} = \frac{1}{2} (E^a \times B^a)^i
\]  

(8)

which, for any unit vector \( e \), obey to the below relation:

\[
|e \cdot \theta^{00}| \leq \theta^{00}.
\]  

(9)

As the space integral of \( \theta^{00} \) and \( \theta^{0i} \) represent respectively the energy \( E \) and the momentum \( P \) of the fields; then we will have:

\[
|P| \leq E
\]  

(10)
This means that the “lump” must travel at a velocity less than or equal to the speed of light in vacuum.

For example for lump traveling at velocity less than the speed of light in vacuum i.e. for:

\[ |\mathbf{P}| < E \quad (11) \]

the field strengths \( G_{\mu\nu}^a \) must go to zero at large distance from the center of energy of the lump; of this fact, it must obey to the linearized field equations far from the lump; what guarantees that it would be the sum of a Coulomb like-field, i.e. an incoming radiation field, and an outgoing radiation fields.

As the outgoing radiation field vanishes by assumption; then uniformly in direction and in time, for any positive number \( \epsilon \) less than \( \frac{1}{2} \) and for positive time \( t \); we will have:

\[ \lim_{x \to \infty} |x|^{\frac{1}{2} + \epsilon} G_{\mu\nu}^a (x,t) = 0 \quad (12) \]

what ensures that there is no outgoing radiation due to the uniformity in \( t \).

Now, let’s define

\[ F(r,t) = -\int_{|\mathbf{r}|} d^3x \Theta^{00}. \quad (13) \]

From the zero divergence and traceless nature of the energy-momentum tensor:

\[ \partial_\mu \Theta^{\mu\nu} = 0 \quad (14) \]

\[ \Theta^{\mu}_{\mu} = 0 \quad (15) \]

We obtain:

\[ \partial_\nu F(r,t) = \int_{|\mathbf{r}|} d^3x \Theta^{00} + \int_{\partial |\mathbf{r}|} d^2S \Theta^{\nu}. \quad (16) \]

with \( d^2S \), the outwardly-directed vector element of surface area.

As \( r \to \infty \), then the assumption that the classical lump has finite-energy and is non-radiating implies that the right-hand side of this above equation goes to \( E \) uniformly in time for positive \( t \).

Therefore, for all positive \( t \), it exists an \( r \) such that:

\[ \partial_0 F(r,t) \geq \frac{E}{2} \quad (17) \]

So that for all positive \( t \) we have:

\[ F(r,t) \geq \frac{Et}{2} + F(r,0). \quad (18) \]

However, the equation (9) requires:

\[ |F(r,t)| \leq r \int_{|\mathbf{r}|} d^3x \Theta^{00} \leq rE. \quad (19) \]

Thus, the only way in which the inequalities (18) and (19) can be consistent for all positive \( t \) is if \( E \) vanishes.

Furthermore, if the lump travels at the velocity of light in vacuum, i.e. if:

\[ E = |\mathbf{P}| \quad (20) \]

Then while choosing \( \mathbf{P} \) to be parallel to the 3-axis; equation (9) would imply:

\[ \Theta^{00} = \delta^{00} \Theta^{00}. \quad (21) \]

Whence:

\[ B^a = k \times E^a \quad (22) \]

\[ E^a = -k \times B^a \quad (23) \]

with \( k \) the unit vector in the 3-direction.

Now, let’s introduce the standard variable of light-cone:
In this case, equations (22) and (23) will be equivalent to:

\[ G_{\mu}^{a} = 0 \]  \hspace{1cm} (25)

and

\[ G_{12}^{a} = 0 \]  \hspace{1cm} (26)

This last, permitting always to have the following gauge:

\[ A^{a}_{\mu} = A_{\tau}^{a} = 0 \]  \hspace{1cm} (27)

it is guarantees that one can always make a gauge transformation that depends only on \( x^{+} \) and \( x^{-} \) such that:

\[ G_{1}^{a} = -\partial_{1} A_{\mu}^{a} \]  \hspace{1cm} (28)

\[ G_{2}^{a} = -\partial_{2} A_{\mu}^{a} \]  \hspace{1cm} (29)

This means that equation (25) implies that \( A_{\mu}^{a} \) is independent of \( x^{1} \) and \( x^{2} \); of this fact, one can gauge it away:

\[ A_{\mu}^{a} = 0 \]  \hspace{1cm} (30)

From there, it is therefore possible to make a gauge transformation that depends only on \( x^{+} \) i.e. only \( A_{\tau}^{a} \) will remain.

