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**Abstract**

We consider a multivariate default system where random environmental information is available. We study the dynamics of the system in a general setting of enlargement of filtrations and adopt the point of view of change of probability measures. We also make a link with the density approach in the credit risk modelling. Finally, we present a martingale characterization result with respect to the observable information filtration on the market.

**MSC:** 91G40, 60G20, 60G44

**Key words:** Multiple defaults, prediction process, product space and product measure, change of probability measure, density hypothesis, martingale characterization

## 1 Introduction

We consider a system of finite default times to study their probability distributions and the dependence between the default system and the environmental market. In the credit risk analysis, the environmental information appears to be an important factor. Besides the dependence structure among the underlying defaults, we also need to investigate the role of other market information upon the system of multiple defaults, and vice versa, the impact of default events on the market. In the credit risk modelling such as in the book of Bielecki and Rutkowski [6] and the paper of Elliott, Jeanblanc and Yor [13], the information structure concerning default times is described by the theory of enlargement of filtrations. In general, we suppose

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that on the market which is represented by a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, the environmental information is modelled by a reference filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ and the default information is then added to form an enlarged filtration $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ which represents the global information of the market. The modelling of the dependence structure of multiple default times is then diversified in two directions by using the bottom-up and top-down models. In the former approach, one starts with a model for the marginal distribution of each default time and then the correlation between them is made precise (see Frey and McNeil [18] for a survey). While in the top-down models which are particularly developed for the portfolio credit derivatives (see for example Arnsdorff and Halperin [2], Bielecki, Crépey and Jeanblanc [5], Cont and Minca [10], Dassios and Zhao [11], Ehlers and Schönbucher [14], Filipović, Overbeck and Schmidt [17], Giesecke, Goldberg and Ding [20], Sidenius, Piterbarg and Andersen [32] among others), we study directly the cumulative loss process and its intensity dynamics.

In this paper, we consider a multi-default system in presence of environmental information by using a general setting of enlargement of filtrations. In order to fully investigate the different key elements in the modelling, we use one random variable $\chi$ on $(\Omega, \mathcal{A})$ valued in a polish space $E$ to describe default risks and to study the dependence between the default system and the remaining market. Given an observation filtration $(\mathcal{N}_t^E)_{t\geq 0}$ on $(E, \mathcal{E})$ with $\mathcal{E} = \mathcal{B}(E)$, the observable information associated to the default system $\chi$ is given by the inverse image filtration $(\mathcal{N}_t := \chi^{-1}(\mathcal{N}_t^E))_{t\geq 0}$ on $(\Omega, \mathcal{A})$. The global market information observed at time $t \geq 0$ is then defined by the enlargement of $\mathcal{F}_t$ as $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{N}_t$. The main advantage of this presentation is twofold. First, the setting is general and can be applied flexibly to diverse situations by suitably choosing the default variable $\chi$ and the observation filtration. For example, for a multi-default system, we can include both ordered (which corresponds to top-down models) and non-ordered (which corresponds to bottom-up models) default times. Second, this framework allows us to distinguish the dependence structures of different nature, notably the correlation within the default system characterized by the so-called prediction process and the dependence between the default system and the environmental market described by a change of probability measure.

The prediction process has initially been introduced in the reliability theory (see for example Norros [29] and Knight [27]) which is defined as the conditional law of $\chi$ with respect to its observation history $\mathcal{N}_t$ and describes the dynamics of the whole default system upon each default event. When we take into account the environmental information $\mathcal{F}_t$, the market information is represented by an enlarged filtration. The main idea is to characterize the dependence between the multi-default system and the remaining market by using a change of probability method with respect to the product probability measure under which the multi-default system $\chi$ and the environmental information $\mathcal{F}_t$ are independent. In this setting, the dependence structure between the default system and the market environment under any arbitrary probability measure can be described in a dynamic manner and represented by the Radon-Nikodym derivative process of the change of probability. We begin our analysis by focusing on the product probability space. In this case, it is easy to apply the change of probability method to deduce estimation and evaluation formulas with market information. The general case where the market is not necessarily represented by a product probability space is more delicate, especially when the Radon-Nikodym derivative is not supposed to be strictly positive as usual. But the previous special case of product space will serve as a useful tool. We show that the key elements in the computation are indeed the prediction process and the Radon-Nikodym derivative.

We establish a link of the change of probability method with the default density approach in El Karoui, Jeanblanc and Jiao [15, 16]. In classic literature on enlargement of filtration theory, the density hypothesis was first introduced by Jacod [24] in an initial enlargement of filtration and is fundamental to ensure the semi-martingale property in the enlarged filtration. We show that the density process and the Radon-Nikodym derivative can be deduced from each other and the two approaches are closed related. Finally, we are interested in the martingale processes in the market filtration $\mathbb{G}$. We present a general martingale characterization result which can be applied to cases such as ordered and non-ordered defaults, which are useful for financial applications. We also discuss the immersion property and show how to construct a $\mathbb{G}$-martingale by using the characterization result.

The following of the paper is organized as follows. In Section 2, we give the general presentation of the multivariate default system and the market filtration. Section 3 focuses on the interaction between the default system and environmental information and presents the change of probability measure methodology in the product probability space setting. In Section 4, we consider the general setting and investigate the link to the density approach in the theory of enlargement of filtrations. Finally in Section 5, we present the martingale characterization results.

## 2 The multi-default system

In this presentation, we introduce a general variable $\chi$ to describe all uncertainty related to the multi-default system such as default or failure times, occurrence orders and associated losses or recovery ratios. We present different levels of information on the market and in particular a general construction of observable information filtration.

## 2.1 Model setting

We fix a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Let us consider a finite family of $n$ underlying firms and describe all default uncertainty by a random variable $\chi$ which takes values in a Polish space $E$. The default times of these firms are represented by a vector of random times $\boldsymbol{\tau} = (\tau_1, \cdots, \tau_n)$. Since $\chi$ contains all information about default uncertainty, there exists a measurable map $f : E \to \mathbb{R}_+^n$ such that $\boldsymbol{\tau} = f(\chi)$ and this map $f$ specifies the default times $\boldsymbol{\tau}$. In this general framework, the random variable $\chi$ can be chosen in a very flexible manner, which allows to consider bottom-up and top-down models in the credit risk literature. For example, one can choose $\chi$ to be the default time vector $\boldsymbol{\tau}$ itself: in this case the Polish space $E$ is just $\mathbb{R}_+^n$ and $f : E \to \mathbb{R}_+^n$ is the identity map. One can also take into account the information of associated losses, namely $E = \mathbb{R}_+^n \times \mathbb{R}^n$ and $\chi = (\tau_i, L_i)_{i=1}^n : \Omega \to \mathbb{R}_+^n \times \mathbb{R}^n$, where $L_i$ denotes the loss induced by the $i^{\text{th}}$ firm at default time. In the top-down models, we consider the ordered default times $\sigma_1 \leq \cdots \leq \sigma_n$. We can choose $E$ to be the subspace $\{\boldsymbol{u} = (u_1, \cdots, u_n) \in \mathbb{R}_+^n \,|\, u_1 \leq \cdots \leq u_n\}$ of $\mathbb{R}_+^n$ and $\chi$ to be the successive default vector $\boldsymbol{\sigma} = (\sigma_1, \cdots, \sigma_n)$. If we intend to take into account the label of each defaulted firm in the top-down setting (in this case, the successive default vector $\boldsymbol{\sigma}$ consists of the order statistics of the random vector $\boldsymbol{\tau}$), we can choose $\chi = (\boldsymbol{\sigma}, J)$ valued in $E = \{\boldsymbol{u} \in \mathbb{R}_+^n \,|\, u_1 \leq \cdots \leq u_n\} \times \mathfrak{S}_n$, where $J$ takes values in the permutation group $\mathfrak{S}_n$ of all bijections from $\{1, \ldots, n\}$ to itself and describes the indices of underlying components for the successive defaults. The default time vector $\boldsymbol{\tau}$ can thus be specified by the measurable map from $E$ to $\mathbb{R}_+^n$ which sends $(u_1, \ldots, u_n, \pi)$ to $(u_{\pi^{-1}(1)}, \cdots, u_{\pi^{-1}(n)})$.

