The Habakkuk hypothesis in a neoclassical framework

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(To make more with less or to make more with more: that is the question)

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Abstract

We present a new way to picture technological change in an otherwise standard Ramsey framework. Technological change takes the form of alterations of the production function itself, rather than changes in total factor productivity. These changes can take two directions that we dub respectively 'complementation' and 'substitution'. Complementation results in a production function that is superior for lower values of capital, while substitution results in a production function that is superior for higher values of capital. Under the most general conditions, when the agent is initially at steady state, both options bring strictly positive utility gains to the agent.

We analyze sequence of steady states with exogenous and endogenous direction of technological change.

With exogenous growth, we prove that when the production functions are Cobb-Douglas or CES (with the same elasticity of substitution), output and consumption grow asymptotically at a common rate and the capital share tends to one under continual substitution; while continual complementation makes output and consumption converge to a common limit and the capital share tend to nil.

With endogenous direction of technological change and under the most general conditions, the agent has a bias towards complementation which brings quicker gains than substitution. We assume that the production functions are Cobb-Douglas and that utility is logarithmic. Then, when the potential rate of complementation is strictly greater than the potential rate of substitution, the labor share oscillates around some endogenous long-run value, determined by the rates of complementation/substitution and by the impatience rate. This growth regime reproduces the Kaldor facts.

Keywords: Economic growth, labor share, capital-labor substitution, endogenous growth theory.

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Introduction

In his famous 1962 essay *American and British Technology in the Nineteenth Century: The Search for Labour-Saving Inventions*, Habakkuk argued that high wages in the USA relatively to Great Britain encouraged American entrepreneurs to take on relatively more labor-saving methods than their British counterparts, which ultimately resulted in the American production system using relatively more capital and less labor.

This point – which was raised by Rothbarth (1946) before – has since then gained considerable influence on the history of technology[1]. The Habakkuk hypothesis is also deeply seated in the field of economic growth, though theorists do not agree on the exact meaning to give to a labor-saving invention, nor on the exact framework to have in mind[2].

Models in neoclassical spirit usually assume that technological change is representable by the alteration of some productivity term (Hicks-neutral, Harrod-neutral, investment-specific, etc.)[3]. The purpose of this paper is to show that by *not* restricting ourselves to a fixed production function, we are able to unveil some new and interesting growth regimes, and shed new light on the issue of how economies switch from one growth regime to another[4].

The structure we use is the simple discrete-time Ramsey model with 100 percent depreciation rate and a strictly positive impatience rate. We suppose that while initially at steady state, the Ramsey agent might rotate the production function *clockwise* or *anticlockwise*. In the first case, the agent switches to a production technique that is more efficient than the old technique only for lower levels of capital; in the second case the agent switches to a production technique that is more efficient than the old technique only for higher levels of capital[5].

Thus, the central assumption we take is that technological change do not affect the product immediately but only with the convergence to the new steady state. We respectively dub

[1]Among many: Hayami and Ruttan (1970) compare the patterns of technical change in Japan and the US between 1880 and 1960 and conclude that factor prices (land and labor) were the main drivers of the direction of innovation; Allen (2009) argues that relatively high wages and low energy prices in Great Britain vis-à-vis the continent explains why the Industrial Revolution (which the author sees as an episode of substitution of capital and power for labor) originated in this country. For a conflicting view, see Temin (1966, 1971) and Field (1983).

[2]Acemoglu (2010) analyzes the link between labor scarcity and the rate of innovation in a range of neoclassical models, and shows that labor scarcity makes innovation accelerate when technology advances are 'strongly labor complementary', i.e. when they make the marginal product of labor decrease.

[3]One reason for the success of the approach based on productivity terms lies in the Uzawa-Robinson, which states that if the growth rate of a neoclassical growth model is constant, then technical change (i) is representable by a steady increase in some productivity term, and (ii) that this productivity term is the Harrod-neutral productivity term. What the traditional interpretation of this result misses is the fact that the growth rate of a growth model does not have to be strictly constant to reproduce the Kaldor facts. We know from Jones (2005) and León-Ledesma and Satchi (2013) that there exist neoclassical growth models where technical change reflects itself in alterations of the production function – rather than in increases of a productivity index of some unaltered production function – that generate the Kaldor facts – see also the parts of Jones (2003) that was not published in Jones (2005), Growiec (2008) and Allen (2012).

[4]We have seen in Senouci (2014) that the switches of regime in 18th and 19th century Great Britain were not consistent with a labor-augmenting vs. investment-specific technical change approach, and that among OECD countries, the relatively less developed countries display higher capital shares than advanced countries. Our approach is consistent with these empirical facts.

On the relationships between capital share and GDP per capita, see Gollin (2002), Rodriguez and Ortega (2006) and Maarek (2012).

[5]Since different production techniques most often make use of *qualitatively* different capital goods, we suppose that each type of capital is produced from the consumption good through a linear technology, then we scale all production functions to define them over the argument 'consumption-goods-expenses on capital goods' rather than 'physical units of capital', in order to have a common unit for inputs. In what follows we simply refer to this variable as 'capital'.

as 'complementation' and 'substitution' the first and second type of technological change[6].

The distinction between capital-saving and labor-saving forms of technical change is common in economic thought[7]. However, neoclassical theory so far has not seen a satisfying, simple way to incorporate these forms of technical change into standard models[8].

Complementation increases the marginal product of labor (without affecting product) around the initial steady state, and triggers convergence towards a new steady state with lower capital expenses and lower product; while substitution increases the marginal product of capital (without affecting product) around the initial steady state and triggers convergence towards a new steady state with higher capital expenses and higher product. Since we assume that the quantity of labor is fixed in the economy, these changes ultimately reflect themselves respectively into lower and higher capital-intensity. In this sense, we can qualify complementation as the capital-saving direction, and substitution as the labor-saving direction of technical change.

Thus, both types of technological change lead to totally different dynamic (and steady-state) outcomes; however, this is very simple to prove that both types of technological change are desirable from the point of view of the agent – both strictly increase his intertemporal utility.

We investigate two types of paths, both from a successive-steady states approach: *(a)* paths driven by exogenous technological change, and *(b)* paths with endogenous choice of the direction of technological change. The first issue needs only the specification of the family of production functions that we consider. The second issue requires to have knowledge of the value function of the Ramsey problem, which (as is well-known) we do only under very narrow assumptions. Thus, at this stage we are able to analyze exogenous-technological change paths under the assumptions that the production functions are either Cobb-Douglas or CES; and we are able to analyze endogenous-direction of technological change paths under the assumptions that the production functions are Cobb-Douglas and that the instantaneous utility function is logarithmic.

We prove the following results.

When considering exogenous technological change, we find that under 'steady comple-

[6]For instance, the early cars constituted a substitution change (away from horses), while later the invention of less expensive cars constituted a complementation change.

[7]See the excellent discussion by Rosenberg (1963).

[8]Atkinson and Stiglitz (1969) first introduced the concept of "localized" technical progress, and qualitatively analyzed its influence on the production function. Barrère (1961), Champernowne (1961) and Zeira (1998, 2006) investigate models where growth is driven by substitution of capital for labor; Seater (2005), Zuleta (2008) and Peretto and Seater (2013) present frameworks where technical change closely resemble the one we consider here. The "putty-clay" literature has also tackled the issue of the determination of capital-intensity as a technology phenomenon – see, for instance, Caballero and Hammour (1998) and Gilchrist and Williams (2000). Allen (2012) investigates the empirical evolution of the World production function since 1820 and concludes that:

> "It is remarkable that countries in 1990 with low capital labor ratios achieved an output per worker that was no higher than countries with the same capital labor ratio in 1820. In the course of the last two hundred years, the rich countries created the production function of the world that defines the growth possibilities of poor countries today." (Allen (2012, abstract))

mentation' consumption and output converge (from above) to constants, while capital expenses and the capital share both tend to zero, in both the Cobb-Douglas and CES cases. In the 'steady substitution' regime, the growth rates of output, capital expenses and consumption converge to some common strictly positive limit, while the capital share tends to one in the limit – in both the Cobb-Douglas and CES cases[9].

When the direction of technological change is endogenous, we find a bias towards complementation whenever the agent is impatient: in the most general case with whatever family of production functions, the agent might increase more steady-state consumption through substitution than through complementation but still choose complementation for the short-term rewards it brings. With the assumptions that the production functions are Cobb-Douglas and that the utility function is logarithmic, we find support for the Habakkuk hypothesis in the sense that the agent never undertakes pure complementation asymptotically: whenever the labor share decreases below some threshold, the agent switches to substitution – which increases capital share. Furthermore, when the potential rate of complementation is greater than the potential rate of substitution, the agent periodically changes the direction of technological change; this regime yields dynamics that fulfill the Kaldor facts while depicting the stationarity of the labor share as an endogenous, technological phenomenon.

The rest of the paper is organized as follows. Section 1 sets the basic Ramsey framework with no technological change. Section 2 introduces the two forms of technological change. Section 3 analyzes the patterns of exogenous growth with Cobb-Douglas and CES production functions. Section 4 treats the endogenous direction of technological change with Cobb-Douglas production functions and logarithmic utility and the resulting growth paths. Section 5 concludes.

1 The Ramsey framework and the value function

We use the discrete-time version of the Ramsey model with total depreciation. In the absence of technological change, the problem of the agent is:

$$\left|\begin{array}{ll} \max_{(c_0,c_1,\ldots)} & \sum_{t=0}^{\infty}\left(\frac{1}{1+\theta}\right)^t u(c_t) \\ \underline{\text{s.t.}} & x_{t+1}=f_\varphi(x_t)-c_t \\ & x_0=\bar{x} \end{array}\right. \tag{1}$$

where f_φ is a neoclassical production function. This is well-known that problem (1) converges through a saddle path to the interior steady state[10]:

$$f'_\varphi\left(x^*_\varphi\right)=1+\theta$$

[9]Of course, these results depend on the specific assumptions we make on the *amount* of technological change – especially on the amount of substitution that we allow for at each period. But we believe that these assumptions deserve special attentions, since they bring asymptotically steady growth.

[10]An interior steady state exists as long as $f'_\varphi(0)>1+\theta$ and $f'_\varphi(\infty)<1+\theta$, which we assume all through the paper for any technique φ.

We call A^*_φ the value of production at steady state:

$$A^*_\varphi \equiv f_\varphi\left(x^*_\varphi\right)$$

Consumption at steady state is then:

$$c^*_\varphi = A^*_\varphi - x^*_\varphi$$

The competitive capital share at any date t is: $\alpha_t = \frac{x_t f'_\varphi(x_t)}{f_\varphi(x_t)}$, so at steady state:

$$\alpha^*_\varphi = \frac{x^*_\varphi f'_\varphi\left(x^*_\varphi\right)}{f_\varphi\left(x^*_\varphi\right)} = \frac{(1+\theta)\, x^*_\varphi}{A^*_\varphi} > 0 \tag{2}$$

Let's denote by $U\left(x_0;\varphi\right)$ the value function corresponding to problem (1) when the production function is f_φ. We already know that:

$$U\left(x^*_\varphi;\varphi\right) = \frac{1+\theta}{\theta} u\left(c^*_\varphi\right) \tag{3}$$

Besides, the recursive formulation of problem (1) is:

$$U\left(x_0;\varphi\right) = \max_{x_1}\left[u\left(f_\varphi(x_0) - x_1\right) + \frac{1}{1+\theta} U\left(x_1;\varphi\right)\right]$$

so the envelope theorem applied to any $x_0 > 0$ yields:

$$\frac{\partial U\left(x_0;\varphi\right)}{\partial x_0} = f'_\varphi(x_0)\; u'\left(f_\varphi(x_0) - x^*_1(x_0)\right) = f'_\varphi(x_0)\; u'\left(c^*_0(x_0)\right), \tag{4}$$

where $c^*_0(x_0)$ stands for the optimal choice of c_0 given x_0 in problem (1) and where $x^*_1(x_0) = f_\varphi(x_0) - c^*_0(x_0)$.

In particular, equation (4) applied at steady state $\left(x_0 = x^*_\varphi\right)$ yields:

$$\left.\frac{\partial U}{\partial x}\right|_{\left(x^*_\varphi;\varphi\right)} = (1+\theta)\, u'\left(c^*_\varphi\right) \tag{5}$$

2 The technological opportunities: complementation vs. substitution

Suppose now that the agent initially stands at the steady state corresponding to technique φ. At date $t = 0$, function f_φ is altered and turned into function $f_{\varphi'}$. The specific assumption we make on $f_{\varphi'}$ is:

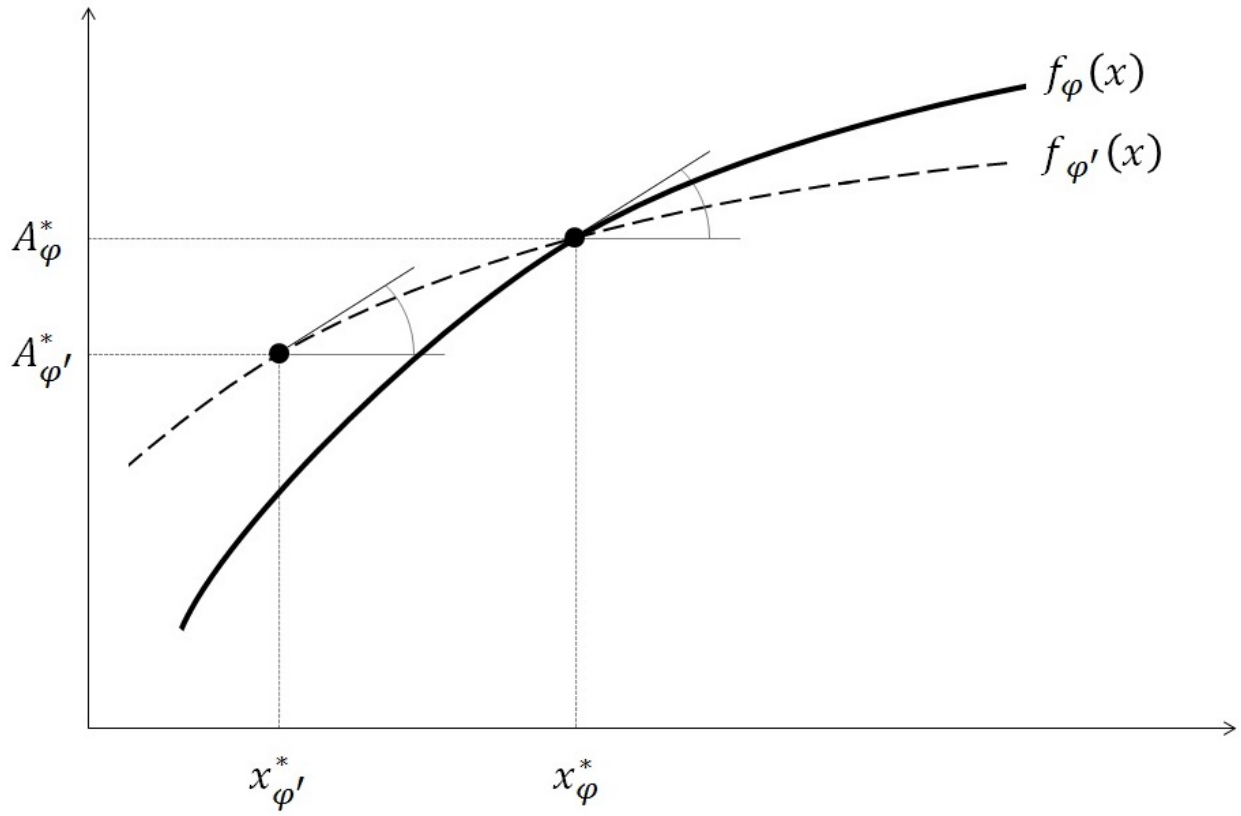

Figure 1: *The first technological opportunity – complementation* $\left(f'_{\varphi'}(x^*_\varphi) < 1+\theta\right)$.

