Bicovariograms and Euler characteristic of random fields excursions

Raphaël Lachièze-Rey

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Abstract

Let f be a $\mathcal{C}^1$ bivariate function with Lipschitz derivatives, and $F = \{x \in \mathbb{R}^2 : f(x) \geqslant \lambda\}$ an upper level set of f, with $\lambda \in \mathbb{R}$. We present a new identity giving the Euler characteristic of F in terms of its three-points indicator functions. A bound on the number of connected components of F in terms of the values of f and its gradient, valid in higher dimensions, is also derived. In dimension 2, if f is a random field, this bound allows to pass the former identity to expectations if f's partial derivatives have Lipschitz constants with finite moments of sufficiently high order, without requiring bounded conditional densities. This approach provides an expression of the mean Euler characteristic in terms of the field's third order marginal. Sufficient conditions and explicit formulas are given for Gaussian fields, relaxing the usual $\mathcal{C}^2$ Morse hypothesis.

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1 Introduction

The geometry of random fields excursion sets has been a subject of intense research over the last two decades. Many authors are concerned with the computation of the mean [3, 4, 5, 8] or variance [12, 20] of the Euler characteristic, denoted by χ here.

As an integer-valued quantity, the Euler characteristic can be easily measured and used in many estimation and modelisation procedures. It is an important indicator of the porosity of a random media [7, 15, 23], it is used in brain imagery [17, 25], astronomy, [20, 21, 22], and many other disciplines. See also [2] for a general review of applied algebraic topology.

Most of the available works on random fields use the results gathered in the celebrated monograph [6], or similar variants. In this case, theoretical computations of the Euler characteristic emanate from Morse theory, where the focus is on the local extrema of the underlying field instead of the set itself. For the theory to be applicable, the functions must be $\mathcal{C}^2$ and satisfy the Morse hypotheses, which conveys some restrictions on the set itself.

The expected Euler characteristic also turned out to be a widely used approximation of the distribution function of the maximum of a Morse random field, and attracted much interest in this direction, see [3, 8, 9, 25]. Indeed, for large $r > 0$, a well-behaved field rarely exceeds r, and if it does, it is likely to have a single highest peak, which yields that the level set of f at

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level r, when not empty, is most often simply connected, and has Euler characteristic 1. In this fashion, $\mathbf{E}\chi(\{f \geqslant r\}) \approx \mathbf{P}(\sup f \geqslant r)$, which provides an additional motivation to compute the mean Euler characteristic of random fields.

Even though [4] provides an asymptotic expression for some classes of infinitely divisible fields, most of the tractable formulae concern Gaussian fields. One of the ambitions of this paper is to provide a formula that is tractable in a rather general setting, and also works in the Gaussian realm. There seems to be no particular obstacle to extend these ideas to higher dimensions in a further work.

Approach and main result

Given a set $A \subset \mathbb{R}^2$, let $\Gamma(A)$ be the class of its bounded connected components (or arc-wise connected components, but these notions coincide in every Euclidean space). We say that a set A is *admissible* if $\Gamma(A)$ and $\Gamma(A^c)$ are finite, and in this case its Euler characteristic is defined by

$$\chi(A) = \#\Gamma(A) - \#\Gamma(A^c),$$

where # denotes the cardinality of a set. The theoretical results of Adler and Taylor [6] regarding the Euler characteristic of random excursions require second order differentiability of the underlying field f, but the expression of the mean Euler characteristic only involves the first-order derivatives, suggesting that second order derivatives do not matter in the computation of the Euler characteristic. In the words of Adler and Taylor (Section 11.7), regarding their Formula (11.7.6), it is a *rather surprising fact that the [mean Euler characteristic of a Gaussian field] depends on the covariance of f only through some of its derivatives at zero*, the latter referring to first-order partial derivatives. We present here a new method for which the second order differentiability is not needed. The results are valid for $\mathcal{C}^1$ fields with locally Lipschitz derivatives, also called $\mathcal{C}^{1,1}$ fields, relaxing slightly the classical $\mathcal{C}^2$ Morse framework.

Our results exploit the findings of [19] connecting smooth sets Euler characteristic and variographic tools. For some $\lambda \in \mathbb{R}$ and a bi-variate function f, define for $x \in \mathbb{R}^2$

$$\delta^\eta(x, f, \lambda) = \mathbf{1}_{\{f(x) \geqslant \lambda, f(x+\eta \mathbf{u}_1) < \lambda, f(x+\eta \mathbf{u}_2) < \lambda\}}, \eta \in \mathbb{R},$$

where $(\mathbf{u}_1, \mathbf{u}_2)$ denotes the canonical basis of $\mathbb{R}^2$, assuming f is defined in these points. When f is a random field, let $\delta^\eta(x, f, \lambda)$ also denote the event $\delta^\eta(x, f, \lambda) = 1$. Let us write a corollary of our main result here (a more general statement can be found in Section 3). Denote by ℓ, or Vol, the Lebesgue measure on $\mathbb{R}^d$.

Corollary 1. *Let $W = [0, a] \times [0, b]$ for some $a, b > 0$, f be a $\mathcal{C}^1$ real random field on $\mathbb{R}^2$ with locally Lipschitz partial derivatives $\partial_1 f, \partial_2 f$, $\lambda \in \mathbb{R}$, and let $F = \{x \in W : f(x) \geqslant \lambda\}$. Assume furthermore that the following conditions are satisfied:*

(i) For some $\kappa > 0$, for $x \in \mathbb{R}^2$, the density of the random vector $(f(x), \partial_1 f(x), \partial_2 f(x))$ is bounded by κ on $\mathbb{R}^3$.

(ii) There is $p > 6$ such that

$$\mathbf{E}[\mathrm{Lip}(f, W)^p] < \infty, \ \mathbf{E}[\mathrm{Lip}(\partial_i f, W)^p] < \infty, i = 1, 2,$$

where $\mathrm{Lip}(g, W)$ *denotes the Lipschitz constant of a function g on W.*

Then $\mathbf{E}\#\Gamma(F)<\infty, \mathbf{E}\#\Gamma(F^c)<\infty,$ *and*

$$\mathbf{E}\chi(F) = \lim_{\varepsilon\to 0}\sum_{x\in\varepsilon\mathbb{Z}^2}[\mathbf{P}(\delta^{\varepsilon}(x,f\mathbf{1}_W,\lambda))-\mathbf{P}(\delta^{-\varepsilon}(x,-f\mathbf{1}_W,-\lambda))] \tag{1}$$

$$= \lim_{\varepsilon\to 0}\varepsilon^{-2}\int_{\mathbb{R}^2}\Big[\mathbf{P}(\delta^{\varepsilon}(x,f\mathbf{1}_W,\lambda))-\mathbf{P}(\delta^{-\varepsilon}(x,-f\mathbf{1}_W,-\lambda))\Big]dx. \tag{2}$$

If f *is furthermore stationary, we have*

$$\mathbf{E}\chi(F) = \overline{\chi}(f,\lambda)\ell(W)+\overline{\mathrm{Per}}(f,\lambda)\mathrm{Per}(W)+\overline{\mathrm{Vol}}(f,\lambda)\chi(W)$$

where the volumic Euler characteristic, perimeter and volume $\overline{\chi},\overline{Per},\overline{Vol}$ *are defined in Theorem 9, they only depend on the behaviour of* f *around the origin.*

The right hand side of (2) is related to the *bicovariogram* of the set F, defined by

$$\delta_0^{x,y}(F) = \ell(F\cap(F+x)^c\cap(F+y)^c), x,y\in\mathbb{R}^2, \tag{3}$$

in that (2) can be reformulated as

$$\mathbf{E}\chi(F) = \lim_{\varepsilon\to 0}\varepsilon^{-2}(\mathbf{E}\delta_0^{-\varepsilon\mathbf{u}_1,-\varepsilon\mathbf{u}_2}(F)-\mathbf{E}\delta_0^{\varepsilon\mathbf{u}_1,\varepsilon\mathbf{u}_2}(F^c)).$$

This approach seems to be new in the literature. It highlights the fact that under suitable conditions, the mean Euler characteristic of random level sets is linear in the field's third order marginal. We also give in Theorem 3 a bound on the number of connected components of the excursion of f, valid in any dimension, which is finer than just bounding by the number of critical points; we could not locate an equivalent result in the literature. This topological estimate is interesting in its own and also applies uniformly to the number of components of 2D-pixel approximations of the excursions of f. We therefore use it here as a majoring bound in the application of Lebesgue's theorem to obtain (1)-(2).