## 2.2 Different information levels and filtrations

We next present different levels of information: the total information and the accessible information. Let $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ be a filtration of $\mathcal{A}$ and represent the environmental information of the market which is not directly related to the default events. We suppose that $\mathbb{F}$ satisfies the usual conditions and that $\mathcal{F}_0$ is the trivial $\sigma$-algebra. The observable default information on $E$ is described by a filtration of the $\sigma$-algebra $\sigma(\chi)$ which we precise below.

The first natural filtration associated with this framework is the regularization of the filtration

$$\mathbb{H} = (\mathcal{H}_t)_{t \geq 0} \text{ where } \mathcal{H}_t := \mathcal{F}_t \vee \sigma(\chi)$$

The $\sigma$-algebra $\mathcal{H}_t$ represents the total information and is not completely observable since the default variable $\chi$ is not known by investors at an arbitrary time $t$. In the literature of enlargement of filtration, this filtration is called the initial enlargement of $\mathbb{F}$ by $\chi$. We recall that (e.g.

Jeulin [25, Lemma 4.4], see also Song [31]) any $\mathcal{H}_t$-measurable random variable can be written in the form $Y_t(\chi)$ where $Y_t(\cdot)$ is an $\mathcal{F}_t \otimes \mathcal{E}$-measurable function.

The total information on $\chi$ is not accessible to all market participants at time $t \geq 0$. We represent the observable information flow on the default events by a filtration $(\mathcal{N}_t)_{t\geq 0}$ on $\mathcal{A}$. Typically it can be chosen as the filtration generated by a counting process. In this paper, we construct the filtration $(\mathcal{N}_t)_{t\geq 0}$ in a more general way. Let $(\mathcal{N}_t^E)_{t\geq 0}$ be a filtration on the Borel $\sigma$-algebra $\mathcal{E} = \mathcal{B}(E)$. If $(\mathcal{N}_t^E)_{t\geq 0}$ is generated by an observation process $(N_t, t \geq 0)$ such as the default counting process or the cumulative loss process defined on the Polish space $E$, then the filtration $(\mathcal{N}_t)_{t\geq 0}$ on $\Omega$ is defined as the inverse image (see e.g. Resnick [30, §3.1]) which is generated by the process $(N_t \circ \chi, t \geq 0)$. More generally, the filtration $(\mathcal{N}_t^E)_{t\geq 0}$ determines a filtration on $(\Omega, \mathcal{A})$ by the inverse image by $\chi$ as

$$\mathcal{N}_t := \chi^{-1}(\mathcal{N}_t^E) = \{\chi^{-1}(A) \,|\, A \in \mathcal{N}_t^E\}, \quad t \geq 0. \tag{1}$$

The global observable market information is then given by the filtration $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ where

$$\mathcal{G}_t = \cap_{s>t} (\mathcal{F}_s \vee \mathcal{N}_s), \quad t \geq 0.$$

Note that $\mathbb{G}$ satisfies the usual conditions, and the following relation holds

$$\mathcal{F}_t \subseteq \mathcal{G}_t \subseteq \mathcal{H}_t.$$

The filtration $\mathbb{G}$ can be viewed as a progressive enlargement but built in a general way. The classic initial and progressive enlargements are included in this framework. Below are some examples.

**Example 2.1** (1) If $\mathcal{N}_t^E = \mathcal{E}$ for all $t$, then $\mathcal{N}_t = \chi^{-1}(\mathcal{E}) = \sigma(\chi)$. So the filtration $\mathbb{G}$ coincides with $\mathbb{H}$, which is the initial enlargement of filtration $\mathbb{F}$ by $\chi$.

(2) In the case $n = 1$ and $E = \mathbb{R}_+$ where $\chi = \tau$ denotes the default time of a single firm, for any $t \geq 0$, let $\mathcal{N}_t^E$ be generated by the functions of the form $\mathbb{1}_{[0,s]}$ with $s \leq t$, then $\mathcal{N}_t = \sigma(\mathbb{1}_{\{\tau \leq s\}}, s \leq t)$. The filtration $\mathbb{G}$ is the classic progressive enlargement of $\mathbb{F}$ by $\tau$.

(3) In general, $(\mathcal{N}_t^E)_{t\geq 0}$ can be any filtration on $(E, \mathcal{E})$ so that $\mathbb{G}$ differs from the initial and progressive enlargements. Consider an insider who has extra information and knows if the firm will default before a deterministic time $t_0$ or not. Let $\chi = \tau$ denote the default time. For any $t \geq 0$, the $\sigma$-algebra $\mathcal{N}_t^E$ is generated by the functions of the form $\mathbb{1}_{[0,s]}$, with $s \leq t$ or $s = t_0$. Then the filtration of the insider is given by $\mathcal{N}_t = \tau^{-1}(\mathcal{N}_t^E) = \sigma(\mathbb{1}_{\{\tau \leq s\}},\ s \leq t \text{ or } s = t_0)$, $t \geq 0$. The filtration $\mathbb{G} = (\mathcal{F}_t \vee \mathcal{N}_t)_{t\geq 0}$ is in general larger than the progressive enlargement of $\mathbb{F}$ by $\tau$.

(4) In the case with advanced or delayed information, if an agent knows the default occurrence with time $\epsilon > 0$ in advance, then $\mathcal{N}_t^E$ is generated by the functions of the form $1\!\!1_{[0,s+\epsilon]}$ with $s \le t$ and $\mathcal{N}_t = \sigma(1\!\!1_{\{\tau \le s+\epsilon\}}, s \le t)$. Similarly, for the delayed information as in Collin-Dufresne, Goldstein and Helwege [9] or Guo, Jarrow and Zeng [22], $\mathcal{N}_t^E$ is generated by $1\!\!1_{[0,(s-\epsilon)^+]}$ with $s \le t$ and $\mathcal{N}_t = \sigma(1\!\!1_{\{\tau \le (s-\epsilon)^+\}}, s \le t)$.

(5) For a family of default times of $n$ non-ordered firms where $\chi = \boldsymbol{\tau}$, we have $E = \mathbb{R}_+^n$. Let $\mathcal{N}_t^E$ be generated by the family of functions $\{1\!\!1_{\{u_i \in [0,s]\}}(\boldsymbol{u}), i \in \{1, \cdots, n\}, s \le t\}$ with $\boldsymbol{u} = (u_1, \cdots, u_n)$. Then $\mathcal{N}_t = \sigma(1\!\!1_{\{\boldsymbol{\tau} \le s\}}, s \le t)$ describes the occurrence of default events. If the default times are ordered, we let $\chi = \boldsymbol{\sigma}$ and $E = \{(u_1, \cdots, u_n) \in \mathbb{R}_+^n \,|\, u_1 \le \cdots \le u_n\}$. In this case, instead of including each default indicator, the counting process gives the information of successive defaults (see [16]). Let $\mathcal{N}_t^E$ be generated by the functions of the form $\sum_{i=1}^n 1\!\!1_{\{u_i \in [0,s]\}}(\boldsymbol{u})$ with $s \le t$. Then $\mathcal{N}_t = \sigma(1\!\!1_{\{\boldsymbol{\sigma} \le s\}}, s \le t)$.