Assumption. *The new production function yields exactly the same output than the old production function at the old steady state:*

$$f_{\varphi'}\left(x^*_\varphi\right) = f_\varphi\left(x^*_\varphi\right) = A^*_\varphi$$

Thus the altering of the production function does not influence production at date $t = 0$ and only does so through capital decumulation or accumulation. The agent will converge to the new steady state characterized by:

$$f'_{\varphi'}\left(x^*_{\varphi'}\right) = 1+\theta$$

As shown on figures 1 and 2, the location of the new steady state depends on the value of the derivative of the new production function $f_{\varphi'}$ at the original capital level x^*_φ. Precisely:

– if $f'_{\varphi'}(x^*_\varphi) < 1+\theta$, then $x^*_{\varphi'} < x^*_\varphi$ and $A^*_{\varphi'} < A^*_\varphi$;

– if $f'_{\varphi'}(x^*_\varphi) > 1+\theta$, then $x^*_{\varphi'} > x^*_\varphi$ and $A^*_{\varphi'} > A^*_\varphi$.

In the first case (figure 1), we say that technological change takes the 'complementation' form; while in the second case (figure 2) we talk about 'substitution'.

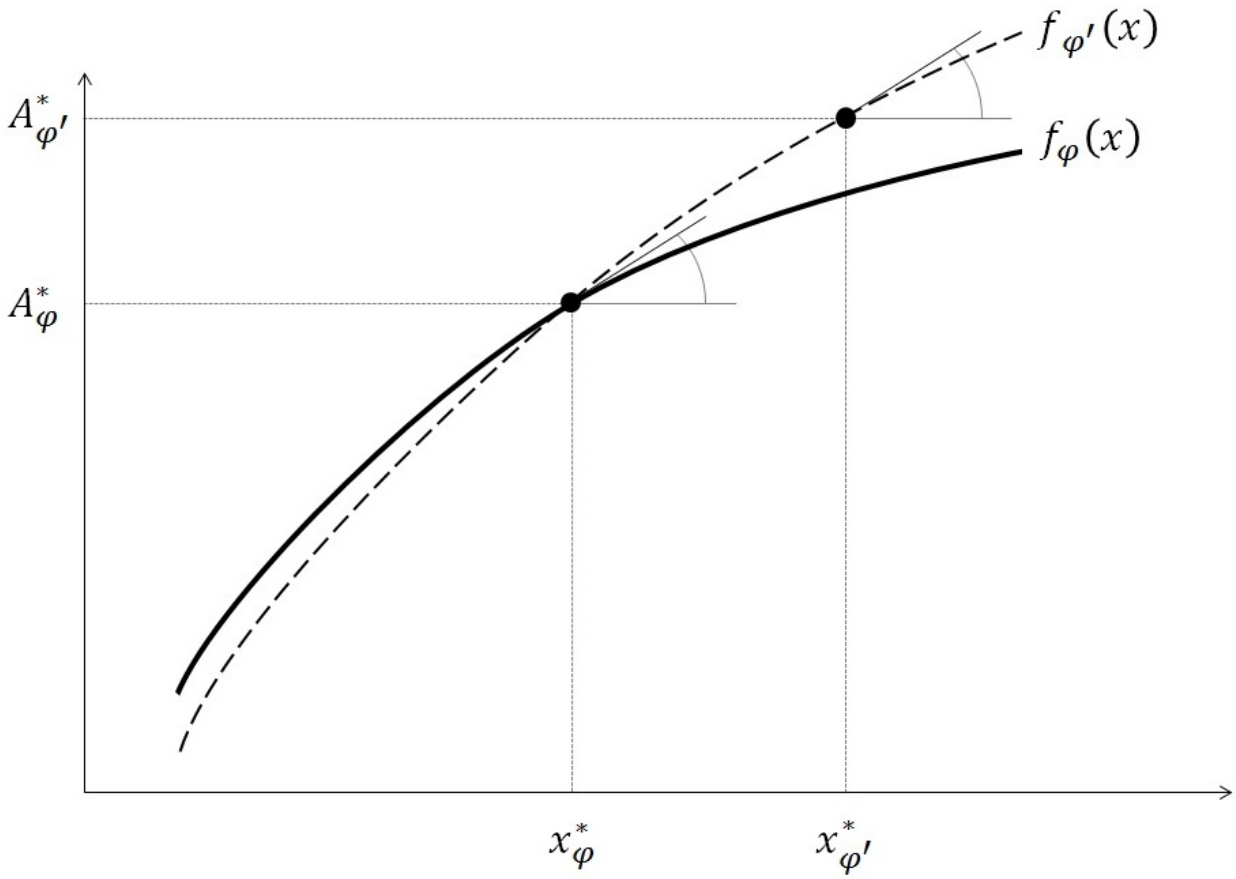

Figure 2: *The second technological opportunity – substitution* $\left(f'_{\varphi'}(x^*_{\varphi}) > 1+\theta\right)$.

The net welfare gains due to the passage from technique φ to φ' while initially at the φ-steady state are:

$$\Delta U_{\varphi\to\varphi'} = U\left(x^*_\varphi, \varphi'\right) - U\left(x^*_\varphi, \varphi\right) = U\left(x^*_\varphi, \varphi'\right) - \frac{1+\theta}{\theta} u\left(c^*_\varphi\right)$$

Notice that if $f'_{\varphi'}(x^*_\varphi) = 1+\theta$, the new steady state coincides with the original one so the altering of the production function has strictly no effect on the consumption path, and in particular $\Delta U_{\varphi\to\varphi'} = 0$.

In all other cases, we can state a strikingly simple result:

Theorem 1. *If Assumption 2 holds and if $f'_{\varphi'}\left(x^*_\varphi\right) \neq 1+\theta$, then:*

$$\Delta U_{\varphi\to\varphi'} > 0.$$

Proof.

- At date $t=0$, the agent owns x^*_φ capital with which (Assumption 2) he manages to produce $A^*_\varphi = f_{\varphi'}(x^*_\varphi)$. If he saves x^*_φ at each period, he will be able to consume c^*_φ at each period and so ensure inter-temporal utility $\frac{1+\theta}{\theta} u\left(c^*_\varphi\right)$. So under Assumption 2, the consumption path $\left[\forall t \geq 0,\ c_t = c^*_\varphi\right]$ is feasible through technique φ' (see figures 1 and 2). Then: $\Delta U_{\varphi\to\varphi'} \geq 0$.
- If further $f'_{\varphi'}(x^*_\varphi) \neq 1+\theta$, then the agent can strictly increase his inter-temporal utility by marginally adjusting the capital level towards $x^*_{\varphi'}$ $\left(\neq x^*_\varphi\right)$. More precisely:
  - If $f'_{\varphi'}(x^*_\varphi) < 1+\theta$, the agent strictly increases inter-temporal utility by consuming marginally more than c^*_φ and decreasing capital level marginally below x^*_φ at date $t=0$;
  - If $f'_{\varphi'}(x^*_\varphi) > 1+\theta$, the agent strictly increases inter-temporal utility by consuming marginally less than c^*_φ and increasing capital level marginally above x^*_φ at date $t=0$.

Thus, whenever $f'_{\varphi'}(x^*_\varphi) \neq 1+\theta$, $\Delta U_{\varphi\to\varphi'} > 0$. □

So the two directions of technological change increase the subjective welfare of the agent. The generality of this result is noteworthy, and indicates that the switch from technique φ to φ' while initially at steady state corresponding to φ must represent an economy on scarce factors of production.

Both technological shocks leave product unchanged at date $t=0$ but influence consumption at date $t=0$. The agent initiates convergence to the new steady state $\left(x^*_{\varphi'}, A^*_{\varphi'}\right)$ by choosing some consumption level $c^*_{0\varphi'}$ which is strictly less than c^*_φ right after a complementary shock, and strictly greater than c^*_φ right after a substitution shock.

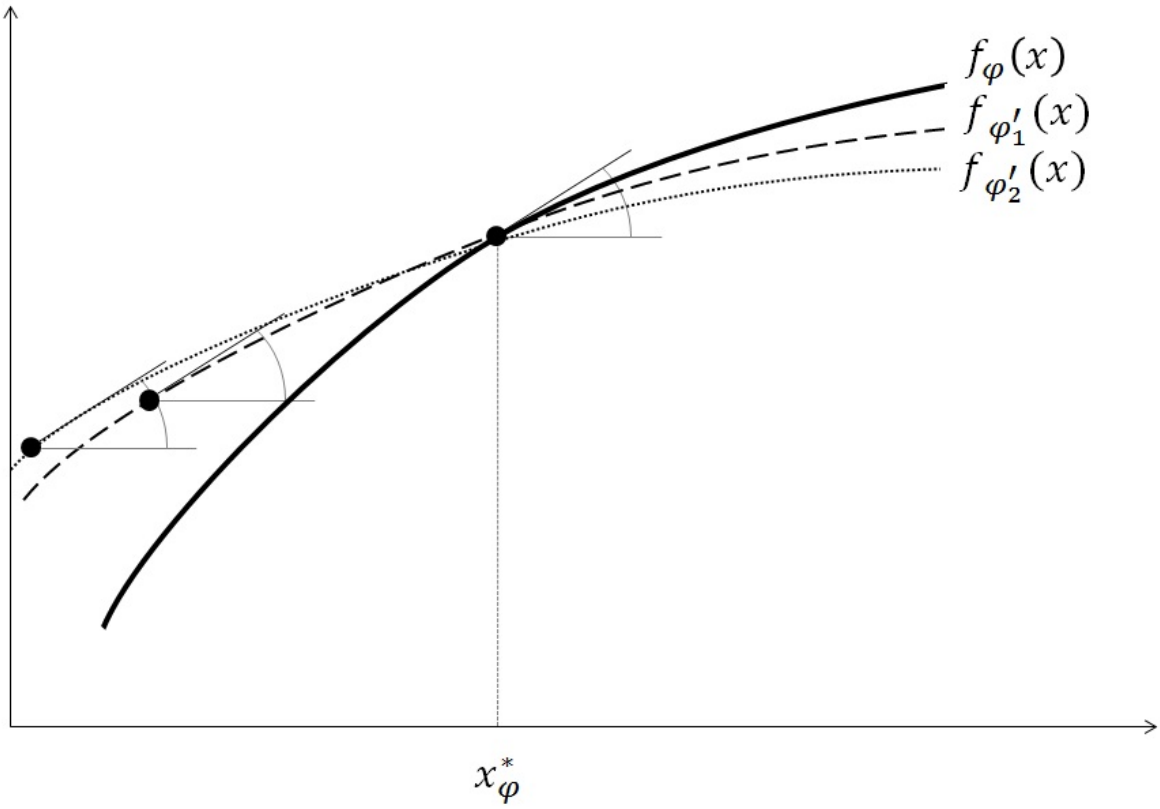

Figure 3: *Ranking technological opportunities – complementation.*

It is also true in some sense that the most 'aggressive' the alteration of the first derivative of the production function around the original steady state, the higher the increase in indirect utility will be. We make it clear in the following proposition:

Proposition 1. *Let $\mathbb{F}_\varphi$ be a family of neoclassical production functions satisfying Assumption 2, and such that for all functions $f_{\varphi_1'}$, $f_{\varphi_2'} \in \mathbb{F}_\varphi$, $f_{\varphi_1'}$ and $f_{\varphi_2'}$* ***only*** *cross at x_φ^*. Then:*

- *If $f'_{\varphi_2'}(x_\varphi^*) < f'_{\varphi_1'}(x_\varphi^*) < 1+\theta$, then: $\Delta U_{\varphi\to\varphi_2'} > \Delta U_{\varphi\to\varphi_1'}$ (>0);*
- *If $f'_{\varphi_2'}(x_\varphi^*) > f'_{\varphi_1'}(x_\varphi^*) > 1+\theta$, then: $\Delta U_{\varphi\to\varphi_2'} > \Delta U_{\varphi\to\varphi_1'}$ (>0).*

Proof. See appendix. □

The situation is depicted on figures 3 and 4. The formal proof reflects the property that, with the notations we have taken, $f_{\varphi_2'}$ always lies above $f_{\varphi_1'}$ in the interval of capital that is reached by the agent after each type of shock: in the complementation case (figure 3), the agent tends to decrease x, but $f_{\varphi_2'}$ lies above $f_{\varphi_1'}$ over the $(0, x_\varphi^*)$ interval so φ_2' is preferred; in the substitution case (figure 4), the agent tends to increase x, but $f_{\varphi_2'}$ lies above $f_{\varphi_1'}$ over the $(x_\varphi^*, +\infty)$ interval so φ_2' is preferred.