It is likely that results presented in Theorem 7 hold in higher dimensions. See for instance [24], that paves the way to an extension of the results of [19] to random fields on spaces with arbitrary dimension. Also, the uniform bounded density hypothesis is relaxed and allows for the density of the $(d+1)$-tuple $(f(x),\partial_1 f(x),\dots,\partial_d f(x))$ to be arbitrarily large in the neighbourhood of $(\lambda,0,\dots,0)$. Theorem 7 features a result where f is defined on the whole plane and the level sets are observed through a bounded window W, as is typically the case for level sets of non-trivial stationary fields, but the intersection with ∂W requires additional notation and care. See Theorem 9 for a result tailored to deal with excursions of stationary fields.

Theorem 11 features the case where f is a Gaussian field assuming only $\mathcal{C}^{1,1}$ regularity (classical literature about random excursions require $\mathcal{C}^2$ Morse fields in dimension $d\geqslant 2$, or $\mathcal{C}^1$ fields in dimension 1). Under the additional hypothesis that f is stationary and isotropic, we retrieve in Theorem 13 the standard results of [6].

Let us explore other consequences of our results. Let $h:\mathbb{R}\to\mathbb{R}$ be a $\mathcal{C}^1$ test function with compact support, and F as in Theorem 1. Using the results of our paper, it is shown in the follow-up article [18] that for any deterministic $\mathcal{C}^2$ Morse function f on $\mathbb{R}^2$,

$$\int_{\mathbb{R}}\chi(F)h(\lambda)d\lambda = -\sum_{i=1}^{2}\int_W \mathbf{1}_{\{\nabla f(x)\in Q_i\}}[h'(f(x))\partial_i f(x)^2+h(f(x))\partial_{ii}f(x)]dx+\text{boundary terms} \tag{4}$$

where

$$Q_1 = \{(x,y) \in \mathbb{R}^2 : y < x < 0\},\ Q_2 = \{(x,y) \in \mathbb{R}^2 : x < y < 0\},$$

yielding applications for instance to shot-noise processes. In the context of random functions, no marginal density hypothesis is required to take the expectation, at the contrary of analogous results, including those from the current paper. Biermé & Desolneux [10, Section 4.1] later gave another interpretation of (4), showing that if it is extended to a random isotropic stationary field, it can be rewritten as a simpler expression, after appropriate integration by parts, namely

$$\mathbf{E}\int_U \chi(\{f \geqslant \lambda\}; U)h(\lambda)d\lambda = -\ell(U)\mathbf{E}\partial_{ii}f(0)h(f(0)),$$

where U is an appropriate open set, and $\chi(\{f \geqslant \lambda; U\})$ is the total curvature of the level set $\{f \geqslant \lambda\}$ within U, generalising the Euler characteristic. They obtained this result by totally different means, via an approach involving Gauss-Bonnet theorem, and giving simpler formulations and proofs, without any requirement on f apart from being $\mathcal{C}^2$.

2 Topological approximation

Let f be a function of class $\mathcal{C}^1$ over some domain $W \subset \mathbb{R}^d$, and $\lambda \in \mathbb{R}$. Define

$$F := F_\lambda(f) = \{x \in W : f(x) \geqslant \lambda\}, \qquad F_{\lambda^+}(f) = \{x \in W : f(x) > \lambda\}.$$

Remark that $F_{\lambda^+}(f) = (F_{-\lambda}(-f))^c$. If we assume that ∇f does not vanish on $\partial F_\lambda(f)$, then $\partial F_\lambda(f) = \partial F_{\lambda^+}(f) = f^{-1}(\{\lambda\})$, and this set is furthermore Lebesgue-negligible, as a $(d-1)$-dimensional manifold.

According to [13, 4.20], $\partial F_\lambda(f)$ is regular in the sense that its boundary is $\mathcal{C}^1$ with Lipschitz normal, if ∇f is locally Lipschitz and does not vanish on $\partial F_\lambda(f)$. This condition is necessary to prevent F from having locally infinitely many connected components, which would make Euler characteristic not properly defined in dimension 2, see [19, Remark 2.11]. We call $\mathcal{C}^{1,1}$ function a differentiable function whose gradient is a locally Lipschitz mapping. Those functions have been mainly used in optimisation problems, and as solutions of some PDEs, see for instance [16]. They can also be characterised as the functions which are locally *semiconvex* and *semiconcave*, see [11].

The results of [19] also yield that the Lipschitzness of ∇f is sufficient for the digital approximation of $\chi(\{f \geqslant \lambda\})$ to be valid. It seems therefore that the $\mathcal{C}^{1,1}$ assumption is the minimal one ensuring the Euler characteristic to be computable in this fashion.

Observation window

An aim of the present paper is to advocate the power of variographic tools for computing the Euler characteristic of random fields excursions. Since many applications are concerned with stationary random fields on the whole plane, we have to study the intersection of excursions with bounded windows, and assess the quality of the approximation.

To this end, call *rectangle* of $\mathbb{R}^d$ any set $W = I_1 \times \cdots \times I_d$ where the I_k are possibly infinite closed intervals of $\mathbb{R}$ with non-empty interiors. Denote by $\partial_k W$ its k-skeleton, as defined for instance in [6, Section 6.2], with $\partial_d W = W$. Let corners$(W) = \partial_0 W$, which number is between 0 and 2^d. Then call *polyrectangle* a finite union $W = \cup_i W_i$ where each W_i is a rectangle, and for $i \neq j$, corners$(W_i) \cap$ corners$(W_j) = \emptyset$. Call $\mathscr{W}_d$ the class of polyrectangles. For $W \in \mathscr{W}_d$, define the k-dimensional skeleton by $\partial_k W = (\cup_i(\partial_k W_i \cap \partial W)) \setminus (\cup_{j<k}\partial_j W), 1 \leqslant k < d, \partial_d W = W$. Note that the closure of W is the disjoint union of the $\partial_k W, 0 \leqslant k \leqslant d$.

For $0 \leqslant k \leqslant d$, denote by $\mathcal{I}_k$ the class of subsets of $\{1, \dots, d\}$ with k elements. For $W \in \mathscr{W}_d, 0 \leqslant k \leqslant d, x \in \partial_k W$, let $I_x(W) \in \mathcal{I}_k$ be the set of indexes such that the tangent space to ∂W in x is spanned by the $\mathbf{u}_i, i \in I_x(W)$. For $d = 2$, and $x \in \partial_1 W$, denote by $\mathbf{n}_W(x)$ the outwards normal unit vector of W in x. We also call *edge of* W a segment $[x, y] \subset \partial_1 W$ that is not strictly contained in another such segment of $\partial_1 W$ (in this case, $x, y \in \text{corners}(W)$).

Definition 2. Let $W \in \mathscr{W}_d$, and $f : W \to \mathbb{R}$ be of class $\mathcal{C}^{1,1}$. Say that the excursion of f at some level $\lambda \in \mathbb{R}$ is *regular within* W if for $0 \leqslant k \leqslant d$, $\{x \in \partial_k W : f(x) = \lambda, \partial_i f(x) = 0, i \in I_x(W)\} = \emptyset$.

For such a function f in dimension 2, it is shown in [19] that the Euler characteristic of its excursion set $F = F_\lambda(f) \cap W$ can be expressed by means of its bicovariograms, defined in (3). For $\varepsilon > 0$ sufficiently small

$$\chi(F) = \varepsilon^{-2}[\delta_0^{-\varepsilon\mathbf{u}_1, -\varepsilon\mathbf{u}_2}(F) - \delta_0^{\varepsilon\mathbf{u}_1, \varepsilon\mathbf{u}_2}(F^c)]. \tag{5}$$

The proof is based on the Gauss approximation of F:

$$F^\varepsilon = \bigcup_{x \in \varepsilon\mathbb{Z}^2 \cap F} \left(x + \varepsilon[-1/2, 1/2)^2\right).$$

According to [19, Theorem 2.7], for ε sufficiently small,

$$\begin{aligned}
\chi(F) &= \chi(F^\varepsilon) \\
&= \sum_{x \in \varepsilon\mathbb{Z}^2} (\delta^\varepsilon(x, f\mathbf{1}_W, \lambda) - \delta^{-\varepsilon}(x, -f\mathbf{1}_W, -\lambda)) \\
&= \varepsilon^{-2} \int_{\mathbb{R}^2} (\delta^\varepsilon(x, f\mathbf{1}_W, \lambda) - \delta^{-\varepsilon}(x, -f\mathbf{1}_W, -\lambda)) dx.
\end{aligned}$$

If f is a random field, the difficulty to pass the result to expectations is to majorize the right hand side uniformly in ε by an integrable quantity, and this goes through bounding the number of connected components of F and its approximation F^ε. This is the object of the next section.

2.1 Topological estimates

The next result, valid in dimension $d \geqslant 1$, does not concern directly the Euler characteristic. Its purpose is to bound the number of connected components of $F_\lambda(f) \cap W$ by an expression depending on f and its partial derivatives. It turns out that a similar bound holds for the excursion approximation $(F_\lambda(f) \cap W)^\varepsilon$ in dimension 2, uniformly in ε, enabling the application of Lebesgue's theorem to the point-wise convergence (5).

Traditionally, see for instance [12, Prop. 1.3], the number of connected components of the excursion set, or its Euler characteristic, is bounded by using the number of critical points, or by the number of points on the level set where f's gradient points towards a predetermined direction. Here, we use another method based on the idea that in a small connected component, a critical point is necessarily close to the boundary, where $f - \lambda$ vanishes. It yields the expression (6) as a bound on the number of connected components. It also allows in Section 3, devoted to random fields, to relax the usual uniform density assumption on the marginals of the $(d+1)$-tuple $(f, \partial_i f, i = 1, \dots, d)$, leaving the possibility that the density is unbounded around $(\lambda, 0, \dots, 0)$.

Denote by $\text{Lip}(g; A) \in \mathbb{R}_+ \cup \{\infty\}$, or just $\text{Lip}(g)$, the Lipschitz constant of a mapping g going from a metric space A to another metric space. Let $W \in \mathscr{W}_d, g : W \to \mathbb{R}$, $\mathcal{C}^1$ with Lipschitz derivatives. Denote by $\mathcal{H}_d^k$ the k-dimensional Hausdorff measure in $\mathbb{R}^d$. Define the possibly infinite quantity, for $1 \leqslant k \leqslant d$,

$$I_k(g; W) := \max(\mathrm{Lip}(g), \mathrm{Lip}(\partial_i g), 1 \leqslant i \leqslant d)^k \int_{\partial_k W} \frac{\mathcal{H}_d^k(dx)}{\max\left(|g(x)|, |\partial_i g(x)|, i \in I_x(W)\right)^k},$$

and $I_0(g) = \#\mathrm{corners}(W)$. Put $I_k(g; W) = 0$ if $\mathrm{Lip}(g) = 0$ and g vanishes, $1 \leqslant k \leqslant d$.

Theorem 3. *Let $W \in \mathscr{W}_d$, and $f : W \to \mathbb{R}$ be a $\mathcal{C}^{1,1}$ function. Let $F = F_\lambda(f)$ or $F = F_{\lambda^+}(f)$ for some $\lambda \in \mathbb{R}$. Assume that ∇f does not vanish on ∂F. We have*

(i)

$$\#\Gamma(F \cap W) \leqslant \sum_{k=0}^{d} 2^k \kappa_k^{-1} I_k(f - \lambda; W), \tag{6}$$

where κ_k is the volume of the k-dimensional unit ball.

(ii) If $d = 2$,

$$\#\Gamma((F \cap W)^\varepsilon) \leqslant C \sum_{k=0}^{2} I_k(f - \lambda; W) \tag{7}$$

for some $C > 0$ not depending on f, λ, or ε.

The proof is given in Section 4.

Remark 4. Theorem 7 gives conditions on the marginal densities of a bivariate random field so that the term on the right hand side has finite expectation.

Remark 5. Similar results hold if partial derivatives of f are only assumed to be Hölder-continuous, i.e. if there is $\delta > 0$ and $H_i > 0, i = 1, \dots, d$, such that $\|\partial_i f(x) - \partial_i f(y)\| \leqslant H_i \|x - y\|^\delta$ for x, y such that $[x, y] \subset W$. Namely, we have to change constants and replace the exponent k in the max by an exponent $k\delta$. We do not treat such cases here because, as noted at the beginning of Section 2, if the partial derivatives are not Lipschitz, the upper level set is not regular enough to compute the Euler characteristic from the bicovariogram, but the proof is similar to the $\mathcal{C}^{1,1}$ case.

Remark 6. Calling B the right hand term of (7) and noticing that $F_{\lambda^+}(f)^c$ is an upper level set of $-f$, an easy reasoning yields (see [19, Remark 2.13])

$$|\chi((F_\lambda(f) \cap W)^\varepsilon)| \leqslant 2B.$$

3 Mean Euler characteristic of random excursions

We call here $\mathcal{C}^1$ random field over a set $\Omega \subseteq \mathbb{R}^d$ a separable random field $(f(x); x \in \Omega)$, such that in each point $x \in \Omega$, the limits

$$\partial_i f(x) := \lim_{s \to 0} \frac{f(x + s\mathbf{u}_i) - f(x)}{s}, \ i = 1, 2,$$

exist a.s., and the fields $(\partial_i f(x), x \in \Omega), i = 1, \dots, d$, are a.s. separable with continuous sample paths. See [1, 6] for a discussion on the regularity properties of random fields. Say that the random field is $\mathcal{C}^{1,1}$ if the partial derivatives are a.s. locally Lipschitz.

Many sets of conditions allowing to take the expectation in (5) can be derived from Theorem 3. We give below a compromise between optimality and compactness.

Theorem 7. *Let $W \in \mathscr{W}_d$ bounded, and let f be a $\mathcal{C}^{1,1}$ random field on W, $\lambda \in \mathbb{R}$, $F = \{x \in W : f(x) \geqslant \lambda\}$. Assume that the following conditions are satisfied:*

(i) For some $\kappa > 0, \alpha > 1$, for $1 \leqslant k \leqslant d, x \in \partial_k W, I \subset \mathcal{I}_k$, the density of the random $(k+1)$-tuple $(f(x) - \lambda, \partial_i f(x), i \in I)$ satisfies

$$\mathbf{P}(|f(x) - \lambda| \leqslant \varepsilon, |\partial_i f(x)| \leqslant \varepsilon, i \in I) \leqslant \kappa \varepsilon^{\alpha k}, \varepsilon > 0,$$

(ii) for some $p > d\alpha(\alpha - 1)^{-1}$,

$$\mathbf{E}[\mathrm{Lip}(f)^p] < \infty,\ \mathbf{E}[\mathrm{Lip}(\partial_i f)^p] < \infty, i = 1, \ldots, d.$$

Then $\mathbf{E}\#\Gamma(F) < \infty, \mathbf{E}\#\Gamma(F^c) < \infty$ and f is a.s. regular within W at level λ. In the context $d = 2$, (1)-(2) *give the mean Euler characteristic.*

Remark 8. In the case where the $\mathrm{Lip}(f), \mathrm{Lip}(\partial_i f), i = 1, \ldots, d$ have a finite moment of order $> d(d+1)$, the hypotheses are satisfied if for instance $(f(x) - \lambda, \partial_i f(x), 1 \leqslant i \leqslant d)$ has a uniformly bounded density, in which case $\alpha = (d+1)/d$ is suitable. If $\alpha < (d+1)/d$, i.e. if the density is unbounded around 0, higher moments for the Lipschitz constants are required.

The proof is deferred to Section 4. We give an explicit expression in the case where f is stationary. Boundary terms involve the perimeter of F, so we introduce the related notation below. Denote by $\mathcal{C}_c^1$ the class of compactly supported $\mathcal{C}^1$ functions on $\mathbb{R}^2$ endowed with the norm $\|\varphi\| = \sup_{x \in \mathbb{R}^d} \|\varphi(x)\|$. For a measurable set A, and $\mathbf{u} \in \mathcal{S}^1$, the unit circle in $\mathbb{R}^2$, define the variational perimeter of A in direction $\mathbf{u}$ by

$$\mathrm{Per}_{\mathbf{u}}(A) = \sup_{\varphi \in \mathcal{C}_c^1 : \|\varphi\| \leqslant 1} \int_A \langle \nabla \varphi(x), \mathbf{u} \rangle dx,$$

Recall that $(\mathbf{u}_1, \mathbf{u}_2)$ is the canonical basis of $\mathbb{R}^2$, and introduce the $\|\cdot\|_\infty$-perimeter

$$\mathrm{Per}_\infty(A) = \mathrm{Per}_{\mathbf{u}_1}(A) + \mathrm{Per}_{\mathbf{u}_2}(A),$$

named so because it is the analogue of the classical perimeter when the Euclidean norm is replaced by the $\|\cdot\|_\infty$-norm, see [14].

Theorem 9. *Let f be a $\mathcal{C}^{1,1}$ stationary random field, $\lambda \in \mathbb{R}$, $W \in \mathscr{W}_2$ bounded. Assume that $(f(0), \partial_1 f(0), \partial_2 f(0))$ has a bounded density, and that there is $p > 6$ such that*

$$\mathbf{E}\Big[\mathrm{Lip}(f; W)^p\Big] < \infty,\ \mathbf{E}\Big[\mathrm{Lip}(\partial_i f; W)^p\Big] < \infty, i = 1, 2.$$

Then the following limits exist:

$$\begin{aligned}
\overline{\chi}(f, \lambda) &:= \lim_{\varepsilon \to 0} \varepsilon^{-2} \Big[\mathbf{P}(\delta^{\varepsilon}(0, f, \lambda)) - \mathbf{P}(\delta^{-\varepsilon}(0, -f, -\lambda))\Big] \\
\overline{\mathrm{Per}_{\mathbf{u}_i}}(f, \lambda) &:= \lim_{\varepsilon \to 0} \varepsilon^{-1} \mathbf{P}(f(0) \geqslant \lambda, f(\varepsilon \mathbf{u}_i) < \lambda), i = 1, 2, \\
\overline{\mathrm{Vol}}(f, \lambda) &:= \mathbf{P}(f(0) \geqslant \lambda),
\end{aligned}$$

and we have, with $\overline{\mathrm{Per}_\infty} = \overline{\mathrm{Per}_{\mathbf{u}_1}} + \overline{\mathrm{Per}_{\mathbf{u}_2}}$,

$$\begin{aligned}
\mathbf{E}\chi(F_\lambda(f) \cap W) = \mathrm{Vol}(W)\overline{\chi}(f, \lambda) + \frac{1}{4}(\mathrm{Per}_{\mathbf{u}_2}(W)\overline{\mathrm{Per}_{\mathbf{u}_1}}(f, \lambda) + \mathrm{Per}_{\mathbf{u}_1}(W)\overline{\mathrm{Per}_{\mathbf{u}_2}}(f, \lambda)) \\
+ \chi(W)\overline{\mathrm{Vol}}(f, \lambda) \quad (8)
\end{aligned}$$

$$\mathbf{E}\mathrm{Per}_\infty(F_\lambda(f) \cap W) = \mathrm{Vol}(W)\overline{\mathrm{Per}_\infty}(f, \lambda) + \mathrm{Per}_\infty(W)\overline{\mathrm{Vol}}(f, \lambda) \quad (9)$$

$$\mathbf{E}\mathrm{Vol}(F_\lambda(f) \cap W) = \mathrm{Vol}(W)\overline{\mathrm{Vol}}(f, \lambda). \quad (10)$$

The proof of Theorem 7 establishes that the expectations contained in [19, (3.2)] are finite. Therefore the result above is a consequence of that proof and [19, Proposition 3.1].

3.1 Gaussian level sets

Let $(f(x), x \in W)$ be a centred Gaussian field on some $W \in \mathscr{W}_d$. Let the covariance function be defined by

$$\sigma(x,y) = \mathbf{E} f(x) f(y),\ x, y \in W.$$

Say that some real function h satisfies the Dudley condition on $D \subset W$ if for some $\alpha > 0$, $|h(x) - h(y)| \leqslant |\log(\|x-y\|)|^{-1-\alpha}$ for $x, y \in W$. We will make the following assumption on σ:

Assumption 10. *Assume that $x \in W \mapsto \partial^2 \sigma(x,x)/\partial x_i \partial y_i$ exists and satisfies the Dudley condition for $i = 1, \dots, d$, that the partial derivatives $\partial^4 \sigma(x,x)/\partial_{x_i}\partial_{x_j}\partial_{y_i}\partial_{y_j}, x \in W$, $1 \leqslant i, j \leqslant d$, exist and that for some finite partition $\{D_k\}$ of W they satisfy the Dudley condition over each D_k.*

Theorem 11. *Let $W \in \mathscr{W}_d$ bounded. Assume that σ satisfies Assumption 10 and that for $x \in W$, $(f(x), \partial_i f(x), i = 1, \dots, d)$ is non-degenerate. Then for any $\lambda \in \mathbb{R}$, $F = F_\lambda(f)$ satisfies the conclusions of Theorem 7.*

Proof. The existence of second order partial derivatives of Assumption 10 and [1, Theorem 2.2.2] yields that for $i = 1, \dots, d$, $(\partial_i f(x); x \in W)$ is well defined in the L^2 sense and is a Gaussian field with covariance functions $\mathbf{E}\partial_i f(x) \partial_i f(y) = \partial^2 \sigma(x,y)/\partial x_i \partial y_i$ for $x, y \in W$. Since the latter covariance functions satisfy Dudley condition, Theorem 1.4.1 in [6] implies the sample-paths continuity of the partial derivatives.

Using again [1, Theorem 2.2.2], for $1 \leqslant i, j \leqslant d$, $(\partial_{i,j} f(x), x \in D)$ is a well-defined Gaussian field with covariance $\mathbf{E}\partial_{i,j} f(x) \partial_{i,j} f(y) = \partial^4 \sigma(x,y)/\partial x_i \partial y_i \partial x_j \partial y_j$. For each k, [6, Theorem 1.4.1] again yields that $\partial_{i,j} f$ is continuous and bounded over D_k, hence $\partial_{i,j} f$ is bounded over W. Finally, formula (2.1.4) in [6] yields that $\mathbf{E}\sup_{x \in W} |\partial_{i,j} f(x)|^p < \infty$ for $p \geqslant 0$. Since $\mathrm{Lip}(\partial_i f) \leqslant d \max_{j=1,\dots,d} \|\partial_{ij} f\|$, Condition (ii) of Theorem 7 is satisfied for any $\alpha > 1$.

Put for notational convenience $f^{(0)} := f, f^{(i)} = \partial_i f, i = 1, \dots, d$. We have for $i, j \in \{0, \dots, d\}$,

$$\begin{aligned}
&|\mathbf{E} f^{(i)}(x) f^{(j)}(x) - f^{(i)}(y) f^{(j)}(y)| \\
&\qquad \leqslant \left| \mathbf{E}\left[\left(f^{(i)}(x) - f^{(i)}(y) \right) f^{(j)}(x) \right] \right| + \left| \mathbf{E}\left[f^{(i)}(y) \left(f^{(j)}(x) - f^{(j)}(y) \right) \right] \right| \\
&\qquad \leqslant \mathbf{E} \sup_W |f^{(j)}| \mathrm{Lip}(f^{(i)}) \|x-y\| + \mathbf{E} \sup_W |f^{(i)}| \mathrm{Lip}(f^{(j)}) \|x-y\|,
\end{aligned}$$

which yields that the covariance function with values in the space of $(d+1) \times (d+1)$ matrices,

$$x \mapsto \Sigma(x) := \mathrm{cov}(f(x), \partial_i f(x), 1 \leqslant i \leqslant d)$$

is Lipschitz on W. In particular, since $\det(\Sigma(x))$ does not vanish on W, it is bounded from below by some $c > 0$, whence the density of $(f(x), \partial_1 f(x), \partial_2 f(x))$, $x \in W$, is uniformly bounded by $(2\pi)^{-d/2} c^{-1/2}$, and assumption (i) from Theorem 7 is satisfied with $\alpha = (d+1)/d$. □

Example 12. Gaussian fields that are $\mathcal{C}^{1,1}$ and not $\mathcal{C}^2$ naturally arise in the context of smooth interpolation. Let E be a locally finite set of points of $\mathbb{R}$. Let $(W(x), x \in E)$ be a Gaussian field on E, and $A_x, B_x, x \in E$ be random variables. For $x \in E$, note $r(x) = \min(E \cap (x, +\infty))$. Define

$$g(y) = \sum_{x \in E} \mathbf{1}_{\{y \in [x, r(x))\}} A_x \left(\frac{y-x}{r(x)-x} \right)^2 + B_x \frac{y-x}{r(x)-x} + W(x).$$

Straightforward computations yield that, with $\Delta_x = W(r(r(x))) - 2W(r(x)) + W(x)$, if

- $A_{r(x)} = \Delta_x - A_x, x \in E$,
- $B_x = W(r(x)) - W(x) - A_x, x \in E$,

then with probability 1, g is $\mathcal{C}^{1,1}$ and not twice differentiable on $(\lim_{x\to-\infty, x\in E}, \lim_{x\to\infty}, x \in E)$. If for some $x_0 \in E$, $(A_{x_0}; W(x), x \in E)$ is a Gaussian process, g is furthermore a Gaussian field.

Given a Gausian process $(g(k); k \in \mathbb{Z}^d)$, it should be possible to carry out a similar approximation scheme in $\mathbb{R}^d$ by defining $g = \sum_{k\in\mathbb{Z}^d} \mathbf{1}_{\{x\in(k+[0,1)^d)\}} g_k$ where g_k is a bicubic polynomial interpolaton of Gaussian variables $W(j), j \in (k + \{0,1\}^d)$ on $k + [0,1)^d$. A possible follow-up on this work could be to investigate the asymptotic properties of topological characteristics of g when it is the smooth interpolation of an irregular Gaussian field as the grid mesh converges to 0.

Let us give the mean Euler characteristic in dimension 2 under the simplifying assumptions that the law of f is invariant under translations and rotations of $\mathbb{R}^2$. This implies for instance that in every $x \in \mathbb{R}^2$, $f(x), \partial_1 f(x)$ and $\partial_2 f(x)$ are independent, see for instance [6] Section 5.6 and (5.7.3). Assumption 10 is simpler to state in this context: $x \mapsto \partial^2\sigma(x,x)/\partial x_i \partial y_i$ and $x \mapsto \partial^4 \sigma(x,x)/\partial x_i \partial x_j \partial y_i \partial y_j$ should exist and satisfy Dudley's condition in 0. It actually yields that f has $\mathcal{C}^2$ sample paths, and it is not clear wether this is equivalent to $\mathcal{C}^{1,1}$ regularity in this framework. For this reason we state the result with the abstract conditions of Theorem 9 .

Theorem 13. *Let $f = (f(x); x \in \mathbb{R}^2)$ be a $\mathcal{C}^{1,1}$ stationary isotropic centred Gaussian field on $\mathbb{R}^2$ with $\mathbf{E}\mathrm{Lip}(\partial_i f)^p < \infty$, for some $p > 6$. Let $\lambda \in \mathbb{R}$, $F = \{x : f(x) \geqslant \lambda\}$, and let $W \in \mathscr{W}_2$ bounded. Let $\mu = \mathbf{E}\partial_1 f(0)^2$, and $\Phi(\lambda) = \frac{1}{\sqrt{2\pi}} \int_\lambda^\infty \exp(-t^2/2)dt$. Then*

$$\mathbf{E}\mathrm{Vol}(F \cap W) = \mathrm{Vol}(W)\Phi(\lambda), \tag{11}$$

$$\mathbf{E}\mathrm{Per}_\infty(F \cap W) = \mathrm{Vol}(W) 2\frac{\sqrt{\mu}}{\pi} \exp(-\lambda^2/2) + \mathrm{Per}_\infty(W)\Phi(\lambda), \tag{12}$$

$$\mathbf{E}\chi(F \cap W) = \left(\mathrm{Vol}(W)\frac{\mu\lambda}{(2\pi)^{3/2}} + \mathrm{Per}_\infty(W)\frac{\sqrt{\mu}}{4\pi}\right) e^{-\lambda^2/2} + \frac{1}{\sqrt{2\pi}}\Phi(\lambda)\chi(W). \tag{13}$$

Remark 14. If W is a square, the relation (13) coincides with [6, (11.7.14)].

Proof. (10) immediately yields (11). To prove (13), first remark that the stationarity of the field and the fact that it is not constant a.s. entail that $(f(0), \partial_1 f(0), \partial_2 f(0)) \stackrel{(d)}{=} (f(x), \partial_1 f(x), \partial_2 f(x)), x \in \mathbb{R}^2$ is non-degenerate. Let us show

$$\overline{\chi}(F) = \lim_{\varepsilon\to 0} \varepsilon^{-2} \left[\mathbf{P}\left(\delta^\varepsilon(0, f, \lambda)\right) - \mathbf{P}\left(\delta^{-\varepsilon}(0, -f, -\lambda)\right)\right] = \frac{\mu\lambda \exp(-\lambda^2/2)}{(2\pi)^{3/2}}. \tag{14}$$

Fix $\varepsilon > 0$. Let M_ε be the 3×3 covariance matrix of $(f(0), f(\varepsilon\mathbf{u}_1), f(\varepsilon\mathbf{u}_2))$. Straightforward computations show that $\det(M_\varepsilon) = \varepsilon^4\mu^2 + o(\varepsilon^4)$, and

$$M_\varepsilon^{-1} = \frac{1}{\det(M_\varepsilon)}\left(\varepsilon^2 W_\varepsilon + \varepsilon^4 D_\varepsilon\right), \tag{15}$$

where the sum of each line and each column of W_ε is 0, for $\varepsilon > 0$, and as $\varepsilon \to 0$

$$W_\varepsilon \to W := \mu \begin{pmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}, D_\varepsilon \to D := \frac{\mu^2}{4}\begin{pmatrix} -4 & 2 & 2 \\ 2 & -1 & 1 \\ 2 & 1 & -1 \end{pmatrix}.$$

Denote by $\mathbf{1}$ the vector $(1,1,1)$, and let $\Lambda = \lambda\mathbf{1}$, $Q = \{(t,s,z) : t \geqslant 0, s < 0, z < 0\}$. Denote by A' the transpose of a matrix (or a vector) A. We have

$$\mathbf{P}(\delta^{\varepsilon}(0,f,\lambda)) = \frac{1}{\sqrt{(2\pi)^3 \det(M_\varepsilon)}} \int_{Q+\Lambda} \exp\left(-\frac{1}{2}(t,s,z)' M_\varepsilon^{-1}(t,s,z)\right) dtdsdz$$

and by isotropy and symmetry, for $\lambda \in \mathbb{R}$,

$$\mathbf{P}(\delta^{-\varepsilon}(0,-f,-\lambda)) = \mathbf{P}(\delta^{\varepsilon}(0,-f,-\lambda)) = \mathbf{P}(\delta^{\varepsilon}(0,f,-\lambda)).$$

Therefore, (8) yields that $\overline{\chi}(F) = \lim_{\varepsilon\to 0} \varepsilon^{-2}\left(\mathbf{P}(\delta^{\varepsilon}(0,f,\lambda)) - \mathbf{P}(\delta^{\varepsilon}(0,f,-\lambda))\right)$. Let $X = (t,s,z) \in Q, Y = \frac{\varepsilon}{\sqrt{\det(M_\varepsilon)}} X$. Since ΛW_ε and $W_\varepsilon \Lambda$ are 0, we have

$$\begin{aligned}
&(X+\Lambda)' M_\varepsilon^{-1}(X+\Lambda) \\
&\qquad = \underbrace{Y'(W_\varepsilon + \varepsilon^2 D_\varepsilon)Y}_{=:\gamma_\varepsilon(Y)} + \frac{2\varepsilon^3}{\sqrt{\det(M_\varepsilon)}} Y' D_\varepsilon \Lambda + \frac{\varepsilon^4}{\det(M_\varepsilon)} \Lambda' D_\varepsilon \Lambda
\end{aligned}$$

$$\mathbf{P}(\delta^{\varepsilon}(0,f,\lambda)) = \frac{\left(\frac{\sqrt{\det(M_\varepsilon)}}{\varepsilon}\right)^3 \exp\left(-\lambda^2 \frac{\varepsilon^4}{2\det(M_\varepsilon)} \mathbf{1}' D_\varepsilon \mathbf{1}\right)}{\sqrt{(2\pi)^3 \det(M_\varepsilon)}} \int_Q \exp\left(-\frac{1}{2}\gamma_\varepsilon(Y) - \varepsilon^3 \frac{Y' D_\varepsilon \Lambda}{\sqrt{\det(M\varepsilon)}}\right) dY$$

and, for some $\theta = \theta(\varepsilon, Y, \lambda) \in [-\varepsilon^3 \det(M_\varepsilon)^{-1/2}, \varepsilon^3 \det(M_\varepsilon)^{-1/2}]$,

$$\exp\left(-\frac{\varepsilon^3 Y' D_\varepsilon \Lambda}{\sqrt{\det(M_\varepsilon)}}\right) - \exp\left(\frac{\varepsilon^3 Y' D_\varepsilon \Lambda}{\sqrt{\det(M_\varepsilon)}}\right) = -\, 2\frac{\varepsilon^3 Y' D_\varepsilon \Lambda}{\sqrt{\det(M_\varepsilon)}} \exp(\theta \Lambda' D_\varepsilon Y)$$

Therefore, as $\varepsilon \to 0$, $\varepsilon^{-2}(\mathbf{P}(\delta^{\varepsilon}(0,f,\lambda)) - \mathbf{P}(\delta^{\varepsilon}(0,f,-\lambda)))$ is equivalent to

$$\begin{aligned}
&\varepsilon^{-2} \frac{\exp(-\lambda^2/2)\det(M_\varepsilon)}{\varepsilon^3 \sqrt{(2\pi)^3}} \int_Q \exp(-\gamma_\varepsilon(Y)/2) \frac{-2\varepsilon^3 Y' D_\varepsilon \Lambda}{\sqrt{\det(M_\varepsilon)}} \exp\left(\theta Y' D_\varepsilon \Lambda\right) dY \\
&\sim \frac{-\exp(-\lambda^2/2)\mu}{\sqrt{2\pi^3}} \int_Q \exp(-\gamma_\varepsilon(Y)/2) Y' D_\varepsilon \Lambda \exp(\theta Y' D_\varepsilon \Lambda) dY. \qquad (16)
\end{aligned}$$

For $Y = (x,y,z) \in Q$, we have

$$\frac{Y'WY}{\mu} = 2x^2 + y^2 + z^2 - 2xy - 2xz = 2x^2 + y^2 + z^2 + 2|xy| + 2|xz| \geqslant \|Y\|^2.$$

Since $W_\varepsilon + \varepsilon^2 D_\varepsilon \to W$ as $\varepsilon \to 0$, $\gamma_\varepsilon(Y) \geqslant \mu\|Y\|^2/2$ for ε sufficiently small, uniformly in $Y \in Q$. This yields a clear majoring bound and Lebesgue's theorem gives $\overline{\chi}(F) = -\mu\exp(-\lambda^2/2) I (2\pi^3)^{-1/2}$ with $I = \int_Q \exp(-\frac{1}{2}Y'WY) Y' D \Lambda dY = 2\lambda J$ where

$$J = \int_Q \exp(-(2t^2 + s^2 + z^2 - 2ts - 2tz))(s+z) dtdsdz = -1/4$$

with the change of variables $u = t - s, v = t - z, w = t$. The statement (14) is therefore proved. The computation of $\overline{\mathrm{Per}_\infty}(F)$ is similar and simpler and is omitted here. $\square$

4 Proofs

4.1 Proof of Theorem 3

(i) Assume without loss of generality $\lambda = 0$ in the proof. Recall that $\Gamma(F \cap W)$ is the collection of bounded connected components of $F \cap W$. For $0 \leqslant k \leqslant d$, denote by $\Gamma_k(F \cap W)$ the elements of $\Gamma(F \cap W)$ that hit $\partial_k W$, and define recursively $\Gamma_k^+(F; W) = \Gamma_k(F \cap W) \setminus \Gamma_{k-1}^+(F; W), 1 \leqslant k \leqslant d$.

Let $1 \leqslant k \leqslant d, C \in \Gamma_k^+(F; W)$, C' arbitrarily chosen in $\Gamma(C \cap \partial_k W)$. Since C' does not touch $\partial_{k-1}W$, it is included in the relative interior of $\partial_k W$ within the affine k-dimensional tangent space to ∂W (or W if $k = d$) that contains C'. Let $I \in \mathcal{I}_k$ such that for $x \in C', I_x(W) = I$. Let $x_C \in \mathrm{cl}(C')$ such that $f(x_C) = \sup_{C'} f$. Since $f \geqslant 0$ on C, and the gradient ∇f does not vanish on ∂C, there is a neighbourhood of x_C that does not touch any other connected component of F, whence x_C is a local maximum of f within $\partial_k W$. The Lagrange multipliers Theorem yields that $\partial_i f(x_C) = 0$ for $i \in I$. Call r_C the maximal radius such that $B_C := (B(x_C, r_C) \cap \partial_k W) \subset C'$. Since B_C touches ∂F, f has a zero on B_C. It follows that $|f(x)| \leqslant 2\mathrm{Lip}(f) r_C$ and $|\partial_i f(x)| \leqslant \mathrm{Lip}(\partial_i f) r_C$ for $x \in B_C, i \in I$. Call I' the set I from which have been removed indexes $i \in I$ such that $\mathrm{Lip}(\partial_i f) = 0$, and hence $\partial_i f = 0$ on B_C. Define

$$M(x) = \max(|f(x)|/2\mathrm{Lip}(f), |\partial_i f(x)|/\mathrm{Lip}(\partial_i f), i \in I') \in \mathbb{R}_+, x \in \partial_k W.$$

We have

$$\begin{aligned} 1 = & \frac{1}{\mathcal{H}_d^k(B_C)} \int_{B_C} \mathbf{1}_{\{M(x) \leqslant r_C\}} \mathcal{H}_d^k(dx) = \kappa_k^{-1} r_C^{-k} \int_{B_C} \mathbf{1}_{\{r_C^{-1} \leqslant M(x)^{-1}\}} \mathcal{H}_d^k(dx) \\ & \leqslant \kappa_k^{-1} \int_{B_C} M(x)^{-k} \mathcal{H}_d^k(dx). \end{aligned}$$

Noticing that

$$M(x) \geqslant \frac{\max(|f(x)|, |\partial_i f(x)|, i \in I')}{\max(2\mathrm{Lip}(f), \mathrm{Lip}(\partial_i f), i \in I')} \geqslant \frac{\max(|f(x)|, |\partial_i f(x)|, i \in I)}{2\max(\mathrm{Lip}(f), \mathrm{Lip}(\partial_i f), i \in I)},$$

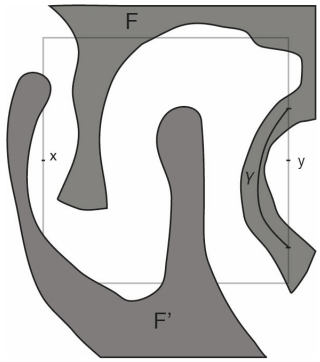
and since the B_C are pairwise disjoint, summing over all the $C \in \Gamma_k^+(F; W)$ and $k \in \{1, \dots, d\}$ gives the result (with $\#\Gamma_0(F; W) \leqslant \#\mathrm{corners}(W)$).

(ii) Theorem 2.12 in the companion paper [19], in the context $d = 2$, features a bound on $\chi((F \cap W)^\varepsilon)$ in terms of the number of occurrences of local configurations called *entanglement points of* F. Roughly speaking, an entanglement point occurs when two close points of F are connected by a tight path in F. As a consequence, if F is sampled with an insufficiently high resolution in this region, the connecting path is not detected, and F looks locally disconnected. For formal definitions, for $x, y \in \varepsilon\mathbb{Z}^2$ at distance ε, introduce $\mathsf{P}_{x,y}$ the closed square with side-length ε such that x and y are the midpoints of two opposite sides. Let $\mathsf{P}'_{x,y} = \partial \mathsf{P}_{x,y} \setminus \{x, y\}$, which has two connected components. Then $\{x, y\}$ is an *entanglement pair of points* of F if $x, y \notin F$ and $(\mathsf{P}'_{x,y} \cup F) \cap \mathsf{P}_{x,y}$ is connected. We call $\mathscr{N}_\varepsilon(F)$ the family of such pairs of points. See Figure 1 for an example.

We introduce the notation $(\!|x, y|\!) = \varepsilon\mathbb{Z}^2 \cap [x, y] \setminus \{x, y\}$, for $x, y \in \varepsilon\mathbb{Z}^2$. For $A \subset \mathbb{R}^2$, note $A^{\oplus\varepsilon} = \{x \in \mathbb{R}^d : d(x, A) \leqslant \varepsilon\}$. To account for boundary effects, we also consider grid points $x, y \in \varepsilon\mathbb{Z}^2 \cap W \cap F$, on the same line or column of $\varepsilon\mathbb{Z}^2$, such that

- x, y are within distance ε from one of the edges of W (the same edge for x and y)
- $(\!|x, y|\!) \neq \emptyset$
- $(\!|x, y|\!) \subseteq \varepsilon\mathbb{Z}^2 \cap F^c \cap F^{\oplus\varepsilon}$.

Figure 1: *Entanglement point:* In this example, $\{x, y\} \in \mathscr{N}_\varepsilon(F)$ because the two connected components of $\mathsf{P}'_{x,y}$, in lighter grey, are connected through $\gamma \subseteq (F \cap \mathsf{P}_{x,y})$. We do not have $\{x, y\} \in \mathscr{N}_\varepsilon(F')$.



The family of such pairs of points $\{x, y\}$ is denoted by $\mathscr{N}_\varepsilon'(F;W)$. It is proved in [19, Theorem 2.12] that

$$\#\Gamma((F \cap W)^\varepsilon) \leqslant 2\#(\mathscr{N}_\varepsilon(F) \cap W^{\oplus\varepsilon}) + 2\#\mathscr{N}_\varepsilon'(F, W) + \#\Gamma(F \cap W) + 2\#\text{corners}(W). \quad (17)$$

It therefore only remains to bound $\#(\mathscr{N}_\varepsilon(F) \cap W^{\oplus\varepsilon})$ and $\#\mathscr{N}_\varepsilon'(F, W)$ to achieve (7). For $m \geqslant 1$ and a function $g : A \subseteq \mathbb{R}^m \to \mathbb{R}$, introduce the continuity modulus

$$\omega(g, A) = \sup_{x \neq y \in A} |g(x) - g(y)|.$$

The bound will follow from the following lemma.

Lemma 15. ***(i)*** *For $\{x, y\} \in \mathscr{N}_\varepsilon(F)$, we have for some $i \in \{1, 2\}$, and $i' = i + 1 \bmod 2$,*

$$\begin{aligned} |f(x)| &\leqslant \omega(f, [x, y]) \leqslant \mathrm{Lip}(f)\varepsilon \\ |\partial_i f(x)| &\leqslant \omega(\partial_i f, [x, y]) \leqslant \mathrm{Lip}(\partial_i f)\varepsilon \\ |\partial_{i'} f(x)| &\leqslant 2\omega(\partial_i f, \mathsf{P}_{x,y}) + \omega(\partial_{i'} f, \mathsf{P}_{x,y}) \leqslant \sqrt{2}\varepsilon(2\mathrm{Lip}(\partial_i f) + \mathrm{Lip}(\partial_{i'} f)), \end{aligned}$$

and idem for y.

(ii)*For $x, y \in \mathscr{N}_\varepsilon'(F, W)$, there is $z = z(x, y) \in (\!|x, y|\!)$, $i \in \{1, 2\}$, such that*

$$\begin{aligned} |f(z)| &\leqslant \mathrm{Lip}(f)\varepsilon \\ |\partial_i f(z)| &\leqslant \mathrm{Lip}(\partial_i f)\varepsilon. \end{aligned}$$

The lemma is proved later for convenience. To obtain the integral upper bounds from (7), note that there is $c > 0$ such that for $\varepsilon > 0$ sufficiently small, for every $x, y \in W$, neighbours in $\varepsilon\mathbb{Z}^2$, $\mathrm{Vol}((B(x, \varepsilon) \cup B(y, \varepsilon)) \cap W) \geqslant \varepsilon^2/c$. Define the possibly infinite quantity, for $z \in W$,

$$M(z) = \max(|f(z)|/2\mathrm{Lip}(f), |\partial_i f(z)|/2\mathrm{Lip}(\partial_i f), |\partial_{i'} f(z)|/(2\sqrt{2}(\mathrm{Lip}(\partial_1 f) + \mathrm{Lip}(\partial_2 f)))).$$

Treating undetermined cases 0/0 can be done like in the proof of **(i)**. Lemma 15 then yields

$$\begin{aligned}
\mathscr{N}_\varepsilon(F) &\leqslant \sum_{x,y\in\mathscr{N}_\varepsilon(F)} \mathbf{1}_{\{\forall z\in B(x,\varepsilon)\cup B(y,\varepsilon)\cap W, M(z)\leqslant\varepsilon\}}\\
&\leqslant \sum_{x,y\in\mathscr{N}_\varepsilon(F)} c\varepsilon^{-2} \int\limits_{(B(x,\varepsilon)\cup B(y,\varepsilon))\cap W} \mathbf{1}_{\{M(z)^{-1}\geqslant\varepsilon^{-1}\}}dz\\
&\leqslant 4c\int_W M(z)^{-2}dz \leqslant c' I_2(f),
\end{aligned}$$

for some $c'>0$, because for every $z\in W$ there are at most 4 couples $\{x,y\}\in\mathscr{N}_\varepsilon(F)$ such that $z\in B(x,\varepsilon)\cup B(y,\varepsilon)$.

Now, given $w\in\partial W$, there can be at most 3 pairs $\{x,y\}\in\mathscr{N}_\varepsilon'(F)$ such that w is on the closest edge of W parallel to $[x,y]$ and $z=z(x,y)$ (defined in Lemma 15) is within distance 3ε from w, and in this case $|f(w)|\leqslant 4\mathrm{Lip}(f)\varepsilon$ and $|\partial_i f(w)|\leqslant 4\mathrm{Lip}(\partial_i f)\varepsilon$ for some $i\in\{1,2\}$. We have $\mathcal{H}_2^1(B(z,3\varepsilon)\cap\partial W)\geqslant\varepsilon$, because z is within distance 2ε from a segment of ∂W parallel to $[x,y]$. It follows that, with $M_i(w)=\max(|f(w)|/4\mathrm{Lip}(f),|\partial_i f(w)|/4\mathrm{Lip}(\partial_i f))$

$$\begin{aligned}
\#\mathscr{N}_\varepsilon'(F,W) &\leqslant \sum_{x,y\in\mathscr{N}_\varepsilon'(F,W)}\sum_{i=1}^2 \mathbf{1}_{\{\forall w\in B(z,3\varepsilon)\cap\partial W, M_i(w)\leqslant\varepsilon\}}\\
&\leqslant \sum_{i=1}^2\sum_{x,y\in\mathscr{N}_\varepsilon'(F)} \frac{1}{\varepsilon}\int_{\partial W\cap B(z,3\varepsilon)} \mathbf{1}_{\{M_i(w)^{-1}\geqslant\varepsilon^{-1}\}}\mathcal{H}_2^1(dw)\\
&\leqslant \sum_{i=1}^2 3\int_{\partial W} M_i(w)^{-1}\mathcal{H}_2^1(dw) = 24 I_1(f;0).
\end{aligned}$$

Proof of Lemma 15. For $x\in\mathbb{R}^2$, denote by $(x_{[1]},x_{[2]})$ its coordinates in the canonical basis, not to be mistaken with a pair of vector of $\mathbb{R}^2$, denoted by (x_1,x_2). If φ is a mapping with values in $\mathbb{R}^2$, denote its coordinates by $(\varphi(\cdot)_{[1]},\varphi(\cdot)_{[2]})$.

(i) Let $x,y\in\mathscr{N}_\varepsilon(F)$. The definition of $\mathscr{N}_\varepsilon(F)$ yields a connected path $\gamma\subseteq(F\cap\mathsf{P}_{x,y})$ going through some $z\in[x,y]$ and connecting the two connected components of $\mathsf{P}_{x,y}'$. Since $f(x)\geqslant 0$ and $f(z)\leqslant 0$, there is a point z' of $[x,y]$ satisfying $f(z')=0$, hence $|f(x)|\leqslant\omega(f,[x,y])$. Note for later that for $t\in\mathsf{P}_{x,y}$ $|f(t)|\leqslant\omega(f,\mathsf{P}_{x,y})\leqslant\mathrm{Lip}(f)\sqrt{2}\varepsilon$.

We assume without loss of generality that $[x,y]$ is horizontal. Let $[z',z'']$ be the (also horizontal) connected component of $F\cap[x,y]$ containing z. After choosing a direction on $[x,y]$, z' and z'' are entry and exit points for F, and their normal vectors $\mathbf{n}_F(z'),\mathbf{n}_F(z'')$ point towards the outside of F. Therefore they satisfy $\mathbf{n}_F(z')_{[1]}\mathbf{n}_F(z'')_{[1]}\leqslant 0$, and so $\partial_1 f(z')\partial_1 f(z'')\leqslant 0$. This gives us by continuity the existence of a point $w\in[x,y]$ such that $0=\partial_1 f(w)$, whence $|\partial_1 f(x)|\leqslant\omega(\partial_1 f,[x,y])$. Note for later that $|\partial_1 f(t)|\leqslant\omega(\partial_1 f,\mathsf{P}_{x,y})$ on $\mathsf{P}_{x,y}$. If $[x,y]$ is vertical, $\partial_2 f$ verifies the inequality instead. Let us keep assuming that $[x,y]$ is horizontal for the sequel of the proof.

We claim that $|\partial_2 f(x)|\leqslant 2\omega(\partial_1 f,\mathsf{P}_{x,y})+\omega(\partial_2 f,\mathsf{P}_{x,y})$, and consider two cases to prove it.

- <u>First case</u> $\partial_2 f(z')\partial_2 f(z'')\leqslant 0$, and by continuity we have $w\in[x,y]$ such that $0=\partial_2 f(w)$, whence $|\partial_2 f(\cdot)|\leqslant\omega(\partial_2 f,\mathsf{P}_{x,y})$ on the whole pixel $\mathsf{P}_{x,y}$. The desired inequality follows.

- <u>Second case</u> $\partial_2 f(z') > 0, \partial_2 f(z'') > 0$ (equivalent treatment if they are both < 0). Assume for instance that z' is the leftmost point, and that $|\partial_2 f(x)| > 2\omega(\partial_1 f, \mathsf{P}_{x,y}) + \omega(\partial_2 f, \mathsf{P}_{x,y})$, otherwise the claim is proved. It implies in particular that $|\partial_2 f(\cdot)| > 2\omega(\partial_1 f, \mathsf{P}_{x,y})$ on the whole pixel $\mathsf{P}_{x,y}$. Since $|\partial_1 f(\cdot)| \leqslant \omega(\partial_1 f, \mathsf{P}_{x,y})$ on $\mathsf{P}_{x,y}$, the implicit function theorem yields a function φ (resp. ψ): $[z'_{[1]}, z''_{[1]}] \to \mathbb{R}$ such that $|\varphi'| \leqslant 1/2$, (resp. $|\psi'| \leqslant 1/2$), $\varphi([z'_{[1]}, z''_{[1]}]) \subset (z'_{[2]} + (-\varepsilon/2, \varepsilon/2))$, (resp. $\psi([z'_{[1]}, z''_{[1]}]) \subset (z'_{[2]} + (-\varepsilon/2, \varepsilon/2))$) and the graph of φ (resp. ψ) coincides with $\partial F \cap ([z'_{[1]}, z''_{[1]}] \times (z'_{[2]} + [-\varepsilon/2, \varepsilon/2])$. In particular, $\varphi = \psi$, and its graph cannot touch the upper half of $\partial \mathsf{P}_{x,y}$. Applying this to every maximal segment $[z', z''] \subset (F \cap [x, y])$, we see that every connected component of F touching $[x, y]$, and hence γ, cannot meet the upper half of $\mathsf{P}_{x,y}$. In particular, it contradicts the definition of $\mathscr{N}_\varepsilon(F)$, whence indeed the assumption is proved by contradiction.

(ii) Let now $\{x, y\}$ be an element of $\mathscr{N}_\varepsilon'(f, W)$. We know that $(\!|x, y|\!) \cap F^c \neq \emptyset$. Let $[z', z''] \subset [x, y]$ be a connected component of $F^c \cap [x, y]$. If $[z', z'']$ is, say, horizontal, since $\mathbf{n}_F(\cdot)_{[1]}$ changes sign between z' and z'', so does $\partial_1 f$, and by continuity there is $w \in [z', z'']$ where $\partial_1 f(w) = 0$. Calling z the closest point from w in $(\!|x, y|\!)$, $\|z - w\| \leqslant \varepsilon$, and by definition of $\mathscr{N}_\varepsilon'(F, W)$, z is also at distance ε from $\partial F = \{f = 0\}$. It follows that $|\partial_1 f(z)| \leqslant \mathrm{Lip}(\partial_1 f)\varepsilon, |f(z)| \leqslant \mathrm{Lip}(f)\varepsilon$. $\square$

4.2 Proof of Theorem 7

Assume without loss of generality $\lambda = 0$. Let us prove that F is a.s. regular within W at level 0. For $0 \leqslant k \leqslant d, I \in \mathcal{I}_k$, define

$$\theta_{k,I} = \{x \in \partial_k W : f(x) = 0, \partial_i f(x) = 0, i \in I\}.$$

There is $c > 0$ such that for $\varepsilon > 0$ sufficiently small, for all $x \in \partial_k W$, $\mathcal{H}^k_d(\partial_k W \cap B(x, \varepsilon)) \geqslant \varepsilon^k / c$. Define $M(y) = \max(|f(y)|/\mathrm{Lip}(f), |\partial_i f(y)|/\mathrm{Lip}(\partial_i f), i \in I)$, with undetermined cases treated like in the proof of Theorem 3-(i). For $x \in \theta_{k,I}, y \in B(x, \varepsilon) \cap \partial_k W$, we have $M(y) \leqslant \varepsilon$. Therefore, for $\eta > 0$,

$$\begin{aligned}
\#\theta_{k,I} =& \liminf_{\varepsilon \to 0} \#\{x \in \partial_k W : \theta_{k,I} \cap B(x, \varepsilon) \neq \emptyset\} \\
\leqslant& \liminf_{\varepsilon \to 0} \int_{\partial_k W} \frac{1}{\mathcal{H}^k_d(B(x, \varepsilon))} \mathbf{1}_{\{M(y) \leqslant \varepsilon\}} \mathcal{H}^k_d(dy) \\
\leqslant& c \liminf_{\varepsilon \to 0} \varepsilon^\eta \int_{\partial_k W} M(y)^{-(k+\eta)} \mathcal{H}^k_d(dy).
\end{aligned}$$

Fatou's lemma yields, with $L := \max(\mathrm{Lip}(f), \mathrm{Lip}(\partial_{i f}), i = 1, \ldots, d), m_y = \max(|f(y)|, |\partial_i f(y)|, i \in I)$,

$$\mathbf{E}\#\theta_{k,I} \leqslant c \liminf_{\varepsilon \to 0} \varepsilon^\eta \int_{\partial_k W} \mathbf{E}\left[\frac{L^{k+\eta}}{m_y^{k+\eta}}\right] \mathcal{H}^k_d(dy).$$

Since $p > \frac{\alpha d}{\alpha - 1} \geqslant \frac{\alpha k}{\alpha - 1}$, we can choose $q > 1$ such that $q < \alpha$ and $q' k < p$, with $q' = (1 - q^{-1})^{-1}$. In particular, if η is chosen sufficiently small, $\mathbf{E}L^{q'(k+\eta)} < \infty$ for $1 \leqslant k \leqslant d$. The fact that $\mathbf{E}\sum_{I \in \mathcal{I}_k} \#\theta_{k,I} = 0$ follows from the bound, uniform in $y \in \partial_k W$,

$$\begin{aligned}
\mathbf{E}L^{q'(k+\eta)} \mathbf{E}m_y^{-q(k+\eta)} \leqslant& C \int_{\mathbb{R}_+} \mathbf{P}(m_y \leqslant t^{-1/(q(k+\eta))}) dt \\
\leqslant& C \int_{\mathbb{R}_+} 1 \vee t^{-\frac{\alpha k}{q(k+\eta)}} dt.
\end{aligned}$$

Assume without loss of generality that η is chosen so that $q(k+\eta) < \alpha k$, so that indeed for all $1 \leqslant k \leqslant d$, the previous bound is finite (uniformly in y).

Therefore, if $d = 2$, using (5), $\chi(F \cap W) = \lim_{\varepsilon \to 0} \chi((F \cap W)^{\varepsilon})$ holds a.s. According to Theorem 3, to apply Lebesgue's theorem and show that this limit can be passed to expectations, yielding (1)-(2), it only remains to show $\mathbf{E}I_1(f;W) < \infty, \mathbf{E}I_2(f;W) < \infty$. This follows from the exact same computation as before, with $k = 1, 2$, and $\eta = 0$.

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