(6) If, besides the default occurrence, we also include a non-zero random mark $L_i$ at each default time such as the loss or recovery given default, we let $\chi = (\boldsymbol{\tau}, \boldsymbol{L})$, which takes value in $E = \mathbb{R}_+^n \times \mathbb{R}^n$. For any $t \ge 0$, let $\mathcal{N}_t^E$ be generated by the functions of the form $\{1\!\!1_{\{u_i \in [0,s]\}}(\boldsymbol{u}) f(l_i) : i \in \{1, \cdots, n\},\ s \le t,\ f$ is any Borel function on $\mathbb{R}\}$. Then the corresponding default filtration on $(\Omega, \mathcal{A})$ is given by $\mathcal{N}_t = \sigma(1\!\!1_{\{\tau_i \le s\}} L_i, i \in \{1, \cdots, n\},\ s \le t)$. Similarly, if the default times are ordered and $\chi = (\boldsymbol{\sigma}, \boldsymbol{L})$, then we can consider the marked point process (see e.g. Jacod [23]). The $\sigma$-algebra $\mathcal{N}_t^E$ is generated by the functions of the form $\sum_{i=1}^n 1\!\!1_{\{u_i \in [0,s]\}}(\boldsymbol{u}) f(l_i), s \le t$. Then $\mathcal{N}_t = \sigma\big(\sum_{i=1}^n 1\!\!1_{\{\sigma_i \le s\}} L_i,\ s \le t\big)$ as the random marks $L_i$ do not take the zero value.

# 3 Prediction process and product space

We are interested in the conditional law of $\chi$ under different information. In this section, we first focus on a situation which only concerns the default information and we introduce the notion of prediction process following Knight [27] and Norros [29]. We then include the environmental information $\mathbb{F}$ in a toy model of product probability space and show that the prediction process plays a key role in this case.

## 3.1 Prediction process

Given the default observation information, the $(\mathcal{N}_t)_{t\ge0}$-conditional probability law of the random variable $\chi$ is given as a $\mathcal{P}(E)$-valued càdlàg $(\mathcal{N}_t)_{t\ge0}$-adapted process $(\eta_t, t \ge 0)$, where $\mathcal{P}(E)$ denotes the set of all Borel probability measures on $E$, equipped with the topology of weak convergence such that for any $\nu \in \mathcal{P}(E)$ and any bounded continuous function $h$ on $E$, the map $\nu \longmapsto \int_E h\, d\nu$ is continuous.

Using the terminology in [27, 29], we call the process $(\eta_t, t \geq 0)$ the *prediction process* of the random variable $\chi$ with respect to the observation filtration $(\mathcal{N}_t)_{t\geq 0}$. We refer the reader to [29, Theorem 1.1] for the existence of a càdlàg version of the process $(\eta_t, t \geq 0)$ and the uniqueness up to indistinguishability. Moreover, the process $(\eta_t, t \geq 0)$ is an $(\mathcal{N}_t)_{t\geq 0}$-martingale with respect to the weak topology in the following sense : for any bounded Borel function $h$ on $E$, the integral process $(\int_E h\, d\eta_t, t \geq 0)$ is an $(\mathcal{N}_t)_{t\geq 0}$-martingale.

**Example of a default counting process with random marks.** We consider the case where the observation filtration $(\mathcal{N}^E_t)_{t\geq 0}$ is generated by the marked point process

$$\sum_{i=1}^{n} \mathbb{1}_{\{u_i \leq t\}} f(l_i),\ t \geq 0$$

which is associated to the occurrence sequence of default events together with the random marks as in Example 2.1 (6). Note that the corresponding $\mathcal{N}_t$ defined in (1) identifies with the $\sigma$-algebra generated by the vector $(\boldsymbol{\sigma}, \boldsymbol{L})_{(i)} := (\sigma_k, L_k)_{k=1}^{i}$ on the set $\{N^{\boldsymbol{\sigma}}_t = i\} = \{\sigma_i \leq t < \sigma_{i+1}\}$, where $N^{\boldsymbol{\sigma}}_t = \sum_{i=1}^{n} \mathbb{1}_{\{\sigma_i \leq t\}}$. Moreover, as we have mentioned, the default vector $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n)$ can be written in the form $\boldsymbol{\sigma} = (u_1(\chi), \ldots, u_n(\chi))$ where $u_1, \ldots, u_n$ are measurable functions on $E$.

The prediction process $(\eta_t, t \geq 0)$ at time $t \geq 0$, i.e. $\eta_t(dx) = \mathbb{P}(\chi \in dx \,|\, \mathcal{N}_t)$ can be calculated by using the Bayesian formula and taking into consideration each event $\{\sigma_i \leq t < \sigma_{i+1}\}$ on which $\eta_t$ is obtained as the conditional distribution of $\chi$ given $(\boldsymbol{\sigma}, \boldsymbol{L})_{(i)}$ restricted on the survival set $\{\sigma_{i+1} > t\}$ and normalized by the conditional survival probability of $\sigma_{i+1}$ given $(\boldsymbol{\sigma}, \boldsymbol{L})_{(i)}$, that is,

$$\eta_t(dx) = \sum_{i=0}^{n} \mathbb{1}_{\{\sigma_i \leq t < \sigma_{i+1}\}} \frac{\eta_{|(\boldsymbol{\sigma},\boldsymbol{L})_{(i)}}(\mathbb{1}_{\{t<u_{i+1}(x)\}} \cdot dx)}{\eta_{|(\boldsymbol{\sigma},\boldsymbol{L})_{(i)}}(\mathbb{1}_{\{t<u_{i+1}(\cdot)\}})} = \frac{\eta_{|(\boldsymbol{\sigma},\boldsymbol{L})_{(N_t)}}(\mathbb{1}_{\{t<u_{N_t+1}(x)\}} \cdot dx)}{\eta_{|(\boldsymbol{\sigma},\boldsymbol{L})_{(N_t)}}(\mathbb{1}_{\{t<u_{N_t+1}(\cdot)\}})}, \tag{2}$$

where $\eta_{|(\boldsymbol{\sigma},\boldsymbol{L})_{(i)}}$ is the conditional law of $\chi$ given $(\boldsymbol{\sigma}, \boldsymbol{L})_{(i)}$, and $\eta_{|(\boldsymbol{\sigma},\boldsymbol{L})_{(i)}}(\mathbb{1}_{\{t<u_{i+1}(x)\}} \cdot dx)$ denotes the random measure on $E$ sending a bounded Borel function $h : E \to \mathbb{R}$ to

$$\int_E h(x)\, \eta_{|(\boldsymbol{\sigma},\boldsymbol{L})_{(i)}}(\mathbb{1}_{\{t<u_{i+1}(x)\}} \cdot dx) := \mathbb{E}[h(\chi)\mathbb{1}_{\{t<\sigma_{i+1}\}} \,|\, (\boldsymbol{\sigma}, \boldsymbol{L})_{(i)}].$$

At each default time, the new arriving default event brings a regime switching to the prediction process, which can be interpreted as the impact of default contagion phenomenon.