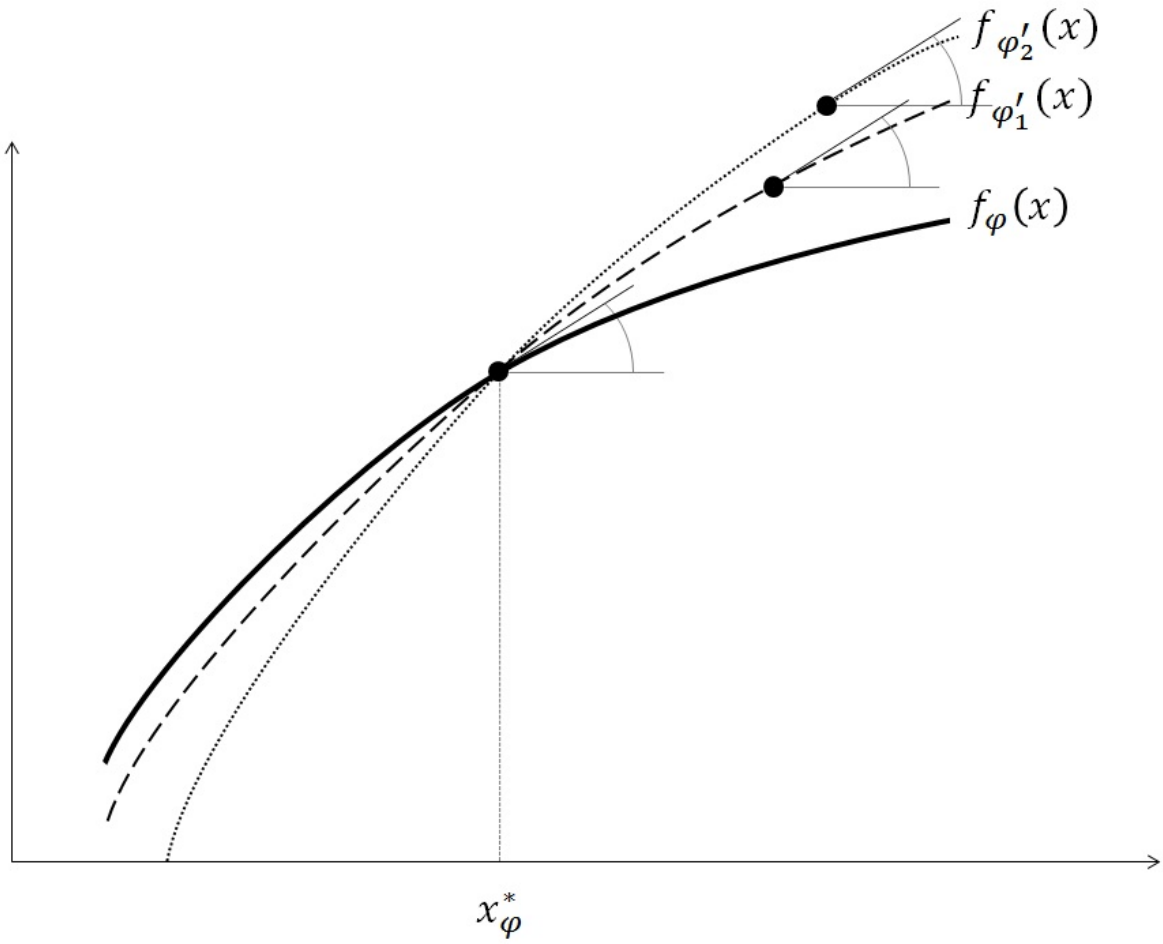

Figure 4: *Ranking technological opportunities – substitution.*

At this level of generality regarding the production functions and the utility function, we have proven through theorem 1 that the agent always benefits from any 'rotation' of the production function around the steady state $(x^*_\varphi, A^*_\varphi)$ corresponding to some technique φ, and through proposition 1 that for any direction, he prefers more rotation than less.

The steady-state consequences of complementation and substitution

Again at this level of generality, we can derive some results on the steady-state consequences of technological change:

Proposition 2. *If $f'_{\varphi'} > 1+\theta$ (substitution case), then:*

$$c^*_{\varphi'} > c^*_\varphi.$$

Proof. In all cases, variation in inter-temporal utility can be written:

$$\begin{aligned}\Delta U_{\varphi\to\varphi'} &= U\left(x^*_{\varphi'};\varphi'\right) - U\left(x^*_\varphi;\varphi\right) - \int_{x^*_\varphi}^{x^*_{\varphi'}} \frac{\partial U(x;\varphi')}{\partial x}\,\mathrm{d}x \\ &= \frac{1+\theta}{\theta}\left[u\left(c^*_{\varphi'}\right) - u\left(c^*_\varphi\right)\right] - \int_{x^*_\varphi}^{x^*_{\varphi'}} f'_{\varphi'}(x)\, u'\left(c^*_{0\varphi'}(x)\right)\,\mathrm{d}x\end{aligned}$$

where $c^*_{\varphi'}$ is the new steady-state value of consumption and where $c^*_{0\varphi'}(x)$ denotes the optimal choice of consumption at date $t=0$ in problem (1) with technique φ' and initial capital $x>0$.

When $f'_{\varphi'} > 1+\theta$, $x^*_\varphi < x^*_{\varphi'}$. So $\int_{x^*_\varphi}^{x^*_{\varphi'}} \underbrace{f'_{\varphi'}(x)\, u'(c^*_{0\varphi'}(x))}_{(>0)}\,\mathrm{d}x > 0$ in this case (substitution).

From theorem 1, $\Delta U_{\varphi\to\varphi'} > 0$.

If $c^*_{\varphi'} \le c^*_\varphi$, then $u(c^*_{\varphi'}) \le u(c^*_\varphi)$ then $\Delta U_{\varphi\to\varphi'} < 0$ from the formula above, which constitutes a contradiction to theorem 1. □

If $f'_{\varphi'} < 1+\theta$ (complementation case), $c^*_{\varphi'}$ might be greater or less c^*_φ. Indeed, when $\theta > 0$, the agent may increase steady-state consumption by economizing on capital, but his strictly positive impatience might also lead him to take advantage of the short-term opportunity of capital decumulation at the expense of steady-state consumption.

3 Exogenous growth with Cobb-Douglas/CES production functions

We now proceed to the analysis of the exogenous growth paths under the assumption that the production functions are all either Cobb-Douglas, or CES.

3.1 The Cobb-Douglas case

In this part we assume that all the production functions f_φ we consider are of Cobb-Douglas type.

For any $x^* > 0$, $A^* > 0$ and $\alpha \in (0,1)$, the only Cobb-Douglas production function of capital share parameter α and for which $f(x^*) = A^*$ and for is given by formula:

$$\forall x > 0, \;\; f(x) = A^* \left(\frac{x}{x^*} \right)^{\alpha}$$

We follow this method and write down all functions f_φ with respect to the steady-state corresponding to problem (1):

$$\forall x > 0, \;\; f_\varphi(x) = A^*_\varphi \left(\frac{x}{x^*_\varphi} \right)^{\alpha_\varphi}$$

with $A^*_\varphi > 0$, $x^*_\varphi > 0$ and $\alpha_\varphi \in (0,1)$.

In both the complementation and the substitution cases, the opportunity that is given to the agent is to alter parameter α_φ, that is, to move to technique φ' such that:

$$\forall x > 0, \;\; f_{\varphi'}(x) = A^*_\varphi \left(\frac{x}{x^*_\varphi} \right)^{\alpha_+} \tag{6}$$

so if $\alpha_+ < \alpha_\varphi$, then $f'_{\varphi'}(x^*_\varphi) < 1+\theta$ so technological change takes the form of complementation, while if $\alpha_+ > \alpha_\varphi$ then $f'_{\varphi'}(x^*_\varphi) > 1+\theta$, which means the agent undergoes substitution.

Production function $f_{\varphi'}$ can also be written with respect to the steady state corresponding to technique φ':

$$\forall x > 0, \;\; f_{\varphi'}(x) = A^*_{\varphi'} \left(\frac{x}{x^*_{\varphi'}} \right)^{\alpha_{\varphi'}} \tag{7}$$

The identification of parameters in (6) and (7) yields:

$$\alpha_{\varphi'} = \alpha_+ \tag{8}$$

$$\frac{A_{\varphi'}}{A_\varphi} = \left(\frac{x_{\varphi'}}{x_\varphi} \right)^{\alpha_{\varphi'}} \tag{9}$$

On the other hand, equation (2) applied to techniques φ and φ' yields:

$$\alpha_\varphi = \frac{(1+\theta)x^*_\varphi}{A^*_\varphi} \tag{10}$$

$$\alpha_{\varphi'} = \frac{(1+\theta)x^*_{\varphi'}}{A^*_{\varphi'}} \tag{11}$$

We get the final result from equations (2), (8) and (9); when the agent alters technique φ by turning α_φ into $\alpha_+ = \alpha_{\varphi'}$, the parameters of the new production function are:

$$x^*_{\varphi'} = x^*_\varphi \left(\frac{\alpha_+}{\alpha_\varphi} \right)^{\frac{1}{1-\alpha_+}} = x^*_\varphi \left(\frac{\alpha_{\varphi'}}{\alpha_\varphi} \right)^{\frac{1}{1-\alpha_{\varphi'}}} \tag{12}$$

$$A^*_{\varphi'} = A^*_\varphi \left(\frac{\alpha_+}{\alpha_\varphi} \right)^{\frac{\alpha_+}{1-\alpha_+}} = A^*_\varphi \left(\frac{\alpha_{\varphi'}}{\alpha_\varphi} \right)^{\frac{\alpha_{\varphi'}}{1-\alpha_{\varphi'}}} \tag{13}$$

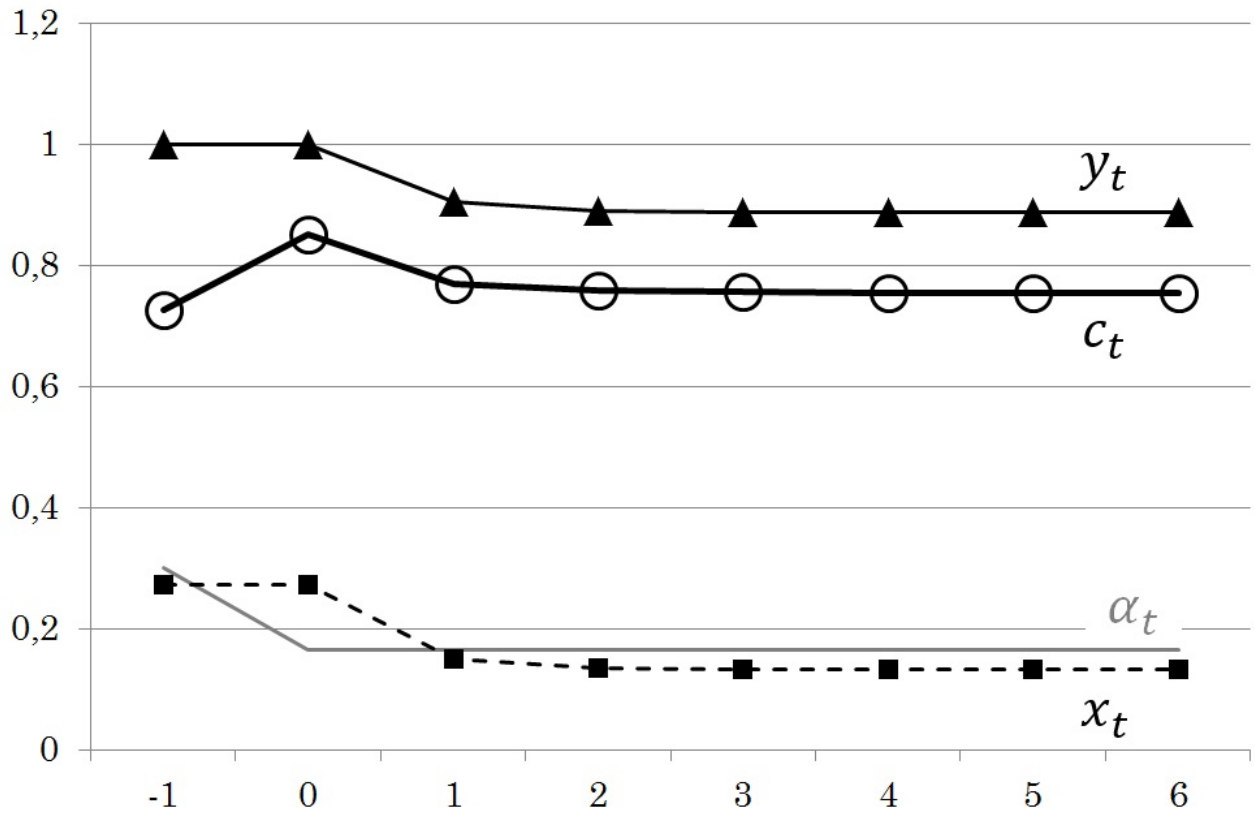

Figure 5: *Post-complementation shock dynamics* (Assumptions: $\alpha_0 = 0.3$; $A^*_\varphi = 1$; $\theta = 0.1$; $\alpha_{\varphi'} = (\alpha_\varphi)^{1.5}$).

We can check that if $\alpha_+ < \alpha_\varphi$, then $x^*_{\varphi'} < x^*_\varphi$ and $A^*_{\varphi'} < A^*_\varphi$; while if $\alpha_+ > \alpha_\varphi$, then $x^*_{\varphi'} > x^*_\varphi$ and $A^*_{\varphi'} > A^*_\varphi$.

Equations (10), (11), (12) and (13) permit to derive the variation of steady-state consumption:

$$c^*_{\varphi'} = A^*_{\varphi'} - x^*_{\varphi'} = A^*_\varphi \left(\frac{\alpha_{\varphi'}}{\alpha_\varphi}\right)^{\frac{\alpha_{\varphi'}}{1-\alpha_{\varphi'}}} - x^*_\varphi \left(\frac{\alpha_{\varphi'}}{\alpha_\varphi}\right)^{\frac{1}{1-\alpha_{\varphi'}}}$$

$$\Longrightarrow \quad c^*_{\varphi'} = c^*_\varphi \frac{1+\theta-\alpha_{\varphi'}}{1+\theta-\alpha_\varphi} \left(\frac{\alpha_{\varphi'}}{\alpha_\varphi}\right)^{\frac{\alpha_{\varphi'}}{1-\alpha_{\varphi'}}} \tag{14}$$

By proposition 2 we know that when $\alpha_{\varphi'} > \alpha_\varphi$ it always holds that $c^*_{\varphi'} > c^*_\varphi$. We can also see why we cannot conclude if $\alpha_{\varphi'} < \alpha_\varphi$.

3.1.1 Post-shock dynamics with Cobb-Douglas production functions and log utility – Solow

When the production function is Cobb-Douglas and when utility is logarithmic, this is well-known that substitution and wealth effects cancel out so the agent simply chooses the constant saving rate $s = \alpha/1+\theta$ at each period. The model is then isomorphic to a Solow model

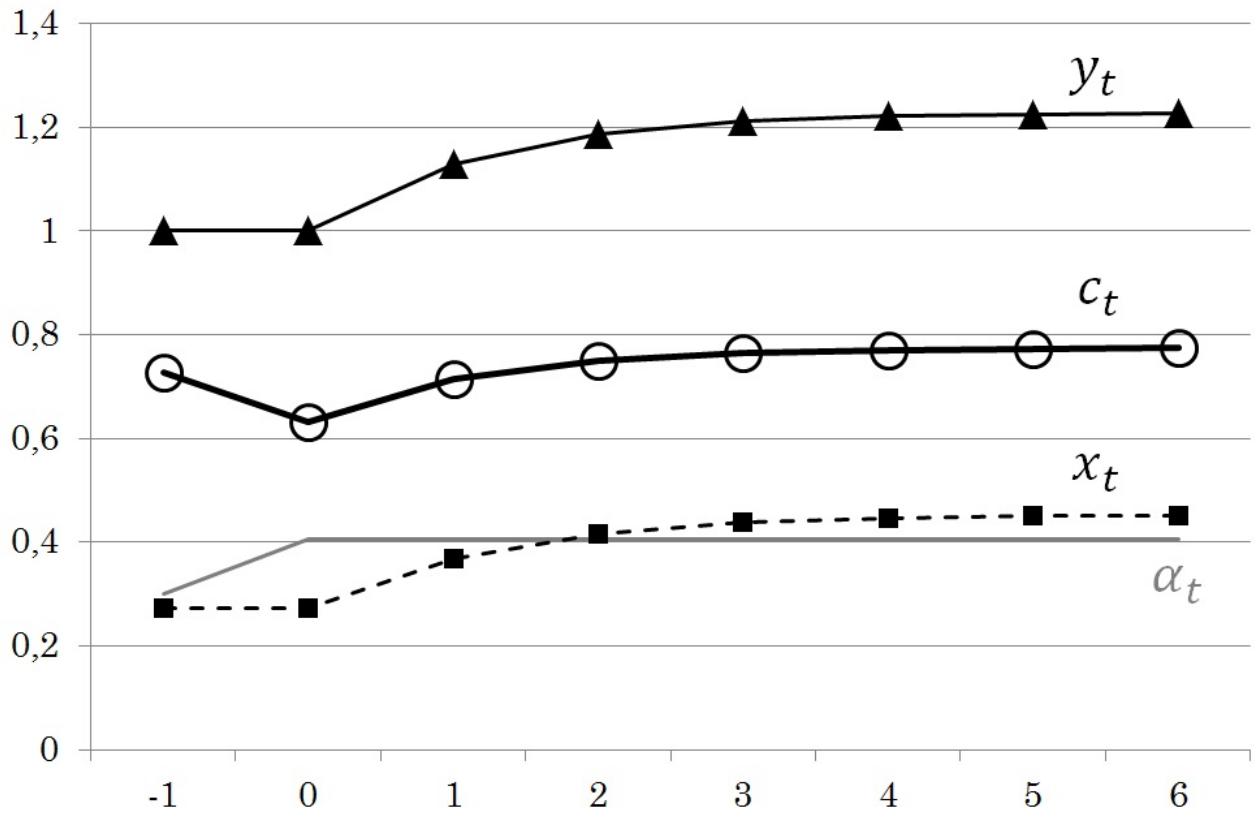

Figure 6: *Post-substitution shock dynamics* (Assumptions: $\alpha_0 = 0.3$; $A^*_\varphi = 1$; $\theta = 0.1$; $\alpha_{\varphi'} = (\alpha_\varphi)^{0.5}$).

and we can easily simulate convergence to the new steady state[11]. Figure 5 shows one convergence path for a complementation case by assuming $\alpha_{\varphi'} = (\alpha_\varphi)^{1.5}$, figure 6 does the same in the substitution case assuming $\alpha_{\varphi'} = (\alpha_\varphi)^{0.5}$. In both cases we assume $\alpha_\varphi = 0.3$, $A^*_\varphi = 1$ and $\theta = 0.1$. We represent the system of the system from date $t = -1$ on, where $t = -1$ that corresponds to the φ-steady state.

In all cases production (y_t) and capital (x_t) remain unaffected at date $t = 0$. However, consumption changes from date $t = 0$.

The complementation shock (figure 5) induces the agent to increase consumption (and so make capital decrease) at $t = 0$; the new steady state features lower GDP, lower capital and, in this case, higher consumption. Figure 7 changes the value of the impatience rate to $\theta = 2$; in this case, the new steady-state consumption is strictly less than the original steady-state consumption.

The substitution shock (figure 6) induces the agent to decrease consumption (and so make capital increase) at $t = 0$ and the new steady state has higher GDP, higher capital and higher consumption (proposition 1).

3.1.2 The sequence of steady states with Cobb-Douglas production functions

[11] The software used is Microsoft Excel 2010.

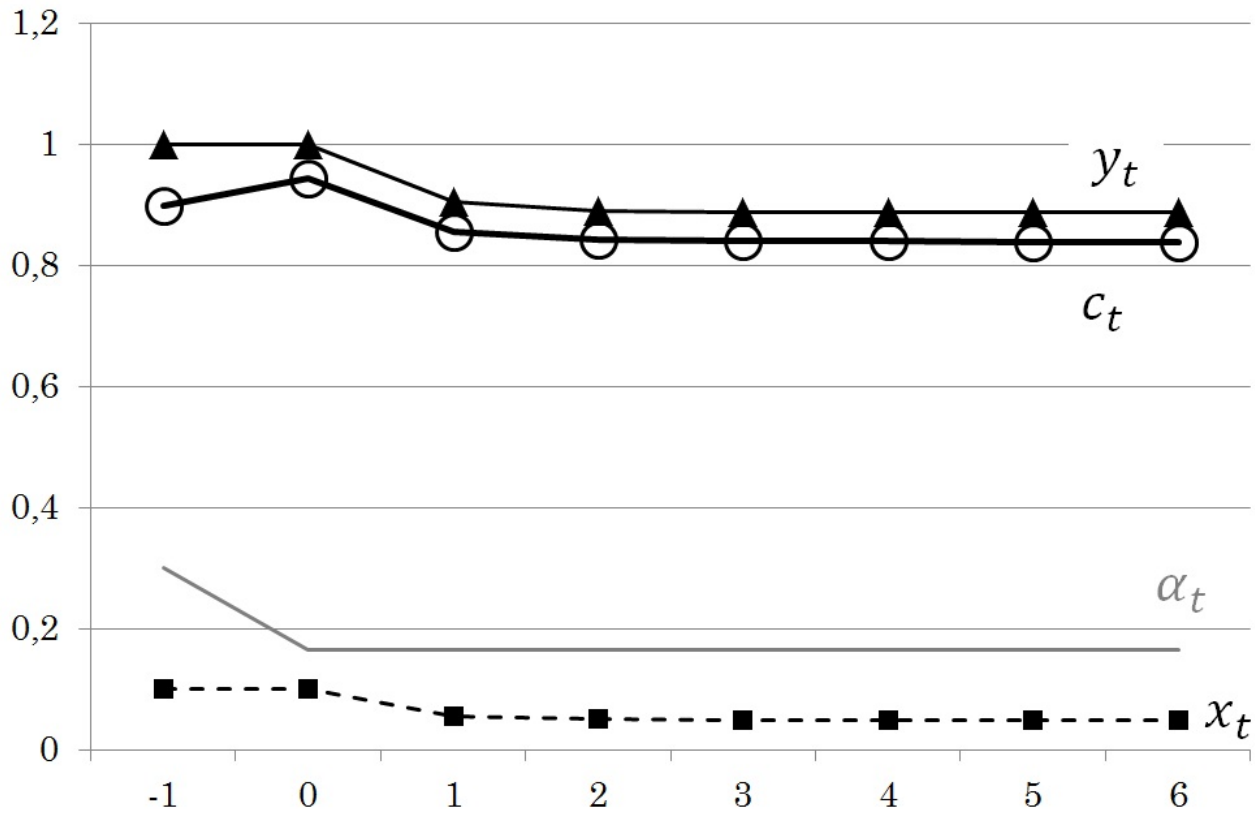

Figure 7: *Post-complementation shock dynamics in a case where steady-state consumption decreases* $\left(c^*_{\varphi'} < c^*_{\varphi}\right)$ (Assumptions: $\alpha_0 = 0.3$; $A^*_{\varphi} = 1$; $\theta = 2$; $\alpha_{\varphi'} = (\alpha_{\varphi})^{1.5}$).

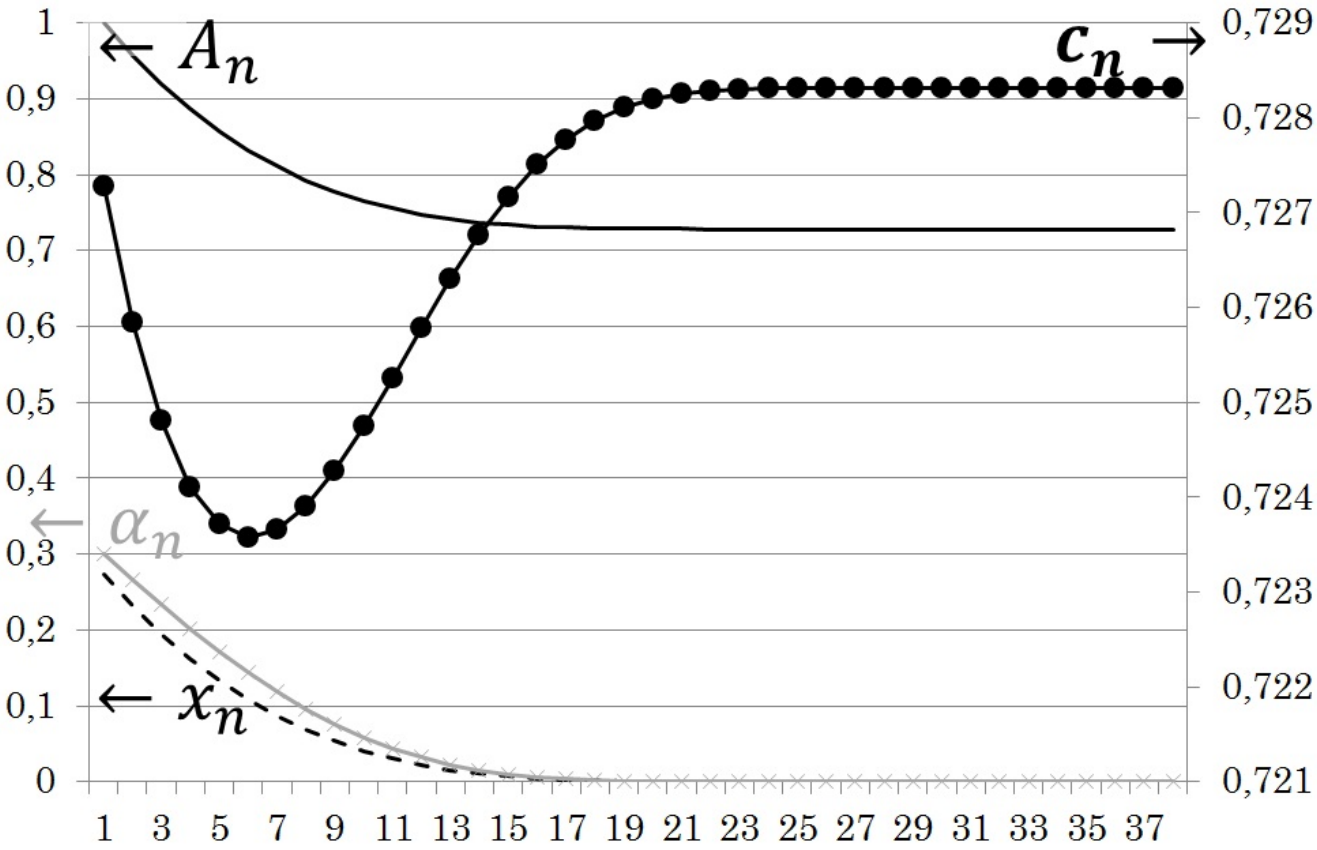

Figure 8: *Successive steady states with steady complementation* (Assumptions: $\alpha_0 = 0.3$; $A_0 = 1$; $\theta = 0.1$; $\mu = 0.1$).

a) The complementation case — Suppose now that the agent undergoes successive technological shocks of the form:

$$\alpha_{+,n} = \alpha_n^{1+\mu}$$

with $\mu > 0$. If we denote respectively by A_n, x_n and c_n the values of production, capital and consumption at each steady state, then from equations (8), (12), (13) and (14):

$$\alpha_{n+1} = \alpha_n^{1+\mu} \tag{15}$$

$$\frac{A_{n+1}}{A_n} = \alpha_n^{\mu \frac{\alpha_n^{1+\mu}}{1-\alpha_n^{1+\mu}}} \tag{16}$$

$$\frac{x_{n+1}}{x_n} = \alpha_n^{\mu \frac{1}{1-\alpha_n^{1+\mu}}} \tag{17}$$

$$\frac{c_{n+1}}{c_n} = \frac{1+\theta-\alpha_n^{1+\mu}}{1+\theta-\alpha_n} \alpha_n^{\mu \frac{\alpha_n^{1+\mu}}{1-\alpha_n^{1+\mu}}} \tag{18}$$

with the initial conditions α_0, A_0 and $x_0 = \frac{\alpha_0 A_0}{1+\theta}$.

From (15), $(\log \alpha_n)$ is a negative geometric sequence with common ratio $1+\mu > 1$, so $\alpha_n = \alpha_0^{(1+\mu)^n} \xrightarrow{n\to\infty} 0$.

Also since $\mu \frac{\alpha_n^{1+\mu}}{1-\alpha_n^{1+\mu}} > 0$ and $\alpha_n \in (0,1)$, then from equation (16): $(0 <)\, A_{n+1} < A_n$. So (A_n) has a positive limit. Since $\alpha_n = \alpha_0^{(1+\mu)^n}$ we can transform equation (16) into:

$$\log \frac{A_{n+1}}{A_n} = -\frac{\alpha_0^{(1+\mu)^{n+1}}}{1-\alpha_0^{(1+\mu)^{n+1}}} \mu(1+\mu)^n \log \frac{1}{\alpha_0}$$

The series $\sum (1+\mu)^n \frac{\alpha_0^{(1+\mu)^{n+1}}}{1-\alpha_0^{(1+\mu)^{n+1}}}$ is convergent, because $(1+\mu)^n \frac{\alpha_0^{(1+\mu)^{n+1}}}{1-\alpha_0^{(1+\mu)^{n+1}}} > 0$ and $n^2 (1+\mu)^n \frac{\alpha_0^{(1+\mu)^{n+1}}}{1-\alpha_0^{(1+\mu)^{n+1}}} \xrightarrow{n\to\infty} 0$ the term in $n^2(1+\mu)^n$ being dominated by the term $\alpha_0^{(1+\mu)^{n+1}}$. Consequently, output converges to A_∞ such that:

$$\ln A_\infty = \ln A_0 - \mu \log \frac{1}{\alpha_0} \sum_{n=0}^{\infty} (1+\mu)^n \frac{\alpha_0^{(1+\mu)^{n+1}}}{1-\alpha_0^{(1+\mu)^{n+1}}} \tag{19}$$

Since $\forall n,\ x_n = \frac{\alpha_n A_n}{1+\theta}$, then $x_n \xrightarrow{n\to\infty} 0$. Consequently, $c_n = A_n - x_n \xrightarrow{n\to\infty} A_\infty\ (> 0)$.

The value of A_n, x_n and c_n at the successive steady states are represented on figure 8, by taking $\alpha_0 = 0.3$, $A_0 = 1$, $\theta = 0.1$ and $\mu = 0.1$. A_n and x_n are on the left-hand scale and c_n is on the right-hand scale. For these values of the parameters, c_n first decreases and then increases, in other cases (notably for high θ), $c_{n+1} < c_n$ for all $n > 0$.

b) The substitution case — Suppose now that the agent undertakes successive steps of substitution, so:

$$\alpha_{+,n} = \alpha_n^{1-\lambda}$$

with $\lambda \in (0,1)$, so now $\alpha_{+,n} > \alpha_n$. Then:

$$\alpha_{n+1} = \alpha_n^{1-\lambda} \tag{20}$$

$$\frac{A_{n+1}}{A_n} = \alpha_n^{-\lambda \frac{\alpha_n^{1-\lambda}}{1-\alpha_n^{1-\lambda}}} \tag{21}$$

$$\frac{x_{n+1}}{x_n} = \alpha_n^{-\lambda \frac{1}{1-\alpha_n^{1-\lambda}}} \tag{22}$$

$$\frac{c_{n+1}}{c_n} = \frac{1+\theta-\alpha_n^{1-\lambda}}{1+\theta-\alpha_n} \alpha_n^{-\lambda \frac{\alpha_n^{1-\lambda}}{1-\alpha_n^{1-\lambda}}} \tag{23}$$

Now $(\log \alpha_n)$ is a negative geometric sequence with common ratio $1-\lambda \in (0,1)$, so:

$$\alpha_n = \alpha_0^{(1-\lambda)^n} \xrightarrow{n\to\infty} 1.$$

To derive the limit growth rate of A, we start by applying the logarithm to (21):

$$\ln \frac{A_{n+1}}{A_n} = \lambda \frac{\alpha_n^{1-\lambda}}{1-\alpha_n^{1-\lambda}} \ln \frac{1}{\alpha_n}$$

Then, we use the identity $\alpha_n = \alpha_0^{(1-\lambda)^n}$ to write down:

$$\ln \frac{A_{n+1}}{A_n} = \lambda \frac{\alpha_0^{(1-\lambda)^{n+1}}}{1-\alpha_0^{(1-\lambda)^{n+1}}} (1-\lambda)^n \ln \frac{1}{\alpha_0}$$

In this last identity, notice that $\alpha_0^{(1-\lambda)^{n+1}} \xrightarrow{n\to\infty} 1$, so:

$$\ln \frac{A_{n+1}}{A_n} \sim_{n\infty} \lambda \frac{(1-\lambda)^n}{1-\alpha_0^{(1-\lambda)^{n+1}}} \ln \frac{1}{\alpha_0} \tag{24}$$

The term in $\frac{(1-\lambda)^n}{1-\alpha_0^{(1-\lambda)^{n+1}}}$ is an indeterminate form. We use de l'Hôpital's rule: if any, $\frac{(1-\lambda)^n}{1-\alpha_0^{(1-\lambda)^{n+1}}}$ has the same limit than:

$$\frac{\ln(1-\lambda)(1-\lambda)^n}{-\ln(1-\lambda)\ln \alpha_0 (1-\lambda)^{n+1} \alpha_0^{(1-\lambda)^{n+1}}}$$

which has limit $\frac{1}{(1-\lambda)\ln \frac{1}{\alpha_0}}$ as n tends to infinity. We inject this result in equation (24) to get the limit growth rate of output:

$$\ln \frac{A_{n+1}}{A_n} \xrightarrow{n\to\infty} \frac{\lambda}{1-\lambda} \tag{25}$$

Since $\forall n,\ x_n = \frac{\alpha_n A_n}{1+\theta}$, equation (25) yields:

$$\ln \frac{x_{n+1}}{x_n} \xrightarrow{n\to\infty} \frac{\lambda}{1-\lambda} \tag{26}$$

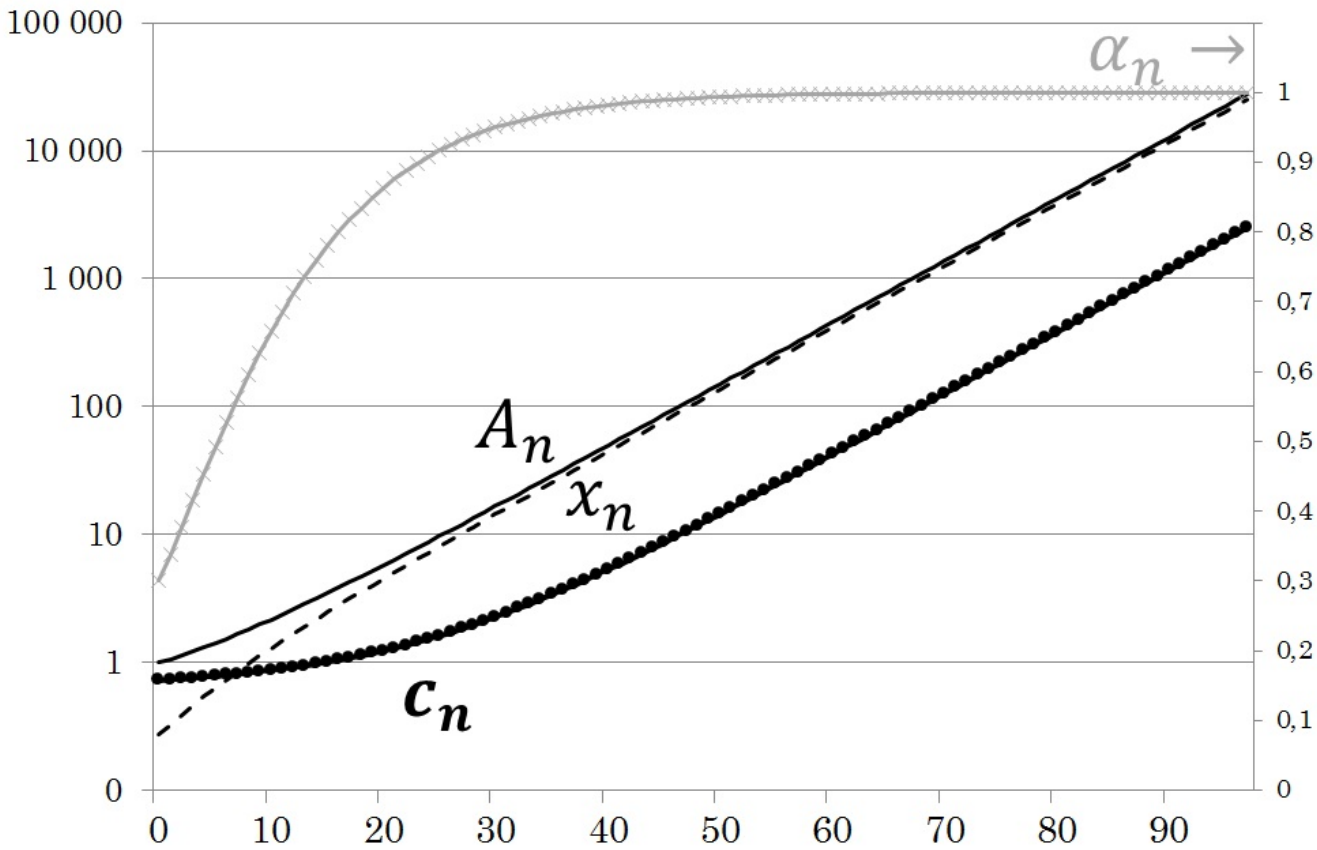

Figure 9: *Successive steady states with steady substitution, log scale* (Assumptions: $\alpha_0 = 0.3$; $A_0 = 1$; $\theta = 0.1$; $\lambda = 0.1$).

Finally, the term $\frac{1+\theta-\alpha_n^{1-\lambda}}{1+\theta-\alpha_n}$ tends to one as n tends to infinity, so from (23) we also have:

$$\ln \frac{c_{n+1}}{c_n} \xrightarrow{n\to\infty} \frac{\lambda}{1-\lambda} \tag{27}$$

Through continuous substitution, output, capital and consumption all grow asymptotically at the same gross rate of $e^{\frac{\lambda}{1-\lambda}} > 1$. The capital share tends to one and, in this framework, substitution possibilities are never exhausted. This pattern is represented on figure 9.

We could have expected the substitution path to meet some bottleneck too[12]. Indeed, the sustainability of economic growth through pure substitution is the result of the assumption that we took on the transition between α_n and α_{n+1}. When assuming that this transition takes the form in (20), α_n converges quickly enough to 1 so growth can be sustained – for some other specifications of $\alpha_{+,n}$ as a function of α_n this result would of course not hold.

Still, we believe that the fact that our specification brings not only sustainability but also asymptotic balance makes it valuable as a reference path.

[12]The presence of growth bottlenecks in the pure-complementation case above is much more general and holds beyond the Cobb-Douglas assumption, since A_n always decreases.

3.2 The CES case

Suppose now that the production functions f_φ, instead of being Cobb-Douglas, are all CES with elasticity of substitution (ES) $\sigma \in (0,1)$.

Moysan and Senouci (2013) have proven that, for any $x^* > 0$, $A^* > 0$ and $\alpha^* \in (0,1)$, the only CES production function with ES σ and for which $f(x^*) = A^*$ and $x^* f'(x^*)/f(x^*) = \alpha^*$ is given by formula:

$$\forall x > 0, \;\; f(x) = A^* \left(\alpha^* \left(\frac{x}{x^*} \right)^{\frac{\sigma-1}{\sigma}} + 1 - \alpha^* \right)^{\frac{\sigma}{\sigma-1}}$$

so we write function f_φ like:

$$f_\varphi(x) = A^*_\varphi \left(\alpha^*_\varphi \left(\frac{x}{x^*_\varphi} \right)^{\frac{\sigma-1}{\sigma}} + 1 - \alpha^*_\varphi \right)^{\frac{\sigma}{\sigma-1}}$$

The relationship $f_\varphi(x^*_\varphi) = 1 + \theta$ is equivalent to: $\alpha^*_\varphi = \frac{(1+\theta)x^*_\varphi}{A^*_\varphi}$.

Like in the Cobb-Douglas case, the agent can only alter parameter α^*_φ. The new production function $f_{\varphi'}$ can first of all be expressed through the old steady state:

$$f_{\varphi'}(x) = A^*_\varphi \left(\alpha_+ \left(\frac{x}{x^*_\varphi} \right)^{\frac{\sigma-1}{\sigma}} + 1 - \alpha_+ \right)^{\frac{\sigma}{\sigma-1}} \tag{28}$$

It can also be expressed through the new steady state:

$$f_{\varphi'}(x) = A^*_{\varphi'} \left(\alpha^*_{\varphi'} \left(\frac{x}{x^*_{\varphi'}} \right)^{\frac{\sigma-1}{\sigma}} + 1 - \alpha^*_{\varphi'} \right)^{\frac{\sigma}{\sigma-1}} \tag{29}$$

The identification of the terms of (28) and (29) yields two indepedent equations in terms of unknown $x^*_{\varphi'}, A^*_{\varphi'}, \alpha^*_{\varphi'}$:

$$A^{*\,\frac{\sigma-1}{\sigma}}_{\varphi'} (1 - \alpha^*_{\varphi'}) = A^{*\,\frac{\sigma-1}{\sigma}}_{\varphi} (1 - \alpha_+) \tag{30}$$

$$\left(\frac{A^*_{\varphi'}}{x^*_{\varphi'}} \right)^{\frac{\sigma-1}{\sigma}} \alpha^*_{\varphi'} = \left(\frac{A^*_{\varphi}}{x^*_{\varphi}} \right)^{\frac{\sigma-1}{\sigma}} \alpha_+ \tag{31}$$

Another independent relationship between $x^*_{\varphi'}, A^*_{\varphi'}$ and $\alpha^*_{\varphi'}$ is given by equation (2) applied to the steady state corresponding to the production function $f_{\varphi'}$:

$$\alpha^*_{\varphi'} = \frac{(1+\theta)x^*_{\varphi'}}{A^*_{\varphi'}} \tag{32}$$

Equations (30), (31) and (32) then constitute a set of three independent equations with three unknowns $x^*_{\varphi'}, A^*_{\varphi'}, \alpha^*_{\varphi'}$. This system solves in:

$$\alpha^*_{\varphi'} = (\alpha_+)^\sigma \, \alpha^{*\,1-\sigma}_\varphi \tag{33}$$

$$A^*_{\varphi'} = A^*_\varphi \left(\frac{1-\alpha_+}{1-\alpha^*_{\varphi'}} \right)^{\frac{\sigma}{\sigma-1}} = A^*_\varphi \left(\frac{1-\alpha_+}{1-(\alpha_+)^\sigma \, \alpha^{*\,1-\sigma}_\varphi} \right)^{\frac{\sigma}{\sigma-1}} \tag{34}$$

$$x^*_{\varphi'} = \frac{\alpha^*_{\varphi'}}{1+\theta} A^*_{\varphi'} \tag{35}$$

Now, $\alpha_+ \neq \alpha^*_{\varphi'}$ because as the agent transits on the new production function towards the new steady state, factor shares are altered. Equation (33) shows that there is however a simple relationship between $\alpha^*_{\varphi'}$ on the one hand and α^*_φ and α_+ on the other hand: $\alpha^*_{\varphi'}$ is the geometric mean of α_+ with weight σ and α^*_φ with weight $1-\sigma$.

By using equation (2) applied to function f_φ we can express $x^*_{\varphi'}$ in equation (35) through x^*_φ:

$$x^*_{\varphi'} = x^*_\varphi \left(\frac{1-\alpha_+}{1-\alpha^*_{\varphi'}} \right)^{\frac{\sigma}{\sigma-1}} \left(\frac{\alpha_+}{\alpha^*_\varphi} \right)^\sigma \quad \left(= x^*_\varphi \left(\frac{1-\alpha_+}{1-(\alpha_+)^\sigma \, \alpha^{*\,1-\sigma}_\varphi} \right)^{\frac{\sigma}{\sigma-1}} \left(\frac{\alpha_+}{\alpha^*_\varphi} \right)^\sigma \right) \tag{36}$$

Since $\sigma \in (0,1)$, we can verify that:

$$\begin{array}{lllll} \alpha_+ < \alpha^*_\varphi & \Longrightarrow & \alpha^*_{\varphi'} < \alpha^*_\varphi & \text{and} & A^*_{\varphi'} < A^*_\varphi \quad \text{and} \quad x^*_{\varphi'} < x^*_\varphi \\ \alpha_+ > \alpha^*_\varphi & \Longrightarrow & \alpha^*_{\varphi'} > \alpha^*_\varphi & \text{and} & A^*_{\varphi'} > A^*_\varphi \quad \text{and} \quad x^*_{\varphi'} > x^*_\varphi \end{array}$$

3.2.1 The sequence of steady states with CES production functions

Like in subsection 3.1.2 in the Cobb-Douglas case, we know turn to the analysis of the sequence of steady states with continual complementation or substitution. From equations (33), (34) and (35), steady-state variables α_n, A_n and x_n evolve according to:

$$\alpha_{n+1} = (\alpha_+(n))^\sigma \alpha_n^{1-\sigma} \tag{37}$$

$$A_{n+1} = A_n \left(\frac{1-\alpha_+(n)}{1-(\alpha_+(n))^\sigma \alpha_n^{1-\sigma}} \right)^{\frac{\sigma}{\sigma-1}} \tag{38}$$

$$x_{n+1} = \frac{\alpha_{n+1}}{1+\theta} A_{n+1} \tag{39}$$

with the initial conditions α_0, A_0 and $x_0 = \frac{\alpha_0 A_0}{1+\theta}$.

a) The complementation case — With continual complementation:

$$\alpha_+(n) = \alpha_n^{1+\mu}$$

So equation (37) becomes:

$$\alpha_{n+1} = \alpha_n^{1+\sigma\mu}$$

so for all n:

$$\alpha_n = \alpha_0^{(1+\sigma\mu)^n} \xrightarrow{n\to\infty} 0$$

Equation (38) yields:

$$A_{n+1} = A_n \left(\frac{1-\alpha_n^{1+\mu}}{1-\alpha_n^{1+\sigma\mu}} \right)^{\frac{\sigma}{\sigma-1}} \tag{40}$$

The series $\sum \ln \frac{1-\alpha_0^{(1+\sigma\mu)^n(1+\mu)}}{1-\alpha_0^{(1+\sigma\mu)^{n+1}}}$ is convergent, since:

$$\ln \frac{1-\alpha_0^{(1+\sigma\mu)^n(1+\mu)}}{1-\alpha_0^{(1+\sigma\mu)^{n+1}}} \sim_{n\infty} \frac{1-\alpha_0^{(1+\sigma\mu)^n(1+\mu)}}{1-\alpha_0^{(1+\sigma\mu)^{n+1}}} - 1 \sim_{n\infty} \alpha_0^{(1+\sigma\mu)^n} \left(\alpha_0^{1+\sigma\mu} - \alpha_0^{1+\mu} \right)$$

Thus, like in the Cobb-Douglas case, through continual complementation output converges to A'_∞ such that:

$$\ln A'_\infty = \ln A_0 + \frac{\sigma}{\sigma-1} \sum_{n=0}^{\infty} \frac{1-\alpha_0^{(1+\sigma\mu)^n(1+\mu)}}{1-\alpha_0^{(1+\sigma\mu)^{n+1}}} \ (< \ln A_0)$$

The capital share tends to zero, capital x_n tends to zero, and steady-state consumption tends to A'_∞.

b) The substitution case — We take:

$$\alpha_+(n) = \alpha_n^{1-\lambda}$$

So equation (37) becomes:

$$\alpha_{n+1} = \alpha_n^{1-\sigma\lambda}$$

Then, for all n:

$$\alpha_n = \alpha_0^{(1-\sigma\lambda)^n} \xrightarrow{n\to\infty} 1$$

Equation (38) in the case of substitution becomes:

$$A_{n+1} = A_n \left(\frac{1-\alpha_n^{1-\lambda}}{1-\alpha_n^{1-\sigma\lambda}} \right)^{\frac{\sigma}{\sigma-1}} = A_n \left(\frac{1-\alpha_0^{(1-\sigma\lambda)^n(1-\lambda)}}{1-\alpha_0^{(1-\sigma\lambda)^{n+1}}} \right)^{\frac{\sigma}{\sigma-1}} \tag{41}$$

The term $\frac{1-\alpha_n^{1-\lambda}}{1-\alpha_n^{1-\sigma\lambda}}$ is an indeterminate form. We apply de l'Hôpital's rule, so this term has the same limit than:

$$\frac{(1-\lambda)\ln\alpha_0 \ \ln(1-\sigma\lambda)\,(1-\sigma\lambda)^n \alpha_0^{(1-\sigma\lambda)^n(1-\lambda)}}{\ln(1-\sigma\lambda)\ \ln\alpha_0\,(1-\sigma\lambda)^{n+1}\alpha_0^{(1-\sigma\lambda)^{n+1}}}$$

which tends to $\frac{1-\lambda}{1-\sigma\lambda}$ when n tends to infinity.

Consequently:

$$\frac{A_{n+1}}{A_n} \xrightarrow{n\to\infty} \left(\frac{1-\lambda}{1-\sigma\lambda}\right)^{\frac{\sigma}{\sigma-1}} \quad (>1)$$

From equation (39) and since $\frac{\alpha_{n+1}}{\alpha_n} \xrightarrow{n\to\infty} 1$ we also have:

$$\frac{x_{n+1}}{x_n} \xrightarrow{n\to\infty} \left(\frac{1-\lambda}{1-\sigma\lambda}\right)^{\frac{\sigma}{\sigma-1}}$$

while from the relationship $c_n = A_n - x_n$ and from (39) we get:

$$\frac{c_{n+1}}{c_n} = \frac{A_{n+1}}{A_n}\frac{1-\frac{\alpha_{n+1}}{1+\theta}}{1-\frac{\alpha_n}{1+\theta}} \xrightarrow{n\to\infty} \left(\frac{1-\lambda}{1-\sigma\lambda}\right)^{\frac{\sigma}{\sigma-1}}$$

So in the CES case (like in the Cobb-Douglas case), continuous substitution leads to asymptotically balanced growth. The limit growth rate of output, capital and consumption is the same and depends on parameters λ and σ only.

4 The endogenous direction of technological change with Cobb-Douglas production functions and logarithmic utility

In the preceding section we have seen that if the production functions were either all Cobb-Douglas or all CES with the same ES, the agent could enjoy steady growth by undertaking substitution.

However there are reasons to think that the agent will always do so, if he has the choice of undertaking either complementation or substitution. We have seen that the increase in subjective welfare due to the passage from technique φ to technique φ' while initially at the φ-steady state is:

$$\Delta U_{\varphi\to\varphi'} = \frac{1+\theta}{\theta}\left[u\left(c^*_{\varphi'}\right) - u\left(c^*_{\varphi}\right)\right] - \int_{x^*_{\varphi}}^{x^*_{\varphi'}} f'_{\varphi'}(x)\, u'\left(c^*_{0\varphi'}(x)\right)\, \mathrm{d}x$$

In a nutshell, the agent does not simply prefer the option that brings the highest increase in steady-state consumption. The term $\left[\int_{x^*_{\varphi}}^{x^*_{\varphi'}} f'_{\varphi'}(x)\, u'(c^*_{0\varphi'}(x))\, \mathrm{d}x\right]$ might be dubbed the cost of convergence to the new steady state, and is >0 in the case of substitution ($x^*_{\varphi'} > x^*_{\varphi}$) and <0 in the case of complementation ($x^*_{\varphi'} < x^*_{\varphi}$). Whenever $\theta > 0$, the agent gives attention to the value of his consumption in these transitional dynamics, which tend to be higher under complementation than under substitution.

In principle, we could solve the problem of the endogenous direction of technological change for any production functions $f_{\varphi'}$, however the computation of $\Delta U_{\varphi\to\varphi'}$ is only possible if we have knowledge of the value function $x \mapsto U(x;\varphi')$ or, equivalently, of the optimal choice function of the Ramsey agent which corresponds to function $x \mapsto c^*_{0\varphi'}(x)$. To the best

of our knowledge, we have an analytical formula for the value function only under the two joint hypotheses that the production function is Cobb-Douglas and that the utility function is logarithmic.

In this case, for any technique φ, the value function corresponding to the problem (1) is logarithmic in the level of capital, so there exist constants M_φ and N_φ such that:

$$\forall x > 0,\ \ U\left(x;\varphi\right) = M_\varphi + N_\varphi \ln \frac{x}{x^*_\varphi}$$

We identify constants M_φ and N_φ through equations (3) and (5):

$$M_\varphi = U\left(x^*_\varphi;\varphi\right) = \frac{1+\theta}{\theta}\ln\left(c^*_\varphi\right) = \frac{\theta}{1+\theta}\ln\left(A^*_\varphi - x^*_\varphi\right)$$

$$N_\varphi = x^*_\varphi \left.\frac{\partial U}{\partial x}\right|_{\left(x^*_\varphi;\varphi\right)} = (1+\theta)x^*_\varphi\ \ln'\left(c^*_\varphi\right) = (1+\theta)\frac{x^*_\varphi}{A^*_\varphi - x^*_\varphi}$$

Thus – under the Cobb-Douglas-logarithmic assumption – for any technique φ, the corresponding value function $U(\cdot,\varphi)$ is:

$$U\left(x;\varphi\right) = \frac{1+\theta}{\theta}\left(\ln\left(A^*_\varphi - x^*_\varphi\right) + \theta\frac{x^*_\varphi}{A^*_\varphi - x^*_\varphi}\ln\frac{x}{x^*_\varphi}\right)$$

Consequently, the passage from technique φ to technique φ' while initially at the φ-steady state brings an increase in utility equal to:

$$\Delta U|_{\varphi\to\varphi'} \equiv U\left(x^*_\varphi;\varphi'\right) - U\left(x^*_\varphi;\varphi\right) = \frac{1+\theta}{\theta}\left\{\ln\left(\frac{A^*_{\varphi'} - x^*_{\varphi'}}{A^*_\varphi - x^*_\varphi}\right) - \theta\frac{x^*_{\varphi'}}{A^*_{\varphi'} - x^*_{\varphi'}}\ln\left(\frac{x^*_{\varphi'}}{x^*_\varphi}\right)\right\} \tag{42}$$

We treat the problem of the endogenous direction of technological change through a steady state-to-steady state approach: that is, we suppose that the agent is always at the steady state corresponding to some technique φ_n when he switches to technique φ_{n+1}, and he does not expect technological change to happen again after that.

We suppose that the agent has the following technological menu:

$$\alpha_{+,n} = \alpha_n^{1-\chi}$$

where $\chi \in (-\mu,\lambda)$ with $\lambda \in (0,1)$ and $\mu > 0$ like in subsection 3.1.2. From proposition 1 we know that the agent will always whoose $\chi = -\mu$ or $\chi = \lambda$; we will verify it by calculus below.

For any $\chi \in (-\mu,\lambda)$, if the agent chooses χ at the nth opportunity, like in subsection 3.1.2 we have:

$$\alpha_{n+1} = \alpha_n^{1-\chi}$$

$$A_{n+1} = A_n \alpha_n^{-\chi\frac{\alpha_n^{1-\chi}}{1-\alpha_n^{1-\chi}}}$$

$$x_{n+1} = x_n \alpha_n^{-\chi\frac{1}{1-\alpha_n^{1-\chi}}}$$

From equation (42), the resulting increase in subjective welfare is, then:

$$\Delta U\left(\alpha_n,\chi\right)=\frac{1+\theta}{\theta}\left\{\chi\frac{\alpha_n^{1-\chi}}{1+\theta-\alpha_n^{1-\chi}}\ \ln\left(\frac{1}{\alpha_n}\right)+\ln\left(\frac{1+\theta-\alpha_n^{1-\chi}}{1+\theta-\alpha_n}\right)\right\}$$

We call Γ the function:

$$\Gamma\ :\ \left|\begin{array}{rcl}(0,1)\times(-\infty,1) & \longrightarrow & \mathbb{R}_+\\ (\alpha,\chi) & \longmapsto & \chi\frac{\alpha^{1-\chi}}{1+\theta-\alpha^{1-\chi}}\ \ln\left(\frac{1}{\alpha}\right)+\ln\left(\frac{1+\theta-\alpha^{1-\chi}}{1+\theta-\alpha}\right)\end{array}\right.$$

Γ is the value of the χ option at the α state, it takes positive values in virtue of theorem 1. The derivative of Γ with respect to χ is:

$$\frac{\partial\Gamma}{\partial\chi}=\left(\ln\frac{1}{\alpha}\right)^2\ \frac{\alpha^{1-\chi}}{1-\alpha^{1-\chi}}(1+\theta)\chi\ \ \gtrless 0\quad\text{when}\ \ \chi\gtrless 0$$

which conforms to proposition 1. Γ has also well-defined limits as χ tends to extreme values:

$$\lim_{\chi\to-\infty}\Gamma\left(\alpha,\chi\right)=\ \ln\left(\frac{1+\theta}{1+\theta-\alpha}\right)\equiv L_C(\alpha)>0$$
$$\lim_{\chi\to 1}\Gamma\left(\alpha,\chi\right)=\ \frac{1}{\theta}\ln\left(\frac{1}{\alpha}\right)+\ln\left(\frac{\theta}{1+\theta-\alpha}\right)\equiv L_S(\alpha)>0$$

$L_C(\alpha)$ and $L_S(\alpha)$ respectively denote the increase in subjective welfare in case of 'total complementation' ($\chi\to-\infty$) and in case of 'complete substitution' ($\chi\to 1$). Remark that L_C increases with α while L_S decreases with α.

Thus, the agent has the choice between choosing $\chi=-\mu$ and getting an increase in utility equal to $\Delta U|_{\text{comp}}$ below, and choosing $\chi=\lambda$ and getting an increase in utility equal to $\Delta U|_{\text{subs}}$, where:

$$\Delta U|_{\text{comp}}=\frac{1+\theta}{\theta}\left\{-\mu\frac{\alpha^{1+\mu}}{1+\theta-\alpha^{1+\mu}}\ln\frac{1}{\alpha}+\ln\left(\frac{1+\theta-\alpha^{1+\mu}}{1+\theta-\alpha}\right)\right\}$$
$$\Delta U|_{\text{subs}}=\frac{1+\theta}{\theta}\left\{\lambda\frac{\alpha^{1-\lambda}}{1+\theta-\alpha^{1-\lambda}}\ln\frac{1}{\alpha}+\ln\left(\frac{1+\theta-\alpha^{1-\lambda}}{1+\theta-\alpha}\right)\right\}$$

Let's define function ψ by:

$$\psi=\frac{\theta}{1+\theta}\left(\Delta U|_{\text{subs}}-\Delta U|_{\text{comp}}\right)$$

so:

$$\psi(\alpha)=\left\{\lambda\frac{\alpha^{1-\lambda}}{1+\theta-\alpha^{1-\lambda}}+\mu\frac{\alpha^{1+\mu}}{1+\theta-\alpha^{1+\mu}}\right\}\ln\frac{1}{\alpha}-\ln\left(\frac{1+\theta-\alpha^{1+\mu}}{1+\theta-\alpha^{1-\lambda}}\right)\tag{43}$$

$\psi(\alpha)$ measures the relative benefits of substitution over complementation when the capital share is originally at α. If $\psi(\alpha) > 0$, then the agent prefers substitution (at rate λ) over complementation (at rate μ).

Function ψ is the guide to the dynamics of the model. If ψ lies everywhere above one, then the agent always chooses substitution and output, capital and consumption grow asymptotically at the gross rate of $e^{\frac{\lambda}{1-\lambda}} > 1$ like in subsection 3.1.2b. If ψ lies everywhere below one, then the agent undertakes continual complementation and ends up consuming A_∞ defined in equation (19). If ψ crosses the abscissae axis from above, then there exists an interior steady state, and the capital share fluctuates around some long-run value α_∞ that depends on θ, μ and λ.

First, remark that:

$$\lim_{\alpha\to 0}\psi \;=\; \lim_{\alpha\to 1}\psi \;=\; 0$$

Secondly, the derivative of ψ is:

$$\psi'(\alpha)=\ln\left(\frac{1}{\alpha}\right)\left(\lambda(1-\lambda)\frac{\alpha^{-\lambda}}{\left(1+\theta-\alpha^{1-\lambda}\right)^2}+\mu(1+\mu)\frac{\alpha^{\mu}}{\left(1+\theta-\alpha^{1+\mu}\right)^2}\right)+\frac{\alpha^{\mu}}{1+\theta-\alpha^{1+\mu}}-\frac{\alpha^{-\lambda}}{1+\theta-\alpha^{1-\lambda}}$$

so:

$$\lim_{\alpha\to 0}\psi' \;=\; +\infty \;\;;\;\; \lim_{\alpha\to 1}\psi' \;=\; 0$$

Thus, function ψ is locally strictly increasing around 0, and since $\lim_{\alpha\to 0}\psi = 0$ we conclude that ψ is locally strictly positive near 0. When the capital share approaches 0, the agent always reacts by undertaking substitution at some date. This property reflects the Habakkuk hypothesis.

The limit of ψ' around one does not allow to conclude on the form of ψ near 1. The formula for ψ'' is:

$$\begin{aligned}\psi''(\alpha)=\ln\left(\frac{1}{\alpha}\right)\Bigg(&\lambda(\lambda-1)\frac{\alpha^{-1-\lambda}\left[\lambda(1+\theta-\alpha^{1-\lambda})-2\alpha^{1-\lambda}\right]}{\left(1+\theta-\alpha^{1-\lambda}\right)^3}\\ &+\mu(\mu+1)\frac{\alpha^{-1+\mu}\left[\mu(1+\theta-\alpha^{1+\mu})+2\alpha^{1+\mu}\right]}{\left(1+\theta-\alpha^{1+\mu}\right)^3}\Bigg)\\ -\frac{1}{\alpha}&\left(\lambda(1-\lambda)\frac{\alpha^{-\lambda}}{\left(1+\theta-\alpha^{1-\lambda}\right)^2}+\mu(1+\mu)\frac{\alpha^{\mu}}{\left(1+\theta-\alpha^{1+\mu}\right)^2}\right)\\ +\frac{\alpha^{-1+\mu}\left(\mu(1+\theta)+\alpha^{1+\mu}\right)}{\left(1+\theta-\alpha^{1+\mu}\right)^2}&+\frac{\alpha^{-1-\lambda}\left(\lambda(1+\theta)-\alpha^{1-\lambda}\right)}{\left(1+\theta-\alpha^{1-\lambda}\right)^2}\end{aligned}$$

so:

$$\lim_{\alpha\to 1}\psi''(\alpha)=\frac{(1+\theta)(\lambda+\mu)}{\theta^2}(\lambda-\mu)$$

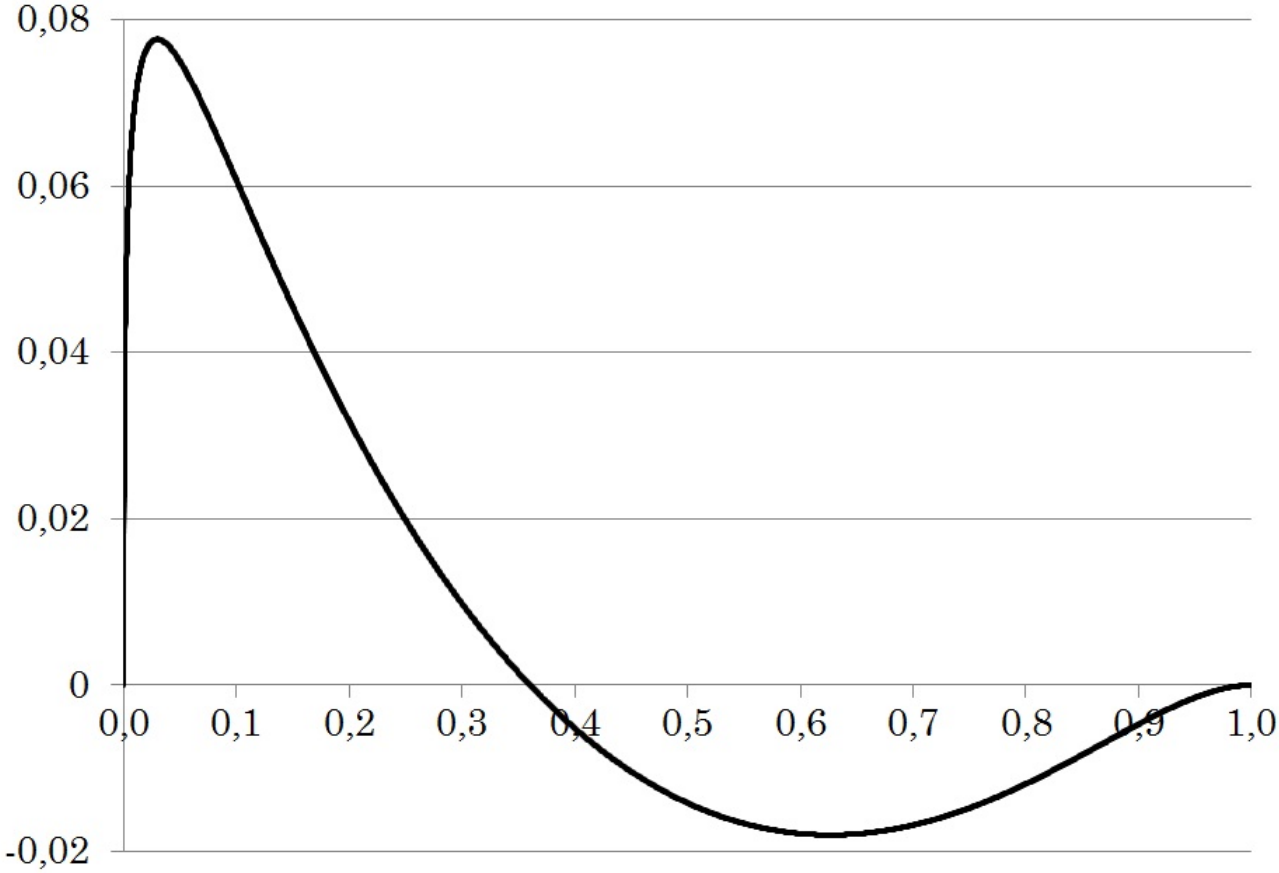

Figure 10: *Function* ψ *for* $\theta = 1$, $\mu = 1$ *and* $\lambda = 0.5$.

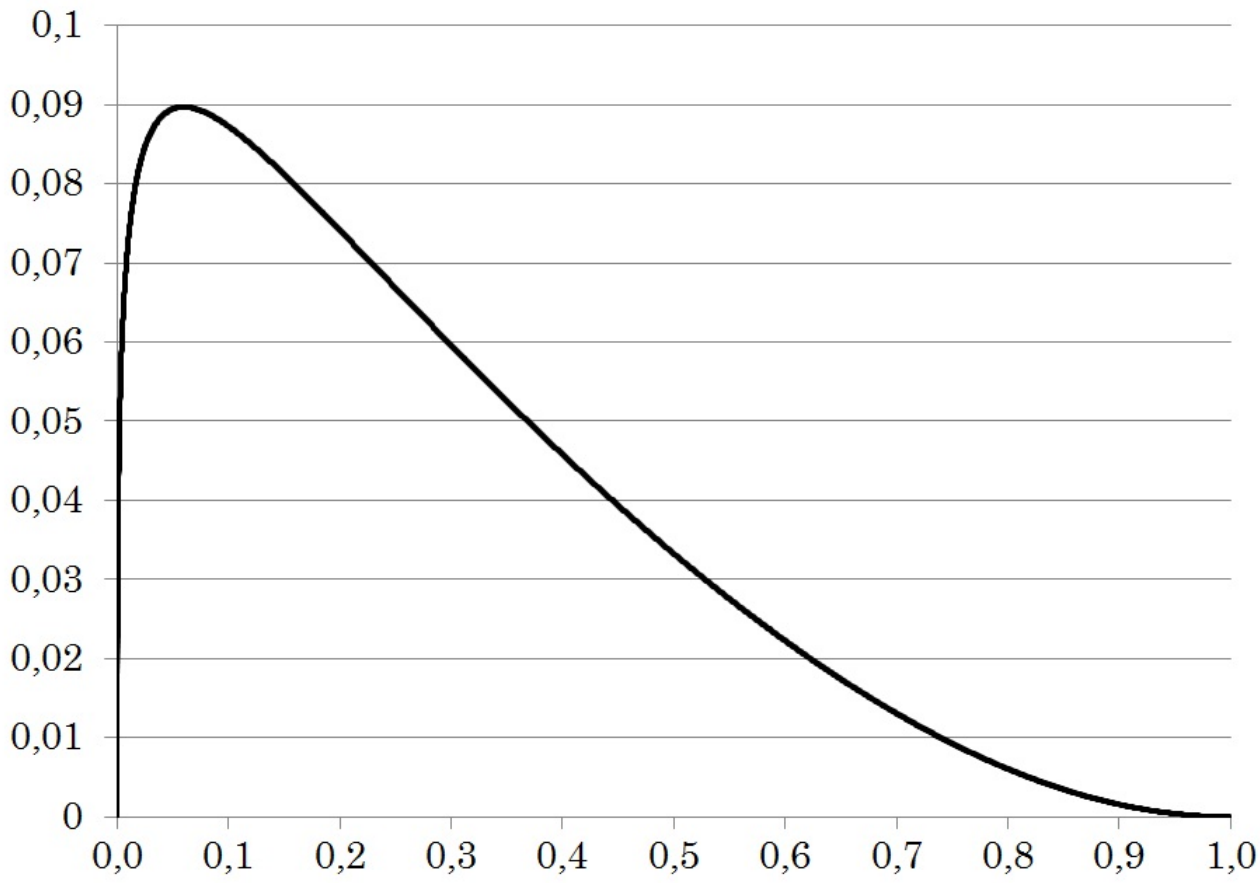

Figure 11: *Function* ψ *for* $\theta = 1$, $\mu = 0.3$ *and* $\lambda = 0.5$.

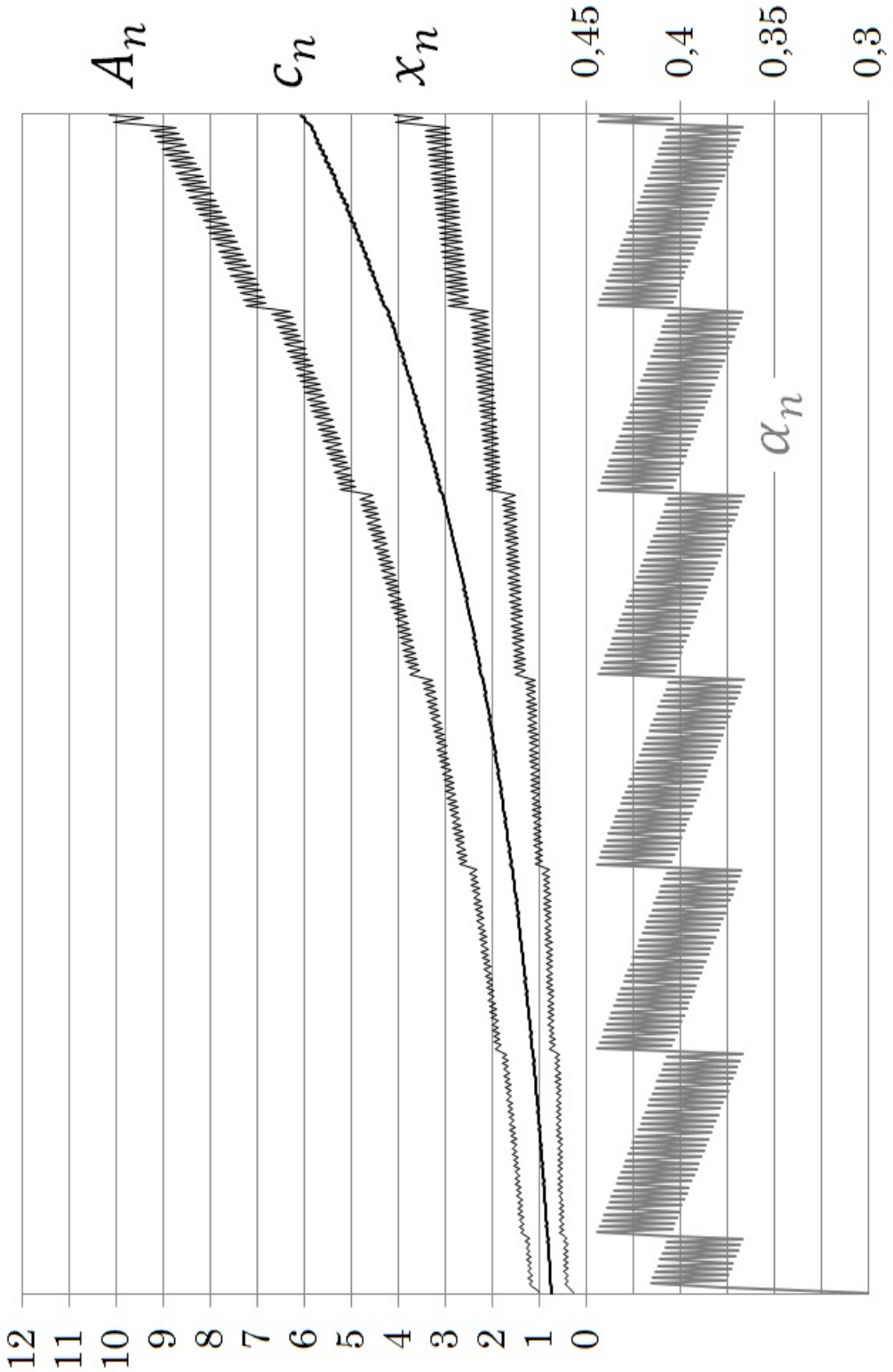

Figure 12: *The growth path with endogenous direction of technological change* (Assumptions: $\alpha_0 = 0.3$; $A_0 = 1$; $\theta = 0.1$; $\lambda = 0.1$; $\mu = 0.115$).