Because of all of these constraint; the motion equations of \( G_{\mu\nu}^{a} \):

\[ \partial^{\mu} G_{\mu\nu}^{a} + f^{abc} A_{\mu}^{b} G_{\mu\nu}^{c} = 0 \]  \hspace{1cm} (3)

will be reduced to:

\[ (\partial^{1} \partial_{1} + \partial^{2} \partial_{2}) A_{\tau}^{a} = 0 \]  \hspace{1cm} (31)

However, the finiteness of the energy ensures that this last equation is only consistent if \( A_{\tau}^{a} \) is independent of \( x^{1} \) and \( x^{2} \); likewise, \( A_{\tau}^{a} \) will depends only on \( x^{+} \); whence one can also gauge it away:

\[ A_{\tau}^{a} = 0 \]  \hspace{1cm} (32)

Thus, it appears clearly that on \( \mathbb{R}^{4} \), any classical lump i.e. any finite-energy (non-singular) solutions of the classical Yang-Mills theory that do not radiate energy out to spatial infinity are forbidden; whence:

**On four-dimensional Minkowski space; any forbidden charge-singlet bounded states of quantum Yang-Mills fields is equivalent to the classical Yang-Mills lumps.**

This means that, if quantum Yang-Mills theory with mass gap \( \Delta > 0 \) which would forbid charge-singlet bounded states having energy inferior to \( \Delta \) exists on \( \mathbb{R}^{4} \); then it will always be possible to defend that charge-singlet bounded states of energy inferior to \( \Delta \) don’t exist because they are classical lumps.

In other words, it will for example always be possible to defend the following statement that we will call “statement 0”:

**On four-dimensional Minkowski space; glueballs of energy inferior to \( \Delta \) are forbidden because they are classical glueballs.**

Therefore, if we put this “statement 0” under mathematical equation i.e. equation that means that glueballs of energy inferior to \( \Delta \) are classical lumps; for example as follow:

\[ \hat{G}_{\mu\nu}^{a} = \begin{cases} 
\hat{G}_{\mu\nu}^{a} & \text{if } E_{G} \geq \Delta \\
\langle \hat{G}_{\mu\nu}^{a} \rangle & \text{if } E_{G} < \Delta
\end{cases} \]  \hspace{1cm} (33)

with \( \hat{G}_{\mu\nu}^{a} \) the fields strength operator; \( \langle \hat{G}_{\mu\nu}^{a} \rangle \) its expected values (a real-valued
function); $E_\alpha$ the eigenvalues of the glueballs Hamiltonian operator and $\Delta$ the mass gap.

Then, proving that quantum SU(3) Yang-Mills theory with mass gap $\Delta > 0$ exists on $\mathbb{R}^4$ would be equivalent to demonstrating or postulating this above equation (33) or analogous.

Likewise, we can show that this following assertion:

**No non-charge singlet massless quanta of Yang-Mills fields can exist on four-dimensional Minkowski space;**

is mathematically equivalent to the following one that we will call “statement 1”:

On Minkowski space; non-charge singlets massless quanta of Yang-Mills fields contained inside a spherical region of the space generate classical fields if the radius of the sphere is superior to $r^C$ the maximum radius of the charge-singlet states of the theory.

Indeed, this “statement 1” affects solely non-charge singlet isolated quanta of the Yang-Mills fields and especially forbids their existence.

To convince us, it is sufficient to consider the definition of classical lump [11]:

*A classical lump is a finite energy solution to the classical field equations in Minkowski space having the property that there exist $\varepsilon$ and $R > 0$ such that for some $t_0 \geq 0$:

$$E_{(R)}(t) = \int_{|x| \leq R} \theta_0(t, \bar{x}) d^{D-1} x \geq \varepsilon; \quad (34)$$

$$\forall \ t \geq t_0$$

$\theta_0(t, \bar{x})$ represents the energy density of the field and $D$ the dimension of space-time.

This definition, characterizes classical lumps by their ability to permanently confine (held together through their own self-interactions) some amount of their energy in a bounded region of space.