In the particular case where $\chi$ coincides with $(\boldsymbol{\sigma}, \boldsymbol{L})$ and the probability law of $(\boldsymbol{\sigma}, \boldsymbol{L})$ has a density $\alpha(\cdot, \cdot)$ with respect to the Lebesgue measure, similar as in [16], we obtain a more explicit form of the prediction process as follows :

$$\eta_t(d(\boldsymbol{v},\boldsymbol{l})) = \sum_{i=0}^{n} \mathbb{1}_{\{\sigma_i \le t < \sigma_{i+1}\}} \delta_{(\boldsymbol{\sigma},\boldsymbol{L})_{(i)}}(d(\boldsymbol{v},\boldsymbol{l})_{(i)}) \mathbb{1}_{\{t < v_{i+1}\}} \frac{\alpha(\boldsymbol{v},\boldsymbol{l})\, d(\boldsymbol{v},\boldsymbol{l})_{(i+1:n)}}{\int_t^\infty \alpha(\boldsymbol{v},\boldsymbol{l})\, d(\boldsymbol{v},\boldsymbol{l})_{(i+1:n)}},$$

where $(\boldsymbol{v},\boldsymbol{l})_{(i)} = (v_k, l_k)_{k=1}^i$, $(\boldsymbol{v},\boldsymbol{l})_{(i+1:n)} = (v_k, l_k)_{k=i+1}^n$, and

$$\int_t^\infty \alpha(\boldsymbol{v},\boldsymbol{l})\, d(\boldsymbol{v},\boldsymbol{l})_{(i+1:n)} := \int_{(]t,\infty[\times\mathbb{R})^{n-i}} \alpha(\boldsymbol{v},\boldsymbol{l})\, d(v_{i+1}, l_{i+1}) \cdots d(v_n, l_n).$$

### 3.2 Conditional default distributions under the product measure

We now consider the conditional law of $\chi$ given the general observation information $\mathbb{G}$ but in a simple case of product space. We assume that the global market $(\Omega, \mathcal{A})$ can be written as $(\Omega^\circ \times E, \mathcal{A}^\circ \otimes \mathcal{E})$ and that the filtration $\mathbb{F}$ is given by $(\mathcal{F}_t^\circ \otimes \{\emptyset, E\})_{t\ge 0}$, where $(\Omega^\circ, \mathcal{A}^\circ)$ is a measurable space and $\mathbb{F}^\circ = (\mathcal{F}_t^\circ)_{t\ge 0}$ is a filtration of $\mathcal{A}^\circ$. The $\mathcal{A}$-measurable random variable $\chi$ is assumed to be given by the second projection, i.e. $\chi(\omega, x) = x$. The filtration $\mathbb{H}$ of total information is then $(\mathcal{H}_t = \mathcal{F}_t^\circ \otimes \mathcal{E})_{t\ge 0}$. We let $\mathbb{P}^\circ$ be the marginal probability measure of $\mathbb{P}$ on $(\Omega^\circ, \mathcal{A}^\circ)$, namely for any bounded $\mathcal{A}^\circ$-measurable $f$ on $\Omega^\circ$, $\int_{\Omega^\circ} f(\omega^\circ)\mathbb{P}^\circ(d\omega^\circ) := \int_\Omega f(\omega^\circ)\mathbb{P}(d\omega^\circ, dx)$. Let $\eta$ be the probability law of $\chi$. In the product space, the probability measure $\eta$ identifies with the marginal measure of $\mathbb{P}$ on $(E, \mathcal{E})$.

The most simple dependence structure between the default variable $\chi$ and the environmental information $\mathbb{F}$ is when they are independent. We introduce the product probability measure $\overline{\mathbb{P}} = \mathbb{P}^\circ \otimes \eta$ on $(\Omega, \mathcal{A})$ under which the two sources of risks $\chi$ and $\mathbb{F}$ are independent. This case will serve as the building stone in our paper. In particular, the calculations are easy applications of Fubini's theorem in this case.

We fix some notation which will be useful in the sequel. If $Y$ is a random variable on the product space $\Omega = \Omega^\circ \times E$, sometimes we omit the first coordinate in the expression of the $\mathcal{A}$-measurable function $Y$ and use the notation $Y(x)$, $x \in E$, which denotes in fact the random variable $Y(\cdot, x)$.

**Default information.** Recall that $\eta_t$ is the conditional probability law of $\chi$ given $\mathcal{N}_t$ and $(\eta_t, t \ge 0)$ is the prediction process of $\chi$. Since $\chi$ is independent of $\mathbb{F}$ under $\overline{\mathbb{P}}$, for any bounded or positive $\mathcal{A}$-measurable function $\Psi$ on $\Omega = \Omega^\circ \times E$, one has

$$\mathbb{E}_{\overline{\mathbb{P}}}[\Psi \,|\, \mathcal{F}_\infty \vee \mathcal{N}_t] = \int_E \Psi(\cdot, x)\eta_t(dx) =: \eta_t(\Psi), \quad \overline{\mathbb{P}}\text{-}a.s.$$

By Dellacherie and Meyer [12, VI.4], there exists a càdlàg version of the martingale $(\eta_t(\Psi), t \ge 0)$ as conditional expectations.

**Total information $\mathbb{H}$.** For the case of total information $\mathbb{H}$, we have to take care about negligible sets. In full generality, the equality $X(\cdot,x)=Y(\cdot,x)$, $\mathbb{P}^\circ$-a.s. for all $x\in E$ does not imply $X(\cdot,\chi)=Y(\cdot,\chi)$, $\overline{\mathbb{P}}$-a.s.. We need a suitable version for such processes. This difficulty can be overcome by Meyer [28] and Stricker and Yor [33].

(i) Given a non-negative $\mathcal{A}$-measurable function $\Psi$ on $\Omega$, from [28, 33], there exists a càdlàg $\mathbb{H}$-adapted process $(\Psi_t^{\mathcal{F}}(\cdot), t\geq 0)$ such that, for any $x\in E$, and for any $t\geq 0$,

$$\Psi_t^{\mathcal{F}}(x)=\mathbb{E}_{\mathbb{P}^\circ}[\Psi(\cdot,x)|\mathcal{F}_t^\circ],\quad \mathbb{P}^\circ\text{-}a.s.$$

In particular, if $X_t$ is an $\mathcal{F}_t^\circ$-measurable random variable valued in $E$, then one has

$$\Psi_t^{\mathcal{F}}(X_t)=\mathbb{E}_{\mathbb{P}^\circ}[\Psi(\cdot,X_t)|\mathcal{F}_t^\circ],\quad \mathbb{P}^\circ\text{-}a.s.$$

We call $(\Psi_t^{\mathcal{F}}(x), t\geq 0)_{x\in E}$ a parametrized $(\mathbb{F}^\circ,\mathbb{P}^\circ)$-martingale depending on a parameter $x\in E$, since the conditional expectation property is valid for all values of $x$ and $t$ outside of a null set.

(ii) This parametrized version of $\mathbb{F}^\circ$-conditional expectation as a function of both variables $(\omega,x)$ is the basic tool for studying projections with respect to $\mathbb{H}$. Under the product measure $\overline{\mathbb{P}}=\mathbb{P}^\circ\otimes\eta$,

$$\mathbb{E}_{\overline{\mathbb{P}}}[\Psi|\mathcal{H}_t]=\Psi_t^{\mathcal{F}}(\chi),\quad \overline{\mathbb{P}}\text{-a.s.}\tag{3}$$

Furthermore, we can extend $\eta_t$ to a $\mathcal{G}_t$-random measure on $(\Omega,\mathcal{H}_t)$ which sends any non-negative $\mathcal{H}_t$-measurable random variable $Y_t(\cdot)$ to the $\mathcal{G}_t$-measurable random variable $\eta_t(Y_t(\cdot))=\int_E Y_t(x)\,\eta_t(dx)$. In the following, by abuse of language, we use $\eta_t$ to denote the conditional laws with respect to both $\mathcal{N}_t$ and $\mathcal{G}_t$ under $\overline{\mathbb{P}}$.

(iii) The above result can be interpreted as a characterization of $(\mathbb{H},\overline{\mathbb{P}})$-martingale in terms of a parametrized $(\mathbb{F}^\circ,\mathbb{P}^\circ)$-martingale depending on a parameter $x\in E$. We shall discuss the martingale properties in more detail in Section 5.

**Accessible information $\mathbb{G}$.** For the observable information $\mathbb{G}$, the projection is firstly made on a larger filtration which includes more information than $\mathbb{G}$ either on $\Omega^\circ$ or on $E$.
(i) Since $\eta_t$ denotes the conditional law of $\chi$ given both $\mathcal{G}_t$ and $\mathcal{N}_t$ under $\overline{\mathbb{P}}$, for any non-negative $\mathcal{H}_t$-measurable random variable $Y_t(\chi)$, we have

$$\mathbb{E}_{\overline{\mathbb{P}}}[Y_t(\chi)|\mathcal{G}_t]=\int_E Y_t(x)\eta_t(dx)=\eta_t(Y_t(\cdot)).\tag{4}$$

(ii) Consider now a non-negative $\mathcal{A}$-measurable random variable $Y$ on $\Omega$. The calculation of its $\mathcal{G}_t$-conditional expectation can be done in two different ways as shown below :

$$\begin{tikzcd} Y \arrow[r, "\mathcal{H}_t=\mathcal{F}_t\vee\sigma(\chi)"] \arrow[d, "\mathcal{F}_\infty\vee\mathcal{N}_t"'] & \mathbb{E}_{\overline{\mathbb{P}}}[Y \mid \mathcal{H}_t] \arrow[d, "\mathcal{G}_t"] \\ \mathbb{E}_{\overline{\mathbb{P}}}[Y \mid \mathcal{F}_\infty \vee \mathcal{N}_t] \arrow[r, "\mathcal{G}_t"'] & \mathbb{E}_{\overline{\mathbb{P}}}[Y \mid \mathcal{G}_t] \end{tikzcd} \tag{5}$$

On the one hand, using the notation introduced in (3),

$$\mathbb{E}_{\overline{\mathbb{P}}}[Y|\mathcal{G}_t] = \mathbb{E}_{\overline{\mathbb{P}}}[\mathbb{E}_{\overline{\mathbb{P}}}[Y|\mathcal{H}_t] \mid \mathcal{G}_t] = \mathbb{E}_{\overline{\mathbb{P}}}[Y_t^{\mathcal{F}}(\chi) \mid \mathcal{G}_t]$$

which, by (4), equals

$$\mathbb{E}_{\overline{\mathbb{P}}}[Y|\mathcal{G}_t] = \eta_t(Y_t^{\mathcal{F}}(\cdot)) = \int_E Y_t^{\mathcal{F}}(x)\eta_t(dx). \tag{6}$$

On the other hand, as shown in (5), the above result can also be obtained by using the intermediary $\sigma$-algebra $\mathcal{F}_\infty \vee \mathcal{N}_t$. Note that by the monotone class theorem, it suffices to consider $Y(\omega, x)$ of the form $Y^\circ(\omega)h(x)$ where $Y^\circ$ is $\mathcal{F}_\infty^\circ$-measurable and $h$ is a Borel function on $E$, then

$$\begin{aligned} \mathbb{E}_{\overline{\mathbb{P}}}[Y|\mathcal{G}_t] &= \mathbb{E}_{\overline{\mathbb{P}}}[\mathbb{E}_{\overline{\mathbb{P}}}[Y^\circ h|\mathcal{F}_\infty \vee \mathcal{N}_t] \mid \mathcal{G}_t] = \mathbb{E}_{\overline{\mathbb{P}}}[Y^\circ \eta_t(h)|\mathcal{G}_t] \\ &= \eta_t(h)\mathbb{E}_{\mathbb{P}^\circ}[Y^\circ|\mathcal{F}_t^\circ] = \eta_t(\mathbb{E}_{\mathbb{P}^\circ}[Y^\circ h|\mathcal{F}_t^\circ]) = \eta_t(\mathbb{E}_{\mathbb{P}^\circ}[Y|\mathcal{F}_t^\circ]) \end{aligned}$$

which is equal to (6).
(iii) By (6) we can characterize a $(\mathbb{G}, \overline{\mathbb{P}})$-martingale as the integral of a parametrized $(\mathbb{F}^\circ, \mathbb{P}^\circ)$-martingale $Y^{\mathcal{F}}(x)$, $x \in E$, with respect to the random measure $\eta_t(dx)$.

### 3.3 Change of probability measures

In this subsection, we consider a probability measure $\mathbb{P}$ on the product space $(\Omega, \mathcal{A}) = (\Omega^\circ \times E, \mathcal{A}^\circ \otimes \mathcal{E})$ which is absolutely continuous with respect to the product measure $\overline{\mathbb{P}}$. We suppose that the Radon-Nikodym derivative of $\mathbb{P}$ with respect to the product measure $\overline{\mathbb{P}}$ is given by

$$\frac{d\mathbb{P}}{d\overline{\mathbb{P}}}\Big|_{\mathcal{H}_T} = \beta_T(\chi) \tag{7}$$

where $T \geq 0$ is a horizon time and $\beta_T(\cdot)$ is a non-negative $\mathcal{F}_T \otimes \mathcal{E}$-measurable random variable. We note that $\mathbb{P}$ is not necessarily an equivalent probability measure of $\overline{\mathbb{P}}$, that is, $\beta_T(\chi)$ is not supposed to be strictly positive $\overline{\mathbb{P}}$-almost surely. The fact that $\eta$ identifies with the probability law of $\chi$ under the probability measure $\mathbb{P}$ implies that

$$\mathbb{E}_{\mathbb{P}^\circ}[\beta_T(x)] = 1, \quad \eta\text{-a.s. for } x \in E. \tag{8}$$

The dependence between the default variable $\chi$ and other market environment $\mathbb{F}$ under $\mathbb{P}$ is characterized by the change of probability in a dynamic manner.

We still examine the different information levels. The conditional law of $\chi$ given $\mathcal{N}_t$ remains invariant under $\mathbb{P}$ and $\overline{\mathbb{P}}$ because of (8), and is denoted as $\eta_t$ under both probability measures.

**Total information $\mathbb{H}$.** The Radon-Nikodym density of $\mathbb{P}$ with respect to $\overline{\mathbb{P}}$ on $\mathcal{H}_t$ is specified by the parametrized $(\mathbb{F}^\circ, \mathbb{P}^\circ)$-martingale $(\beta_t^{\mathcal{F}}(x) = \mathbb{E}_{\mathbb{P}^\circ}[\beta_T(x)|\mathcal{F}_t^\circ],\, t \in [0,T])$ depending on the parameter $x \in E$. For any $t \in [0,T]$, $\beta_t^{\mathcal{F}}(\chi)$ is the Radon-Nikodym density of $\mathbb{P}$ with respect to $\overline{\mathbb{P}}$ on $\mathcal{H}_t$. We use the notation $\beta_t$ to denote $\beta_t^{\mathcal{F}}(\chi)$ when there is no ambiguity.

(i) For a non-negative $\mathcal{H}_T$-measurable random variable $Y$ on $\Omega$, by the change of probability and Fubini's theorem under $\overline{\mathbb{P}}$, we have

$$\mathbb{E}_{\mathbb{P}}[Y|\mathcal{H}_t] = \frac{\mathbb{E}_{\overline{\mathbb{P}}}[Y\beta_T|\mathcal{H}_t]}{\beta_t}1\!\!1_{\{\beta_t>0\}} = \frac{(Y\beta)_t^{\mathcal{F}}}{\beta_t}1\!\!1_{\{\beta_t>0\}}, \quad \mathbb{P}\text{-}a.s. \tag{9}$$

where the last equality comes from (3).

**Remark 3.1** Note that we only suppose that $\mathbb{P}$ is absolutely continuous w.r.t. $\overline{\mathbb{P}}$. In particular, any $\overline{\mathbb{P}}$-negligible set is $\mathbb{P}$-negligible. However, the converse statement is not necessarily true. In particular, although $\beta_t$ is not necessarily strictly positive $\overline{\mathbb{P}}$-a.s., the set $\{\beta_t = 0\}$ is negligible under $\mathbb{P}$. In fact, one has

$$\mathbb{P}(\beta_t = 0) = \mathbb{E}_{\overline{\mathbb{P}}}[\beta_t 1\!\!1_{\{\beta_t=0\}}] = 0$$

since $\beta_t$ is the Radon-Nikodym derivative $d\mathbb{P}/d\overline{\mathbb{P}}$ on $\mathcal{H}_t$ (note that this formula does not signify $\overline{\mathbb{P}}(\beta_t = 0) = 0$). In the following, to simplify the presentation, we omit the indicator $1\!\!1_{\{\beta_t>0\}}$ and the equality (9) can be written as $\mathbb{E}_{\mathbb{P}}[Y|\mathcal{H}_t] = (Y\beta)_t^{\mathcal{F}}/\beta_t$.

In addition, under the probability $\overline{\mathbb{P}}$, $\beta_T = 0$ a.s. on the set $\{\beta_t = 0\}$ since $\beta$ is a non-negative $(\mathbb{H}, \overline{\mathbb{P}})$-martingale. In fact, let $T^x = \inf\{t \geq 0 : \beta_{t-}(x) = 0\}$, then $\beta_t(x) > 0$ on $[\![0, T^x]\!]$ and $\beta_t(x) = 0$ on $[\![T^x, \infty]\!]$.

(ii) By (9), an $\mathbb{H}$-adapted process is an $(\mathbb{H}, \mathbb{P})$-martingale if and only if its product with $\beta$ is an $(\mathbb{H}, \overline{\mathbb{P}})$-martingale, or equivalently, a parametrized $(\mathbb{F}^\circ, \mathbb{P}^\circ)$-martingale depending on a parameter $x \in E$.

**Accessible information $\mathbb{G}$.** We use the notation $\eta^{\mathcal{G}}$ for the conditional law of $\chi$ given $\mathcal{G}_t$ under the probability $\mathbb{P}$.

(i) The Bayes formula allows to calculate directly the conditional law $\eta^{\mathcal{G}}$ by

$$\eta_t^{\mathcal{G}}(dx) = \frac{\eta_t(\beta_t(x) \cdot dx)}{\eta_t(\beta_t(\cdot))} \tag{10}$$

where for a non-negative $\mathcal{A}$-measurable function $\Psi$ on $\Omega$, the notation $\eta_t(\Psi(x) \cdot dx)$ denotes the $\mathcal{A}$-random measure on $E$ which sends a non-negative Borel function $f$ on $E$ to

$$\int_E f(x)\eta_t(\Psi(x) \cdot dx) = \eta_t(f(\cdot)\Psi(\cdot)) \tag{11}$$

(ii) The $\mathcal{G}_t$-conditional expectation of a non-negative $\mathcal{H}_t$-measurable random variable $Y_t(\chi)$ is

$$\mathbb{E}_{\mathbb{P}}[Y_t(\chi)|\mathcal{G}_t] = \int_E Y_t(x)\eta_t^{\mathcal{G}}(dx) =: \eta_t^{\mathcal{G}}(Y_t(\cdot)) \tag{12}$$

(iii) For a non-negative $\mathcal{H}_T$-measurable random variable $Y$ on $\Omega$, we first project on the larger $\sigma$-algebra $\mathcal{H}_t$ and then use (9) and (12) to obtain

$$\mathbb{E}_{\mathbb{P}}[Y|\mathcal{G}_t] = \mathbb{E}_{\mathbb{P}}\big[\mathbb{E}_{\mathbb{P}}[Y|\mathcal{H}_t]|\mathcal{G}_t\big] = \int_E \frac{(Y\beta)_t^{\mathcal{F}}(x)}{\beta_t(x)}\eta_t^{\mathcal{G}}(dx). \tag{13}$$

An equivalent form can be obtained by using the Bayes formula as

$$\mathbb{E}_{\mathbb{P}}[Y \mid \mathcal{G}_t] = \frac{\eta_t((Y\beta)_t^{\mathcal{F}}(\cdot))}{\eta_t(\beta_t(\cdot))} \tag{14}$$

The equality between (13) and (14) can also be shown by (10).

(iv) Accordingly, a $(\mathbb{G}, \mathbb{P})$-martingale can be characterized as follows: a $\mathbb{G}$-adapted process is a $(\mathbb{G}, \mathbb{P})$-martingale if and only if its product with $\eta_t(\beta_t(\cdot))$ is an integral of the parametrized $(\mathbb{F}^\circ, \mathbb{P}^\circ)$-martingale $(Y\beta)^{\mathcal{F}}(x)$, $x \in E$ with respect to $\eta(dx)$, or alternatively, if and only if it can be written as the integral of the quotient of two parametrized $(\mathbb{F}^\circ, \mathbb{P}^\circ)$-martingales $(Y\beta)_t^{\mathcal{F}}(x)/\beta_t(x)$ with respect to $\eta_t^{\mathcal{G}}(dx)$.

# 4 Interaction with the environment information

In this section, we come back to the general financial market setting modelled in Section 2.1, where the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ is not necessarily given in a product form. However, the toy model of product space we discussed in the previous section will provide useful tools for the general case. From the financial point of view, the method of change of probability measure allows to describe the dynamic dependence structure between the default risk variable $\chi$ and the underlying market information filtration $\mathbb{F}$. In a recent work, Coculescu [8] has constructed an explicit non-Markovian default contagion model where the change of probability method is adopted. From the mathematical point of view, we obtain computational results where a standard hypothesis that the Radon-Nikodym derivative is supposed to be strictly positive can be relaxed.

In the enlargement of filtrations, Jacod's density hypothesis (see [24]), which is originally introduced in the setting of initial enlargement of filtrations, plays an essential role. In the credit risk analysis, the density approach of default has been adopted in progressive enlargement of filtrations to study the impact of a default event and a family of ordered multiple defaults (c.f. [15, 16], Kchia, Larsson and Protter [26] and Gapeev, Jeanblanc, Li and Rutkowski [19]). We present the link between the conditional density and the Radon-Nikodym derivative of change of probability in the general setting of a multi-default system.

### 4.1 Basic hypothesis

We first make precise the density hypothesis in our model setting.

**Hypothesis 4.1** The conditional probability law of $\chi$ given the filtration $\mathbb{F}$ admits a density with respect to a $\sigma$-finite measure $\nu$ on $(E, \mathcal{E})$, i.e., for any $t \geq 0$, there exists an $\mathcal{F}_t \otimes \mathcal{E}$-measurable function $(\omega, x) \to \alpha_t(\omega, x)$ such that for any non-negative Borel function $f$,

$$\mathbb{E}_{\mathbb{P}}[f(\chi)|\mathcal{F}_t] = \int_E f(x)\alpha_t(x)\nu(dx), \quad \mathbb{P}\text{-}a.s.$$

In [24], $\nu$ is chosen to be $\eta$ which is the probability law of $\chi$. In practice, another choice of $\nu$ is often the Lebesgue measure, especially for the multi-default models.

Under the probability $\mathbb{P}$, for any $x \in E$, the process $(\alpha_t(x), t \geq 0)$ is an $\mathbb{F}$-martingale. The probability law $\eta$ of $\chi$ is absolutely continuous with respect to the measure $\nu$ and is given by

$$\eta(dx) = \alpha_0(x)\,\nu(dx).$$

In the case where $\nu$ coincides with $\eta$, we have $\alpha_0(x) = 1$. The $\mathcal{F}_t$-conditional probability law of $\chi$ is absolutely continuous with respect to $\eta$. Let

$$\mathbb{P}(\chi \in dx|\mathcal{F}_t) =: \beta_t(x)\eta(dx), \tag{15}$$

then the following relation holds

$$\alpha_t(x) = \alpha_0(x)\beta_t(x), \quad \mathbb{P} \otimes \eta\text{-a.s.}.$$

By [24, Lemma 1.8], there exists a non-negative càdlàg version of $(\omega, t, x) \mapsto \beta_t(\omega, x)$ such that for any $x \in E$, $\beta_t(x)$ is an $\mathbb{F}$-martingale.

We can interpret the density by using the language of change of probability measure. In a setting of initial enlargement of filtrations, i.e. with respect to the filtration $\mathbb{H}$, where $\nu$

coincides with $\eta$, it is proved in Grorud and Pontier [21] (see also Amendinger, Becherer and Schweizer [4]) that if $\beta_t(\cdot)$ is strictly positive, $\mathbb{P}$-a.s., there exists a probability measure $\widehat{\mathbb{P}}$ on the measurable space $(\Omega, \mathcal{A})$ which is equivalent to $\mathbb{P}$ such that $\chi$ is independent of $\mathbb{F}$ under the probability measure $\widehat{\mathbb{P}}$, and that $\widehat{\mathbb{P}}$ coincides with $\mathbb{P}$ on $\mathbb{F}$ and $\sigma(\chi)$ respectively. In fact, the process $(\frac{1}{\beta_t(\chi)}, t \geq 0)$ is a $(\mathbb{H}, \mathbb{P})$-martingale of expectation 1 and the probability $\widehat{\mathbb{P}}$ is characterized by the Radon-Nikodym derivative (c.f. [21, Lemma 3.1])

$$\frac{d\widehat{\mathbb{P}}}{d\mathbb{P}}\Big|_{\mathcal{H}_t} = \frac{\alpha_0(\chi)}{\alpha_t(\chi)} = \frac{1}{\beta_t(\chi)}. \tag{16}$$

The above result is very useful for studying insider's information. Inspired by this idea, we study processes in the observation filtration $\mathbb{G}$ (in the general setting which is not necessarily the initial or progressive enlargement of filtrations) by combining density and change of probability measure.

**Remark 4.2** We note that the probability $\widehat{\mathbb{P}}$ does not exist in general on the probability space $(\Omega, \mathcal{A})$, notably when $\eta_t^{\mathcal{F}}$ is absolutely continuous but not equivalent to $\eta$. In fact, although $\beta_t(\chi) > 0$, $\mathbb{P}$-a.s. (see Remark 3.1), in general, $\mathbb{E}_{\mathbb{P}}[\frac{1}{\beta_t(\chi)}]$ can be strictly smaller than 1, so that we can't use (16) to define an equivalent probability measure $\widehat{\mathbb{P}}$. We provide a simple counter-example. Let $\Omega = \{\omega_0, \omega_1\}$, with $\mathbb{P}(\{\omega_0\}) = \mathbb{P}(\{\omega_1\}) = \frac{1}{2}$ and $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_1 = \{\emptyset, \{\omega_0\}, \{\omega_1\}, \Omega\}$. Let $\chi : \Omega \to E = \{0, 1\}$ be defined by $\chi(\omega_0) = 0$ and $\chi(\omega_1) = 1$. The $\mathcal{F}_1$-conditional law of $\chi$ is absolutely continuous with respect to $\eta$ with the density $\beta_1(\omega, x) : \Omega \times E \to \mathbb{R}_+$ given by $2\mathbb{1}_{\{\omega_0, 0\}} + 2\mathbb{1}_{\{\omega_1, 1\}}$. However, $\beta_1(\cdot)$ is not strictly positive on all senarios, and there does not exist a decoupling probability measure $\widehat{\mathbb{P}}$ under which $\chi$ is independent of $\mathbb{F}$ and that $\mathbb{P}$ is absolutely continuous with respect to $\widehat{\mathbb{P}}$. See Aksamit, Choulli and Jeanblanc [1] for a further discussion on this subject.

In the general case where $\beta_t(\cdot)$ is not necessarily strictly positive, that is, the $\mathcal{F}_t$-conditional probability law of $\chi$ is absolutely continuous but not equivalent with respect to its probability law $\eta$, we can no longer use the approach in [21] since the change of probability (16) is not well defined. To overcome this difficulty, we propose to use a larger product measurable space constructed from the initial probability space $(\Omega, \mathcal{A})$. The results established in the previous section 3.2 will be useful, as we shall explain in the next subsection.

## 4.2 Conditional expectations in the general setting

This subsection focuses on the computations with respect to the observable information $\mathbb{G}$ which is a general filtration as defined in Section 2.2. Recall that $\mathbb{G}$ is an enlargement of $\mathbb{F}$ as $\mathcal{G}_t = \cap_{s>t} (\mathcal{F}_s \vee \mathcal{N}_s)$ where $(\mathcal{N}_t)_{t \geq 0}$ is the inverse image given by $\mathcal{N}_t = \chi^{-1}(\mathcal{N}_t^E)$, and satisfies the usual conditions. We still assume Hypothesis 4.1. Note that $\beta_t(\cdot)$ is not supposed to be strictly positive.

To establish the main computation result, the idea in [21] is to use the decoupling probability measure where $\chi$ and $\mathbb{F}$ are independent. However, in the original space $(\Omega, \mathcal{A})$, this probability measure $\widehat{\mathbb{P}}$ does not necessarily exist (see Remark 4.2). Our method consists in extending the original probability space by introducing an auxiliary product space $(\Omega \times E, \mathcal{A} \otimes \mathcal{E})$ which is equipped with a product probability measure $\overline{\mathbb{P}} = \mathbb{P} \otimes \eta$. We consider the graph map of $\chi$, which is by definition the map $\Gamma_\chi : \Omega \to \Omega \times E$ sending $\omega \in \Omega$ to $(\omega, \chi(\omega))$. Viewed as a random variable on $(\Omega, \mathcal{A})$ valued in the product space $\Omega \times E$, the map $\Gamma_\chi$ admits a probability law $\mathbb{P}'$ under $\mathbb{P}$. More precisely, $\mathbb{P}'$ is the probability measure on the product space $(\Omega \times E, \mathcal{A} \otimes \mathcal{E})$ such that, for any non-negative $\mathcal{A} \otimes \mathcal{E}$-measurable function $f$ on $\Omega \times E$,

$$\int_{\Omega \times E} f(\omega, x)\mathbb{P}'(d\omega, dx) = \int_\Omega (f \circ \Gamma_\chi)(\omega)\mathbb{P}(d\omega) = \mathbb{E}_\mathbb{P}[f(\chi)]. \tag{17}$$

Note that $\mathbb{P}'$ is absolutely continuous with respect to the product probability $\overline{\mathbb{P}}$, and the corresponding Radon-Nikodym derivative is given by $\beta_t(\cdot)$ on $\mathcal{F}_t \otimes \mathcal{E}$ under Hypothesis 4.1 since

$$\mathbb{E}_\mathbb{P}[f(\chi)] = \mathbb{E}_\mathbb{P}[\mathbb{E}_\mathbb{P}[f(\chi)|\mathcal{F}_t]] = \int_\Omega \int_E f(\omega, x)\beta_t(x)\eta(dx)\mathbb{P}(d\omega) = \int_{\Omega \times E} f(\omega, x)\beta_t(x)\overline{\mathbb{P}}(d\omega, dx)$$

if $f$ is $\mathcal{F}_t \otimes \mathcal{E}$-measurable. Moreover, the composition of $\Gamma_\chi$ with the second projection $\Omega \times E \to E$ coincides with $\chi$. Thus we can use the method developed in Section 3.3. In the particular case where $\beta(\cdot)$ is strictly positive and the probability measure $\widehat{\mathbb{P}}$ in (16) exists, then the probability law of $\Gamma_\chi$ under $\widehat{\mathbb{P}}$ coincides with the product probability measure $\overline{\mathbb{P}}$.