Thus, when $\mu > \lambda$, 1 is a local maximum of ψ. Since ψ has limit 0 near 1, this means that ψ is locally negative at the left of 1: *in this case,* the agent is always chooses to undertake complementation at some point and does not let capital share diverge to one like in the pure-substitution case. We could not prove formally that in this case the (interior) steady state was unique, although simulations strongly suggest that this is the case.

In the contrary, when $\mu < \lambda$, $\lim_{\alpha \to 1} \psi''(\alpha) < 0$, so 1 is a local minimum of ψ, which means that ψ is locally strictly positive near 1: the agent then undertakes substitution, and make the capital share diverge to one. We have been unable to prove formally that in this case no interior equilibrium exists[13], but simulations for a wide range of parameters strongly suggest that this is the case.

Figures 10 and 11 present the two cases. The first one assumes $\theta = 1$, $\mu = 1$ and $\lambda = 0.5$ while the second assumes μ to 0.3. In the first case, the interior steady state is around $\alpha_\infty = 0.36$, in the second case ψ lies everywhere above zero.

So when $\mu > \lambda$, the agent chooses substitution for $\alpha < \alpha_\infty$ and complementation for $\alpha > \alpha_\infty$.

When $(1-\lambda)(1+\mu) = 1$ – for example if $\mu = 1$ and $\lambda = 0.5$ –, the capital share only takes two values asymptotically, since in this case $\left(\alpha^{1+\mu}\right)^{1-\lambda} = \alpha$ for all α, with α_∞ standing somewhere between these two values.

If $(1-\lambda)(1+\mu) > 1$ (which is equivalent to $\frac{\mu}{1+\mu} > \lambda$), then $\left(\alpha^{1+\mu}\right)^{1-\lambda} < \alpha$ for all α. Then, if the agent always alternated complementation and substitution, the capital share would tend to zero; consequently, the agent sometimes undertakes substitution twice in a row. Between two such episodes, the economy alternates complementation and substitution and the capital share has a downward trend.

Finally, If $(1-\lambda)(1+\mu) > 1$ (or equivalently $\frac{\mu}{1+\mu} < \lambda$), then $\left(\alpha^{1+\mu}\right)^{1-\lambda} > \alpha$ so the alternation of complementation and substitution leads to an upward trend in capital share, so the agent sometimes has to undertake complementation twice in a row.

Figure 12 shows the first terms of the sequence of steady states with endogenous direction of technological change, for $\alpha_0 = 0.3$, $A_0 = 1$, $\theta = 0.1$, $\lambda = 0.1$ and $\mu = 0.115$. This case pertains to the second case above, where $(1-\lambda)(1+\mu) > 1$ (and $\mu > \lambda$). The capital share oscillates around a long term value in the neighborhood of 0.405, between 0.366 and 0.444.

This economic growth regime thus quits the Kaldor facts: GDP per capita fluctuates around some exponential pattern and the capital share is stationary. But, in contrast to neoclassical theory based on exogenous production functions, this latter conclusion is endogenous (and only valid in the case $\lambda < \mu$) and is consistent with the observation that the labor share tends to peak during recessions during the whole US postwar history. Another empirical regularity emerges from figure 12: consumption is pro cyclical and less volatile than investment (x) and than overall GDP (A); interestingly the mechanism at work here is not through demand but through supply.

[13] This could happen if function ψ crossed the abcissae axis twice over $(0,1)$.

5 Conclusion

In this paper, we have investigated a simple Ramsey model with an unfamiliar way of introducing technical change. In doing so, we have depicted capital-labor substitution (and its opposite, labor-capital substitution) as a technological change move, rather than an accident of the production function.

We have done so for several reasons. First, the traditional productivity-linked picture of technology is a limit that the theory should get rid of, for the production function can credibly evolve in ways that cannot be represented by an increase of any productivity term – indeed, our economic growth regime reproduces the same Kaldor facts as the neoclassical model under labor-augmenting technical change, but also explains the constancy of the capital share. Secondly, to contribute to the strand of research already cited in the introduction that sees – either through theory or through empirics – the production function as being written dynamically, rather than given. Thirdly, to argue that a simple Ramsey framework might not be such a bad guide to the analysis of long-run growth[14]. Fourth and finally, to make sense of Habakkuk's hypothesis under minimal hypotheses – in this framework, high-wage share countries have more chance to undertake capital-labor substitution, and so have also more chance to experience rapid GDP growth[15].

The approach taken here predicts that poor countries should display a lower capital-output ratio, and a lower labor share. The former point seemingly enters in contradiction with the often seen empirical result that the correlation between the capital-output ratio and GDP per capita is weak[16], the relevance of cross-country data on relative values of capital stocks (prices and quantities) is highly doubtful. Data on labor shares are more reliable as they do not rely on estimates related to capital, and we have shown in Senouci (2014) 4 the relationship between GDP per capita and the labor among OECD countries and in the OECD database clearly displays an inverted-U pattern.

Although in closed economy, the model also carries an answer to the question: why do not all the countries adopt the more advanced techniques? Imagine that some country has undertaken complementation for a long time so that savings are approximately nil there and GDP per capita is very low. Now suppose that this country starts to observe some rich country, which has undertaken, say, only substitution in the past, so GDP is very high and the capital-output ratio is also very high. This would be very costly in terms of present and near future consumption if the poor country adopted the rich country's more capital-intensive method. Moreover, if the poor country has initially a small tradable sector and is credit-constrained in the international markets, he cannot import foreign capital goods up to the

[14]We have found some conditions under which the impatient agent would still enjoy some technological change that makes his steady-state consumption *decrease*. Without assuming any externality (e.g. of Malthusian form), we have seen that a Ramsey agent can rationally impoverish himself.

[15]We have departed from the original Habakkuk (1962) argument by assuming that wages were equal to the marginal product of labor, rather than depending on some exogenous option.

[16]See Easterly and Levine (2001).

point where the rich country's technology is more profitable than his own[17]. The poor country would then rather wait that the rich country operates some complementation, making the sacrifice less painful.

Indeed, there exists some empirical support for this view. Comin and Hobijn (2004), Comin et al. (2008) and Comin et al. (2010) investigated the cross-country pattern of technology adoption and found significant and variable lags of adoption, and also detected some form of temporal complementarity: quick past adoption predicts quick future adoption. Comin and Hobijn (2010) investigated the level of use of technologies (instead of only mere qualitative 'adoption' variable) and have shown that although adoption lags have shortened all over the twentieth century, the asymptotic of use of technologies (once adopted) is much less than in the part. This result is understandable in our terms: today's modern technologies (mobile phones, etc.) are much less expensive than yesterday's modern technologies (factories, etc.) so countries tend to adopt them more easily; however, in developed countries these technologies come to complement some big stock of capital, while in developing countries they are not part of such a great capital stock – so the use rates are not as high as in developed countries.

To constitute a more satisfying framework of analysis of medium to long-run growth, the skills part of technology obviously misses from the approach. The interaction between skills and tools is probably of considerable importance to understand the dynamic patterns of economies, and one can legitimately object that our approach completely neglects labor. The evolution of skills might even be as important as the evolution of tools to understand the dynamics of factor shares and growth[18]. If the word 'technology' refers to some set of capital goods, and if a technique is the sum of a technology and some set of skills, then this study investigated technological change but not technical change. In future research, we hope to analyze the relationships between skills and tools along this framework.

[17]This view is reflected in the United Nations Conference on Trade And Development 2013 report on LDCs: "The choice of technology often creates a conflict between the objective of achieving competitiveness by acquiring advanced technology (which invariably tends to be capital-intensive) and the objective of creating decent jobs in sufficient quantity. (...) The LDCs' manufacturing export sector is included in regional and global value chains, and it must accordingly apply the international standards of quality and production processes in which those chains operate. Still, the segments of these chains that are located in LDCs are mainly the labour-intensive ones (...) Clearly, the LDCs cannot afford to ignore the fact that they need foreign exchange to import capital goods, technology and other inputs required to build their productive capacities. They must also bear in mind the need to maintain or increase their export capacity. To be able to export, they may need to attract FDI, which typically chooses capital-intensive technologies that do not generate much employment. " [UNCTAD (2013, chapter 4)].

[18]Among many studies focusing on the role of human capital in long-run growth, see Bessen (1997, 2003, 2011, 2012), Caselli (1999), Caselli & Coleman (2001, 2006) and Beaudry, Doms & Lewis (2010).

Appendix: Proof of proposition 1

When $f_{\varphi_1'}$ and $f_{\varphi_2'}$ meet only at x_φ^*, two cases are conceivable: either $[f_{\varphi_2'}(x) > f_{\varphi_1'}(x)$ for $x \in (0, x_\varphi^*)$ and $f_{\varphi_2'}(x) < f_{\varphi_2'}(x)$ for $x \in (x_\varphi^*, +\infty)]$ or $[f_{\varphi_2'}(x) < f_{\varphi_1'}(x)$ for $x \in (0, x_\varphi^*)$ and $f_{\varphi_2'}(x) > f_{\varphi_2'}(x)$ for $x \in (x_\varphi^*, +\infty)]$. Let's denote by $\left(c_{\varphi_1',0}^*, c_{\varphi_1',1}^*, \cdots\right)$ the optimal consumption path from $t = 0$ on if the agent incurs technical change from technique φ to technique φ_1' at date $t = 0$ (before consumption decision) while initially at the steady state corresponding to φ. Let's also denote by $\left(c_{\varphi_2',0}^*, c_{\varphi_2',1}^*, \cdots\right)$ the corresponding consumption path resulting from technical change from φ to φ_2'.

- When $f_{\varphi_2'}'(x_\varphi^*) < f_{\varphi_1'}'(x_\varphi^*) < 1 + \theta$ (complementation case), suppose that the agent undergoes φ_2' technical change but, instead of choosing the optimal $c_{\varphi_2',0}^*$, he consumes $c_{\varphi_1',0}^*$ at date $t = 0$. Then, at the beginning of date $t = 1$ he holds $\tilde{x}_1 = \underbrace{f_{\varphi_2'}(x_\varphi^*)}_{=f_{\varphi_1'}(x_\varphi^*)=A_\varphi^*} - c_{\varphi_1',0}^* = x_{\varphi_1'}^* > x_\varphi^*$ units of capital.

  Suppose now that at date $t = 1$ he chooses to consume $c_{\varphi_1',1}^*$; then, he ends up with $\tilde{x}_2$ units of capital at the beginning of date $t = 1$, with $\tilde{x}_2 = f_{\varphi_2'}(\tilde{x}_1) - c_{\varphi_1',1}^* > f_{\varphi_1'}(\tilde{x}_1) - c_{\varphi_1',1}^* = x_{\varphi_1',2}^* > x_\varphi^*$. So he can *throw away* $\tilde{x}_2 - x_{\varphi_1',2}^*$ units of capital, produce $f_{\varphi_2'}(x_{\varphi_1',2}^*) > f_{\varphi_1'}(x_{\varphi_1',2}^*)$, consume $c_{\varphi_1',2}^*$: he will end up holding $\tilde{x}_3 = f_{\varphi_2'}(x_{\varphi_1',2}^*) - c_{\varphi_1',2}^* > f_{\varphi_1'}(x_{\varphi_1',2}^*) - c_{\varphi_1',2}^* = x_{\varphi_1',3}^*$. By a clear recurrence, this shows that the consumption path $\left(c_{\varphi_1',0}^*, c_{\varphi_1',1}^*, \cdots\right)$ is feasible through technique φ_2'. So $U\left(c_{\varphi_2',0}^*, c_{\varphi_2',1}^*, \cdots\right) \geq U\left(c_{\varphi_1',0}^*, c_{\varphi_1',1}^*, \cdots\right)$. And since there is some strictly positive waste in the proof, we also have: $U\left(c_{\varphi_2',0}^*, c_{\varphi_2',1}^*, \cdots\right) > U\left(c_{\varphi_1',0}^*, c_{\varphi_1',1}^*, \cdots\right)$, which amounts to $\Delta U_{\varphi \to \varphi_2'} > \Delta U_{\varphi \to \varphi_1'}$.

- When $f_{\varphi_2'}'(x_\varphi^*) > f_{\varphi_1'}'(x_\varphi^*) > 1 + \theta$ (substitution case), a symmetric reasoning shows that, also here, the consumption path $\left(c_{\varphi_1',0}^*, c_{\varphi_1',1}^*, \cdots\right)$ is feasible through technique φ_2', and then that $\Delta U_{\varphi \to \varphi_2'} > \Delta U_{\varphi \to \varphi_1'}$. In this case, the production function $f_{\varphi_2'}$ happens to be strictly above $f_{\varphi_1'}$ for levels of capital above x_φ^*, which are the levels of capital which are reached after both φ_1' and φ_2' shocks.

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