As on $\mathbb{R}^4$, non-charge singlet isolated quanta of Yang-Mills fields could always be considered as contained into a sphere of radius superior to $r^C$; then “statement 1” is equivalent to sustain that non-charge singlet isolated quanta of Yang-Mills fields generate classical fields with which they can be held through their own self-interactions.

However, quanta of Yang-Mills fields holding with some amount classical Yang-Mills fields through their own self-interactions satisfies clearly the definition of classical lump; whence, “statement 1” is equivalent to sustain that on $\mathbb{R}^4$, non-charge singlet isolated quanta of Yang-Mills fields are classical lumps.

Classical lumps being forbidden (see [12; 11; 10]); “statement 1” is therefore mathematically equivalent to:

**No non-charge singlet massless quanta of Yang-Mills fields can exist on four-dimensional Minkowski space.**

This means that if we express “statement 1” through mathematical equation, for example as follow:

$$\hat{G}_\mu^a(x) = \begin{cases} \hat{G}_\mu^a(x) & \text{if } x \leq r^C \\ \langle \hat{G}_\mu^a(x) \rangle & \text{if } x > r^C \end{cases} \quad (35)$$

with $\hat{G}_\mu^a$ the fields strength operator; $\langle \hat{G}_\mu^a \rangle$ its expected values and $r^C$ the
radius beyond of which, fields generated by non-charge singlet quanta of Yang-Mills fields become classical fields.

Then, proving that quantum Yang-Mills theory with mass gap $\Delta > 0$ exists on $\mathbb{R}^4$, would also be equivalent to demonstrating or postulating this equation (35) or analogous; after all, mass gap forbids the existence of isolated non-charge singlet massless quanta of the Yang-Mills fields.

Let’s emphasize that the fact that, $\mathbb{R}^4$ quantum Yang-Mills theory with mass gap $\Delta > 0$ allows or even is equivalent to the “statements 0 and 1” has a big consequence.

Indeed, these statements sustain that the quantum statute i.e. the statute of operator-valued tempered distribution of the Yang-Mills fields strength depends on their energy and/or range, what come out of Wightman’s axioms framework, which assume that the fields of the theory are operator-valued tempered distributions, independently of their energy and/or range.

Of this fact, contrary to $\mathbb{R}^4$ quantum Yang-Mills theory with mass gap $\Delta > 0$; Wightman’s axioms not allow “statements 0 and 1”; whence:

**On $\mathbb{R}^4$; quantum Yang-Mills theory with mass gap cannot exist following the strict sense of Wightman’s axioms.**

Let’s notice that the simplest manner to avoid or overcome this result is to insert at least in the W 1 Wightman axiom the following supplementary assumption:

**The statute of operator-valued tempered distributions of the fields can depend on their energy and/or their range.**

In this case, $\mathbb{R}^4$ quantum Yang-Mills theory with mass gap would be well definable mathematically.

Be that as it may; while allowing or even being equivalent to “statement 0 and 1”; $\mathbb{R}^4$ quantum Yang-Mills theory with mass gap constrains the type of interpretation to which Quantum Mechanics must obey.

Indeed, if the quantum statute of the fields of the theory depends on their energy and/or their range; then, the Quantum Mechanics concept of wave-function could not be interpreted as being real physical object describing the ontological reality of the field’s quanta since following the energy and/or range of these fields their description can be made without invoke this concept of wave-function.

This means that $\mathbb{R}^4$ quantum Yang-Mills theory with mass gap is no compatible with interpretations like Bohmian Mechanics [13]; Many-worlds interpretation [14]; Transactional interpretation [15]; Consistent histories [16], etc. which consider the wave-function as being real physical object describing the ontological reality of the field’s quanta insofar as this not allows quite “statement 0 and 1”.

Let’s mention that as (theoretical physically) Quantum ChromoDynamics (QCD) [17] founded on SU(3) Yang-Mills theory is admitted to exhibit at least confinement; then “statement 1” is usable; whence, QCD disqualifies interpretations that consider the wave-function as being real physical object describing the ontological reality of the field’s quanta.

**3. Conclusion:**

The proof that on four-dimensional Minkowski space; quantum Yang-Mills
theory with mass gap $\Delta > 0$ cannot exist following the strict sense of the Wightman’s axioms, reveal that it is necessary to modify at least slightly the W1 Wightman axiom if we want that mathematically, $\mathbb{R}^4$ quantum Yang-Mills theories with mass $\Delta > 0$ be well definable.

References:


