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A BOUNDARY DRIVEN GENERALISED CONTACT PROCESS WITH EXCHANGE OF PARTICLES: HYDRODYNAMICS IN INFINITE VOLUME

KEVIN KUOCH, MUSTAPHA MOURRAGUI, AND ELLEN SAADA

Abstract. We consider a two species process which evolves in a finite or infinite domain in contact with particle reservoirs at different densities, according to the superposition of a generalised contact process and a rapid-stirring dynamics in the bulk of the domain, and a creation/annihilation mechanism at its boundaries. For this process, we study the law of large numbers for densities and current. The limiting equations are given by a system of non-linear reaction-diffusion equations with Dirichlet boundary conditions.

1. Introduction

In this paper, we consider the evolution on a lattice of two types of populations, according to a boundary driven generalised contact process with exchange of particles. This process is the superposition of a *contact process with random slowdowns* (or CPRS) and a rapid-stirring dynamics in the bulk of the domain, and a creation/annihilation mechanism at its boundaries, due to stochastic reservoirs.

The CPRS was introduced in [22] to model the *sterile insect technique*, developed by E. Knipling and R. Bushland (see [20, 8]) in the fifties to control the New World screw worm, a serious threat to warm-blooded animals. This pest has been eradicated from the USA and Mexico only in recent decades. The technique works as follows: Screw worms are reared in captivity and exposed to gamma rays. The male screw worms become sterile. If a sufficient number of sterile males are released in the wild then enough female screw worms are mated by sterile males so that the number of offspring will decrease generation after generation. This technique is well suited for screw worms, because female apparently mate only once in their lifetime; but it is also being tried for a large variety of pests, including current projects to fight dengue in South America (Brazil, Panama).

The particle system $(\eta_t)_{t\geq 0}$ we look at has state space $\{0,1,2,3\}^S$, for $S \subset \mathbb{Z}^d$ (we refer to [24] for interacting particle systems). Each site of S is either empty (we say it is in state 0), occupied by wild screw worms only (state 1), by sterile screw worms only (state 2), or by wild and sterile screw worms together (state 3). On each site, we only keep track of the presence or not of the type of the male screw worms (and not of their number), and we assume that enough female screw worms are around as not to limit mating.

For the CPRS dynamics, we introduce a release rate r and growth rates λ_1, λ_2. A site gets sterile males at rate r independently of everything else (this corresponds to an artificial introduction of sterile males). The rate at which wild males give birth (to wild males) on neighbouring sites is λ_1 at sites in state 1, and λ_2 at sites in state 3. Sterile males do not give birth. We assume that $\lambda_2 < \lambda_1$ to reflect the fact that at sites in state 3 the fertility is decreased. Deaths for each type of male screw worms occur at all sites at rate 1, they are mutually independent.

Date: October 31, 2016.
2000 *Mathematics Subject Classification.* Primary 82C22; Secondary 60F10, 82C35.
Key words and phrases. generalised contact process, two species process, hydrodynamic limit, specific entropy, stationary nonequilibrium states, reservoirs, infinite volume, current.

For a configuration η, the transitions of the CPRS at a site $x \in S$ are summarized as follows:

$$\begin{array}{llll} 0 \to 1 & \text{at rate } \lambda_1 n_1(x,\eta) + \lambda_2 n_3(x,\eta) & 1 \to 0 & \text{at rate } 1 \\ 0 \to 2 & \text{at rate } r & 2 \to 0 & \text{at rate } 1 \\ 1 \to 3 & \text{at rate } r & 3 \to 1 & \text{at rate } 1 \\ 2 \to 3 & \text{at rate } \lambda_1 n_1(x,\eta) + \lambda_2 n_3(x,\eta) & 3 \to 2 & \text{at rate } 1 \end{array} \tag{1.1}$$

where $n_i(x,\eta)$ is the number of nearest neighbours of x in state i for $i = 1, 3$.
In [22], a phase transition in r is exhibited for the CPRS in $S = \mathbb{Z}^d$: Assuming that $\lambda_2 \leq \lambda_c < \lambda_1$, where λ_c denotes the critical value of the d-dimensional basic contact process (see [24] on the basic contact process), there exists a critical value r_c such that wild male screw worms (that is, states 1 and 3) survive for $r < r_c$, and die out for $r \geq r_c$.

Our goal in the present paper is, for a given infinite volume S with boundaries (this expression seems puzzling, but we typically think of a piece of land whose width is much smaller than its length; the latter is thought as infinite, and the former as having boundaries), to add to the above dynamics displacements within S, as well as departures from S and immigrations to S. We model them respectively by an exchange dynamics in the bulk, and by a creation/annihilation mechanism at the boundaries of S due to the presence of stochastic reservoirs. For the superposition of the CPRS with these two dynamics, we are interested in the evolution of the empirical densities and currents corresponding to wild and sterile screw worms, for which we establish hydrodynamic limits. The limiting equations are given by systems of non-linear reaction-diffusion equations, with Dirichlet boundary conditions. We also obtain hydrodynamic limits when S is either a finite volume in contact with reservoirs, or the infinite volume $\mathbb{Z}^d$.

Hydrodynamic limits investigate the macroscopic properties of interacting particle systems (we refer to the books [27, 19] for a thorough presentation). From a probabilistic point of view, it corresponds to a law of large numbers for the evolution of the spatial density of particles in a given system. After the results of [16, 18], where the intensive use of large deviation techniques led to a robust proof of the hydrodynamic behaviour of a large class of finite volume gradient equilibrium systems, the method has been extended to nonequilibrium systems in a bounded domain in [11, 9, 10], as well as in an infinite volume without boundaries for conservative dynamics in [12, 13, 14, 28, 23].
In the last years, many papers have been devoted to systems in contact with reservoirs in a bounded domain; we just quote a few of them, [1, 3, 2, 4] and references therein. The nonequilibrium systems considered there were provided by lattice gas models also submitted to an external mechanism of creation and annihilation of particles, modelling exchange reservoirs at the boundaries. Even though the stochastic dynamics describing the evolution in the bulk was conservative and in equilibrium, the action of the reservoirs made the full process non reversible.
The hydrodynamic limit of a class of jump, birth and death processes has been studied in [6, 7]: the combination of the Symmetric Simple Exclusion Process and of a Glauber dynamics in dimension 1 to model the annihilation and creation of particles led to a reaction-diffusion equation. The density and current large deviations have then been proved for the dynamics evolving on a torus and on a one-dimensional bounded interval in contact with reservoirs respectively in [17, 4]; there, the lower bound of the large deviations principle was obtained only for smooth trajectories, and for birth and the death rates that were monotone, concave functions.

To our knowledge, the present paper is the first work about hydrodynamics of an interacting particle system evolving in an *infinite* volume with boundary stochastic reservoirs and leading to *a system* of reaction-diffusion equations. The natural questions that emerge after the hydrodynamic limit are fluctuations and large deviations with respect to the expected limit. Nonequilibrium fluctuations of interacting particle systems have only been derived for few one-dimensional dynamics. It is one of the main open problems in the field. We now plan to study phase transitions, hydrostatics, and large deviations for our model.

Our set-up is the following. The non-conservative dynamics that we consider evolves in the infinite cylinder

$$\Lambda_N = \{-N, \cdots, N\} \times \mathbb{Z}^{d-1}. \tag{1.2}$$

In the bulk of Λ_N, particles evolve according to the superposition of the CPRS (which is non-conservative) and of an exchange dynamics (which is conservative). The latter satisfies a detailed balance condition with respect to a family of Gibbs measures. The reservoirs defining the movements of populations at the infinite boundary

$$\Gamma_N := \{-N, N\} \times \mathbb{Z}^{d-1} \tag{1.3}$$

of Λ_N are modelled by a reversible birth and death process. The full dynamics keeps fixed the value of the density at the boundary. We have therefore to face the difficulty of non reversibility in the bulk combined with the fact that the stationary measures are not explicitly known.

Our key tools to establish hydrodynamic limits will be first the analysis of the specific entropy and the specific Dirichlet form *in infinite volume* (to our knowledge, such an analysis is carried out in infinite volume with reservoirs for the first time), then the use of couplings to derive hydrodynamics by going from systems evolving in a large but finite volume to systems in infinite volume. Here by a large finite volume, we mean the cylinder with a length M_N (to be precised later on) large with respect to N. The definition of our dynamics will enable us to use basic coupling, which is not always possible (we refer to [15, 5] for dynamics requiring more intricate couplings). Finally we prove uniqueness of weak solutions to the limiting system of reaction-diffusion equations.

The paper is organized as follows. In Section 2, we detail our model. In Section 3, we state our results: the hydrodynamic limit of the boundary driven process in infinite volume is stated in Theorem 3.1; we also state two related results, hydrodynamics in the full space $\mathbb{Z}^d$ in Theorem 3.2, and in a bounded space with boundary conditions in Theorem 3.3. Lastly the law of large numbers for the conservative and non-conservative currents is stated in Proposition 3.4. For the proofs of Theorems 3.2 and 3.3 we refer to [21]. Sections 4 to 8 are devoted to the proof of Theorem 3.1: it is outlined in Section 4; Section 5 deals with specific entropy and Dirichlet forms, Section 6 with hydrodynamics in large finite volume, Section 7 with couplings to derive the boundary conditions in infinite volume; uniqueness of solutions is proved in Section 8. In Section 9 we prove Proposition 3.4.

2. Description of the model

2.1. **Equivalent formulations for configurations.** Rather than studying directly the process $(\eta_t)_{t\geq 0}$ describing the evolution of states $1, 2, 3$, we introduce another interpretation for the model. The corresponding configuration space is

$$\widehat{\Sigma}_N := \big(\{0,1\} \times \{0,1\}\big)^{\Lambda_N}. \tag{2.1}$$

(In the sequel, we shall denote with a "hat" everything related to product spaces and to vectors). Elements of $\widehat{\Sigma}_N$ are denoted by (ξ, ω), where ξ-particles represent the wild screw worms, while ω-particles represent the sterile ones. The correspondence with $(\eta_t)_{t\geq 0}$ is given by the following relations: For $x \in \Lambda_N$,

$$\begin{array}{lll} \eta(x) = 0 & \iff & (1-\xi(x))(1-\omega(x)) = 1\,, \\ \eta(x) = 1 & \iff & \xi(x)(1-\omega(x)) = 1\,, \\ \eta(x) = 2 & \iff & (1-\xi(x))\omega(x) = 1\,, \\ \eta(x) = 3 & \iff & \xi(x)\omega(x) = 1\,. \end{array} \tag{2.2}$$

In other words, $\xi(x) = 1$ (resp. $\omega(x) = 1$) if wild (resp. sterile) screw worms are present on x. Both can be present, giving the state 3 for $\eta(x)$, or only one of them, giving the states 1 or 2 for $\eta(x)$. We may also express the correspondence (2.2) by an application from $\widehat{\Sigma}_N$ to $\{1,2,3\}^{\Lambda_N}$, that is, we write

$$\eta = \eta((\xi,\omega)), \quad \text{where, for any } x \in \Lambda_N, \quad \eta(x) = 2\omega(x) + \xi(x)\,. \tag{2.3}$$

Moreover, in order to describe the evolution of densities for states 1, 2, 3, we also define, for $x \in \Lambda_N$,

$$\begin{cases} \eta_1(x) = \xi(x)(1-\omega(x)) \equiv \mathbf{1}_{\{\eta(x)=1\}}\,, \\ \eta_2(x) = (1-\xi(x))\omega(x) \equiv \mathbf{1}_{\{\eta(x)=2\}}\,, \\ \eta_3(x) = \xi(x)\omega(x) \equiv \mathbf{1}_{\{\eta(x)=3\}}\,. \end{cases} \tag{2.4}$$

By abuse of language, when $\eta_i(x) = 1$ for $i = 1, 2, 3$, we say that there is a particle of type i at x. It is convenient to complete (2.4) by defining empty occupation of site x by

$$\eta_0(x) = (1-\xi(x))(1-\omega(x)) = \mathbf{1}_{\{\eta(x)=0\}} = 1 - \eta_1(x) - \eta_2(x) - \eta_3(x)\,. \tag{2.5}$$

In view of going from finite to infinite volume, for each positive integer n, denote by

$$\Lambda_{N,n} = \{-N, \cdots, N\} \times \{-n, \cdots, n\}^{d-1} \tag{2.6}$$

the sub-lattice of size $(2N+1) \times (2n+1)^{d-1}$ of Λ_N, and by

$$\widehat{\Sigma}_{N,n} = \big(\{0,1\} \times \{0,1\}\big)^{\Lambda_{N,n}} \tag{2.7}$$

the corresponding state space.

2.2. **The infinitesimal generator.** The boundary driven generalised contact process with exchange of particles in Λ_N is the Markov process on $\widehat{\Sigma}_N$ whose generator $\mathfrak{L}_N := \mathfrak{L}_{\lambda_1,\lambda_2,r,\widehat{b},N}$ can be decomposed as

$$\mathfrak{L}_N := N^2 \mathcal{L}_N + \mathbb{L}_N + N^2 L_{\widehat{b},N}\,, \tag{2.8}$$

with the generators $\mathcal{L}_N$ for the exchanges of particles, $\mathbb{L}_N$ for the CPRS, and $L_{\widehat{b},N}$ for the boundary dynamics. We now detail those dynamics and their properties. For the existence of the Markov process with generator $\mathfrak{L}_N$ in infinite volume, we refer to [24, Chapter 1], since we consider a compact state space and bounded rates (defined below). Cylinder functions are a core for the generator $\mathfrak{L}_N$.

• For the exchange dynamics, the action of $\mathcal{L}_N$ on cylinder functions $f : \widehat{\Sigma}_N \to \mathbb{R}$ is

$$\mathcal{L}_N f(\xi,\omega) = \sum_{k=1}^{d} \sum_{x,x+e_k \in \Lambda_N} \mathcal{L}^{x,x+e_k} f(\xi,\omega) \text{ with} \tag{2.9}$$

$$\mathcal{L}^{x,x+e_k} f(\xi,\omega) = f(\xi^{x,x+e_k}, \omega^{x,x+e_k}) - f(\xi,\omega)\,, \tag{2.10}$$

where $(e_1, \dots, e_d)$ denotes the canonical basis of $\mathbb{R}^d$, and for any $\zeta \in \Sigma_N := \{0,1\}^{\Lambda_N}$, $\zeta^{x,y}$ is the configuration obtained from ζ by exchanging the occupation variables $\zeta(x)$ and $\zeta(y)$, i.e.

$$(\zeta^{x,y})(z) := \begin{cases} \zeta(y) & \text{if } \ z = x\,, \\ \zeta(x) & \text{if } \ z = y\,, \\ \zeta(z) & \text{if } \ z \neq x, y\,. \end{cases}$$

Since $(\xi,\omega) \in \widehat{\Sigma}_N$, these exchanges correspond to jumps between sites x and y for ξ-particles and ω-particles, which do not influence each other.

• We now define a family of *invariant probability measures* for $\mathcal{L}_N$, which are product, and parametrized by three chemical potentials, since the exchange dynamics conserves, in each transition, the number of particles of each type. We denote by Λ the macroscopic open cylinder

$$\Lambda = (-1,1) \times \mathbb{R}^{d-1}\,. \tag{2.11}$$

For a vector-valued function $\widehat{m} = (m_1, m_2, m_3) : \Lambda \to \mathbb{R}^3$, let $\bar{\nu}^N_{\widehat{m}(\cdot)}$ be the product measure on Λ_N with varying chemical potential $\widehat{m}$ such that, for all positive integers n, the restriction $\bar{\nu}^N_{\widehat{m}(\cdot),n}$ of $\bar{\nu}^N_{\widehat{m}(\cdot)}$ to $\widehat{\Sigma}_{N,n}$ is given by

$$\bar{\nu}^N_{\widehat{m}(\cdot),n}(\xi,\omega) = \widehat{Z}^{-1}_{\widehat{m},n} \exp\Big\{ \sum_{i=1}^{3} \sum_{x \in \Lambda_{N,n}} m_i(x/N)\eta_i(x) \Big\}, \tag{2.12}$$

where $\widehat{Z}_{\widehat{m},n}$ is the normalization constant:

$$\widehat{Z}_{\widehat{m},n} = \prod_{x\in\Lambda_{N,n}} \Big\{ 1 + \sum_{i=1}^{3} \exp(m_i(x/N)) \Big\}. \tag{2.13}$$

Notice that the family of measures $\big\{\bar\nu^N_{\widehat{m}}\,,\ \ \widehat{m}\in\mathbb{R}^3\big\}$ with constant parameters is reversible with respect to the generator $\mathcal{L}_N$. For $\widehat{m}\in\mathbb{R}^3$ and $1\le i\le 3$, let $\psi_i(\widehat{m})$ be the expectation of $\eta_i(0)$ under $\bar\nu^N_{\widehat{m}}$:

$$\psi_i(\widehat{m}) \;=\; \mathbf{E}^{\bar\nu^N_{\widehat{m}}}\big[\eta_i(0)\big]\,.$$

Observe that the function Ψ defined on $(0,+\infty)^3$ by $\Psi(\widehat{m}) = (\psi_1(\widehat{m}),\psi_2(\widehat{m}),\psi_3(\widehat{m}))$ is a bijection from $(0,+\infty)^3$ to $(0,1)^3$. We shall therefore do a change of parameter: For every $\widehat{\rho}=(\rho_1,\rho_2,\rho_3)\in(0,1)^3$, we denote by $\nu^N_{\widehat{\rho}}$ the product measure $\overline{\nu}^N_{\widehat{m}}$ such that $\Psi(\widehat{m})=\widehat{\rho}$, hence

$$\rho_i \;=\; \mathbf{E}^{\nu^N_{\widehat{\rho}}}\big[\eta_i(0)\big]\,,\quad i=1,2,3\,. \tag{2.14}$$

From now on, we work with the representation $\nu^N_{\widehat{\rho}(\cdot),n}$ of the measure $\bar\nu^N_{\widehat{m}(\cdot),n}$.

Finally, note that in view of the diffusive scaling limit, the generator $\mathcal{L}_N$ has been speeded up by N^2 in (2.8).

• According to (1.1), the generator $\mathbb{L}_N := \mathbb{L}_{N,\lambda_1,\lambda_2,r}$ of the CPRS is given by

$$\begin{aligned}
\mathbb{L}_N f(\xi,\omega) &= \sum_{x\in\Lambda_N} \mathbb{L}^x_{\Lambda_N} f(\xi,\omega) \quad \text{ where, for any } \mathcal{A}\subset\mathbb{Z}^d,\ x\in\mathcal{A} && (2.15)\\
\mathbb{L}^x_{\mathcal{A}} f(\xi,\omega) &= \Big(r(1-\omega(x))+\omega(x)\Big)\Big[f(\xi,\sigma^x\omega)-f(\xi,\omega)\Big] && (2.16)\\
&\quad + \Big(\beta_{\mathcal{A}}(x,\xi,\omega)\big(1-\xi(x)\big)+\xi(x)\Big)\Big[f(\sigma^x\xi,\omega)-f(\xi,\omega)\Big] \quad \text{ with}\\
\beta_{\mathcal{A}}(x,\xi,\omega) &= \lambda_1 \sum_{\substack{y\in\mathcal{A}\\ \|y-x\|=1}} \xi(y)(1-\omega(y)) + \lambda_2 \sum_{\substack{y\in\mathcal{A}\\ \|y-x\|=1}} \xi(y)\omega(y)\,, && (2.17)
\end{aligned}$$

where $\|\cdot\|$ denotes the norm in $\mathbb{R}^d$, $\|u\| = \sqrt{\sum_{i=1}^d |u_i|^2}$ for $u\in\mathbb{R}^d$, and for $\zeta\in\Sigma_N$, $\sigma^x\zeta$ is the configuration obtained from ζ by flipping the configuration at x, i.e.

$$(\sigma^x\zeta)(z) := \begin{cases} 1-\zeta(x) & \text{if } \ z=x\,,\\ \zeta(z) & \text{if } \ z\neq x\,. \end{cases}$$

The representation (2.2) sheds light on the fact that (2.15) corresponds to a contact process (the ξ-particles) in a dynamic random environment, namely the ω-particles. Indeed, the ω-particles move on their own and are not influenced by ξ-particles, while ξ-particles have birth rates whose value depends on the presence or not of ω-particles. In [22] a variant of the CPRS dynamics in a quenched random environment is also considered, with the (ξ,ω)-formalism. On the other hand, we noticed previously that ω-particles can also be considered as an environment for the exchange dynamics.

• We now turn to the dynamics at the boundaries of the domain. We denote respectively the closure and the boundary of Λ (see (2.11)) by

$$\overline{\Lambda} = [-1,1]\times\mathbb{R}^{d-1}, \qquad \Gamma = \{(u_1,\dots,u_d)\in\overline{\Lambda} : u_1=\pm1\}. \tag{2.18}$$

For a metric space E and any integer $1\le m\le+\infty$, denote by $\mathcal{C}^m(\overline{\Lambda};E)$ (resp. $\mathcal{C}^m_c(\Lambda;E)$) the space of m-continuously differentiable functions on $\overline{\Lambda}$ (resp. with compact support in Λ) with values in E, and by $\mathcal{C}(\overline{\Lambda};E)$ (resp. $\mathcal{C}(\Lambda;E)$, $\mathcal{C}_c(\Lambda;E)$) the space of continuous functions on $\overline{\Lambda}$ (resp. on Λ, with compact support in Λ) with values in E.

Fix a positive function $\widehat{b} = (b_1, b_2, b_3) : \Gamma \to \mathbb{R}_+^3$. Assume that there exists a neighbourhood V of $\overline{\Lambda}$ in $\mathbb{R}^d$ and a smooth function $\widehat{\theta} = (\theta_1, \theta_2, \theta_3) : V \to (0,1)^3$ in $\mathcal{C}^2(V;\mathbb{R}^3)$ such that

$$0 < c^* \le \min_{1\le i\le 3} |\theta_i| \le \max_{1\le i\le 3} |\theta_i| \le C^* < 1 \tag{2.19}$$

for two positive constants c^*, C^*, and such that the restriction of $\widehat{\theta}$ to Γ is equal to $\widehat{b}$:

$$\widehat{\theta}(\cdot)\big|_{\Gamma} = \widehat{b}(\cdot)\,. \tag{2.20}$$

The boundary dynamics acts as a birth and death process on the boundary Γ_N of Λ_N (see (1.3)) described by the generator $L_{\widehat{b},N}$ defined by

$$\begin{aligned} L_{\widehat{b},N} f(\xi,\omega) &= \sum_{x\in\Gamma_N} L^x_{\widehat{b},N} f(\xi,\omega)\,, \qquad \text{where} && (2.21)\\ L^x_{\widehat{b},N} f(\xi,\omega) &= c_x\big(\widehat{b}(x/N),\xi,\sigma^x\omega\big)\Big[f(\xi,\sigma^x\omega) - f(\xi,\omega)\Big] \\ &\quad + c_x\big(\widehat{b}(x/N),\sigma^x\xi,\omega\big)\big[f(\sigma^x\xi,\omega) - f(\xi,\omega)\big] \\ &\quad + c_x\big(\widehat{b}(x/N),\sigma^x\xi,\sigma^x\omega\big)\big[f(\sigma^x\xi,\sigma^x\omega) - f(\xi,\omega)\big]\,, && (2.22) \end{aligned}$$

where the rates $c_x\big(\widehat{b}(x/N),\xi,\omega\big)$ are given for $x\in\Gamma_N$ and $(\xi,\omega)\in\widehat{\Sigma}_N$ by

$$\begin{aligned} c_x\big(\widehat{b}(x/N),\xi,\omega\big) &= \sum_{i=0}^{3} b_i(x/N)\eta_i(x)\,, && (2.23)\\ \text{with} \quad b_0(x/N) &= 1 - \sum_{i=1}^{3} b_i(x/N) && (2.24) \end{aligned}$$

where $\eta_i(x)$, $i\in\{0,1,2,3\}$ are defined in (2.4)–(2.5). In other words, according to $\widehat{b}(.)$, a site $x\in\Gamma_N$ goes from state $i\in\{0,1,2,3\}$ to state $j\in\{0,1,2,3\}$ $(j\ne i)$ at rate $b_j(x/N)$ (see (7.8) below, where this interpretation is used).

We have that by (2.20), the boundary dynamics is such that the measure $\nu^N_{\widehat{\theta}}$ is reversible with respect to the operator $L_{\widehat{b},N}$ (see Consequences 5.3*(ii)*). Note also that the generator $L_{\widehat{b},N}$ has been speeded up by N^2 in (2.8).

3. The results

3.1. **Notation.** We fix $T>0$. We denote by $(\xi_t,\omega_t)_{t\in[0,T]}$ the Markov process on $\widehat{\Sigma}_N$ with generator $\mathfrak{L}_N$ defined in (2.8). Given a probability measure μ on $\widehat{\Sigma}_N$, the probability measure $\mathbb{P}^{N,\widehat{b}}_{\mu}$ on the path space $D([0,T],\widehat{\Sigma}_N)$, endowed with the Skorohod topology and the corresponding Borel σ-algebra, is the law of $(\xi_t,\omega_t)_{t\in[0,T]}$ with initial distribution μ. The associated expectation is denoted by $\mathbb{E}^{N,\widehat{b}}_{\mu}$.

We denote by $(S^{\widehat{b}}_N(t))_{t\in[0,T]}$ the semigroup associated to the generator $\mathfrak{L}_N$. We shall denote by $\mu(t)$ the time evolution of the measure μ under the semigroup $S^{\widehat{b}}_N$: $\mu(t) = \mu S^{\widehat{b}}_N(t)$.

We denote by $\mathcal{M}$ the space of finite signed measures on Λ, endowed with the weak topology. For $m\in\mathcal{M}$ and a function $F\in\mathcal{C}(\Lambda;\mathbb{R})$, we let $\langle m,F\rangle$ be the integral of F with respect to m. For each configuration $(\xi,\omega)\in\widehat{\Sigma}_N$, let $\widehat{\pi}^N = \widehat{\pi}^N(\xi,\omega) = (\pi^{N,1},\pi^{N,2},\pi^{N,3})\in\mathcal{M}^3$, where for $i\in\{1,2,3\}$, the positive measure $\pi^{N,i}$ is obtained by assigning mass N^{-d} to each particle of type i (cf. (2.4)):

$$\pi^{N,i} = N^{-d}\sum_{x\in\Lambda_N} \eta_i(x)\,\delta_{x/N}\ ,$$

where δ_u is the Dirac measure concentrated on u. For any function $\widehat{G} = (G_1, G_2, G_3) \in \mathcal{C}(\Lambda; \mathbb{R}^3)$, the integral of $\widehat{G}$ with respect to $\widehat{\pi}^N$, denoted by $\langle \widehat{\pi}^N, \widehat{G} \rangle$, is given by

$$\langle \widehat{\pi}^N, \widehat{G} \rangle = \sum_{i=1}^{3} \langle \pi^{N,i} \, , \, G_i \rangle \, .$$

We also denote by $\widehat{\pi}^N$ the map from $D([0,T], \widehat{\Sigma}_N)$ to $D([0,T], \mathcal{M}^3)$ defined by $\widehat{\pi}^N(\xi_\cdot, \omega_\cdot)_t = \widehat{\pi}^N(\xi_t, \omega_t)$ and we denote by $Q_\mu^{N,\widehat{b}} = \mathbb{P}_\mu^{N,\widehat{b}} \circ (\widehat{\pi}^N)^{-1}$ the law of the process $\big(\widehat{\pi}^N(\xi_t, \omega_t)\big)_{t \in [0,T]}$.
Let $\mathcal{M}_+^1$ be the subset of $\mathcal{M}$ of all positive measures absolutely continuous with respect to the Lebesgue measure with positive density bounded by 1:

$$\mathcal{M}_+^1 = \big\{ \pi \in \mathcal{M} : \pi(du) = \rho(u) du \quad \text{and} \quad 0 \le \rho(u) \le 1 \ \text{ a.e.} \big\} \, .$$

Let $D([0,T], (\mathcal{M}_+^1)^3)$ be the set of right continuous trajectories with left limits with values in $(\mathcal{M}_+^1)^3$, endowed with the Skorohod topology and equipped with its Borel $\sigma-$ algebra.
For a metric space E, and integers $1 \le m, k \le +\infty$, we denote by $\mathcal{C}_c^{k,m}([0,T] \times \overline{\Lambda}; E)$ (resp. $\mathcal{C}_{c,0}^{k,m}([0,T] \times \overline{\Lambda}; E)$) the space of functions from $[0,T] \times \overline{\Lambda}$ to E that are k-continuously differentiable in time and m-continuously differentiable in space, with compact support in $[0,T] \times \overline{\Lambda}$ (resp. and vanishing at the boundary Γ of Λ). Similarly, we define $\mathcal{C}_c^{k,m}([0,T] \times \Lambda; E)$ to be the subspace of $\mathcal{C}_c^{k,m}([0,T] \times \overline{\Lambda}; E)$ of functions with compact support in $[0,T] \times \Lambda$.
For $\widehat{G} = (G_1, G_2, G_3), \widehat{H} = (H_1, H_2, H_3) \in \big(L^2(\Lambda)\big)^3$, $\langle \widehat{G}(\cdot), \widehat{H}(\cdot) \rangle$ is the scalar product:

$$\langle \widehat{G}(\cdot), \widehat{H}(\cdot) \rangle = \sum_{i=1}^{3} \langle G_i(\cdot), H_i(\cdot) \rangle = \sum_{i=1}^{3} \int_\Lambda G_i(u) H_i(u) du \, .$$

For a smooth function $G : [0,T] \times \Lambda \to \mathbb{R}$, $\partial_s G(s,u)$ represents the partial derivative with respect to the time variable s and for $1 \le j \le d, k \ge 1$, $\partial_{e_j}^k G(s,u)$ stands for the k-th partial derivative in the direction e_j with respect to the space variable u. The discrete gradient $\partial_{e_1}^N$ in the direction e_1 is defined, for $x, x + e_1 \in \Lambda_N$, $G : \Lambda \to \mathbb{R}$, by

$$\partial_{e_1}^N G(x/N) = N\Big(G\Big(\frac{x+e_1}{N}\Big) - G\Big(\frac{x}{N}\Big)\Big) \, .$$

The discrete Laplacian Δ_N and the Laplacian Δ are respectively defined for $G \in \mathcal{C}^2(\Lambda; \mathbb{R})$, if $x, x \pm e_j \in \Lambda_N$ for $1 \le j \le d$ and $u \in \Lambda \setminus \Gamma$, by

$$\Delta_N G(x/N) = N^2 \sum_{j=1}^{d} \Big[G\Big(\frac{x+e_j}{N}\Big) + G\Big(\frac{x-e_j}{N}\Big) - 2G\Big(\frac{x}{N}\Big)\Big] \quad \text{and} \quad \Delta G(u) = \sum_{j=1}^{d} \partial_{e_j}^2 G(u).$$

We have now all the material to state our results.

3.2. **Hydrodynamics in infinite volume with reservoirs.** We first describe *the hydrodynamic equations*. Let $\widehat{\gamma} = (\gamma_1, \gamma_2, \gamma_3) : \overline{\Lambda} \to [0,1]^3$ be a smooth initial profile, and denote by $\widehat{\rho} = (\rho_1, \rho_2, \rho_3) : [0,T] \times \overline{\Lambda} \to [0,1]^3$ a typical macroscopic trajectory. We shall prove in Theorem 3.1 below that the macroscopic evolution of the local particle density $\widehat{\pi}^N$ is described by the following system of non-linear reaction-diffusion equations

$$\begin{cases} \partial_t \widehat{\rho} & = \ \Delta \widehat{\rho} + \widehat{F}(\widehat{\rho}) \quad \text{in } \Lambda \times (0,T), \\ \widehat{\rho}(0, \cdot) & = \ \widehat{\gamma}(\cdot) \quad \text{in } \Lambda, \\ \widehat{\rho}(t, \cdot)|_\Gamma & = \ \widehat{b}(\cdot) \quad \text{for } \ 0 \le t \le T \, , \end{cases} \tag{3.1}$$

where $\widehat{F} = (F_1, F_2, F_3) : [0,1] \to \mathbb{R}^3$ is given by

$$\begin{cases} F_1(\rho_1, \rho_2, \rho_3) & = \ 2d(\lambda_1 \rho_1 + \lambda_2 \rho_3)\rho_0 + \rho_3 - \rho_1(r+1) \, , \\ F_2(\rho_1, \rho_2, \rho_3) & = \ r\rho_0 + \rho_3 - 2d(\lambda_1 \rho_1 + \lambda_2 \rho_3)\rho_2 - \rho_2 \, , \\ F_3(\rho_1, \rho_2, \rho_3) & = \ 2d(\lambda_1 \rho_1 + \lambda_2 \rho_3)\rho_2 + r\rho_1 - 2\rho_3 \, , \end{cases} \tag{3.2}$$

where $\rho_0 = 1 - \rho_1 - \rho_2 - \rho_3$.

A weak solution of (3.1) is a function $\widehat{\rho}(\cdot,\cdot) = (\rho_1, \rho_2, \rho_3) : [0,T] \times \Lambda \to [0,1]^3$ satisfying (IB1), (IB2) and (IB3) below:

(IB1) For any $i \in \{1,2,3\}$, $\rho_i \in L^\infty((0,T) \times \Lambda)$.

(IB2) For any function $\widehat{G}(t,u) = \widehat{G}_t(u) = (G_{1,t}(u), G_{2,t}(u), G_{3,t}(u))$ in $\mathcal{C}^{1,2}_{c,0}\big([0,T] \times \overline{\Lambda}; \mathbb{R}^3\big)$, writing similarly the density as $\widehat{\rho}_t(u) = \widehat{\rho}(t,u)$, we have

$$\begin{aligned} \langle \widehat{\rho}_T(\cdot), \widehat{G}_T(\cdot)\rangle - \langle \widehat{\rho}_0(\cdot), \widehat{G}_0(\cdot)\rangle - \int_0^T ds\, \langle \widehat{\rho}_s(\cdot), \partial_s \widehat{G}_s(\cdot)\rangle = \int_0^T ds\, \langle \widehat{\rho}_s(\cdot), \Delta \widehat{G}_s(\cdot)\rangle \\ + \int_0^T ds\, \langle \widehat{F}(\rho_s)(\cdot), \widehat{G}_s(\cdot)\rangle - \sum_{i=1}^3 \int_0^T ds \int_\Gamma \mathbf{n}_1(r)\, b_i(r) (\partial_{e_1} G_{i,s})(r)\, dS(r)\,, \end{aligned} \tag{3.3}$$

where $\mathbf{n}=(\mathbf{n}_1, \dots, \mathbf{n}_d)$ stands for the outward unit normal vector to the boundary surface Γ and dS for an element of surface on Γ.

(IB3) $\widehat{\rho}(0,u) = \widehat{\gamma}(u)$ a.e.

This system of equations (3.1) has a unique weak solution (see Section 8).

Our main result is:

Theorem 3.1. *For each $N \geq 1$, let μ_N be a probability measure on $\widehat{\Sigma}_N$. The sequence of probability measures $(Q^{N,\widehat{b}}_{\mu_N})_{N\geq 1}$ is weakly relatively compact and all its converging subsequences converge to some limit $Q^{\widehat{b},*}$ which is concentrated on absolutely continuous paths in $\mathcal{C}([0,T], (\mathcal{M}^1_+)^3)$ whose density $\widehat{\rho}$ satisfies (IB1) and (IB2).*

Moreover, if for any $\delta > 0$ and for any function $\widehat{G} \in \mathcal{C}_c(\overline{\Lambda}; \mathbb{R}^3)$,

$$\lim_{N\to\infty} \mu_N \Big\{ \Big| \langle \widehat{\pi}_N(\xi,\omega), \widehat{G}(\cdot)\rangle \; - \; \langle \widehat{\gamma}(\cdot), \widehat{G}(\cdot)\rangle \Big| \geq \delta \Big\} = 0\,, \tag{3.4}$$

for an initial continuous profile $\widehat{\gamma} : \Lambda \to [0,1]^3$, then the sequence $(Q^{N,\widehat{b}}_{\mu_N})_{N\geq 1}$ converges to the Dirac measure concentrated on the unique weak solution $\widehat{\rho}(\cdot,\cdot)$ of the boundary value problem (3.1). Accordingly, for any $t \in [0,T]$, any $\delta > 0$ and any function $\widehat{G} \in \mathcal{C}^{1,2}_c\big([0,T] \times \overline{\Lambda}; \mathbb{R}^3\big)$,

$$\lim_{N\to\infty} \mathbb{P}^{N,\widehat{b}}_{\mu_N} \Big\{ \Big| \langle \widehat{\pi}_N(\xi_t,\omega_t), \widehat{G}(\cdot)\rangle \; - \; \langle \widehat{\rho}_t(\cdot), \widehat{G}(\cdot)\rangle \Big| \geq \delta \Big\} = 0\,.$$

The proof of Theorem 3.1 will be outlined in Section 4, and done in the following Sections 5, 6, 7, 8.

3.3. **Hydrodynamics in $\mathbb{Z}^d$.** Our approach enables us to derive as well the hydrodynamic limit in the full volume $\mathbb{Z}^d$. There, the reaction-diffusion process $(\xi_t, \omega_t)_{t\in[0,T]}$ on $\big(\{0,1\} \times \{0,1\}\big)^{\mathbb{Z}^d}$ has generator

$$N^2 \mathcal{L} + \mathbb{L} = N^2 \sum_{k=1}^d \sum_{x \in \mathbb{Z}^d} \mathcal{L}^{x,x+e_k} + \sum_{x\in\mathbb{Z}^d} \mathbb{L}^x_{\mathbb{Z}^d}$$

(see (2.10), (2.16)–(2.17)) with law $\mathbb{P}^N_{\mu_N}$ when the initial distribution is μ_N. It satisfies the following theorem, proved in [21].

Theorem 3.2. *Consider a sequence of probability measures $(\mu_N)_{N\geq 1}$ on $\big(\{0,1\} \times \{0,1\}\big)^{\mathbb{Z}^d}$ associated to a continuous profile $\widehat{\gamma} : \mathbb{R}^d \to [0,1]^3$, that is, for any function $\widehat{G} = (G_1, G_2, G_3) \in \mathcal{C}_c(\mathbb{R}^d; \mathbb{R}^3)$,*

$$\lim_{N\to\infty} \mu_N \Big\{ \Big| \langle \widehat{\pi}_N(\xi,\omega), \widehat{G}(\cdot)\rangle \; - \; \langle \widehat{\gamma}(\cdot), \widehat{G}(\cdot)\rangle \Big| \geq \delta \Big\} = 0\,,$$

for all $\delta > 0$. Then for any $t \in [0,T]$, for any function $\widehat{G} = (G_1, G_2, G_3) \in \mathcal{C}_c(\mathbb{R}^d; \mathbb{R}^3)$,

$$\lim_{N\to\infty} \mathbb{P}^N_{\mu_N} \Big\{ \Big| \langle \widehat{\pi}_N(\xi_t,\omega_t), \widehat{G}(\cdot)\rangle \; - \; \langle \widehat{\rho}_t(\cdot), \widehat{G}(\cdot)\rangle \Big| \geq \delta \Big\} = 0$$

for all $\delta > 0$, where $\widehat{\rho}(t,u)$ is the unique weak solution of the system

$$\begin{cases} \partial_t \widehat{\rho} = \Delta\widehat{\rho} + \widehat{F}(\widehat{\rho}) & in \;\; \mathbb{Z}^d \times (0,T), \\ \widehat{\rho}(0,\cdot) = \widehat{\gamma}(\cdot) & in \;\; \mathbb{Z}^d, \end{cases}$$

that is, $\widehat{\rho}(\cdot,\cdot)$ satisfies the following assertions:

(I1) *For any $i \in \{1,2,3\}$, $\rho_i \in L^\infty\big([0,T]\times\mathbb{R}^d\big)$,*

(I2) *For any function $\widehat{G}_t(u) = \widehat{G}(t,u)$ in $\mathcal{C}_c^{1,2}\big([0,T]\times\mathbb{R}^d;\mathbb{R}^3\big)$, we have*

$$\begin{aligned} \langle \widehat{\rho}_T(\cdot), \widehat{G}_T(\cdot)\rangle - \langle \widehat{\rho}_0(\cdot), \widehat{G}_0(\cdot)\rangle - \int_0^T ds\, \langle \widehat{\rho}_s(\cdot), \partial_s \widehat{G}_s(\cdot)\rangle \\ = \int_0^T ds\, \langle \widehat{\rho}_s(\cdot), \Delta \widehat{G}_s(\cdot)\rangle \;+\; \int_0^T ds\, \langle \widehat{F}(\rho_s)(\cdot), \widehat{G}_s(\cdot)\rangle\,, \end{aligned}$$

(I3) *$\widehat{\rho}(0,u) = \widehat{\gamma}(u)$ a.e.*

3.4. **Hydrodynamic limit in finite volume with reservoirs.** As Theorem 3.1 deals with an infinite bulk, we are consequently able to derive the limit in a finite volume in contact with reservoirs. We denote $B_N = \{-N,\cdots,N\}\times\mathbb{T}_N^{d-1}$, $B = (-1,1)\times\mathbb{T}^{d-1}$ and its closure $\overline{B} = [-1,1]\times\mathbb{T}^{d-1}$, where $\mathbb{T}_N^d$ is the d-dimensional microscopic torus of length N and $\mathbb{T}^d$ is the d-dimensional torus $[0,1)^d$. In finite volume with stochastic reservoirs, the reaction-diffusion process $(\xi_t,\omega_t)_{t\in[0,T]}$ on $\widetilde{\Sigma}_N = \big(\{0,1\}\times\{0,1\}\big)^{B_N}$, with generator $N^2\widetilde{\mathcal{L}}_N + \widetilde{\mathbb{L}}_N + N^2\widetilde{L}_{\widehat{b},N}$, given by formulas (2.9), (2.15), (2.16), (2.17), (2.21), where we replace Λ_N by B_N, and Γ_N by $\widetilde{\Gamma}_N = \{-N,N\}\times\mathbb{T}_N^{d-1}$, satisfies the following theorem, proved in [21].

Theorem 3.3. *If for any $\delta > 0$ and for any function $\widehat{G} \in \mathcal{C}(\overline{B};\mathbb{R}^3)$,*

$$\lim_{N\to\infty} \mu_N\Big\{ \Big| \langle \widehat{\pi}_N(\xi,\omega), \widehat{G}(\cdot)\rangle \;-\; \langle \widehat{\gamma}(\cdot), \widehat{G}(\cdot)\rangle \Big| \geq \delta \Big\} = 0\,,$$

for an initial continuous profile $\widehat{\gamma} : B \to [0,1]^3$, then the sequence of probability measures $(Q_{\mu_N}^{N,\widehat{b}})_{N\geq 1}$ converges to the Dirac measure concentrated on the unique weak solution $\widehat{\rho}(\cdot,\cdot)$ of the boundary value problem

$$\begin{cases} \partial_t \widehat{\rho} & = & \Delta\widehat{\rho} \;+\; \widehat{F}(\widehat{\rho}) & in & B\times(0,T), \\ \widehat{\rho}(0,\cdot) & = & \widehat{\gamma}(\cdot) & in & B, \\ \widehat{\rho}(t,\cdot)|_\Gamma & = & \widehat{b}(\cdot) & for \;\; 0 \leq t \leq T\,, \end{cases}$$

that is, $\widehat{\rho}(\cdot,\cdot) : [0,T]\times B \to [0,1]^3$ satisfies

(B1) *For any $i \in \{1,2,3\}$, $\rho_i \in L^2\big((0,T);H^1(B)\big)$, that is, ρ_i admits a generalised derivative such that, if $\nabla\rho_i = (\partial_{e_1}\rho_i,\cdots,\partial_{e_d}\rho_i)$ stands for the gradient of ρ_i,*

$$\int_0^T ds\Big(\int_B \parallel \nabla\rho_i(s,u) \parallel^2 du\Big) < \infty\,.$$

(B2) *For any function $\widehat{G}_t(u) = \widehat{G}(t,u) \in \mathcal{C}_0^{1,2}\big([0,T]\times\overline{B};\mathbb{R}^3\big)$, we have*

$$\begin{aligned} \langle \widehat{\rho}_T(\cdot), \widehat{G}_T(\cdot)\rangle - \langle \widehat{\rho}_0(\cdot), \widehat{G}_0(\cdot)\rangle - \int_0^T ds\, \langle \widehat{\rho}_s(\cdot), \partial_s \widehat{G}_s(\cdot)\rangle = \int_0^T ds\, \langle \widehat{\rho}_s(\cdot), \Delta \widehat{G}_s(\cdot)\rangle \\ + \int_0^T ds\, \langle \widehat{F}(\rho_s)(\cdot), \widehat{G}_s(\cdot)\rangle - \sum_{i=1}^3 \int_0^T ds \int_{\widetilde{\Gamma}} \mathbf{n}_1(r)\, b_i(r)(\partial_{e_1} G_{i,s})(r)\, dS(r)\,, \end{aligned}$$

where $\widetilde{\Gamma}$ denotes here the boundary of B.

(B3) *$\widehat{\rho}(0,u) = \widehat{\gamma}(u)$ a.e.*

From now on, we are back to the set-up of Theorem 3.1.

3.5. **Currents.** In this subsection, we study the evolution of the empirical currents, namely the conservative one (cf. [1]) and the non-conservative one (cf. [4]). For $t \geq 0$, $1 \leq i \leq 3$ and any $y, z \in \Lambda_N$ such that $\|y - z\| = 1$, denote by $J_t^{y,z}(\eta_i)$ the total number of particles of type i that jumped from y to z before time t and by

$$W_t^{x,x+e_j}(\eta_i) = J_t^{x,x+e_j}(\eta_i) - J_t^{x+e_j,x}(\eta_i),\ 1 \leq j \leq d$$

the conservative current of particles of type i across the bond $\{x, x+e_j\}$ before time t. The corresponding conservative empirical measure $\mathbb{W}_t^N$ is the product finite signed measure on Λ_N defined as $\mathbb{W}_t^N(\eta_i) = (W_{1,t}^N(\eta_i), \dots, W_{d,t}^N(\eta_i)) \in \mathcal{M}^d$, where for $1 \leq j \leq d$, $1 \leq i \leq 3$,

$$W_{j,t}^N(\eta_i) \;=\; N^{-(d+1)} \sum_{x,x+e_j \in \Lambda_N} W_t^{x,x+e_j}(\eta_i)\delta_{x/N}\,.$$

For a continuous vector field $\mathbf{G} = (G_1, \dots, G_d) \in \mathcal{C}_c(\Lambda; \mathbb{R}^d)$ the integral of $\mathbf{G}$ with respect to $\mathbb{W}_t^N(\eta_i)$, also denoted by $\langle \mathbb{W}_t^N(\eta_i), \mathbf{G}\rangle$, is given by

$$\langle \mathbb{W}_t^N(\eta_i), \mathbf{G}\rangle \;=\; \sum_{j=1}^d \langle W_{j,t}^N(\eta_i), G_j\rangle\,. \tag{3.5}$$

For $x \in \Lambda_N$, $1 \leq i \leq 3$, we denote by $Q_t^x(\eta_i)$ the total number of particles of type i created minus the total number of particles of type i annihilated at site x before time t. The corresponding non-conservative empirical measure is

$$Q_t^N(\eta_i) = \frac{1}{N^d} \sum_{x \in \Lambda_N} Q_t^x(\eta_i)\delta_{x/N}.$$

We can now state the law of large numbers for the current:

Proposition 3.4. *Fix a smooth initial profile $\widehat{\gamma} : \Lambda \to [0,1]^3$. Let $(\mu_N)_{N\geq 1}$ be a sequence of probability measures on $\widehat{\Sigma}_N$ satisfying (3.4) and $\widehat{\rho}$ be the weak solution of the system of equations (3.1). Then, for each $T > 0$, $\delta > 0$, $\widehat{\mathbf{G}} = (\mathbf{G}_1, \mathbf{G}_2, \mathbf{G}_3) \in \mathcal{C}_c^1(\Lambda; (\mathbb{R}^d)^3)$ and $\widehat{H} = (H_1, H_2, H_3) \in \mathcal{C}_c^1(\Lambda; \mathbb{R}^3)$,*

$$\lim_{N\to\infty} \mathbb{P}_{\mu_N}^{N,\widehat{b}}\Big[\Big|\sum_{i=1}^3 \langle \mathbb{W}_t^N(\eta_i), \mathbf{G}_i\rangle \;-\; \int_0^T dt\,\big\langle\big\{-\nabla\widehat{\rho}_t\big\}\,,\,\widehat{\mathbf{G}}\big\rangle\Big| > \delta\Big] \;=\; 0\,, \tag{3.6}$$

$$\lim_{N\to\infty} \mathbb{P}_{\mu_N}^{N,\widehat{b}}\Big[\Big|\sum_{i=1}^3 \langle Q_t^N(\eta_i), H_i\rangle \;-\; \int_0^T dt\,\big\langle \widehat{F}(\widehat{\rho}_t)\,,\,\widehat{H}\big\rangle\Big| > \delta\Big] \;=\; 0\,. \tag{3.7}$$

We shall prove Proposition 3.4 in Section 9.

4. Outline of the proof of Theorem 3.1

The proof is divided essentially in two parts. In the first one, we prove the hydrodynamic limit for the system evolving in a large finite volume. There, by large we just mean a volume of size M_N such that $\lim_{N\to+\infty} M_N/N = +\infty$. In the second part, from the first one and coupling arguments, we shall derive the result in infinite volume. The coupling will allow us to prove that by taking M_N appropriately large enough, particles outside the cylinder of length M_N do not affect enough the number of particles in a box with length of order KN for any $K > 0$ (cf. Propositions 4.3 and 4.4 below). For all these requirements on M_N to be fulfilled we take

$$M_N = N^{1+\frac{1}{d}}. \tag{4.1}$$

• For the first part of the proof of Theorem 3.1, we consider a Markov process with state space $\widehat{\Sigma}_{N,M_N}$ (cf. (2.7)) and generator $\mathfrak{L}_{N,M_N}$, where for any positive integer $n > 1$, $\mathfrak{L}_{N,n}$ denotes the restriction of the generator $\mathfrak{L}_N$ to the box $\Lambda_{N,n}$:

$$\mathfrak{L}_{N,n} = N^2\mathcal{L}_{N,n} \,+\, \mathbb{L}_{N,n} \,+\, N^2 L_{\widehat{b},N,n}\,, \tag{4.2}$$

with, for $\mathcal{L}^{x,x+e_k}$ defined in (2.10), $\mathbb{L}^x_{\Lambda_{N,n}}$ in (2.16), and $L^x_{\widehat{b},N}$ in (2.22),

$$\mathcal{L}_{N,n} = \sum_{k=1}^{d} \sum_{x,x+e_k\in\Lambda_{N,n}} \mathcal{L}^{x,x+e_k}\,, \quad \mathbb{L}_{N,n} = \sum_{x\in\Lambda_{N,n}} \mathbb{L}^x_{\Lambda_{N,n}}\,, \quad L_{\widehat{b},N,n} = \sum_{x\in\Lambda_{N,n}\cap\Gamma_N} L^x_{\widehat{b},N}\,. \tag{4.3}$$

Observe that this finite volume dynamics can be seen as a dynamics $(\zeta_t,\chi_t)_{t\in[0,T]}$ evolving in the infinite volume Λ_N, where transitions taking place outside Λ_{N,M_N}, or involving particles outside Λ_{N,M_N}, are suppressed. For the exchange part of the dynamics, it means that particles outside Λ_{N,M_N} do not move and particles inside Λ_{N,M_N} jump as in the original infinite volume process in Λ_N, with the restriction that jumps off Λ_{N,M_N} are suppressed. For the CPRS part, it means that transitions outside Λ_{N,M_N}, as well as births in Λ_{N,M_N} induced by particles outside Λ_{N,M_N}, are suppressed. For the boundary dynamics part, it means that transitions outside Λ_{N,M_N} are suppressed. By abuse of language, we still denote the generator of $(\zeta_t,\chi_t)_{t\in[0,T]}$ by $\mathfrak{L}_{N,M_N}$. Given a probability measure μ_N on $\widehat{\Sigma}_N$, we denote by $\widetilde{\mathbb{P}}^{M_N,\widehat{b}}_{\mu_N}$ the law of the process $(\zeta_t,\chi_t)_{t\in[0,T]}$ with initial distribution μ_N, by $\widetilde{\mathbb{E}}^{M_N,\widehat{b}}_{\mu_N}$ the corresponding expectation, and by $\widetilde{Q}^{M_N,\widehat{b}}_{\mu_N} = \widetilde{\mathbb{P}}^{M_N,\widehat{b}}_{\mu_N}\circ(\widehat{\pi}^N)^{-1}$ the law of the process $\big(\widehat{\pi}^N(\zeta_t,\chi_t)\big)_{t\in[0,T]}$. We define $(\widetilde{\eta}_1,\widetilde{\eta}_2,\widetilde{\eta}_3)$ then $\widetilde{\eta}_0$ associated to (ζ,χ) as we did for (η_1,η_2,η_3) and η_0 w.r.t. (ξ,ω) in (2.4)–(2.5).

Following the Guo, Papanicolaou and Varadhan method [16], to derive the hydrodynamic behaviour of our system in large finite volume, we divide the proof into several steps. We first prove through the martingales associated to the process $(\zeta_t,\chi_t)_{t\in[0,T]}$ that the limit is the Dirac mass concentrated on the set of weak solutions to the system of equations (3.1):

(1) tightness of the measures $(\widetilde{Q}^{M_N,\widehat{b}}_{\mu_N})_{N\geq 1}$ in $D([0,T],(\mathcal{M}^1_+)^3)$,
(2) identification of the limit points of $(\widetilde{Q}^{M_N,\widehat{b}}_{\mu_N})_{N\geq 1}$ as absolutely continuous paths whose density satisfies conditions (IB1) and (IB2).

Then condition (3.4) implies (IB3), and we have to prove

(3) uniqueness of a weak solution to the hydrodynamic equations (3.1),

which allows to conclude that the limit is the Dirac measure associated to the unique solution of (3.1).

The proof of (1) and of part of (2) is by now standard and left to the reader (we refer to [19] for details). We postpone the proof of (3) to Section 8. As $M_N \gg N$, the hydrodynamic limit we obtain is the system of equations (3.1) in infinite volume with reservoirs. To understand how (3.1) appears as limit point in (2), let us consider, for any function $\widehat{G} = (G_1,G_2,G_3)\in\mathcal{C}^{1,2}_{c,0}([0,T]\times\overline{\Lambda};\mathbb{R}^3)$, the mean-zero martingale

$$\begin{aligned}\widehat{M}^N_T(\widehat{G}) &= \sum_{i=1}^{3} M^{N,i}_T(G_i) \qquad \text{where, for } 1\leq i\leq 3,\\ M^{N,i}_T(G_i) &= \langle\pi^{N,i}_T,G_{i,T}\rangle - \langle\pi^N_0,G_{i,0}\rangle - \int_0^T \langle\pi^{N,i}_s,\partial_s G_{i,s}\rangle ds - \int_0^T \mathfrak{L}_{N,M_N}\langle\pi^{N,i}_s,G_{i,s}\rangle ds\,. \end{aligned}\tag{4.4}$$

To establish the convergence of $\widehat{M}^N_T(\widehat{G})$ and exhibit a limit point, we compute, for $1\leq i\leq 3$:

$$\begin{aligned}N^2\mathcal{L}_{N,M_N}\langle\pi^{N,i}_s,G_{i,s}\rangle &= \langle\pi^{N,i}_s,\Delta_N G_{i,s}\rangle - \frac{1}{N^{d-1}}\sum_{x\in\Gamma^+_N}\partial^N_{e_1}G_{i,s}((x-e_1)/N)\widetilde{\eta}_{i,s}(x)\\ &\quad + \frac{1}{N^{d-1}}\sum_{x\in\Gamma^-_N}\partial^N_{e_1}G_{i,s}(x/N)\widetilde{\eta}_{i,s}(x)\,,\end{aligned}\tag{4.5}$$

where $\Gamma^{\pm}_N = \{(u_1,\dots,u_d)\in\overline{\Lambda}_N : u_1=\pm N\}$. Indeed, since $M_N\gg N$ and G has compact support, for N large enough M_N does not appear on the r.h.s. of (4.5).

We also compute $\widehat{f} = (f_1, f_2, f_3) : \widehat{\Sigma}_N \to \mathbb{R}^3$:

$$\begin{cases} f_1(\zeta,\chi) &= \mathbb{L}_{N,M_N}\widetilde{\eta}_1(0) = \beta_{\Lambda_{N,M_N}}(0,\zeta,\chi)\widetilde{\eta}_0(0) + \widetilde{\eta}_3(0) - (r+1)\widetilde{\eta}_1(0),\\ f_2(\zeta,\chi) &= \mathbb{L}_{N,M_N}\widetilde{\eta}_2(0) = r\widetilde{\eta}_0(0) + \widetilde{\eta}_3(0) - \beta_{\Lambda_{N,M_N}}(0,\zeta,\chi)\widetilde{\eta}_2(0) - \widetilde{\eta}_2(0),\\ f_3(\zeta,\chi) &= \mathbb{L}_{N,M_N}\widetilde{\eta}_3(0) = \beta_{\Lambda_{N,M_N}}(0,\zeta,\chi)\widetilde{\eta}_2(0) + r\widetilde{\eta}_1(0) - 2\widetilde{\eta}_3(0)\,, \end{cases} \tag{4.6}$$

so that

$$\mathbb{L}_{N,M_N}\langle \pi_s^{N,i}, G_{i,s}\rangle = \frac{1}{N^d}\sum_{x\in\Lambda_{N,M_N}} G_{i,s}(x/N)\tau_x f_i(\zeta_s,\chi_s), \tag{4.7}$$

where $\tau.$ denotes the shift operator, that is, for $x \in \Lambda_{N,M_N}$, $\tau_x\widetilde{\eta}_1(0) = \mathbf{1}_{\{\widetilde{\eta}(x)=1\}} = \zeta(x)(1-\chi(x))$.
Again, for N large enough, M_N does not play any role on the r.h.s. of (4.7).
Since $\widehat{G}$ vanishes at the boundaries on $\overline{\Lambda}$, the generator $L_{\widehat{b},N,M_N}$ is not needed.
Therefore, to close the equations in $\widehat{M}_t^N(\widehat{G})$, we need to do two replacements, stated in Lemma 4.2 and Proposition 4.3 below: we have to replace local functions in the bulk by a function of the empirical density in (4.7), and to replace the density at the boundary by a value of the function $\widehat{b}$ in (4.5). Lemma 4.2 and Proposition 4.3 are the main steps to show that any limit point of the sequence $(\widetilde{Q}_{\mu_N}^{M_N,\widehat{b}})_{N\geq 1}$ is concentrated on trajectories that are weak solutions of the system of equations (3.1). The proof of Lemma 4.2 relies on uniform upper bounds on the entropy production and the Dirichlet form stated in Subsection 5.1 (Theorem 5.1) and proved in Subsection 5.2. The proof of Proposition 4.3 relies on the properties of the boundary dynamics.

Remark 4.1. *Except for the replacement lemma at the boundary (that is, Proposition 4.3), all the results needed in steps (1), (2) and (3) are valid both in infinite volume and in a large finite volume with length $M_N = N^{1+\frac{1}{d}}$. Therefore, we shall state and prove all these results in infinite volume.*

For any smooth profile $\widehat{\theta} = (\theta_1, \theta_2, \theta_3) : \overline{\Lambda} \to (0,1)^3$ satisfying (2.19) and (2.20), and for any cylinder function $\phi(\xi,\omega)$, denote by $\widetilde{\phi}(\widehat{\theta})$ the expectation of ϕ with respect to $\nu^N_{\widehat{\theta}}$. For any $\ell \in \mathbb{N}$, define the empirical mean densities in a box of size $(2\ell+1)^d$ centered at x by $\widehat{\eta}^\ell(x) = (\eta_1^\ell(x), \eta_2^\ell(x), \eta_3^\ell(x))$:

$$\eta_i^\ell(x) = \frac{1}{(2\ell+1)^d}\sum_{\substack{\|y-x\|\leq \ell\\ y\in\Lambda_N}} \eta_i(y), \text{ for } 1 \leq i \leq 3.$$

so that we can define for any $\epsilon > 0$ small enough (as usual we omit to write integer parts in bounds of intervals: ϵN replaces $\lfloor \epsilon N \rfloor$),

$$V_{\epsilon N}(\xi,\omega) = \Big|\frac{1}{(2\epsilon N+1)^d}\sum_{\|y\|\leq\epsilon N}\tau_y\phi(\xi,\omega) - \widetilde{\phi}(\widehat{\eta}^{\epsilon N}(0))\Big|. \tag{4.8}$$

Lemma 4.2 (replacement in the bulk)**.** *For any $G \in \mathcal{C}_c^\infty([0,T]\times\Lambda,\mathbb{R})$,*

$$\varlimsup_{\epsilon\to 0}\varlimsup_{N\to\infty}\mathbb{E}_{\mu_N}^{N,\widehat{b}}\Big(\frac{1}{N^d}\sum_{x\in\Lambda}\int_0^T |G_s(x/N)|\tau_x V_{\epsilon N}(\xi_s,\omega_s)ds\Big) = 0.$$

The proof of Lemma 4.2 is postponed to Subsection 5.3.
We now state that the limiting trajectories for the system in large finite volume satisfy the Dirichlet boundary conditions with value $\widehat{b}(\cdot)$. The proof of Proposition 4.3 is postponed to Section 6.

Proposition 4.3 (replacement at the boundary)**.** *For any bounded function $H : [0,T]\times\Gamma \to \mathbb{R}$ with compact support in Γ, for any $\delta > 0$, for all $i \in \{1,2,3\}$,*

$$\varlimsup_{N\to\infty}\widetilde{\mathbb{P}}_{\mu_N}^{M_N,\widehat{b}}\Big(\Big|\int_0^T\frac{1}{N^{d-1}}\sum_{x\in\Lambda_{N,M_N}\cap\Gamma_N} H_t(x/N)\Big(\widetilde{\eta}_{i,t}(x) - b_i(x/N)\Big)dt\Big| > \delta\Big) = 0.$$

• For the second part of the proof of Theorem 3.1, we couple the original process $(\xi_t,\omega_t)_{t\in[0,T]}$ in infinite volume with $(\zeta_t,\chi_t)_{t\in[0,T]}$. Let $\overline{\mu}_N$ be the measure on $\widehat{\Sigma}_N\times\widehat{\Sigma}_N$ concentrated on its diagonal and with marginals equal to μ_N. Denote by $\overline{\mathbb{P}}^{M_N,\widehat{b}}_{\overline{\mu}_N}$ the law of the coupled process $((\xi_t,\omega_t),(\zeta_t,\chi_t))_{t\in[0,T]}$ with initial distribution $\overline{\mu}_N$, and by $\overline{\mathbb{E}}^{M_N,\widehat{b}}_{\overline{\mu}_N}$ the corresponding expectation. By Tchebycheff inequality, for all $\delta>0$ and $t\geq 0$,

$$
\begin{aligned}
&\mathbb{P}^{N,\widehat{b}}_{\mu_N}\Big(\Big|\frac{1}{N^d}\sum_{x\in\Lambda_N}G_{i,t}(x/N)\Big(\eta_{i,t}(x)-\rho_i(t,x/N)\Big)\Big|>\delta\Big)\\
&\leq \widetilde{\mathbb{P}}^{M_N,\widehat{b}}_{\mu_N}\Big(\Big|\frac{1}{N^d}\sum_{x\in\Lambda_N}G_{i,t}(x/N)\Big(\widetilde{\eta}_{i,t}(x)-\rho_i(t,x/N)\Big)\Big|>\frac{\delta}{2}\Big) \qquad (4.9)\\
&\quad+\frac{2}{\delta}\overline{\mathbb{E}}^{M_N,\widehat{b}}_{\overline{\mu}_N}\Big(\Big|\frac{1}{N^d}\sum_{x\in\Lambda_N}G_{i,t}(x/N)\Big(\eta_{i,t}(x)-\widetilde{\eta}_{i,t}(x)\Big)\Big|\Big). \qquad (4.10)
\end{aligned}
$$

The hydrodynamic result in large finite volume enables to deal with (4.9). For (4.10), we have to prove the following coupling result, which will conclude the proof of Theorem 3.1.

Proposition 4.4. *For any bounded function $\widehat{G}=(G_1,G_2,G_3):[0,T]\times\Lambda\to\mathbb{R}^3$ with compact support in Λ, for all $i\in\{0,1,2,3\}$,*

$$
\overline{\lim_{N\to\infty}}\;\overline{\mathbb{E}}^{M_N,\widehat{b}}_{\overline{\mu}_N}\Big(\Big|\frac{1}{N^d}\sum_{x\in\Lambda_N}G_{i,t}(x/N)\Big(\widetilde{\eta}_{i,t}(x)-\eta_{i,t}(x)\Big)\Big|\Big)=0\,.
$$

In Section 7 we shall define the appropriate coupling between $(\xi_t,\omega_t)_{t\in[0,T]}$ and $(\zeta_t,\chi_t)_{t\in[0,T]}$, which turns out to be basic coupling, and prove Proposition 4.4.

5. Specific entropy, Dirichlet forms and proof of Lemma 4.2

5.1. **Specific entropy: Definitions and results.** We start by defining the two main ingredients needed in the proof of the hydrodynamic limit: the specific entropy and the specific Dirichlet form of a measure on $\widehat{\Sigma}_N$ with respect to some reference product measure. For each positive integer n and a measure μ on $\widehat{\Sigma}_N$, we denote by μ_n the marginal of μ on $\widehat{\Sigma}_{N,n}$: For each $(\zeta,\chi)\in\widehat{\Sigma}_{N,n}$,

$$
\mu_n(\zeta,\chi)=\mu\big\{(\xi,\omega)\;:\;(\xi(x),\omega(x))=(\zeta(x),\chi(x))\;\text{ for } x\in\Lambda_{N,n}\big\}\,. \qquad (5.1)
$$

We fix as *reference measure* a product measure $\nu^N_{\widehat{\theta}}:=\nu^N_{\widehat{\theta}(\cdot)}$, where $\widehat{\theta}=(\theta_1,\theta_2,\theta_3):\Lambda\to(0,1)^3$ is a smooth function satisfying (2.19) and (2.20). In other words (recall (2.12), (2.14)), introducing the function $\theta_0(.)=1-\theta_1(.)-\theta_2(.)-\theta_3(.)$, we have

$$
\begin{aligned}
\nu^N_{\widehat{\theta}(\cdot),n}(\xi,\omega)&=\widehat{Z}^{-1}_{\widehat{\theta},n}\exp\Big\{\sum_{i=1}^{3}\sum_{x\in\Lambda_{N,n}}\Big(\log\frac{\theta_i(x/N)}{\theta_0(x/N)}\Big)\eta_i(x)\Big\} \qquad (5.2)\\
\text{with}\qquad \widehat{Z}^{-1}_{\widehat{\theta},n}&=\prod_{x\in\Lambda_{N,n}}\theta_0(x/N).
\end{aligned}
$$

We denote by $s_n(\mu_n|\nu^N_{\widehat{\theta},n})$ the relative entropy of μ_n with respect to $\nu^N_{\widehat{\theta},n}$ defined by

$$
s_n(\mu_n|\nu^N_{\widehat{\theta},n})=\sup_{U\in C_b(\widehat{\Sigma}_{N,n})}\Big\{\int U(\xi,\omega)d\mu_n(\xi,\omega)-\log\int e^{U(\xi,\omega)}d\nu^N_{\widehat{\theta},n}(\eta,\xi)\Big\}. \qquad (5.3)
$$

In this formula $C_b(\widehat{\Sigma}_{N,n})$ stands for the space of all bounded continuous functions on $\widehat{\Sigma}_{N,n}$. Since the measure $\nu^N_{\widehat{\theta},n}$ gives a positive probability to each configuration, all the measures on $\widehat{\Sigma}_{N,n}$ are absolutely continuous with respect to $\nu^N_{\widehat{\theta},n}$ and we have an explicit formula for the entropy :

$$
s_n(\mu_n|\nu^N_{\widehat{\theta},n})=\int\log\left(f_n(\xi,\omega)\right)d\mu_n(\xi,\omega), \qquad (5.4)
$$

where f_n is the probability density of μ_n with respect to $\nu^N_{\widehat{\theta},n}$.

Define the Dirichlet forms

$$D_n(\mu_n|\nu^N_{\widehat{\theta},n}) = \mathcal{D}^0_n(\mu_n|\nu^N_{\widehat{\theta},n}) \,+\, D^{\widehat{b}}_n(\mu_n|\nu^N_{\widehat{\theta},n})\,, \tag{5.5}$$

with

$$\mathcal{D}^0_n(\mu_n|\nu^N_{\widehat{\theta},n}) \;=\; \sum_{k=1}^d \sum_{x:(x,x+e_k)\in\Lambda_{N,n}\times\Lambda_{N,n}} (\mathcal{D}^0_n)^{x,x+e_k}(\mu_n|\nu^N_{\widehat{\theta},n})\,, \tag{5.6}$$

$$D^{\widehat{b}}_n(\mu_n|\nu^N_{\widehat{\theta},n}) \;=\; \sum_{x\in\Lambda_{N,n}\cap\Gamma_N} (D^{\widehat{b}}_n)^x(\mu_n|\nu^N_{\widehat{\theta},n})\,, \tag{5.7}$$

where, writing $y = x+e_k$,

$$(\mathcal{D}^0_n)^{x,y}(\mu_n|\nu^N_{\widehat{\theta},n}) = \int \Big(\sqrt{f_n(\xi^{x,y},\omega^{x,y})} - \sqrt{f_n(\xi,\omega)}\Big)^2 d\nu^N_{\widehat{\theta},n}(\xi,\omega), \tag{5.8}$$

$$\begin{aligned}(D^{\widehat{b}}_n)^x(\mu_n|\nu^N_{\widehat{\theta},n}) = &\int c_x\big(\widehat{b}(x/N),\xi,\sigma^x\omega\big)\Big(\sqrt{f_n(\xi,\sigma^x\omega)} - \sqrt{f_n(\xi,\omega)}\Big)^2 d\nu^N_{\widehat{\theta},n}(\xi,\omega)\\ &+\int c_x\big(\widehat{b}(x/N),\sigma^x\xi,\omega\big)\Big(\sqrt{f_n(\sigma^x\xi,\omega)} - \sqrt{f_n(\xi,\omega)}\Big)^2 d\nu^N_{\widehat{\theta},n}(\xi,\omega)\\ &+\int c_x\big(\widehat{b}(x/N),\sigma^x\xi,\sigma^x\omega\big)\Big(\sqrt{f_n(\sigma^x\xi,\sigma^x\omega)} - \sqrt{f_n(\xi,\omega)}\Big)^2 d\nu^N_{\widehat{\theta},n}(\xi,\omega)\,.\end{aligned} \tag{5.9}$$

We shall also need

$$\mathbb{D}_n(\mu_n|\nu^N_{\widehat{\theta},n}) = \sum_{x\in\Lambda_{N,n}} (\mathbb{D}_n)^x(\mu_n|\nu^N_{\widehat{\theta},n})\,, \tag{5.10}$$

where

$$\begin{aligned}(\mathbb{D}_n)^x(\mu_n|\nu^N_{\widehat{\theta},n}) = &\int \Big(r(1-\omega(x))+\omega(x)\Big)\Big(\sqrt{f_n(\xi,\sigma^x\omega)} - \sqrt{f_n(\xi,\omega)}\Big)^2 d\nu^N_{\widehat{\theta},n}(\xi,\omega)\\ &+\int \Big(\beta_{\Lambda_{N,n}}(x,\xi,\omega)(1-\xi(x))+\xi(x)\Big)\Big(\sqrt{f_n(\sigma^x\xi,\omega)} - \sqrt{f_n(\xi,\omega)}\Big)^2 d\nu^N_{\widehat{\theta},n}(\xi,\omega)\,.\end{aligned} \tag{5.11}$$

Define the *specific entropy* $\mathcal{S}(\mu|\nu^N_{\widehat{\theta}})$ and the *Dirichlet form* $\mathfrak{D}(\mu|\nu^N_{\widehat{\theta}})$ of a measure μ on $\widehat{\Sigma}_N$ with respect to $\nu^N_{\widehat{\theta}}$ as

$$\mathcal{S}(\mu|\nu^N_{\widehat{\theta}}) = N^{-1}\sum_{n\geq 1} s_n(\mu_n|\nu^N_{\widehat{\theta},n})e^{-n/N}, \tag{5.12}$$

$$\mathfrak{D}(\mu|\nu^N_{\widehat{\theta}}) = N^{-1}\sum_{n\geq 1} D_n(\mu_n|\nu^N_{\widehat{\theta},n})e^{-n/N}. \tag{5.13}$$

Notice that by the entropy convexity and since $\sup_{x\in\Lambda_N}\{\xi(x)+\omega(x)\}$ is finite, for any positive measure μ on $\widehat{\Sigma}_N$ and any integer n, we have

$$s_n(\mu_n|\nu^N_{\widehat{\theta},n}) \;\leq\; C_0 N n^{d-1}\,, \tag{5.14}$$

for some constant C_0 that depends on $\widehat{\theta}$ (cf. comments following Remark V.5.6 in [19]). Moreover there exists a positive constant $C_0' \equiv C(\widehat{\theta})$ such that for any positive measure μ on $\widehat{\Sigma}_N$,

$$\mathcal{S}(\mu|\nu^N_{\widehat{\theta}}) \;\leq\; C_0' N^d\,. \tag{5.15}$$

Indeed, by (5.12) and (5.14) we have

$$\mathcal{S}(\mu|\nu^N_{\widehat{\theta}}) \;\leq\; N^d C_0 \frac{1}{N}\sum_{n\geq 1} e^{-n/N}\Big(\frac{n}{N}\Big)^{d-1} \tag{5.16}$$

that we bound comparing the Riemann sum with an integral.

The goal of this section is to prove the following appropriate bounds on the entropy production and the Dirichlet form.

Theorem 5.1. *Let $\widehat{\theta} = (\theta_1, \theta_2, \theta_3) : \overline{\Lambda} \to (0,1)^3$ be a smooth function satisfying* (2.19) *and* (2.20). *For any time $t \geq 0$, there exists a positive finite constant $C_1 \equiv C(t, \widehat{\theta}, \lambda_1, \lambda_2, r)$, so that*

$$\int_0^t \mathfrak{D}(\mu(s)|\nu^N_{\widehat{\theta}})\, ds \;\leq\; C_1 N^{d-2}\,.$$

To get this result, we need to bound the entropy production in terms of the Dirichlet form. This is given by the following lemma.

Lemma 5.2. *There exist positive constants A_0, A_1 such that for any $t > 0$,*

$$\partial_t \mathcal{S}(\mu(t)|\nu^N_{\widehat{\theta}}) \leq -A_0 N^2 \mathfrak{D}(\mu(t)|\nu^N_{\widehat{\theta}}) + A_1 N^d\,. \tag{5.17}$$

Note that since we do not know invariant measures for our dynamics, we control the entropy production with respect to a product measure which is not stationary, which leads to the correction term $A_1 N^d$ in (5.17).

5.2. **Specific entropy: Proof of Theorem 5.1.** We now prove Theorem 5.1 and Lemma 5.2.

Proof of Theorem 5.1. Integrate the expression (5.17) from 0 to t and use (5.15). □

Proof of Lemma 5.2. For a measure μ_n on $\widehat{\Sigma}_{N,n}$, denote by f^t_n the density of $\mu_n(t)$ with respect to $\nu^N_{\widehat{\theta},n}$. By the definition (5.12) of specific entropy, we need to bound $\partial_t s_n(\mu_n(t)|\nu^N_{\widehat{\theta},n})$. For any subset $\mathcal{A} \subset \Lambda$ and any function $f \in L^1(\nu^N_{\widehat{\theta}})$, denote by $\langle f \rangle_{\mathcal{A}}$ the function on $(\{0,1\} \times \{0,1\})^{\Lambda \setminus \mathcal{A}}$ obtained by integrating f with respect to $\nu^N_{\widehat{\theta}}$ over the coordinates $\{(\xi(x), \omega(x)), x \in \mathcal{A}\}$. We denote

$$\Lambda^c_{N,n} = \Lambda_{N,n+1} \setminus \Lambda_{N,n}. \tag{5.18}$$

In the case where $\mathcal{A} = \Lambda^c_{N,n}$, we simplify the notation $\langle f \rangle_{\mathcal{A}}$ by $\langle f \rangle_{n+1}$. Note that

$$\langle f^t_{n+1} \rangle_{n+1} = f^t_n\,. \tag{5.19}$$

Following the Kolmogorov forward equation, one has

$$\partial_t f^t_n = \langle \mathfrak{L}^*_{N,n+1} f^t_{n+1} \rangle_{n+1}, \tag{5.20}$$

where $\mathfrak{L}^*_{N,n}$ stands for the adjoint operator of $\mathfrak{L}_{N,n}$ in $L^2(\nu^N_{\widehat{\theta},n})$. By (5.20),

$$\begin{aligned}
\partial_t s_n(\mu_n(t)|\nu^N_{\widehat{\theta},n}) &= \partial_t \int f^t_n \big(\log f^t_n\big) d\nu^N_{\widehat{\theta},n} = \int \big(\log f^t_n\big) \mathfrak{L}^*_{N,n+1} f^t_{n+1} d\nu^N_{\widehat{\theta},n+1} \\
&= N^2 \int f^t_{n+1} \mathcal{L}_{N,n+1}\big(\log f^t_n\big) d\nu^N_{\widehat{\theta},n+1} + \int f^t_{n+1} \mathbb{L}_{N,n+1}\big(\log f^t_n\big) d\nu^N_{\widehat{\theta},n+1} \\
&\quad + N^2 \int f^t_{n+1} L_{\widehat{b},N,n+1}\big(\log f^t_n\big) d\nu^N_{\widehat{\theta},n+1} \qquad (5.21)\\
&=: \Omega_1 + \Omega_2 + \Omega_3\,. \qquad (5.22)
\end{aligned}$$

We shall first derive useful tools (in Step 1 below), then obtain (in Steps 2,3,4) the following bounds on the three integrals Ω_1, Ω_2 and Ω_3 in terms of the entropy and Dirichlet forms. There exists positive constants C, C_1'', K_2 such that

$$\begin{aligned}
\Omega_1 \quad &\leq \quad -\frac{N^2}{2}\mathcal{D}_n^0(\mu_n(t)|\nu^N_{\widehat{\theta},n}) + (Cn + C_1''AN)Nn^{d-2} \\
&\quad + \frac{N^2}{A}\sum_{\ell=1}^{N}\sum_{k=2}^{d}\sum_{(x,x+e_k)\in(\Lambda_{N,n}\times\Lambda^c_{N,n})\cup(\Lambda^c_{N,n}\times\Lambda_{N,n})}(\mathcal{D}^0_{n+\ell})^{x,x+e_k}(\mu_{n+\ell}(t)|\nu^N_{\widehat{\theta},n+\ell}), \qquad (5.23)\\
\Omega_2 \quad &\leq \quad -\mathbb{D}_n(\mu_n(t)|\nu^N_{\widehat{\theta},n}) + K_2Nn^{d-1}, \qquad (5.24)\\
\Omega_3 \quad &\leq \quad -N^2 D_n^{\widehat{b}}(\mu_n(t)|\nu^N_{\widehat{\theta},n}). \qquad (5.25)
\end{aligned}$$

The constant A comes from (5.37) in Step 1 below. Gathering (5.23), (5.24) and (5.25) gives

$$\begin{aligned}
\partial_t s_n(\mu_n(t)|\nu^N_{\widehat{\theta},n}) \quad &\leq \quad -\frac{N^2}{2}\mathcal{D}_n^0(\mu_n(t)|\nu^N_{\widehat{\theta}(\cdot),n}) \\
&\quad + \frac{N^2}{A}\sum_{\ell=1}^{N}\sum_{k=2}^{d}\sum_{(x,x+e_k)\in(\Lambda_{N,n}\times\Lambda^c_{N,n})\cup(\Lambda^c_{N,n}\times\Lambda_{N,n})}(\mathcal{D}^0_{n+\ell})^{x,x+e_k}(\mu_{n+\ell}(t)|\nu^N_{\widehat{\theta},n+\ell}) \\
&\quad -\mathbb{D}_n(\mu_n(t)|\nu^N_{\widehat{\theta},n}) - N^2 D_n^{\widehat{b}}(\mu_n(t)|\nu^N_{\widehat{\theta},n}) + \big((C+K_2)n + C_1''AN\big)Nn^{d-2}. \qquad (5.26)
\end{aligned}$$

Note that for any $M \gg N$ large enough, for some positive constant K_1',

$$\begin{aligned}
&\sum_{n=1}^{M}\frac{1}{N}e^{-n/N}\sum_{\ell=1}^{N}\sum_{k=2}^{d}\sum_{\substack{(x,y)\in\Lambda_{N,n}\times\Lambda^c_{N,n}\\ y\in\{x+e_k,x-e_k\}}}(\mathcal{D}^0_{n+\ell})^{x,y}(\mu_{n+\ell}(t)|\nu^N_{\widehat{\theta},n+\ell}) \\
&\qquad \leq K_1'\sum_{n=1}^{M+N}\frac{1}{N}e^{-n/N}\mathcal{D}^0_n(\mu_n(t)|\nu^N_{\widehat{\theta},n}) \ \leq K_1'\mathfrak{D}(\mu(t)|\nu^N_{\widehat{\theta}}).
\end{aligned} \qquad (5.27)$$

To conclude the proof of the Lemma it remains to multiply (5.26) by $N^{-1}\exp(-n/N)$ and sum over $n \in \mathbb{N}$ to write an upper bound for $\partial_t\mathcal{S}(\mu(t)|\nu^N_{\widehat{\theta}})$. Using (5.27) and choosing (for instance) $A = 4K_1'$ on one hand, and dealing with the last term on the r.h.s. of (5.26) as in (5.16) on the other hand, we get (5.17).

Step 1: Tools. To do changes of variables, it is convenient to write (5.2) as follows:

$$\nu^N_{\widehat{\theta}(\cdot),n}(\xi,\omega) = \exp\Big\{\sum_{j=0}^{3}\sum_{x\in\Lambda_{N,n}}\vartheta_j(x/N)\eta_j(x)\Big\} \qquad \text{with} \qquad \vartheta_j(x/N) = \log\theta_j(x/N). \qquad (5.28)$$

- *Changes of variables formulas*: For a cylinder function f on $\widehat{\Sigma}_N$, and $x, y \in \Lambda_N$, we have
  (i) for $(i,j) \in \{0,1,2,3\}^2$ such that $i \neq j$,

$$\int \eta_i(x)\eta_j(y)f(\xi^{x,y},\omega^{x,y})d\nu^N_{\widehat{\theta}}(\xi,\omega) = \int \eta_j(x)\eta_i(y)(R^{x,y}_{i,j}(\widehat{\theta})+1)f(\xi,\omega)d\nu^N_{\widehat{\theta}}(\xi,\omega) \qquad (5.29)$$

$$\text{where} \qquad R^{x,y}_{i,j}(\widehat{\theta}) = \exp\Big((\vartheta_j(y/N)-\vartheta_j(x/N)) - (\vartheta_i(y/N)-\vartheta_i(x/N))\Big) - 1 \qquad (5.30)$$

$$\text{satisfies} \qquad R^{x,y}_{i,j}(\widehat{\theta}) = O(N^{-1}). \qquad (5.31)$$

  (ii) for $(i,j) \in \{(1,2),(2,1),(3,0),(0,3)\}$,

$$\int \eta_i(x)f(\sigma^x\xi,\sigma^x\omega)d\nu^N_{\widehat{\theta}}(\xi,\omega) = \int \eta_j(x)e^{\vartheta_i(x/N)-\vartheta_j(x/N)}f(\xi,\omega)d\nu^N_{\widehat{\theta}}(\xi,\omega), \qquad (5.32)$$

  (iii) for $(i,j) \in \{(1,0),(0,1),(3,2),(2,3)\}$,

$$\int \eta_i(x)f(\sigma^x\xi,\omega)d\nu^N_{\widehat{\theta}}(\xi,\omega) = \int \eta_j(x)e^{\vartheta_i(x/N)-\vartheta_j(x/N)}f(\xi,\omega)d\nu^N_{\widehat{\theta}}(\xi,\omega), \qquad (5.33)$$

(iv) for $(i,j) \in \{(1,3),(3,1),(2,0),(0,2)\}$,

$$\int \eta_i(x) f(\xi, \sigma^x \omega) d\nu_{\widehat{\theta}}^N(\xi,\omega) = \int \eta_j(x) e^{\vartheta_i(x/N)-\vartheta_j(x/N)} f(\xi,\omega) d\nu_{\widehat{\theta}}^N(\xi,\omega) \,. \tag{5.34}$$

• *Inequalities*: For any positive a, b, A,

$$\begin{aligned} a(\log b - \log a) &\le -\big(\sqrt{b}-\sqrt{a}\big)^2 + (b-a), && (5.35)\\ \log a &\le 2(\sqrt{a}-1), && (5.36)\\ 2ab &\le \frac{N}{A}a^2 + \frac{A}{N}b^2. && (5.37)\end{aligned}$$

Consequences 5.3. *(i) For all x, $1 \le k \le d$ such that $x, x+e_k \in \Lambda_{N,n}$ we have,*

$$\begin{aligned}
&\int \mathcal{L}^{x,x+e_k} f(\xi,\omega) d\nu_{\widehat{\theta},n}^N(\xi,\omega) = \frac{1}{2}\int \mathcal{L}^{x,x+e_k} f(\xi,\omega) d\nu_{\widehat{\theta},n}^N(\xi,\omega)\\
&\quad + \frac{1}{2}\sum_{0\le i\ne j\le 3}\int \eta_j(x)\eta_i(x+e_k)(R_{i,j}^{x,x+e_k}(\widehat{\theta})+1)\Big(f(\xi,\omega) - f(\xi^{x,x+e_k},\omega^{x,x+e_k})\Big) d\nu_{\widehat{\theta},n}^N(\xi,\omega)\\
&= -\frac{1}{2}\sum_{0\le i\ne j\le 3}\int \eta_j(x)\eta_i(x+e_k) R_{i,j}^{x,x+e_k}(\widehat{\theta})\Big(f(\xi^{x,x+e_k},\omega^{x,x+e_k}) - f(\xi,\omega)\Big) d\nu_{\widehat{\theta},n}^N(\xi,\omega)\\
&\le \frac{1}{2}(\mathcal{D}_n^0)^{x,x+e_k}(f)\\
&\quad + \frac{1}{8}\sum_{0\le i\ne j\le 3}\int \eta_j(x)\eta_i(x+e_k)\big(R_{i,j}^{x,x+e_k}(\widehat{\theta})\big)^2\Big(\sqrt{f(\xi^{x,x+e_k},\omega^{x,x+e_k})} + \sqrt{f(\xi,\omega)}\Big)^2 d\nu_{\widehat{\theta},n}^N(\xi,\omega)\\
&\le \frac{1}{2}(\mathcal{D}_n^0)^{x,x+e_k}(f)\\
&\quad + \frac{1}{8}\sum_{0\le i\ne j\le 3}\int \eta_j(x)\eta_i(x+e_k)\big(R_{i,j}^{x,x+e_k}(\widehat{\theta})\big)^2\Big(3f(\xi^{x,x+e_k},\omega^{x,x+e_k}) + \frac{3}{2}f(\xi,\omega)\Big) d\nu_{\widehat{\theta},n}^N(\xi,\omega)\\
&\le \frac{1}{2}(\mathcal{D}_n^0)^{x,x+e_k}(f)\\
&\quad + \frac{3}{8}\sum_{0\le i\ne j\le 3}\int \eta_j(x+e_k)\eta_i(x)\big(R_{i,j}^{x,x+e_k}(\widehat{\theta})\big)^2(R_{j,i}^{x+e_k,x}(\widehat{\theta})+1) f(\xi,\omega) d\nu_{\widehat{\theta},n}^N(\xi,\omega)\\
&\quad + \frac{3}{16}\sum_{0\le i\ne j\le 3}\int \eta_j(x)\eta_i(x+e_k)\big(R_{i,j}^{x,x+e_k}(\widehat{\theta})\big)^2 f(\xi,\omega) d\nu_{\widehat{\theta},n}^N(\xi,\omega)\\
&\le \frac{1}{2}(\mathcal{D}_n^0)^{x,x+e_k}(f) + CN^{-2}\|\sqrt{f}\|^2_{L^2(\nu_{\widehat{\theta},n}^N)}
\end{aligned}$$

for some positive constant C, where we have used the change of variables formula (5.29) *for the first equality and last but one inequality,* (5.37) *with $A = N/2$ for the first and second inequalities (there we have first expanded the square), and* (5.31) *twice with* (2.19) *for the last inequality.*

(ii) Because the restriction of $\widehat{\theta}$ to Γ is equal to $\widehat{b}$ (see (2.20)*),* (5.32)–(5.34) *yield that the measure $\nu_{\widehat{\theta}}^N$ is reversible with respect to the operator $L_{\widehat{b},N}$.*

Step 2: Bound on Ω_1. We decompose the generator $\mathcal{L}_{N,n+1}$ into a part associated to exchanges within $\Lambda_{N,n}$ and a part associated to exchanges at the boundaries, that is,

$$\begin{aligned}
\Omega_1 &= N^2\sum_{k=1}^{d}\sum_{(x,x+e_k)\in\Lambda_{N,n}\times\Lambda_{N,n}}\int f^t_{n+1}\mathcal{L}^{x,x+e_k}(\log f^t_n)d\nu^N_{\widehat{\theta},n+1}\\
&\quad +N^2\sum_{k=2}^{d}\sum_{(x,x+e_k)\in(\Lambda_{N,n}\times\Lambda^c_{N,n})\cup(\Lambda^c_{N,n}\times\Lambda_{N,n})}\int f^t_{n+1}\mathcal{L}^{x,x+e_k}(\log f^t_n)d\nu^N_{\widehat{\theta},n+1}\\
&= N^2\sum_{k=1}^{d}\sum_{(x,x+e_k)\in\Lambda_{N,n}\times\Lambda_{N,n}}\Omega_1^{(1)}(x,x+e_k) \qquad (5.38)\\
&\quad +N^2\sum_{k=2}^{d}\sum_{(x,x+e_k)\in(\Lambda_{N,n}\times\Lambda^c_{N,n})\cup(\Lambda^c_{N,n}\times\Lambda_{N,n})}\Omega_1^{(2)}(x,x+e_k)\,. \qquad (5.39)
\end{aligned}$$

Successively, for the term (5.38), writing $y=x+e_k$,

$$\begin{aligned}
\Omega_1^{(1)}(x,y) &= \int f^t_{n+1}(\xi,\omega)\Big(\log f^t_n(\xi^{x,y},\omega^{x,y})-\log f^t_n(\xi,\omega)\Big)d\nu^N_{\widehat{\theta},n+1}(\xi,\omega)\\
&= \int\langle f^t_{n+1}(\xi,\omega)\rangle_{n+1}\log\frac{f^t_n(\xi^{x,y},\omega^{x,y})}{f^t_n(\xi,\omega)}d\nu^N_{\widehat{\theta},n}(\xi,\omega)\\
&\le -(\mathcal{D}^0_n)^{x,y}(\mu_n(t)|\nu^N_{\widehat{\theta},n})+\int\mathcal{L}^{x,y}f^t_n(\xi,\omega)d\nu^N_{\widehat{\theta},n}(\xi,\omega)\\
&\le -\frac{1}{2}(\mathcal{D}^0_n)^{x,y}(\mu_n(t)|\nu^N_{\widehat{\theta},n})+\frac{C}{N^2}\ , \qquad (5.40)
\end{aligned}$$

where we used (5.19) and (5.35) for the first inequality and Consequences 5.3*(i)* combined with the fact that f^t_n is a probability density for the second one.

For the part (5.39) associated to the boundaries, we shall write for each pair $(x,y)=(x,x+e_k)\in(\Lambda_{N,n}\times\Lambda^c_{N,n})\cup(\Lambda^c_{N,n}\times\Lambda_{N,n})$,

$$\mathcal{L}^{x,y}=\sum_{0\le i\ne j\le 3}\mathcal{L}^{x,y}_{i\leftrightarrow j} \qquad (5.41)$$

where $\mathcal{L}^{x,y}_{i\leftrightarrow j}$ stands for the exchange of values i and j between sites x and y.

$$\mathcal{L}^{x,y}_{i\leftrightarrow j}f(\xi,\omega)=\eta_i(x)\eta_j(y)\Big(f(\xi^{x,y},\omega^{x,y})-f(\xi,\omega)\Big)+\eta_j(x)\eta_i(y)\Big(f(\xi^{x,y},\omega^{x,y})-f(\xi,\omega)\Big). \qquad (5.42)$$

So that,

$$\begin{aligned}
\Omega_1^{(2)}(x,y) &= \sum_{0\le i\ne j\le 3}\int\eta_i(x)\eta_j(y)f^t_{n+1}(\xi,\omega)\log\frac{f^t_n(\xi^{x,y},\omega^{x,y})}{f^t_n(\xi,\omega)}d\nu^N_{\widehat{\theta},n+1}(\xi,\omega)\\
&\quad +\sum_{0\le i\ne j\le 3}\int\eta_j(x)\eta_i(y)f^t_{n+1}(\xi,\omega)\log\frac{f^t_n(\xi^{x,y},\omega^{x,y})}{f^t_n(\xi,\omega)}d\nu^N_{\widehat{\theta},n+1}(\xi,\omega)\,. \qquad (5.43)
\end{aligned}$$

Let us detail the computation for $(x,y)\in\Lambda_{N,n}\times\Lambda^c_{N,n}$ and $i=1$, $j=3$, the other values would be deduced in a similar way. By a change of variables $(\xi',\omega')=(\xi^{x,y},\omega^{x,y})$ in the integral corresponding to $i=1,j=3$ in the second term of the r.h.s. (5.43), using (5.29),(5.30), and noticing that if $\eta_1(x)\eta_3(y)=1$ then $\xi^{x,y}=\xi$, and that since f^t_n does not depend on y, $f^t_n(\xi,\omega^{x,y})=f^t_n(\xi,\sigma^x\omega)$, we have

$$\begin{aligned}
&\int f^t_{n+1}(\xi,\omega)\mathcal{L}^{x,y}_{1\leftrightarrow 3}(\log f^t_n(\xi,\omega))d\nu^N_{\widehat{\theta},n+1}(\xi,\omega) = \int \eta_1(x)\eta_3(y) f^t_{n+1}(\xi,\omega)\log\frac{f^t_n(\xi^{x,y},\omega^{x,y})}{f^t_n(\xi,\omega)}d\nu^N_{\widehat{\theta},n+1}(\xi,\omega)\\
&\quad + \int \eta_1(x)\eta_3(y)(R^{x,y}_{1,3}(\widehat{\theta})+1) f^t_{n+1}(\xi^{x,y},\omega^{x,y})\log\frac{f^t_n(\xi,\omega)}{f^t_n(\xi^{x,y},\omega^{x,y})}d\nu^N_{\widehat{\theta},n+1}(\xi,\omega)\\
&= \int \eta_1(x)\langle\eta_3(y)f^t_{n+1}(\xi,\omega)\rangle_{n+1}\log\frac{f^t_n(\xi,\sigma^x\omega)}{f^t_n(\xi,\omega)}d\nu^N_{\widehat{\theta},n}(\xi,\omega) + O(N^{-1})\\
&\quad + \int \eta_1(x)\langle\eta_3(y)f^t_{n+1}(\xi,\omega^{x,y})\rangle_{n+1}\log\frac{f^t_n(\xi,\omega)}{f^t_n(\xi,\sigma^x\omega)}d\nu^N_{\widehat{\theta},n}(\xi,\omega)\\
&= \int \eta_1(x)\Big(\langle F^{(1)}_{1,3}(\xi,\omega)\rangle_{n+1} - \langle F^{(2)}_{1,3}(\xi,\omega)\rangle_{n+1}\Big)\log\frac{f^t_n(\xi,\sigma^x\omega)}{f^t_n(\xi,\omega)}d\nu^N_{\widehat{\theta},n}(\xi,\omega) + O(N^{-1})\,, \qquad (5.44)
\end{aligned}$$

where we used (5.31) in the second equality, and where

$$F^{(1)}_{i,j}(\xi,\omega) = \eta_j(y)f^t_{n+1}(\xi,\omega)\,,\quad F^{(2)}_{i,j}(\xi,\omega) = \eta_j(y)f^t_{n+1}(\xi^{x,y},\omega^{x,y})\,. \qquad (5.45)$$

If we now define

$$\begin{aligned}
E_1(i,j) &:= \{(\xi,\omega) : \langle F^{(1)}_{i,j}(\xi,\omega)\rangle_{n+1} \ge \langle F^{(2)}_{i,j}(\xi,\omega)\rangle_{n+1}, f^t_n(\xi,\sigma^x\omega)\ge f^t_n(\xi,\omega)\} \qquad (5.46)\\
E_2(i,j) &:= \{(\xi,\omega) : \langle F^{(1)}_{i,j}(\xi,\omega)\rangle_{n+1} \le \langle F^{(2)}_{i,j}(\xi,\omega)\rangle_{n+1}, f^t_n(\xi,\sigma^x\omega)\le f^t_n(\xi,\omega)\} \qquad (5.47)
\end{aligned}$$

the integral in the r.h.s. of (5.44) is non-negative on $E_1(1,3)\cup E_2(1,3)$. Then, thanks to the inequalities (5.36)–(5.37), for A to be chosen later, the integral in the r.h.s. of (5.44) is bounded by

$$\begin{aligned}
&\int_{E_1(1,3)\cup E_2(1,3)} \eta_1(x)\Big(\langle F^{(1)}_{1,3}(\xi,\omega)\rangle_{n+1} - \langle F^{(2)}_{1,3}(\xi,\omega)\rangle_{n+1}\Big)\log\frac{f^t_n(\xi,\sigma^x\omega)}{f^t_n(\xi,\omega)}d\nu^N_{\widehat{\theta},n}(\xi,\omega)\\
&\le 2\int_{E_1(1,3)\cup E_2(1,3)} \eta_1(x)\Big(\langle F^{(1)}_{1,3}(\xi,\omega)\rangle_{n+1} - \langle F^{(2)}_{1,3}(\xi,\omega)\rangle_{n+1}\Big)\times\Big(\sqrt{\frac{f^t_n(\xi,\sigma^x\omega)}{f^t_n(\xi,\omega)}}-1\Big)d\nu^N_{\widehat{\theta},n}(\xi,\omega)\\
&\le \frac{N}{A}\int_{E_1(1,3)\cup E_2(1,3)} \eta_1(x)\Big(\sqrt{\langle F^{(1)}_{1,3}(\xi,\omega)\rangle_{n+1}} - \sqrt{\langle F^{(2)}_{1,3}(\xi,\omega)\rangle_{n+1}}\Big)^2 d\nu^N_{\widehat{\theta},n}(\xi,\omega)\\
&+ \frac{A}{N}\int_{E_1(1,3)\cup E_2(1,3)} \Big(\sqrt{\langle F^{(1)}_{1,3}(\xi,\omega)\rangle_{n+1}} + \sqrt{\langle F^{(2)}_{1,3}(\xi,\omega)\rangle_{n+1}}\Big)^2\times\Big(\sqrt{\frac{f^t_n(\xi,\sigma^x\omega)}{f^t_n(\xi,\omega)}}-1\Big)^2 d\nu^N_{\widehat{\theta},n}(\xi,\omega)\\
&=: I_1 + I_2\,. \qquad (5.48)
\end{aligned}$$

In order to get rid of N in I_1, we use (5.19) to introduce a new sum in m, and to rewrite I_1 as

$$\begin{aligned}
I_1 = \frac{N}{A}\frac{1}{N}\sum_{m=n+1}^{n+N}\int_{E_1(1,3)\cup E_2(1,3)} \eta_1(x)\Big(&\sqrt{\langle\eta_3(y)f^t_m(\xi^{x,y},\omega^{x,y})\rangle_{\Lambda_{N,m}\setminus\Lambda_{N,n}}}\\
&-\sqrt{\langle\eta_3(y)f^t_m(\xi,\omega)\rangle_{\Lambda_{N,m}\setminus\Lambda_{N,n}}}\Big)^2 d\nu^N_{\widehat{\theta},m}(\xi,\omega). \qquad (5.49)
\end{aligned}$$

We now apply Cauchy-Schwarz inequality to bound I_1 by a piece of the specific Dirichlet form,

$$\begin{aligned}
I_1 &\le \frac{1}{A}\sum_{m=n+1}^{n+N}\int_{E_1(1,3)\cup E_2(1,3)} \eta_1(x)\Big\langle\eta_3(y)\Big(\sqrt{f^t_m(\xi^{x,y},\omega^{x,y})} - \sqrt{f^t_m(\xi,\omega)}\Big)^2\Big\rangle_{\Lambda_{N,m}\setminus\Lambda_{N,n}} d\nu^N_{\widehat{\theta},m}(\xi,\omega)\\
&= \frac{1}{A}\sum_{m=n+1}^{n+N}\int_{E_1(1,3)\cup E_2(1,3)} \eta_1(x)\eta_3(y)\Big(\sqrt{f^t_m(\xi^{x,y},\omega^{x,y})} - \sqrt{f^t_m(\xi,\omega)}\Big)^2 d\nu^N_{\widehat{\theta},m}(\xi,\omega)\\
&\le \frac{1}{A}\sum_{m=n+1}^{n+N}\int \eta_1(x)\eta_3(y)\Big(\sqrt{f^t_m(\xi^{x,y},\omega^{x,y})} - \sqrt{f^t_m(\xi,\omega)}\Big)^2 d\nu^N_{\widehat{\theta},m}(\xi,\omega)\,. \qquad (5.50)
\end{aligned}$$

Now, to bound I_2, we separate the integrations on $E_1(1,3)$ and on $E_2(1,3)$. We first look at the integral on $E_1(1,3)$, to get

$$\begin{aligned}
&\frac{A}{N}\int_{E_1(1,3)}\eta_1(x)\Big(\sqrt{\langle F^{(1)}_{1,3}(\xi,\omega)\rangle_{n+1}}+\sqrt{\langle F^{(2)}_{1,3}(\xi,\omega)\rangle_{n+1}}\Big)^2\Big(\sqrt{\frac{f^t_n(\xi,\sigma^x\omega)}{f^t_n(\xi,\omega)}}-1\Big)^2 d\nu^N_{\widehat{\theta},n}(\xi,\omega)\\
&\le \frac{4A}{N}\int_{E_1(1,3)}\eta_1(x)\frac{\langle F^{(1)}_{1,3}(\xi,\omega)\rangle_{n+1}}{f^t_n(\xi,\omega)}\Big(\sqrt{f^t_n(\xi,\sigma^x\omega)}-\sqrt{f^t_n(\xi,\omega)}\Big)^2 d\nu^N_{\widehat{\theta},n}(\xi,\omega)\\
&\le \frac{4A}{N}\int_{E_1(1,3)}\eta_1(x)\Big(f^t_n(\xi,\sigma^x\omega)-2\sqrt{f^t_n(\xi,\sigma^x\omega)}\sqrt{f^t_n(\xi,\omega)}+f^t_n(\xi,\omega)\Big)d\nu^N_{\widehat{\theta},n}(\xi,\omega)\\
&\le \frac{4A}{N}\int_{E_1(1,3)}\eta_1(x)\Big(f^t_n(\xi,\sigma^x\omega)-f^t_n(\xi,\omega)\Big)d\nu^N_{\widehat{\theta},n}(\xi,\omega)\\
&\le \frac{4A}{N}\int\eta_1(x)f^t_n(\xi,\sigma^x\omega)d\nu^N_{\widehat{\theta},n}(\xi,\omega)=\frac{4A}{N}\int\eta_3(x)e^{(\vartheta_1(x/N)-\vartheta_3(x/N))}f^t_n(\xi,\omega)d\nu^N_{\widehat{\theta},n}(\xi,\omega)\\
&\le \frac{AC_1}{N}
\end{aligned}\tag{5.51}$$

for some positive constant C_1. We have used the definition (5.46) of $E_1(1,3)$ for the first and third inequalities, the definition (5.45) of $F^{(1)}_{1,3}(\xi,\omega)$ with the bound $\langle F^{(1)}_{1,3}(\xi,\omega)\rangle_{n+1}\le\langle f^t_{n+1}(\xi,\omega)\rangle_{n+1}=f^t_n(\xi,\omega)$ for the second inequality, (5.34) for the equality, (5.28), (2.19) and that f^t_n is a probability density to conclude.

We now look at the integral on $E_2(1,3)$, to get

$$\begin{aligned}
&\frac{A}{N}\int_{E_2(1,3)}\eta_1(x)\Big(\sqrt{\langle F^{(1)}_{1,3}(\xi,\omega)\rangle_{n+1}}+\sqrt{\langle F^{(2)}_{1,3}(\xi,\omega)\rangle_{n+1}}\Big)^2\Big(\sqrt{\frac{f^t_n(\xi,\sigma^x\omega)}{f^t_n(\xi,\omega)}}-1\Big)^2 d\nu^N_{\widehat{\theta},n}(\xi,\omega)\\
&\le \frac{4A}{N}\int_{E_2(1,3)}\eta_1(x)\frac{\langle F^{(2)}_{1,3}(\xi,\omega)\rangle_{n+1}}{f^t_n(\xi,\omega)}\Big(\sqrt{f^t_n(\xi,\sigma^x\omega)}-\sqrt{f^t_n(\xi,\omega)}\Big)^2 d\nu^N_{\widehat{\theta},n}(\xi,\omega)\\
&\le \frac{8A}{N}\int_{E_2(1,3)}\eta_1(x)\frac{\langle F^{(2)}_{1,3}(\xi,\omega)\rangle_{n+1}}{f^t_n(\xi,\omega)}f^t_n(\xi,\omega)d\nu^N_{\widehat{\theta},n}(\xi,\omega)\\
&\le \frac{8A}{N}\int\eta_3(x)\eta_1(y)(R^{x,y}_{1,3}(\widehat{\theta})+1)f^t_{n+1}(\xi,\omega)d\nu^N_{\widehat{\theta},n+1}(\xi,\omega)\\
&\le \frac{AC_1'}{N}
\end{aligned}\tag{5.52}$$

for some positive constant C_1'. We have used the definition (5.47) of $E_2(1,3)$ for the first and second inequalities, the definition (5.45) of $F^{(2)}_{1,3}(\xi,\omega)$ with (5.29) for the third inequality, and (5.28), (2.19) and finally that f^t_n is a probability density.
Combining (5.40) with (5.50), (5.51), (5.52), we get the upper bound (5.23) of Ω_1.

Step 3: Bound on Ω_2. We decompose the generator of the reaction part into a part involving only sites within $\Lambda_{N,n}$ and a part involving sites in $\Lambda^c_{N,n}$. Recalling (4.2), (4.3), we have

$$\Omega_2 \;=\; \int f^t_{n+1}\mathbb{L}_{N,n+1}(\log f^t_n)d\nu^N_{\widehat{\theta},n+1}=\int f^t_{n+1}\mathbb{L}_{N,n}(\log f^t_n)d\nu^N_{\widehat{\theta},n+1}+\Omega^{(1)}_2\,.$$

Proceeding as for (5.40), we get

$$\int f^t_{n+1}\mathbb{L}_{N,n}\log f^t_n d\nu^N_{\widehat{\theta},n+1}\le -\mathbb{D}_n(\mu_n(t)|\nu_{\widehat{\theta},n})+\int\mathbb{L}_{N,n}f^t_n d\nu^N_{\widehat{\theta},n}\,.\tag{5.53}$$

The second term on the r.h.s. is of order $O(Nn^{d-1})$ since the rates $\beta_{\Lambda_{N,n}}(.,.)$ are bounded. Moreover, denoting $\partial\Lambda_{N,n} = \{x \in \Lambda_{N,n} : \exists y \in \Lambda_{N,n}^c, \|y-x\| = 1\}$,

$$\begin{aligned}\Omega_2^{(1)} \quad = \quad & \sum_{x\in\partial\Lambda_{N,n}} \int f_{n+1}^t(\xi,\omega)\Big(\lambda_1 \sum_{\substack{y\in\Lambda_{N,n}^c\\ \|y-x\|=1}} \xi(y)(1-\omega(y)) \\ & +\lambda_2 \sum_{\substack{y\in\Lambda_{N,n}^c\\ \|y-x\|=1}} \xi(y)\omega(y)\Big)(1-\xi(x))\log\frac{f_n^t(\sigma^x\xi,\omega)}{f_n^t(\xi,\omega)} d\nu_{\widehat{\theta},n+1}^N(\xi,\omega)\end{aligned}$$

which can be proved to be of order $O(Nn^{d-2})$ in an analogous way to the computation done for $\Omega_1^{(2)}$, and using that the rates $\beta_{\Lambda_{N,n}}(.,.)$ are bounded, inequalities (5.36)–(5.37). Combining this with (5.53) yields the upper bound (5.24) for Ω_2.

Step 4: Bound on Ω_3. It is in this step that the reversibility of the measure $\nu_{\widehat{\theta},n}^N$ with respect to the generator $L_{\widehat{b},N,n}$ plays a crucial role. It implies that, for any $x \in \Lambda_{N,n} \cap \Gamma_N$,

$$\int L_{\widehat{b},N}^x f_n^t d\nu_{\widehat{\theta},n}^N = 0. \tag{5.54}$$

Since $L_{\widehat{b},N,n+1} = \sum_{x\in\Lambda_{N,n+1}\cap\Gamma_N} L_{\widehat{b},N}^x$, using (5.19) and inequality (5.35) we derive (5.25) as follows.

$$\begin{aligned}\Omega_3 \quad &= \quad N^2 \sum_{x\in\Lambda_{N,n+1}\cap\Gamma_N} \int f_{n+1}^t L_{\widehat{b},N}^x(\log f_n^t) d\nu_{\widehat{\theta},n+1}^N \\ &= \quad N^2 \sum_{x\in\Lambda_{N,n}\cap\Gamma_N} \int \langle f_{n+1}^t(\xi,\omega)\rangle_{n+1} L_{\widehat{b},N}^x(\log f_n^t) d\nu_{\widehat{\theta},n}^N \\ &\le \quad -N^2 D_n^{\widehat{b}}(\mu_n(t)|\nu_{\widehat{\theta},n}^N) + N^2 \sum_{x\in\Lambda_{N,n}\cap\Gamma_N} \int L_{\widehat{b},N}^x f_n^t d\nu_{\widehat{\theta},n}^N \\ &= \quad -N^2 D_n^{\widehat{b}}(\mu_n(t)|\nu_{\widehat{\theta},n}^N)\,,\end{aligned}$$

where we used that f_n^t does not depend on $\Lambda_{N,n}^c$ for the second equality. Thanks to (5.54), in the last equality we got rid of a term with an order too large in N. □

5.3. **Replacement lemma in the bulk (proof of Lemma 4.2).** Fix $G \in \mathcal{C}_c^\infty([0,T]\times\Lambda,\mathbb{R})$, let $K > 0, \delta > 0$ be such that the (compact) support of G is contained in the box $\Lambda(1-\delta,K) := [-1+\delta, 1-\delta]\times[-K,K]^{d-1}$. Let $0 < a < \delta/2$, and let $\widehat{\theta}_a = (\theta_{a,1},\theta_{a,2},\theta_{a,3}) : \Lambda \to (0,1)^3$ be a smooth function, equal in $\Lambda(1-a,K)$ to some constant, say $\widehat{\alpha}$, and to $\widehat{b}$ at the boundaries. Therefore

$$\mathbb{E}_{\mu_N}^{N,\widehat{b}}\Big(\frac{1}{N^d}\sum_{x\in\Lambda_N}\int_0^T |G_s(x/N)|\tau_x V_{\epsilon N}(\xi_s,\omega_s)ds\Big) \le \|G\|_\infty \mathbb{E}_{\mu_N}^{N,\widehat{b}}\left(\frac{1}{N^d}\sum_{x\in\Lambda_{\lfloor N(1-\delta)\rfloor,NK}}\int_0^T \tau_x V_{\epsilon N}(\xi_s,\omega_s)ds\right).$$

Denote $\bar{f}^T = T^{-1}\int_0^T f_{N(K+2)}^s ds$. By Theorem 5.1, there exists some positive constant C_1 such that the expectation on the above r.h.s. is bounded by

$$\frac{T}{N^d}\int \sum_{x\in\Lambda_{\lfloor N(1-\delta)\rfloor,NK}} \tau_x V_{\epsilon N}(\xi,\omega)\bar{f}^T(\xi,\omega)d\nu_{\widehat{\theta}_a,N(K+2)}^N(\xi,\omega) - \gamma T N^{2-d} D_{N(K+2)}(\bar{f}^T) + \gamma C_1,$$

for all positive γ. To prove the Lemma, it thus remains to show that for all positive γ, a,

$$\overline{\lim_{\epsilon\to 0}}\ \overline{\lim_{N\to\infty}}\ \sup_f \left(\frac{1}{N^d}\int \sum_{x\in\Lambda_{\lfloor N(1-\delta)\rfloor,NK}} \tau_x V_{\epsilon N}(\xi,\omega) f(\xi,\omega) d\nu_{\widehat{\alpha},N(K+2)}^N(\xi,\omega) - \gamma N^{2-d} D_{N(K+2)}(f)\right) = 0,$$

where the supremum is carried over all densities f with respect to $\nu^N_{\widehat{\alpha},N(K+2)}$ such that $D_{N(K+2)}(f) \leq CN^{d-2}$. This is a consequence of the one and two block estimates stated below (see [16, 19] for the now standard proofs). The one block estimate ensures the average of local functions in some large microscopic boxes can be replaced by its mean with respect to the grand-canonical measure parametrized by the particle density in these boxes. While the two block estimate ensures the particle density over large microscopic boxes is close to the one over small macroscopic boxes:

Lemma 5.4 (One block estimate)**.** *Given a constant profile* $\widehat{\rho} = (\rho_1, \rho_2, \rho_3) \in (0,1)^3$*,*

$$\overline{\lim_{k\to\infty}}\ \overline{\lim_{N\to\infty}} \sup_{f:\mathcal{D}^0_{N(K+2)}(f)\leq CN^{d-2}} \int \frac{1}{N^d} \sum_{x\in\Lambda_{N,NK}} \tau_x V_k(\xi,\omega) f(\xi,\omega) d\nu^N_{\widehat{\rho},N(K+2)}(\xi,\omega) = 0\,,$$

where for $k \in \mathbb{N}$*,* $V_k(\xi,\omega)$ *was defined in* (4.8)*.*

Lemma 5.5 (Two block estimate)**.** *Given a constant profile* $\widehat{\rho} = (\rho_1, \rho_2, \rho_3) \in (0,1)^3$*, for* $i \in \{1,2,3\}$*,*

$$\overline{\lim_{k\to\infty}}\ \overline{\lim_{\epsilon\to 0}}\ \overline{\lim_{N\to\infty}} \sup_{f:\mathcal{D}^0_{N(K+2)}(f)\leq CN^{d-2}}$$
$$\sup_{|h|\leq\epsilon N} \frac{1}{N^d} \int \sum_{x\in\Lambda_{N,NK}} \left|\eta^k_i(x+h) - \eta^{\epsilon N}_i(x)\right| f(\xi,\omega) d\nu^N_{\widehat{\rho},N(K+2)}(\xi,\omega) = 0.$$

6. Hydrodynamic limit in large finite volume: Proof of Proposition 4.3

In this section, we prove the last result to derive the hydrodynamic limit in large finite volume (that is of size $M_N = N^{1+\frac{1}{d}}$), Proposition 4.3. As mentioned in Remark 4.1, Proposition 4.3 is the only difference in our proof of hydrodynamics with the case of infinite volume dynamics.

6.1. **Estimates for finite volume.** Next estimates will be useful to prove Proposition 4.3.

Lemma 6.1. *For a smooth profile* $\widehat{\theta} = (\theta_1, \theta_2, \theta_3) : \overline{\Lambda} \to (0,1)^3$ *satisfying* (2.19) *and* (2.20)*, there exist positive constants* A_0*,* A'_0 *and* A_1 *depending only on* $\widehat{\theta}$ *such that for any* $c > 0$*, for any cylinder function* $f \in L^2(\nu^N_{\widehat{\theta},M_N})$*,*

$$\langle L_{\widehat{b},N,M_N}\sqrt{f}, \sqrt{f}\rangle = -D^{\widehat{b}}_{M_N}(f), \tag{6.1}$$

$$\langle \mathcal{L}_{N,M_N}\sqrt{f}, \sqrt{f}\rangle \leq -A_0\mathcal{D}^0_{M_N}(f) + A'_0|\Lambda_{N,M_N}|N^{-2}\|\sqrt{f}\|^2_{L^2(\nu^N_{\widehat{\theta},M_N})}, \tag{6.2}$$

$$\langle \mathbb{L}_{N,M_N}\sqrt{f}, \sqrt{f}\rangle \leq A_1|\Lambda_{N,M_N}|\|\sqrt{f}\|^2_{L^2(\nu^N_{\widehat{\theta},M_N})}, \tag{6.3}$$

where $|\Lambda_{N,M_N}| = (2N+1)(2M_N+1)^{d-1}$ *stands for the cardinality of the set* Λ_{N,M_N}*.*

Proof. The reversibility of $\nu^N_{\widehat{\theta},M_N}$ with respect to the generator $L_{\widehat{b},N,M_N}$ yields (6.1).

To prove (6.2), observe that for all $A, B > 0$, $A(B-A) = -\dfrac{1}{2}(B-A)^2 + \dfrac{1}{2}(B^2 - A^2)$, so that

$$\langle \mathcal{L}_{N,M_N}\sqrt{f}, \sqrt{f}\rangle = \sum_{k=1}^d \sum_{x,x+e_k\in\Lambda_{N,M_N}} \int \sqrt{f(\xi,\omega)}\Big(\sqrt{f(\xi^{x,x+e_k},\omega^{x,x+e_k})} - \sqrt{f(\xi,\omega)}\Big) d\nu^N_{\widehat{\theta},M_N}(\xi,\omega)$$

$$= -\frac{1}{2}\mathcal{D}^0_{M_N}(f) + \frac{1}{2}\sum_{k=1}^d \sum_{x,x+e_k\in\Lambda_{N,M_N}} \int \mathcal{L}^{x,x+e_k} f(\xi,\omega) d\nu^N_{\widehat{\theta},M_N}(\xi,\omega)\,. \tag{6.4}$$

By Consequences 5.3*(i)*, we have

$$\frac{1}{2}\int \mathcal{L}_{N,M_N} f(\xi,\omega) d\nu^N_{\widehat{\theta},M_N}(\xi,\omega) \leq \frac{1}{4}\mathcal{D}^0_{M_N}(f) + C|\Lambda_{N,M_N}|N^{-2}\|\sqrt{f}\|^2_{L^2(\nu^N_{\widehat{\theta},M_N})} \tag{6.5}$$

for some positive constant C. Putting together (6.4) and (6.5) we obtain (6.2).

To prove (6.3), we have

$$\begin{aligned}
&\langle \mathbb{L}_{N,M_N}\sqrt{f},\sqrt{f}\rangle = I_1+I_2\\
&:= \sum_{x\in\Lambda_{N,M_N}}\int \Big(\beta_{\Lambda_{N,M_N}}(x,\xi,\omega)(1-\xi(x))+\xi(x)\Big)\sqrt{f(\xi,\omega)}\Big(\sqrt{f(\sigma^x\xi,\omega)}-\sqrt{f(\xi,\omega)}\Big)d\nu^N_{\widehat{\theta},M_N}(\xi,\omega)\\
&\quad+\sum_{x\in\Lambda_{N,M_N}}\int \Big(r(1-\omega(x))+\omega(x)\Big)\sqrt{f(\xi,\omega)}\Big(\sqrt{f(\xi,\sigma^x\omega)}-\sqrt{f(\xi,\omega)}\Big)d\nu^N_{\widehat{\theta},M_N}(\xi,\omega)\,.
\end{aligned}$$

For I_1, using first that all the rates are bounded, then (5.37) with $A=2N$, we have

$$\begin{aligned}
I_1 &\le C(\lambda_1,\lambda_2,r)\sum_{x\in\Lambda_{N,M_N}}\int\Big(\sqrt{f(\xi,\omega)}\sqrt{f(\sigma^x\xi,\omega)}+f(\xi,\omega)\Big)d\nu^N_{\widehat{\theta},M_N}(\xi,\omega)\\
&\le C(\lambda_1,\lambda_2,r)\sum_{x\in\Lambda_{N,M_N}}\int\Big(f(\xi,\omega)+\frac14 f(\sigma^x\xi,\omega)+f(\xi,\omega)\Big)d\nu^N_{\widehat{\theta},M_N}(\xi,\omega)
\end{aligned}$$

for some constant $C(\lambda_1,\lambda_2,r)$. We conclude using the change of variables (5.33) and (2.19). We proceed similarly for I_2, using the change of variables (5.34). □

6.2. **Boundary conditions in large finite volume.** Recall that μ_N stands for the initial measure and μ_{N,M_N} for the marginal of μ_N on $\widehat{\Sigma}_{N,M_N}$ (cf. (5.1)). Recall from Section 4 that we denoted by $\widetilde{\mathbb{P}}^{M_N,\widehat{b}}_{\mu_N}$ the law of the finite volume process $(\zeta_t,\chi_t)_{t\in[0,T]}$, by $\widetilde{\mathbb{E}}^{M_N,\widehat{b}}_{\mu_N}$ the corresponding expectation and by $(\widetilde{\eta}_1,\widetilde{\eta}_2,\widetilde{\eta}_3)$ then $\widetilde{\eta}_0$ the associated conserved quantities for the exchange dynamics as in (2.4)–(2.5).

Proof of Proposition 4.3. Let $\widehat{\theta}=(\theta_1,\theta_2,\theta_3):\Lambda\to(0,1)^3$ be a smooth profile satisfying (2.19) and (2.20). Denote by $\nu^N_{\widehat{\theta},\Lambda_N\setminus\Lambda_{N,M_N}}$ the marginal of $\nu^N_{\widehat{\theta}}$ on $\big(\{0,1\}\times\{0,1\}\big)^{\Lambda_N\setminus\Lambda_{N,M_N}}$, and by $\widetilde{\mu}^{N,M_N}_{\widehat{\theta}}$ the measure on $\widehat{\Sigma}_N$ given by

$$\widetilde{\mu}^{N,M_N}_{\widehat{\theta}}=\mu_{N,M_N}\otimes\nu^N_{\widehat{\theta},\Lambda_N\setminus\Lambda_{N,M_N}}\,. \tag{6.6}$$

Let $H:[0,T]\times\Gamma\to\mathbb{R}$ be a bounded function with compact support contained in $\{-1,1\}\times[-K,K]^{d-1}\subset\Gamma$ for some $K>0$; let $\delta>0$, and $i\in\{1,2,3\}$. To shorten the notation, we denote

$$\begin{aligned}
B_i^{H_t}(\zeta_t,\chi_t) &= \frac{1}{N^{d-1}}\sum_{x\in\Lambda_{N,M_N}\cap\Gamma_N}H_t(x/N)\Big(\widetilde{\eta}_{i,t}(x)-b_i(x/N)\Big)\\
&= \frac{1}{N^{d-1}}\sum_{x\in\Lambda_{N,NK}\cap\Gamma_N}H_t(x/N)\Big(\widetilde{\eta}_{i,t}(x)-b_i(x/N)\Big)\,.
\end{aligned}$$

Because H has compact support in Γ, we have (cf. (6.6)),

$$\widetilde{\mathbb{P}}^{M_N,\widehat{b}}_{\mu_N}\Big(\Big|\int_0^T B_i^{H_t}(\zeta_t,\chi_t)dt\Big|>\delta\Big)=\widetilde{\mathbb{P}}^{M_N,\widehat{b}}_{\widetilde{\mu}^{N,M_N}_{\widehat{\theta}}}\Big(\Big|\int_0^T B_i^{H_t}(\zeta_t,\chi_t)dt\Big|>\delta\Big)\,.$$

To prove the proposition it is enough to show that

$$\varlimsup_{N\to\infty}\frac{1}{|\Lambda_{N,M_N}|}\log\widetilde{\mathbb{P}}^{M_N,\widehat{b}}_{\widetilde{\mu}^{N,M_N}_{\widehat{\theta}}}\Big(\Big|\int_0^T B_i^{H_t}(\zeta_t,\chi_t)dt\Big|>\delta\Big)=-\infty\,.$$

Since the Radon-Nikodym derivative $\dfrac{d\mu_{N,M_N}}{d\nu^N_{\widehat{\theta},M_N}}$ is bounded by $\exp(NM_N^{d-1}K_1)$ for some positive constant K_1, by (6.6) it is enough to show that

$$\varlimsup_{N\to\infty}\frac{1}{|\Lambda_{N,M_N}|}\log\widetilde{\mathbb{P}}^{M_N,\widehat{b}}_{\nu^N_{\widehat{\theta}}}\Big(\Big|\int_0^T B_i^{H_t}(\zeta_t,\chi_t)dt\Big|>\delta\Big)=-\infty\,.$$

By exponential Chebyshev's inequality, the expression in the last limit is bounded above by

$$-a\delta + \frac{1}{|\Lambda_{N,M_N}|} \log \widetilde{\mathbb{E}}^{M_N,\widehat{b}}_{\nu^N_{\widehat{\theta}}} \exp\Big(a|\Lambda_{N,M_N}|\Big|\int_0^T B_i^{H_t}(\zeta_t,\chi_t)dt\Big|\Big)$$

for any $a>0$. Using that $e^{|\alpha|} \le e^{\alpha}+e^{-\alpha}$ and that, for two generic sequences $(a_L)_{L\in\mathbb{N}}, (b_L)_{L\in\mathbb{N}}$ we have

$$\overline{\lim_L} L^{-1}\log\big(a_L+b_L\big) \le \max\Big(\overline{\lim_L} L^{-1}\log a_L, \overline{\lim_L} L^{-1} b_L\Big),$$

one can pull off the absolute value even if it means replacing H by $-H$. Therefore, to prove the proposition, we have to show that, for any bounded function H, there exists a positive constant $C>0$, such that for any $a>0$,

$$\overline{\lim_{N\to+\infty}} \frac{1}{|\Lambda_{N,M_N}|} \log \widetilde{\mathbb{E}}^{M_N,\widehat{b}}_{\nu^N_{\widehat{\theta}}} \exp\Big(a|\Lambda_{N,M_N}|\int_0^T B_i^{H_t}(\zeta_t,\chi_t)dt\Big) \le C$$

and then to let $a\uparrow+\infty$.

By Feynman-Kac formula (cf. Appendix 1, Section 7 in [19]),

$$\begin{aligned}&\frac{1}{|\Lambda_{N,M_N}|} \log \widetilde{\mathbb{E}}^{M_N,\widehat{b}}_{\nu^N_{\widehat{\theta}}} \exp\Big(a|\Lambda_{N,M_N}|\int_0^T B_i^{H_t}(\zeta_t,\chi_t)dt\Big)\\ &\le \int_0^T \sup_f\Big\{\int aB_i^{H_t}(\zeta,\chi)f(\zeta,\chi)d\nu^N_{\widehat{\theta},M_N}(\zeta,\chi) + |\Lambda_{N;M_N}|^{-1}\langle \mathfrak{L}_{N,M_N}\sqrt{f},\sqrt{f}\rangle\Big\}dt\,,\end{aligned} \tag{6.7}$$

where the supremum is carried over all densities f with respect to $\nu^N_{\widehat{\theta},M_N}$. By Lemma 6.1,

$$\langle \mathfrak{L}_{N,M_N}\sqrt{f},\sqrt{f}\rangle \le -N^2 D_{M_N}(f) + A_0|\Lambda_{N,M_N}| \le -N^2 D^{\widehat{b}}_{M_N}(f) + A_0|\Lambda_{N,M_N}| \tag{6.8}$$

for some positive constant A_0, where D_{M_N} and $D^{\widehat{b}}_{M_N}$ are defined in (5.5)–(5.10).

To evaluate $\int aB_i^{H_t}(\zeta,\chi)f(\zeta,\chi)d\nu^N_{\widehat{\theta},M_N}(\zeta,\chi)$, observe first that for all $x\in\Lambda_{N,M_N}$, $1\le i\le 3$,

$$\begin{aligned}\widetilde{\eta}_i(x) - b_i(x/N) &= \widetilde{\eta}_i(x)(1-b_i(x/N)) - b_i(x/N)(1-\widetilde{\eta}_i(x))\\ &= \sum_{0\le j\ne i\le 3}\Big(\widetilde{\eta}_i(x)b_j(x/N) - b_i(x/N)\widetilde{\eta}_j(x/N)\Big).\end{aligned} \tag{6.9}$$

We detail for instance the case $i=1$, the others follow the same way.

$$\int aB_1^{H_t}(\zeta,\chi)f(\zeta,\chi)d\nu^N_{\widehat{\theta},M_N}(\zeta,\chi)=\frac{a}{N^{d-1}}\sum_{x\in\Lambda_{N,NK}\cap\Gamma_N}\int H_t(x/N)\Bigg(\Big(\widetilde{\eta}_1(x)b_0(x/N)-b_1(x/N)\widetilde{\eta}_0(x)\Big)$$

$$+\Big(\widetilde{\eta}_1(x)b_3(x/N)-b_1(x/N)\widetilde{\eta}_3(x)\Big)+\Big(\widetilde{\eta}_1(x)b_2(x/N)-b_1(x/N)\widetilde{\eta}_2(x)\Big)\Bigg)f(\zeta,\chi)d\nu^N_{\widehat{\theta},M_N}(\zeta,\chi)$$

$$=\frac{a}{N^{d-1}}\sum_{x\in\Lambda_{N,NK}\cap\Gamma_N}\int H_t(x/N)\Bigg(b_1(x/N)\widetilde{\eta}_0(x)\Big(f(\sigma^x\zeta,\chi)-f(\zeta,\chi)\Big)$$

$$+b_1(x/N)\widetilde{\eta}_3(x)\Big(f(\zeta,\sigma^x\chi)-f(\zeta,\chi)\Big)+b_1(x/N)\widetilde{\eta}_2(x)\Big(f(\sigma^x\zeta,\sigma^x\chi)-f(\zeta,\chi)\Big)\Bigg)d\nu^N_{\widehat{\theta},M_N}(\zeta,\chi)$$

$$\le\frac{N^2}{2|\Lambda_{N,M_N}|}\sum_{x\in\Lambda_{N,NK}\cap\Gamma_N}\int b_1(x/N)\Bigg(\widetilde{\eta}_0(x)\Big(\sqrt{f(\sigma^x\zeta,\chi)}-\sqrt{f(\zeta,\chi)}\Big)^2$$

$$+\widetilde{\eta}_3(x)\Big(\sqrt{f(\zeta,\sigma^x\chi)}-\sqrt{f(\zeta,\chi)}\Big)^2+\widetilde{\eta}_2(x)\Big(\sqrt{f(\sigma^x\zeta,\sigma^x\chi)}-\sqrt{f(\zeta,\chi)}\Big)^2\Bigg)d\nu^N_{\widehat{\theta},M_N}(\zeta,\chi)$$

$$+\frac{C_1'a^2}{2}\frac{|\Lambda_{N,M_N}|}{N^{d+1}}||H^2||_\infty\,, \tag{6.10}$$

where C_1' is some positive constant. We used (6.9), changes of variables given by (5.32)–(5.34) with (2.20) for the two equalities. Next, we used (5.37) with $A=|\Lambda_{N,M_N}|aN^{-d}$, and that f is a probability density, with a computation similar to the one done for Consequences 5.3*(i)*.
Note that the last sum above is a part of the Dirichlet form for the boundaries $D^{\widehat{b}}_{M_N}(f)$ (see (5.7), (5.9)).

Collecting all previous estimates, that is (6.7), (6.8), (6.10) and the similar ones for $B_i, i\in\{0,2,3\}$, we proved that

$$\frac{1}{|\Lambda_{N,M_N}|}\log\widetilde{\mathbb{E}}^{M_N,\widehat{b}}_{\mu_N}\exp\Big(a|\Lambda_{N,M_N}|\Big|\int_0^T B_i^{H_t}(\zeta_t,\chi_t)dt\Big|\Big)\le T\frac{C_1'a^2}{2}\frac{|\Lambda_{N,M_N}|}{N^{d+1}}||H^2||_\infty+TA_0\,. \tag{6.11}$$

Because $M_N=N^{1+\frac{1}{d}}$, the r.h.s. of (6.11) goes to TA_0 when $N\uparrow+\infty$, which concludes the proof. □

7. Hydrodynamic limit in infinite volume

7.1. **The coupled process.** We couple $(\zeta_t,\chi_t)_{t\in[0,T]}$ to our original dynamics in infinite volume $(\xi_t,\omega_t)_{t\in[0,T]}$. We denote with a "bar" everything related to this coupling. As explained in the introduction, our model allows the use of basic coupling for each of the involved dynamics; namely, the coupled particles try to behave similarly as much as possible. The generator $\overline{\mathfrak{L}}_N$ of the coupled process is given by

$$\overline{\mathfrak{L}}_N=N^2\overline{\mathcal{L}}_N+\overline{\mathbb{L}}_N+N^2\overline{L}_{\widehat{b},N}\,.$$

For this coupling, we shall come back to the initial equivalent formulation of configurations: $\eta\in\{0,1,2,3\}^{\Lambda_N}$ corresponds to (ξ,ω), and $\widetilde{\eta}\in\{0,1,2,3\}^{\Lambda_N}$ to (ζ,χ) via the application (2.3), that is $\eta=\eta((\xi,\omega)),\widetilde{\eta}=\widetilde{\eta}((\zeta,\chi))$. For $x\in\Lambda_N$, $k\in\{1,...,d\}$, define $\eta^{x,x+e_k}$ to be the configuration obtained from η by exchanging the values of η at x and $x+e_k$. Notice that via (2.2), $\eta^{x,x+e_k}$ is equivalent to $(\xi^{x,x+e_k},\omega^{x,x+e_k})$.

Define the coupled generator for the exchange part by

$$\overline{\mathcal{L}}_N=\overline{\mathcal{L}}^1_N+\overline{\mathcal{L}}^2_N\,, \tag{7.1}$$

where, for a cylinder function f on $(\{0,1,2,3\}^{\Lambda_N})^2$, and $(\xi,\omega)\in\widehat{\Sigma}_N,(\zeta,\chi)\in\widehat{\Sigma}_N$, abbreviating $\widetilde{\eta}$ for $\widetilde{\eta}((\zeta,\chi))$, and η for $\eta((\xi,\omega))$ (which will be done in all this section), we have

$$\begin{aligned}
\overline{\mathcal{L}}^1_N f(\widetilde{\eta},\eta) &= \sum_{k=1}^d \sum_{x,x+e_k\in\Lambda_{N,M_N}} \overline{\mathcal{L}}^{1,(x,x+e_k)} f(\widetilde{\eta},\eta) \qquad \text{with}\\
\overline{\mathcal{L}}^{1,(x,x+e_k)} f(\widetilde{\eta},\eta) &= f(\widetilde{\eta}^{x,x+e_k},\eta^{x,x+e_k}) - f(\widetilde{\eta},\eta)\,, \qquad (7.2)\\
\overline{\mathcal{L}}^2_N f(\widetilde{\eta},\eta) &= \sum_{k=2}^d \sum_{x\in\mathcal{A}(k)} \overline{\mathcal{L}}^{2,(x,x+e_k)} f(\widetilde{\eta},\eta) \qquad \text{with}\\
\overline{\mathcal{L}}^{2,(x,x+e_k)} f(\widetilde{\eta},\eta) &= f(\widetilde{\eta},\eta^{x,x+e_k}) - f(\widetilde{\eta},\eta)\,, \qquad (7.3)
\end{aligned}$$

where $\mathcal{A}(k)=\{x\in\Lambda_N:(x,x+e_k)\notin\Lambda_{N,M_N}\times\Lambda_{N,M_N}\}$. This means that either $(x,x+e_k)\in(\Lambda_{N,M_N}\times\Lambda^c_{N,M_N})\cup(\Lambda^c_{N,M_N}\times\Lambda_{N,M_N})$, or ($x\notin\Lambda_{N,M_N}$ and $x+e_k\notin\Lambda_{N,M_N}$) (recall that Λ^c_{N,M_N} was defined in (5.18)).

For $1\le j\le 3$ and $x\in\Lambda_N$, denote by η^j_x the configuration obtained from η by flipping the state of x to j. A (basic) coupling for the reaction part is given by $\overline{\mathbb{L}}_N$ (see [22, Proposition 4.2] or [21] for details and justifications for this coupling):

$$\overline{\mathbb{L}}_N = \overline{\mathbb{L}}^{1,a}_N + \overline{\mathbb{L}}^{1,b}_N + \overline{\mathbb{L}}^2_N, \qquad (7.4)$$

where $\overline{\mathbb{L}}^{1,a}_N$ stands for the generator with coupled flips, $\overline{\mathbb{L}}^{1,b}_N$ for uncoupled flips within Λ_{N,M_N} and $\overline{\mathbb{L}}^2_N$ for uncoupled flips occurring at the boundary of Λ_{N,M_N} or outside Λ_{N,M_N}, that is,

$$\begin{aligned}
&\overline{\mathbb{L}}^{1,a}_N f(\widetilde{\eta},\eta) = \sum_{x\in\Lambda_{N,M_N}} \overline{\mathbb{L}}^{1,a,x}_N f(\widetilde{\eta},\eta) \quad \text{with}\\
&\overline{\mathbb{L}}^{1,a,x}_N f(\widetilde{\eta},\eta) = \Big\{\big(\widetilde{\eta}_0(x)\eta_0(x)\overline{\beta}_{M_N}(x,\widetilde{\eta},\eta) + \eta_3(x)\widetilde{\eta}_3(x)\big)\Big\}\big[f(\widetilde{\eta}^1_x,\eta^1_x) - f(\widetilde{\eta},\eta)\big]\\
&+\Big\{\widetilde{\eta}_1(x)\eta_1(x) + \widetilde{\eta}_2(x)\eta_2(x)\Big\}\big[f(\widetilde{\eta}^0_x,\eta^0_x) - f(\widetilde{\eta},\eta)\big] + \Big\{r\widetilde{\eta}_0(x)\eta_0(x) + \widetilde{\eta}_3(x)\eta_3(x)\Big\}\big[f(\widetilde{\eta}^2_x,\eta^2_x) - f(\widetilde{\eta},\eta)\big]\\
&+\Big\{\widetilde{\eta}_2(x)\eta_2(x)\overline{\beta}_{M_N}(x,\widetilde{\eta},\eta) + r\widetilde{\eta}_1(x)\eta_1(x)\Big\}\big[f(\widetilde{\eta}^3_x,\eta^3_x) - f(\widetilde{\eta},\eta)\big]\\
&+\Big\{\widetilde{\eta}_2(x)\eta_0(x)\overline{\beta}_{M_N}(x,\widetilde{\eta},\eta)\Big\}\big[f(\widetilde{\eta}^3_x,\eta^1_x) - f(\widetilde{\eta},\eta)\big] + \Big\{\widetilde{\eta}_0(x)\eta_2(x)\overline{\beta}_{M_N}(x,\widetilde{\eta},\eta)\Big\}\big[f(\widetilde{\eta}^1_x,\eta^3_x) - f(\widetilde{\eta},\eta)\big]\\
&+\Big\{\widetilde{\eta}_2(x)\eta_3(x)\Big\}\big[f(\widetilde{\eta}^0_x,\eta^1_x) - f(\widetilde{\eta},\eta)\big] + \Big\{\widetilde{\eta}_3(x)\eta_2(x)\Big\}\big[f(\widetilde{\eta}^1_x,\eta^0_x) - f(\widetilde{\eta},\eta)\big]\\
&+\Big\{\widetilde{\eta}_3(x)\eta_1(x)\Big\}\big[f(\widetilde{\eta}^2_x,\eta^0_x) - f(\widetilde{\eta},\eta)\big] + \Big\{\widetilde{\eta}_1(x)\eta_3(x)\Big\}\big[f(\widetilde{\eta}^0_x,\eta^2_x) - f(\widetilde{\eta},\eta)\big]\\
&+\Big\{\widetilde{\eta}_1(x)\eta_0(x)r\Big\}\big[f(\widetilde{\eta}^3_x,\eta^2_x) - f(\widetilde{\eta},\eta)\big] + \Big\{\widetilde{\eta}_0(x)\eta_1(x)r\Big\}\big[f(\widetilde{\eta}^2_x,\eta^3_x) - f(\widetilde{\eta},\eta)\big]\,. \qquad (7.5)
\end{aligned}$$

Let $\beta_{M_N}(\cdot,\cdot)$ be the growth rate on Λ_{N,M_N} defined via (2.3), (2.4) and (2.17) by

$$\beta_{M_N}(x,\eta) = \beta_{M_N}(x,\eta((\xi,\omega))) = \beta_{\Lambda_{N,M_N}}(x,\xi,\omega) = \sum_{\substack{y\in\Lambda_{N,M_N}\\ \|y-x\|=1}} \Big\{\lambda_1\eta_1(y) + \lambda_2\eta_3(y)\Big\},$$

and

$$\begin{aligned}
\overline{\beta}_{M_N}(x,\widetilde{\eta},\eta) &= \overline{\beta}_{M_N}(x,\widetilde{\eta}((\zeta,\chi)),\eta((\xi,\omega))) = \beta_{M_N}(x,\widetilde{\eta}) \wedge \beta_{M_N}(x,\eta),\\
\overline{\beta}^{(1)}_{M_N}(x,\widetilde{\eta},\eta) &= \beta_{M_N}(x,\widetilde{\eta}) - \overline{\beta}_{M_N}(x,\widetilde{\eta},\eta), \quad \overline{\beta}^{(2)}_{M_N}(x,\widetilde{\eta},\eta) = \beta_{M_N}(x,\eta) - \overline{\beta}_{M_N}(x,\widetilde{\eta},\eta).
\end{aligned}$$

Then

$$
\begin{aligned}
&\overline{\mathbb{L}}_N^{1,b} f(\widetilde{\eta},\eta) = \sum_{x\in\Lambda_{N,M_N}} \overline{\mathbb{L}}_N^{1,b,x} f(\widetilde{\eta},\eta) \quad \text{with}\\
&\overline{\mathbb{L}}_N^{1,b,x} f(\widetilde{\eta},\eta) = \Big\{\widetilde{\eta}_0(x)\Big(\eta_0(x)\overline{\beta}^{(1)}_{M_N}(x,\widetilde{\eta},\eta) + \big(\eta_1(x)+\eta_3(x)+\eta_2(x)\big)\beta_{M_N}(x,\widetilde{\eta})\Big)\\
&\qquad\qquad + \widetilde{\eta}_3(x)\Big(\eta_0(x)+\eta_1(x)\Big)\Big\}\left[f(\widetilde{\eta}^1_x,\eta) - f(\widetilde{\eta},\eta)\right]\\
&+ \Big\{\eta_0(x)\Big(\big(\widetilde{\eta}_0(x)+\widetilde{\eta}_2(x)\big)\overline{\beta}^{(2)}_{M_N}(x,\widetilde{\eta},\eta) + \big(\widetilde{\eta}_1(x)+\widetilde{\eta}_3(x)\big)\beta_{M_N}(x,\eta)\Big) + \widetilde{\eta}_0(x)\eta_3(x)\Big\}\\
&\quad \times \left[f(\widetilde{\eta},\eta^1_x) - f(\widetilde{\eta},\eta)\right]\\
&+ \Big\{\widetilde{\eta}_2(x)\Big(\big(\eta_2(x)+\eta_0(x)\big)\overline{\beta}^{(1)}_{M_N}(x,\widetilde{\eta},\eta) + \big(\eta_1(x)+\eta_3(x)\big)\beta_{M_N}(x,\widetilde{\eta})\Big) + \widetilde{\eta}_1(x)\Big(\eta_2(x)+\eta_3(x)\Big)r\Big\}\\
&\quad \times \left[f(\widetilde{\eta}^3_x,\eta) - f(\widetilde{\eta},\eta)\right]\\
&+ \Big\{\eta_2(x)\Big(\big(\widetilde{\eta}_0(x)+\widetilde{\eta}_2(x)\big)\overline{\beta}^{(2)}_{M_N}(x,\widetilde{\eta},\eta) + \big(\widetilde{\eta}_1(x)+\widetilde{\eta}_3(x)\big)\beta_{M_N}(x,\eta)\Big) + \Big(\widetilde{\eta}_2(x)+\widetilde{\eta}_3(x)\Big)\eta_1(x)r\Big\}\\
&\quad \times \left[f(\widetilde{\eta},\eta^3_x) - f(\widetilde{\eta},\eta)\right]\\
&+ \Big\{\widetilde{\eta}_2(x)\Big(\eta_0(x)+\eta_1(x)\Big) + \widetilde{\eta}_1(x)\Big(\eta_0(x)+\eta_2(x)\Big)\Big\}\left[f(\widetilde{\eta}^0_x,\eta) - f(\widetilde{\eta},\eta)\right]\\
&+ \Big\{\Big(\widetilde{\eta}_2(x)+\widetilde{\eta}_3(x)\Big)\eta_0(x)r + \Big(\widetilde{\eta}_0(x)+\widetilde{\eta}_2(x)\Big)\eta_3(x)\Big\}\left[f(\widetilde{\eta},\eta^2_x) - f(\widetilde{\eta},\eta)\right]\\
&+ \Big\{\Big(\widetilde{\eta}_0(x)+\widetilde{\eta}_2(x)\Big)\eta_1(x) + \Big(\widetilde{\eta}_0(x)+\widetilde{\eta}_1(x)\Big)\eta_2(x)\Big\}\left[f(\widetilde{\eta},\eta^0_x) - f(\widetilde{\eta},\eta)\right]\\
&+ \Big\{\big(\widetilde{\eta}_0(x)\Big(\eta_3(x)+\eta_2(x)\Big)r + \widetilde{\eta}_3(x)\Big(\eta_0(x)+\eta_2(x)\Big)\Big\}\left[f(\widetilde{\eta}^2_x,\eta) - f(\widetilde{\eta},\eta)\right]. \qquad (7.6)
\end{aligned}
$$

Finally

$$
\overline{\mathbb{L}}_N^{2} f(\widetilde{\eta},\eta) = \sum_{x\in\Lambda_{N,M_N}} \overline{\mathbb{L}}_N^{2,a,x} f(\widetilde{\eta},\eta) + \sum_{x\notin\Lambda_{N,M_N}} \overline{\mathbb{L}}_N^{2,b,x} f(\widetilde{\eta},\eta), \quad \text{with} \qquad (7.7)
$$

$$
\overline{\mathbb{L}}_N^{2,a,x} f(\widetilde{\eta},\eta) = \eta_0(x)\beta^{out}_{N,M_N}(x,\eta)\big[f(\widetilde{\eta},\eta^1_x) - f(\widetilde{\eta},\eta)\big] + \eta_0(x)\beta^{out}_{N,M_N}(x,\eta)\big[f(\widetilde{\eta},\eta^3_x) - f(\widetilde{\eta},\eta)\big],
$$

where, for $x\in\Lambda_{N,M_N}$, $\beta^{out}_{N,M_N}(x,\eta) = \sum_{\substack{y\in\Lambda^c_{N,M_N}\\ \|y-x\|=1}} \Big\{\lambda_1\eta_1(y)+\lambda_2\eta_3(y)\Big\}$; and, for $x\notin\Lambda_{N,M_N}$, the transitions produce changes only on the second configuration η, in a similar way as in $\overline{\mathbb{L}}_N^{1,a}+\overline{\mathbb{L}}_N^{1,b}$, but with β_{M_N} replaced by β_{Λ_N}. Since we shall not use this second part of the generator $\overline{\mathbb{L}}_N^{2}$ in our computations, we do not detail it.

Note that the generator $L_{\widehat{b},N}$ (see (2.21)) can be rewritten as, for a cylinder function g on $\{0,1,2,3\}^{\Lambda_N}$, and $\eta=\eta((\xi,\omega))$,

$$
\begin{aligned}
L_{\widehat{b},N}\, g(\eta) &= \sum_{x\in\Gamma_N}\sum_{j=0}^{3} b_j(x/N)\big(1-\eta_j(x)\big)\big(g(\eta^j_x)-g(\eta)\big)\\
&= \sum_{x\in\Gamma_N}\sum_{j=0}^{3} b_j(x/N)\big(g(\eta^j_x)-g(\eta)\big). \qquad (7.8)
\end{aligned}
$$

We construct now a coupled dynamics in which, on each site x, the state in both η and $\widetilde{\eta}$ changes to the same state j at rate $b_j(x/N)$ when possible, that is within Λ_{N,M_N}:

$$\overline{L}_{\widehat{b},N} = \overline{L}^1_{\widehat{b},N} + \overline{L}^2_{\widehat{b},N} \qquad \text{where} \tag{7.9}$$

$$\overline{L}^1_{\widehat{b},N} = \sum_{x\in\Gamma_N\cap\Lambda_{N,M_N}} \overline{L}^{1,x}_{\widehat{b},N} \qquad \text{with} \qquad \overline{L}^{1,x}_{\widehat{b},N} f(\widetilde{\eta},\eta) = \sum_{i=0}^{3} b_i(x/N)\big(f(\widetilde{\eta}^i_x,\eta^i_x) - f(\widetilde{\eta},\eta)\big) \tag{7.10}$$

$$\overline{L}^2_{\widehat{b},N} = \sum_{\substack{x: x\in\Gamma_N,\\ x\notin\Lambda_{N,M_N}}} \overline{L}^{2,x}_{\widehat{b},N} \qquad \text{with} \qquad \overline{L}^{2,x}_{\widehat{b},N} f(\widetilde{\eta},\eta) = \sum_{i=0}^{3} b_i(x/N)\big(f(\widetilde{\eta},\eta^i_x) - f(\widetilde{\eta},\eta)\big). \tag{7.11}$$

We now have all the material to investigate the specific entropy and Dirichlet form for the coupled process. Recall from Section 5 that $\nu^N_{\widehat{\theta}} := \nu^N_{\widehat{\theta}(\cdot)}$ is a product probability measure on $\widehat{\Sigma}_N$, where $\widehat{\theta} = (\theta_1,\theta_2,\theta_3) : \Lambda \to (0,1)^3$ is a smooth function such that $\widehat{\theta}(\cdot)\big|_\Gamma = \widehat{b}(\cdot)$. Let μ be a probability measure on $\widehat{\Sigma}_N$ and denote by $\overline{\mu} = \mu\otimes\mu$, $\overline{\nu}^N_{\widehat{\theta}} = \nu^N_{\widehat{\theta}}\otimes\nu^N_{\widehat{\theta}}$ the product measures on $\widehat{\Sigma}_N\times\widehat{\Sigma}_N$. As in (5.3) and (5.4), for a positive integer $n>1$, we define the entropy of $\overline{\mu}$ with respect to $\overline{\nu}^N_{\widehat{\theta}}$ by

$$\overline{s}_n(\overline{\mu}_n|\overline{\nu}^N_{\widehat{\theta},n}) = \int \log\big(\overline{f}_n((\zeta,\chi),(\xi,\omega))\big)\, d\overline{\mu}_n((\zeta,\chi),(\xi,\omega)), \tag{7.12}$$

where $\overline{f}_n$ is the probability density of $\overline{\mu}_n$ with respect to $\overline{\nu}^N_{\widehat{\theta},n}$, and $\overline{\mu}_n$ (resp. $\overline{\nu}^N_{\widehat{\theta},n}$) stands for the marginal of $\overline{\mu}$ (resp. $\overline{\nu}^N_{\widehat{\theta}}$) on $\widehat{\Sigma}_{N,n}\times\widehat{\Sigma}_{N,n}$ (see (5.1)).

Let $\overline{\mathfrak{L}}_{N,n}$ denote the restriction of the generator $\overline{\mathfrak{L}}_N$ to the box $\Lambda_{N,n}$:

$$\overline{\mathfrak{L}}_{N,n} = N^2\overline{\mathcal{L}}_{N,n} + \overline{\mathbb{L}}_{N,n} + N^2\overline{L}_{\widehat{b},N,n} \qquad \text{with} \tag{7.13}$$

$$\overline{\mathcal{L}}_{N,n} = \sum_{k=1}^{d}\sum_{x,x+e_k\in\Lambda_{N,n}} \big\{\mathbf{1}_{\{x,x+e_k\in\Lambda_{N,M_N}\}}\overline{\mathcal{L}}^{1,(x,x+e_k)} + \mathbf{1}_{\{x\in\mathcal{A}(k)\}}\overline{\mathcal{L}}^{2,(x,x+e_k)}\big\} \tag{7.14}$$

$$\overline{\mathbb{L}}_{N,n} = \sum_{x\in\Lambda_{N,n}} \big\{\mathbf{1}_{\{x\in\Lambda_{N,M_N}\}}(\overline{\mathbb{L}}^{1,a,x}_N + \overline{\mathbb{L}}^{1,b,x}_N + \overline{\mathbb{L}}^{2,a,x}_N) + \mathbf{1}_{\{x\notin\Lambda_{N,M_N}\}}\overline{\mathbb{L}}^{2,b,x}_N\big\} \tag{7.15}$$

$$\overline{L}_{\widehat{b},N,n} = \sum_{x\in\Lambda_{N,n}\cap\Gamma_N} \big\{\mathbf{1}_{\{x\in\Gamma_N\cap\Lambda_{N,M_N}\}}\overline{L}^{1,x}_{\widehat{b},N} + \mathbf{1}_{\{x\in\Gamma_N, x\notin\Lambda_{N,M_N}\}}\overline{L}^{2,x}_{\widehat{b},N}\big\}, \tag{7.16}$$

Define the Dirichlet forms

$$\overline{D}_n(\overline{\mu}_n|\overline{\nu}^N_{\widehat{\theta},n}) = \overline{D}^0_n(\overline{\mu}_n|\overline{\nu}^N_{\widehat{\theta},n}) + \overline{D}^{\widehat{b}}_n(\overline{\mu}_n|\overline{\nu}^N_{\widehat{\theta},n}), \tag{7.17}$$

with each one defined similarly to (5.6)–(5.11), but relatively to (7.14)–(7.16).
Define the specific entropy $\overline{\mathcal{S}}(\overline{\mu}|\overline{\nu}^N_{\widehat{\theta}})$ and the Dirichlet form $\overline{\mathfrak{D}}(\overline{\mu}|\overline{\nu}^N_{\widehat{\theta}})$ of $\overline{\mu}$ with respect to $\overline{\nu}^N_{\widehat{\theta}}$ as

$$\overline{\mathcal{S}}(\overline{\mu}|\overline{\nu}^N_{\widehat{\theta}}) = N^{-1}\sum_{n\geq 1}\overline{s}_n(\overline{\mu}_n|\overline{\nu}^N_{\widehat{\theta},n})e^{-n/N}, \tag{7.18}$$

$$\overline{\mathfrak{D}}(\overline{\mu}|\overline{\nu}^N_{\widehat{\theta}}) = N^{-1}\sum_{n\geq 1}\overline{D}_n(\overline{\mu}_n|\overline{\nu}^N_{\widehat{\theta},n})e^{-n/N}. \tag{7.19}$$

Since the product measure $\overline{\nu}^N_{\widehat{\theta}}$ is reversible for the boundary generator $\overline{L}_{\widehat{b},N}$, next lemma has a proof similar to the one of Theorem 5.1 (which is therefore omitted).

Lemma 7.1. *For any time $t\geq 0$, there exists a positive finite constant $\overline{C}_1 \equiv \overline{C}(t,\widehat{\theta},\lambda_1,\lambda_2,r)$, so that*

$$\int_0^t \overline{\mathfrak{D}}(\overline{\mu}(s)|\overline{\nu}^N_{\widehat{\theta}})\, ds \leq \overline{C}_1 N^{d-2}.$$

7.2. **Proof of Proposition 4.4.** For $i \in \{0,1,2,3\}$, let $h_{M_N,i}$ be the function on $\{0,1,2,3\}^{\Lambda_N} \times \{0,1,2,3\}^{\Lambda_N}$ given by, for $((\zeta,\chi),(\xi,\omega)) \in \widehat{\Sigma}_N \times \widehat{\Sigma}_N$ and $\eta = \eta((\xi,\omega)), \widetilde{\eta} = \widetilde{\eta}((\zeta,\chi))$, using (2.3), (2.4),

$$h_{M_N,i}(\widetilde{\eta},\eta) \;=\; N^{-d-1}\sum_{n=1}^{M_N-1} e^{-n/N} H_{i,n}(\widetilde{\eta},\eta), \quad \text{with} \quad H_{i,n}(\widetilde{\eta},\eta) \;=\; \sum_{x\in\Lambda_{N,n}} \big|\widetilde{\eta}_i(x)-\eta_i(x)\big|\,. \tag{7.20}$$

Let K be such that the (compact) support of $\widehat{G}$ is a subset of $[-1,1]\times[-K,K]^{d-1}$. For any $i \in \{0,1,2,3\}$ and $t\geq 0$, we have

$$\begin{aligned}\frac{1}{N^d}\sum_{x\in\Lambda_N}\big|G_{i,t}(x/N)\big|\big|\widetilde{\eta}_{i,t}(x)-\eta_{i,t}(x)\big| &\leq A_0\frac{1}{N^d}\sum_{n=1}^{N} e^{-(n+KN)/N}\,\frac{1}{N}\sum_{x\in\Lambda_{N,n+KN}}\big|\widetilde{\eta}_{i,t}(x)-\eta_{i,t}(x)\big| \\ &\leq A_0 h_{M_N,i}(\widetilde{\eta}_t,\eta_t)\,,\end{aligned}$$

for some positive constant $A_0 = A_0(\widehat{G},K)$. Therefore, in order to prove the proposition it is enough to show that,

$$\lim_{N\to+\infty}\overline{\mathbb{E}}^{M_N,\widehat{b}}_{\overline{\mu}_N}\left[\sum_{i=0}^{3} h_{M_N,i}(\widetilde{\eta}_t,\eta_t)\right] \;=\; 0\,.$$

We start by splitting the quantity $h_{M_N,i}(\widetilde{\eta}_t,\eta_t)$ into two parts: The sum over all sites n, such that $M_N - N \leq n \leq M_N - 1$ and the sum over sites $n < M_N - N$. By Hille-Yosida Theorem

$$\begin{aligned}\partial_t\overline{\mathbb{E}}^{M_N,\widehat{b}}_{\overline{\mu}_N}\left[\sum_{i=0}^{3} h_{M_N,i}(\widetilde{\eta}_t,\eta_t)\right] \;=\;& N^{-1-d}\sum_{n=M_N-N}^{M_N-1} e^{-n/N}\overline{\mathbb{E}}^{M_N,\widehat{b}}_{\overline{\mu}_N}\left[\sum_{i=0}^{3}\overline{\mathfrak{L}}_N H_{i,n}(\widetilde{\eta}_t,\eta_t)\right] \\ &+ N^{-1-d}\sum_{n=1}^{M_N-N-1} e^{-n/N}\overline{\mathbb{E}}^{M_N,\widehat{b}}_{\overline{\mu}_N}\left[\sum_{i=0}^{3}\overline{\mathfrak{L}}_N H_{i,n}(\widetilde{\eta}_t,\eta_t)\right]\,.\end{aligned} \tag{7.21}$$

The first part is bounded by a quantity vanishing when $N\uparrow\infty$. Indeed, since for each $x\in\Lambda_N$, $\overline{\mathfrak{L}}_N|\widetilde{\eta}_{i,t}(x)-\eta_{i,t}(x)|\leq CN^2$ for some positive constant C, we have

$$N^{-1-d}\sum_{n=M_N-N}^{M_N-1} e^{-n/N}\overline{\mathbb{E}}^{M_N,\widehat{b}}_{\overline{\mu}_N}\left[\sum_{i=0}^{3}\overline{\mathfrak{L}}_N H_{i,n}(\widetilde{\eta}_t,\eta_t)\right] \leq K_1 N^{3-d}(M_N)^{d-1}e^{-N^{1/d}}, \tag{7.22}$$

for some positive constant K_1.
We now split the second term of the r.h.s. of (7.21) according to the decomposition of the generator in (7.13). Recalling (7.1)-(7.3), we have

$$\begin{aligned}&\overline{\mathbb{E}}^{M_N,\widehat{b}}_{\overline{\mu}_N}\left[\overline{\mathcal{L}}_N H_{i,n}(\widetilde{\eta}_t,\eta_t)\right] \\ &= \sum_{k=2}^{d}\sum_{\substack{(x,y)\in\Lambda_{N,n}\times\Lambda^c_{N,n}\\ y\in\{x+e_k,x-e_k\}}}\int\Big[|\widetilde{\eta}_i(y)-\eta_i(y)|-|\widetilde{\eta}_i(x)-\eta_i(x)|\Big]\overline{f}^{\,t}_{n+1}((\zeta,\chi),(\xi,\omega))d\overline{\nu}^N_{\widehat{\theta}}((\zeta,\chi),(\xi,\omega)) \\ &= \frac{1}{N}\sum_{\ell=1}^{N}\sum_{k=2}^{d}\sum_{\substack{(x,y)\in\Lambda_{N,n}\times\Lambda^c_{N,n}\\ y\in\{x+e_k,x-e_k\}}}\int\Big[|\widetilde{\eta}_i(y)-\eta_i(y)|-|\widetilde{\eta}_i(x)-\eta_i(x)|\Big]\overline{f}^{\,t}_{n+\ell}((\zeta,\chi),(\xi,\omega))d\overline{\nu}^N_{\widehat{\theta}}((\zeta,\chi),(\xi,\omega))\,,\end{aligned}$$

where we introduced a new sum in the spirit of (5.49). Using the equality, for $j\in\{1,2,3\}$ and $y\in\Lambda_N$,

$$\eta_j(y)(1-\widetilde{\eta}_j(y)) + (1-\eta_j(y))\widetilde{\eta}_j(y) = |\eta_j(y)-\widetilde{\eta}_j(y)|\,, \tag{7.23}$$

as well as the change of variables (5.29) for $\overline{f}^{\,t}_{n+\ell}$, and analogous computations as for Consequences 5.3*(i)*, using (5.37) with $A = N/a$ for some $a>0$ to be chosen later, (5.29) and (5.31) (we do not give details as this is similar to what was done in the proof of Lemma 5.2), we obtain, for $n \le M_N - N - 1$,

$$\overline{\mathbb{E}}^{M_N,\widehat{b}}_{\overline{\mu}_N}\left[\overline{\mathcal{L}}_N \sum_{i=0}^{3} H_{i,n}(\widetilde{\eta}_t,\eta_t)\right] \le \frac{a}{N}\sum_{\ell=1}^{N}\sum_{k=2}^{d}\sum_{\substack{(x,y)\in\Lambda_{N,n}\times\Lambda^c_{N,n}\\ y\in\{x+e_k,x-e_k\}}} (\overline{\mathcal{D}}^0_{n+\ell})^{x,y}(\overline{\mu}_{n+\ell}(t)|\overline{\nu}^N_{\widehat{\theta},n+\ell}) + \frac{1}{a}C_2 n^{d-2}, \tag{7.24}$$

for some positive constant C_2.

Now, gathering (7.22) with (7.24), multiplying by $N^{-d-1}\exp(-n/N)$, summing over $1\le n\le M_N-1$ and using (5.27), we get (as in the transition from (5.26) to (5.17))

$$\overline{\mathbb{E}}^{M_N,\widehat{b}}_{\overline{\mu}_N}\left[N^2\overline{\mathcal{L}}_N \sum_{i=0}^{3} h_{M_N,i}(\widetilde{\eta}_t,\eta_t)\right] \le K_1 N^{3-d}(M_N)^{d-1}e^{-N^{1/d}} + aN^{1-d}\overline{\mathfrak{D}}(\overline{\mu}(t)|\overline{\nu}^N_{\widehat{\theta}}) + \frac{C_2'}{a} \tag{7.25}$$

for some positive constant C_2'. Then, recalling (7.4)–(7.7), we have

$$\begin{aligned}\overline{\mathbb{L}}_N|\eta_i(x)-\widetilde{\eta}_i(x)| &\le -2C_2''|\eta_i(x)-\widetilde{\eta}_i(x)| + 34C_2''\sum_{j=0}^{3}(1-\eta_j(x))\widetilde{\eta}_j(x)\\ &\le -2C_2''|\eta_i(x)-\widetilde{\eta}_i(x)| + 34C_2''\sum_{j=0}^{3}|\eta_j(x)-\widetilde{\eta}_j(x)| \end{aligned}\tag{7.26}$$

with $C_2''=\max\big(r,2d\lambda_1\big)$, where we used (7.23) for the second inequality.
Finally, recalling (7.10)–(7.11), we have

$$\overline{L}^1_{\widehat{b},N}|\widetilde{\eta}_i(x)-\eta_i(x)| = -\Big(\sum_{j=0}^{3} b_j(x/N)\Big)|\widetilde{\eta}_i(x)-\eta_i(x)| \le 0. \tag{7.27}$$

Collecting all the above estimates, we obtain, for some positive constant A_1,

$$\begin{aligned}\partial_t\overline{\mathbb{E}}^{M_N,\widehat{b}}_{\overline{\mu}_N}\left[\sum_{i=0}^{3} h_{M_N,i}(\widetilde{\eta}_t,\eta_t)\right] &\le A_1\overline{\mathbb{E}}^{M_N,\widehat{b}}_{\mu_N}\left[\sum_{i=0}^{3} h_{M_N,i}(\widetilde{\eta}_t,\eta_t)\right]\\ &\quad + K_1 N^{3-d}(M_N)^{d-1}e^{-N^{1/d}} + aN^{1-d}\overline{\mathfrak{D}}(\overline{\mu}(t)|\overline{\nu}^N_{\widehat{\theta}}) + \frac{C_2'}{a}\,.\end{aligned}$$

Integrating in time this inequality and using Lemma 7.1 gives, for some positive constant A_2,

$$\overline{\mathbb{E}}^{M_N,\widehat{b}}_{\overline{\mu}_N}\left[\sum_{i=0}^{3} h_{M_N,i}(\widetilde{\eta}_t,\eta_t)\right] \le A_1\int_0^t \overline{\mathbb{E}}^{M_N,\widehat{b}}_{\overline{\mu}_N}\left[\sum_{i=0}^{3} h_{M_N,i}(\widetilde{\eta}_s,\eta_s)\right]ds + A_2\left[N^{3-d}(M_N)^{d-1}e^{-N^{1/d}} + \frac{a}{N} + \frac{1}{a}\right].$$

Applying now Gronwall lemma, and choosing $a=\sqrt{N}$, we obtain

$$\overline{\mathbb{E}}^{M_N,\widehat{b}}_{\overline{\mu}_N}\left[\sum_{i=0}^{3} h_{M_N,i}(\widetilde{\eta}_t,\eta_t)\right] \le A_2\left[N^{3-d}(M_N)^{d-1}e^{-N^{1/d}} + \frac{2}{\sqrt{N}}\right]e^{Ct}\,t\,.$$

To conclude the proof of the proposition, we just have to recall that by our choice of M_N in (4.1), the sequence M_N is such that $N^{3-d}(M_N)^{d-1}e^{-N^{1/d}}$ decreases to 0 as $N\uparrow\infty$.

8. Uniqueness of weak solutions of equation (3.1)

This section is devoted to the proof of uniqueness of weak solutions of equation (3.1) in infinite volume with reservoirs.

We need to introduce the following notation and tools. Denote by $L^2((-1,1))$ the Hilbert space on the one-dimensional bounded interval $(-1,1)$ equipped with the inner product,

$$\langle\varphi,\psi\rangle_2 = \int_{-1}^{1}\varphi(u_1)\,\overline{\psi(u_1)}\,du_1\ ,$$

where, for $z \in \mathbb{C}$, $\bar{z}$ is the complex conjugate of z and $|z|^2 = z\bar{z}$. The norm in $L^2((-1,1))$ is denoted by $\|\cdot\|_2$.

Let $H^1((-1,1))$ be the Sobolev space of functions φ with generalised derivatives $\partial_{u_1}\varphi$ in $L^2((-1,1))$. Endowed with the scalar product $\langle\cdot,\cdot\rangle_{1,2}$, defined by

$$\langle\varphi,\psi\rangle_{1,2} = \langle\varphi,\psi\rangle_2 + \langle\partial_{u_1}\varphi\,,\,\partial_{u_1}\psi\rangle_2\ ,$$

$H^1((-1,1))$ is a Hilbert space. The corresponding norm is denoted by $\|\cdot\|_{1,2}$. Denote by $H^1_0((-1,1))$ the closure of $\mathcal{C}^\infty_c((-1,1);\mathbb{R})$ in $H_1((-1,1))$.

Consider the following classical boundary-eigenvalue problem for the Laplacian:

$$\begin{cases} -\Delta\varphi = \alpha\varphi\,, \\ \varphi \in H^1_0((-1,1))\,. \end{cases} \tag{8.1}$$

From the Sturm–Liouville theorem (cf. [29]), one can construct for the problem (8.1) a countable system $\{\varphi_n, \alpha_n : n \geq 1\}$ of eigensolutions which contains all possible eigenvalues. The set $\{\varphi_n : n \geq 1\}$ of eigenfunctions forms a complete orthonormal system in the Hilbert space $L^2((-1,1))$. Moreover each φ_n belongs to $H^1_0((-1,1))$ and the set $\{\varphi_n/\alpha_n^{1/2} : n \geq 1\}$ is a complete orthonormal system in the Hilbert space $H^1_0((-1,1))$. Hence, a function ψ belongs to $L^2((-1,1))$ if and only if

$$\psi \;=\; \lim_{n\to\infty}\sum_{k=1}^{n}\langle\psi,\varphi_k\rangle_2\,\varphi_k \quad \text{in} \quad L^2((-1,1)).$$

In this case, for all $\psi_1,\psi_2 \in L^2((-1,1))$

$$\langle\psi_1,\psi_2\rangle_2 \;=\; \sum_{k=1}^{\infty}\langle\psi_1,\varphi_k\rangle_2\,\overline{\langle\psi_2,\varphi_k\rangle_2}\,.$$

Furthermore, a function ψ belongs to $H^1_0((-1,1))$ if and only if

$$\psi \;=\; \lim_{n\to\infty}\sum_{k=1}^{n}\langle\psi,\varphi_k\rangle_2\,\varphi_k \quad \text{in} \quad H^1_0((-1,1)), \quad \text{and}$$

$$\langle\psi_1,\psi_2\rangle_{1,2} \;=\; \sum_{k=1}^{\infty}\alpha_k\langle\psi_1,\varphi_k\rangle_2\,\overline{\langle\psi_2,\varphi_k\rangle_2} \quad \text{for all} \quad \psi_1,\psi_2 \in H^1_0((-1,1)). \tag{8.2}$$

One can check that since we work with $(-1,1)$, $\alpha_n = n^2\pi^2$ and $\varphi_n(u_1) = \sin(n\pi u_1)$, $n \in \mathbb{N}$.

Proposition 8.1. *For any $T > 0$, the system of equations* (3.1) *has a unique weak solution in the class* $\big(L^\infty\big([0,T]\times\Lambda\big)\big)^3$.

Proof of Proposition 8.1. We follow the arguments in [25] adapted to the our case.

Fix $T > 0$, define the heat kernel on the time interval $(0,T]$ by the following expression

$$p_1(t,u_1,v_1) \;=\; \sum_{n\geq 1} e^{-\alpha_n t}\varphi_n(u_1)\overline{\varphi_n(v_1)}\,,\ t \in [0,T]\,,\ u_1,v_1 \in [-1,1]\,.$$

Let $g \in \mathcal{C}^0_c((-1,1);\mathbb{R})$ and denote by $\delta_\cdot$ the Dirac function. The heat kernel p_1 is such that $p_1(0,u_1,v_1) = \delta_{u_1-v_1}$, $p_1 \in \mathcal{C}^\infty((0,T]\times(-1,1)\times(-1,1);\mathbb{R})$ and the function defined via the convolution operator:

$$\varphi_1(t,u_1) := (p_1 \star g)(t,u_1) = \int_{-1}^{1} p_1(t,u_1,v_1)g(v_1)dv_1$$

solves the following boundary value problem

$$\begin{cases} \partial_t\varphi = \partial^2_{u_1}\varphi\,, \\ \varphi(0,\cdot) = \;g(\cdot)\,, \\ \varphi(t,\cdot) \in H^1_0((-1,1)) \;\text{ for } 0 < t \leq T\ . \end{cases} \tag{8.3}$$

Let $\check{p}$ be the heat kernel for $(t,\check{u},\check{v})\in(0,T)\times\mathbb{R}^{d-1}\times\mathbb{R}^{d-1}$

$$\check{p}(t,\check{u},\check{v})=\big(4\pi t\big)^{-(d-1)/2}\exp\bigg\{-\frac{1}{4t}\sum_{k=2}^{d}(u_k-v_k)^2\bigg\}.$$

For each function $\check{f}\in\mathcal{C}_c(\mathbb{R}^{d-1};\mathbb{R})$, it is known that

$$\check{h}_t^{\check{f}}(t,\check{u}):=(\check{p}\star\check{f})(t,\check{u})=\int_{\mathbb{R}^{d-1}}\check{p}(t,\check{u},\check{v})\check{f}(\check{v})d\check{v}\,.$$

solves the equation $\partial_t\check{\rho}=\Delta\check{\rho}$, $\check{\rho}_0=f$, on $(0,t]\times\mathbb{R}^{d-1}$. Moreover $\check{h}_t^f\in\mathcal{C}^\infty((0,T]\times\mathbb{R}^{d-1};\mathbb{R})$.

For $t\in(0,T]$, $\widehat{f}=(f_1,f_2,f_3)\in\mathcal{C}_c(\overline{\Lambda};\mathbb{R}^3)$ and $\varepsilon>0$ small enough, let $\mathcal{H}_{t,\varepsilon}^{\widehat{f}}:[0,t]\times\Lambda\longrightarrow\mathbb{R}$ be defined by

$$\mathcal{H}_{t,\varepsilon}^{\widehat{f}}(s,u):=\sum_{i=1}^{3}\mathcal{H}_{t,\varepsilon}^{f_i}(s,u):=\sum_{i=1}^{3}\big(p*f_i\big)(t+\epsilon-s,u),$$

where p is the heat kernel on $(0,T]\times\Lambda\times\Lambda$ given by

$$p(t,u,v)=p_1(t,u_1,v_1)\check{p}(t,\check{u},\check{v}).$$

Then $\mathcal{H}_{t,\varepsilon}^{f}$ solves the equation $\partial_t\rho=\Delta\rho$ on $(0,t]\times\mathbb{R}^d$, $\rho_0=f$.

Consider $\widehat{\rho}^{(1)}=(\rho_1^{(1)},\rho_2^{(1)},\rho_3^{(1)})$ and $\widehat{\rho}^{(2)}=(\rho_1^{(2)},\rho_2^{(2)},\rho_3^{(2)})$ two weak solutions of (3.1) associated to an initial profile $\widehat{\gamma}=(\gamma_1,\gamma_2,\gamma_3):\Lambda\to[0,1]^3$. Set $\overline{m}_i=\rho_i^{(1)}-\rho_i^{(2)}$, $1\le i\le 3$. We shall prove below that for any function $m(\cdot,\cdot)\in L^\infty([0,T]\times\Lambda)$ and each $i\le i\le d$,

$$\int_0^t ds\left|\int_\Lambda m(s,u)\mathcal{H}_{t,\varepsilon}^{f_i}(s,v)dv\right|ds\;\le\;C_1t\,\|m\|_\infty\|f_i\|_1, \tag{8.4}$$

for some positive constant C_1, where for a trajectory $m:[0,t]\times\Lambda\to\mathbb{R}$, $\|m\|_\infty=\|m\|_{L^\infty([0,t]\times\Lambda)}$ stands for the infinite norm in $L^\infty([0,t]\times\Lambda)$.

On the other hand, from the fact that $\rho_i^{(1)},\rho_i^{(2)}$, $1\le i\le 3$ are in $L^\infty([0,T]\times\Lambda)$, it follows that there exists a positive constant C_2 such that, for almost every $(s,u)\in[0,t]\times\Lambda$, for every $1\le i\le 3$,

$$\big|F_i(\rho_i^{(1)}(s,u))-F_i(\rho_i^{(2)}(s,u))\big|\le C_2\sum_{i=1}^{3}\|\rho_i^{(1)}-\rho_i^{(2)}\|_\infty\,.$$

Since $\widehat{\rho}^{(1)}$ and $\widehat{\rho}^{(2)}$ are two weak solutions of (3.1), we obtain by (8.4) that for all $0\le\tau\le t$, $1\le i,k\le 3$

$$\begin{aligned}\Big|\big\langle\overline{m}_i(\tau,.),\mathcal{H}_{\tau,\varepsilon}^{f_k}(\tau,.)\big\rangle\Big|&=\sum_{i=1}^{3}\Big|\int_0^\tau\big\langle F_i(\widehat{\rho}^{(1)})-F_i(\widehat{\rho}^{(2)}),\mathcal{H}_{\tau,\varepsilon}^{f_k}(\tau,.)\big\rangle\Big|\\&\le C_1't\Big(\sum_{i=1}^{3}\|\rho_i^{(1)}-\rho_i^{(2)}\|_\infty\Big)\|f_k\|_1\,,\end{aligned}$$

for $C_1'=C_1C_2$.

By observing that $p(\varepsilon,\cdot,\cdot)$ is an approximation of the identity in ε, we obtain by letting $\varepsilon\downarrow 0$,

$$\Big|\big\langle\overline{m}_i(\tau,.),f_k\big\rangle\Big|\;\le\;C_1't\Big(\sum_{i=1}^{3}\|\rho_i^{(1)}-\rho_i^{(2)}\|_\infty\Big)\|f_k\|_1\,. \tag{8.5}$$

We claim that $\overline{m}_i\in L^\infty([0,t]\times\Lambda)$ and

$$\|\overline{m}_i\|_\infty\;\le\;C_1'\,t\Big(\sum_{i=1}^{3}\|\rho_i^{(1)}-\rho_i^{(2)}\|_\infty\Big)\,. \tag{8.6}$$

Indeed (cf. [26], [25]), denote by $R(t) = \sum_{i=1}^{3} \|\rho_i^{(1)} - \rho_i^{(2)}\|_\infty$, by (8.5), for any open set U of Λ with finite Lebesgue measure $\lambda(U)$, we have for all $0 \le \tau \le t$,

$$\int_U \overline{m}_i(\tau, u) du \le C_1'\, t\, R(t) \lambda(U). \tag{8.7}$$

Fix $0 < \delta < 1$. For any open set U of Λ with finite Lebesgue measure and for $0 \le \tau \le t$ let

$$B_{\delta,\tau}^U = \Big\{ u \in U \ : \ \overline{m}_i(\tau, u) > C_1'\, t\, R(t)(1+\delta) \Big\}.$$

Suppose that $\lambda(B_{\delta,\tau}^U) > 0$, there exists an open set V, such that, $B_{\delta,\tau}^U \subset V$ and $\lambda\big(V \setminus B_{\delta,\tau}^U\big) \le \lambda(V)\frac{\delta}{2}$ and we have

$$\begin{aligned}
\lambda(V)\big(C_1'\, t\, R(t)\big) &< \lambda(V)\big(C_1'\, t\, R(t)\big)(1+\delta)(1-\delta/2) = \big(C_1'\, t\, R(t)\big)(1+\delta)\big(\lambda(V) - \lambda(V)\delta/2\big) \\
&\le \big(C_1' t\, R(t)\big)(1+\delta)\big(\lambda(V) - \lambda\big(V \setminus B_{\delta,\tau}^U\big)\big) = \big(C_1' \sqrt{t} R(t)\big)(1+\delta)\lambda\big(B_{\delta,\tau}^U\big) \\
&< \int_{B_{\delta,\tau}^U} \overline{m}_i(\tau, x) dx \,.
\end{aligned}$$

Thus, from (8.7) and since $B_{\delta,\tau}^U \subset V$, we get

$$\lambda(V)\big(C_1'\, t\, R(t)\big) < \int_V \overline{m}_i(\tau, x) dx \quad \le \big(C_1'\, t\, R(t)\big)\lambda(V)\,,$$

which leads to a contradiction.

By the arbitrariness of $0 < \delta < 1$ we obtain that if U is any open set of Λ with $\lambda(U) < \infty$,

$$\lambda\Big(\Big\{ u \in U \ : \ \overline{m}_i(\tau, u) > C_1'\, t\, R(t) \Big\}\Big) = 0.$$

This implies

$$\overline{m}_i(\tau, x) \le C_1'\, t\, R(t) \qquad \text{a.e. in } \Lambda$$

and concludes the proof of (8.6) by the arbitrariness of $\tau \in [0, t]$.

We now turn to the proof of the uniqueness, from (8.6),

$$\|\overline{m}_i\|_\infty \ \le \ C_1'\, t \Big(\sum_{j=1}^{3} \|\overline{m_j}\|_\infty \Big)\,,$$

and then

$$R(t) \le 3C_1'\, t\, R(t)\,.$$

Choosing $t = t_0$ such that $3C_1'\, t_0 \ < 1$, this gives uniqueness in $[0, t_0] \times \Lambda$. To conclude the proof we just have to repeat the same arguments in $[t_0, 2t_0]$, and in each interval $[kt_0, (k+1)t_0]$, $k \in \mathbb{N}$, $k > 1$. It remains to prove inequality (8.4). From Fubini's Theorem, we have

$$\begin{aligned}
&\int_0^t \left| \int_\Lambda m(s,u) \mathcal{H}_{t,\varepsilon}^{f_i}(s,u) du \right| ds \\
&\le \int_0^t ds \int_{\mathbb{R}^{d-1}} d\check{v} \int_{\mathbb{R}^{d-1}} d\check{u} \Bigg| \sum_{n \ge 1} e^{-n^2\pi^2(t+\varepsilon-s)} \int_{-1}^{1} dv_1 \Big\{ \sin(n\pi v_1) f_i(v_1, \check{v}) \Big\} \\
&\qquad\qquad \times \int_{-1}^{1} du_1 \Big\{ \sin(n\pi u_1) \check{p}(t+\varepsilon-s, \check{u}, \check{v}) m(s, u_1, \check{u}) \Big\} \Bigg| \\
&\le \int_0^t ds \int_{\mathbb{R}^{d-1}} d\check{v} \int_{\mathbb{R}^{d-1}} d\check{u}\; \check{p}(t+\varepsilon-s, \check{u}, \check{v}) \left| \sum_{n \ge 1} \Big\langle \varphi_n, m(s, (\cdot, \check{u})) \Big\rangle \times \Big\langle \varphi_n, f_i(\cdot, \check{v}) \Big\rangle \right| \\
&\le \int_0^t ds \int_\Lambda du \int_\Lambda dv \Big\{ |m(s,u)|\, |f_i(v)|\, \check{p}(t+\varepsilon-s, \check{u}, \check{v}) \Big\} \\
&\le \ 4\, t\, \|m\|_\infty \|f_i\|_1\,,
\end{aligned}$$

where we used the fact that $\check{p}(s,\cdot,\cdot)$ is a probability kernel in $\mathbb{R}^{d-1}$ for all $s>0$. $\square$

9. Empirical currents

In this section, we derive the law of large numbers for the empirical currents stated in Proposition 3.4. Recall that for $x\in\Lambda_N$, $1\le i\le 3$, and $j=1,\dots,d$, $W_t^{x,x+e_j}(\eta_i)$ stands for the conservative current of particles of type i across the edge $\{x,x+e_j\}$, and $Q_t^x(\eta_i)$ for the total number of particles of type i created minus the total number of particles of type i annihilated at site x before time t. We have the following families of jump martingales:

$$\widetilde{W}_t^{x,x+e_j}(\eta_i)=W_t^{x,x+e_j}(\eta_i)-N^2\int_0^t\Big(\eta_{i,s}(x)(1-\eta_{i,s}(x+e_j))-(1-\eta_{i,s}(x))\eta_{i,s}(x+e_j)\Big)ds \quad (9.1)$$

with quadratic variation (because $J_t^{x,x+e_j}(\eta_i)$ and $J_t^{x+e_j,x}(\eta_i)$ have no common jump)

$$\langle\widetilde{W}^{x,x+e_j}(\eta_i)\rangle_t \;=\; N^2\int_0^t\Big(\eta_{i,s}(x)(1-\eta_{i,s}(x+e_j))+(1-\eta_{i,s}(x))\eta_{i,s}(x+e_j)\Big)ds \quad (9.2)$$

and, for $\widehat{f}=(f_1,f_2,f_3):\widehat{\Sigma}_N\to\mathbb{R}^3$ defined in (4.6),

$$\widetilde{Q}_t^x(\eta_i)=Q_t^x(\eta_i)-\int_0^t\tau_x f_i(\xi_s,\omega_s)ds \quad (9.3)$$

with quadratic variations

$$\begin{cases}\langle\widetilde{Q}^x(\eta_1)\rangle_t &= \displaystyle\int_0^t\tau_x\Big(\beta_N(0,\xi_s,\omega_s)\eta_{0,s}(0)+\eta_{3,s}(0)+(r+1)\eta_{1,s}(0)\Big)ds,\\ \langle\widetilde{Q}^x(\eta_2)\rangle_t &= \displaystyle\int_0^t\tau_x\Big(r\eta_{0,s}(0)+\eta_{3,s}(0)+\beta_N(0,\xi_s,\omega_s)\eta_{2,s}(0)+\eta_{2,s}(0)\Big)ds,\\ \langle\widetilde{Q}^x(\eta_3)\rangle_t &= \displaystyle\int_0^t\tau_x\Big(\beta_N(0,\xi_s,\omega_s)\eta_{2,s}(0)+r\eta_{1,s}(0)+2\eta_{3,s}(0)\Big)ds.\end{cases} \quad (9.4)$$

Proof of Proposition 3.4. Given a smooth continuous vector field $\mathbf{G}=(G_1,...,G_d)\in\mathcal{C}_c^\infty(\Lambda,\mathbb{R}^d)$, after definition (3.5), sum the martingale (9.1) over $\{x,x+e_j\in\Lambda_N\}$ to get the martingale $\widetilde{\mathbf{M}}_t^G$, given by

$$\begin{aligned}\widetilde{\mathbf{M}}_t^G(\eta_i) \;&=\; \sum_{j=1}^d\bigg(\langle W_{j,t}^N(\eta_i),G_j\rangle-\frac{N^2}{N^{d+1}}\sum_{x,x+e_j\in\Lambda_N}\int_0^t G_j(x/N)\Big(\eta_{i,s}(x)-\eta_{i,s}(x+e_j)\Big)ds\bigg)\\ &=\; \langle\mathbb{W}_t^N(\eta_i),\mathbf{G}\rangle-\frac{1}{N^d}\sum_{j=1}^d\sum_{x\in\Lambda_N}\int_0^t\partial_{x_j}G_j(x/N)\eta_{i,s}(x)ds+O(N^{-1})\\ &=\; \langle\mathbb{W}_t^N(\eta_i),\mathbf{G}\rangle-\sum_{j=1}^d\langle\pi_s^{N,i},\partial_{x_j}G_j\rangle+O(N^{-1}),\end{aligned}$$

where we did a Taylor expansion. Relying on (9.2), the expectation of $\langle\widetilde{\mathbf{M}}^G(\eta_i)\rangle_t$ vanishes when $N\to\infty$, so that by Doob's martingale inequality, for any $\delta>0$,

$$\lim_{N\to\infty}\mathbb{P}_{\mu_N}^{N,\widehat{b}}\Big[\sup_{0\le t\le T}\big|\sum_{i=1}^3\widetilde{\mathbf{M}}_t^G(\eta_i)\big|>\delta\Big]\;=\;0.$$

Using that the empirical density $\widehat{\pi}^N$ converges towards the solution of (3.1), this concludes the law of large numbers (3.6) for the current $\mathbb{W}_T^N$.

Fix a smooth vector field $\widehat{H}=(H_1,H_2,H_3)\in\mathcal{C}_c^\infty(\Lambda,\mathbb{R}^3)$. Sum (9.3) over $x\in\Lambda_N$ to get the martingale

$$\widetilde{\mathbf{N}}_t^H(\eta_i)=\langle Q_t^N(\eta_i),H_i\rangle-\frac{1}{N^d}\sum_{x\in\Lambda_N}\int_0^t H_i(x/N)\tau_x f_i(\xi_s,\omega_s)ds$$

Relying on (9.4), the expectation of its quadratic variation vanishes as $N \to \infty$ as well. Use the replacement lemma 4.2 to express $\widetilde{\mathbf{N}}_t^H(\eta_i)$ with functionals of the density fields and conclude (3.7) by Doob's martingale inequality: For any $\delta > 0$,

$$\lim_{N\to\infty} \mathbb{P}^{N,\widehat{b}}_{\mu_N}\Big[\sup_{0\le t\le T}\Big|\sum_{i=1}^{3}\widetilde{\mathbf{N}}_t^H(\eta_i)\Big| > \delta\Big] \;=\; 0.$$

□

Acknowledgements

We thank the anonymous referees for their careful reading of the manuscript and their comments.

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