We first describe processes in the filtration $\mathbb{G}$. Note that if $Y(\cdot)$ is a function on $\Omega \times E$, then the expression $Y(\chi)$ denotes actually $Y(\cdot) \circ \Gamma_\chi$ as a function on $\Omega$. We make precise in the following lemma the measurability of the application:

$$\begin{tikzcd} \Omega \arrow[r, "\Gamma_\chi"] \arrow[rr, bend right, "Y(\chi)"'] & \Omega \times E \arrow[r, "Y(\cdot)"] & \mathbb{R} \end{tikzcd} . \tag{18}$$

**Lemma 4.3** *Let $\mathcal{F}$ be a sub-$\sigma$-algebra of $\mathcal{A}$ on $\Omega$ and $\mathcal{E}_0$ be a sub-$\sigma$-algebra of $\mathcal{E}$ on $E$. Then*

*1) the map $\Gamma_\chi : (\Omega, \mathcal{F} \vee \chi^{-1}(\mathcal{E}_0)) \to (\Omega \times E, \mathcal{F} \otimes \mathcal{E}_0)$ is measurable, where $\chi^{-1}(\mathcal{E}_0) = \{\chi^{-1}(B) \,|\, B \in \mathcal{E}_0\}$ is a $\sigma$-algebra on $\Omega$;*

*2) if the map $Y(\cdot) : \Omega \times E \to \mathbb{R}$ is $\mathcal{F} \otimes \mathcal{E}_0$-measurable, then $Y(\chi) : \Omega \to \mathbb{R}$ is $\mathcal{F} \vee \chi^{-1}(\mathcal{E}_0)$-measurable.*

Proof: For 1), it suffices to prove that for all $A \in \mathcal{F}$ and $B \in \mathcal{E}_0$, one has

$$\Gamma_\chi^{-1}(A \times B) \in \mathcal{F} \vee \chi^{-1}(\mathcal{E}_0)$$

since $\mathcal{F} \otimes \mathcal{E}_0$ is generated by the sets of the form $A \times B$. Indeed,

$$\begin{aligned}\Gamma_\chi^{-1}(A \times B) &= \{\omega \in \Omega \,|\, (\omega, \chi(\omega)) \in A \times B\} = \{\omega \in A \,|\, \chi(\omega) \in B\} \\ &= A \cap \chi^{-1}(B) \in \mathcal{F} \vee \chi^{-1}(\mathcal{E}_0)\end{aligned}$$

which implies the first assertion. The second assertion 2) results from the fact that the composition of two measurable maps is still measurable. □

The above Lemma 4.3 implies directly the following result.

**Corollary 4.4** *Let $(Y_t(\cdot), t \geq 0)$ be a process adapted to the filtration $\mathbb{F} \otimes \mathcal{N}^E$, then $(Y_t(\chi), t \geq 0)$ is a $\mathbb{G}$-adapted process.*

The proposition below calculates the $\mathbb{G}$-conditional expectations, which generalizes [15, Theorem 3.1] for classic progressive enlargement of filtration and [16, Proposition 2.2] for successive multiple default times. It provides a very concise formula for computations and applications (the formula has the same form as in the particular case of product space in Section 3.2). Moreover, we show that to make estimations with respect to the filtration $\mathbb{G}$, the key terms are the prediction process and the Radon-Nikodym derivative.

**Proposition 4.5** *Let $Y_T(\cdot)$ be a non-negative $\mathcal{F}_T \otimes \mathcal{E}$-measurable function on $\Omega \times E$ and $t \leq T$. Then*

$$\mathbb{E}_{\mathbb{P}}[Y_T(\chi)|\mathcal{G}_t] = \frac{\int_E \mathbb{E}_{\mathbb{P}}[Y_T(x)\beta_T(x)|\mathcal{F}_t]\,\eta_t(dx)}{\int_E \beta_t(x)\,\eta_t(dx)}, \tag{19}$$

*where $\eta_t$ is the conditional law of $\chi$ given $\mathcal{N}_t$ and $\beta_t(x)$ is as in* (15).

Proof: Recall that $\overline{\mathbb{P}}$ denotes the product probability measure $\mathbb{P} \otimes \eta$ on $(\Omega \times E, \mathcal{A} \otimes \mathcal{E})$. Then for any $t \geq 0$, the probability $\mathbb{P}'$ defined in (17) is absolutely continuous with respect to $\overline{\mathbb{P}}$ on $\mathcal{F}_t \otimes \mathcal{E}$ with the Radon-Nikodym derivative given by $\beta_t(\cdot)$. Indeed, if $f$ is a non-negative $\mathcal{F}_t \otimes \mathcal{E}$-measurable function, then by definition (17), the expectation of $f$ with respect to the probability measure $\mathbb{P}'$ is

$$\begin{aligned}&\int_{\Omega \times E} f(\omega, x)\mathbb{P}'(d\omega, dx) = \mathbb{E}_{\mathbb{P}}[f(\cdot, \chi)] = \mathbb{E}_{\mathbb{P}}[\mathbb{E}_{\mathbb{P}}[f(\cdot, \chi)|\mathcal{F}_t]] \\ &= \mathbb{E}_{\mathbb{P}}\Big[\int_E f(\cdot, x)\beta_t(x)\,\eta(dx)\Big] = \int_{\Omega \times E} f(\omega, x)\beta_t(x)\,\overline{\mathbb{P}}(d\omega, dx).\end{aligned} \tag{20}$$

We next consider a non-negative $\mathcal{A} \otimes \mathcal{E}$-measurable random variable $Y(\cdot)$ on $\Omega \times E$. By Lemma 4.3 and the definition of $\mathbb{P}'$, for any sub-$\sigma$-algebra $\mathcal{F}$ of $\mathcal{A}$ and any sub-$\sigma$-algebra $\mathcal{E}_0$ of $\mathcal{E}$, we have

$$\mathbb{E}_{\mathbb{P}}[Y(\chi)|\mathcal{F} \vee \chi^{-1}(\mathcal{E}_0)] = \mathbb{E}_{\mathbb{P}'}[Y(\cdot)|\mathcal{F} \otimes \mathcal{E}_0](\chi) \tag{21}$$

where the expression $\mathbb{E}_{\mathbb{P}'}[Y(\cdot)|\mathcal{F}\otimes\mathcal{E}_0](\chi)$ denotes the composition $\mathbb{E}_{\mathbb{P}'}[Y(\cdot)|\mathcal{F}\otimes\mathcal{E}_0]\circ\Gamma_\chi$ as indicated by (18). Hence, we obtain for $\mathcal{G}_t=\mathcal{F}_t\vee\chi^{-1}(\mathcal{N}_t^E)$ the equality

$$\mathbb{E}_{\mathbb{P}}[Y(\chi)|\mathcal{G}_t]=\mathbb{E}_{\mathbb{P}'}[Y(\cdot)|\mathcal{F}\otimes\mathcal{N}_t^E](\chi). \tag{22}$$

Finally, we obtain by (22) and (20) that

$$\mathbb{E}_{\mathbb{P}}[Y_T(\chi)|\mathcal{G}_t]=\mathbb{E}_{\mathbb{P}'}[Y_T(\cdot)|\mathcal{F}_t\otimes\mathcal{N}_t^E](\chi)=\frac{\mathbb{E}_{\overline{\mathbb{P}}}\big[Y_T(\cdot)\beta_T(\cdot)|\mathcal{F}_t\otimes\mathcal{N}_t^E\big]}{\mathbb{E}_{\overline{\mathbb{P}}}\big[\beta_t(\cdot)|\mathcal{F}_t\otimes\mathcal{N}_t^E\big]}(\chi),\quad \mathbb{P}\text{-a.s.}$$

which implies the equality (19) since $\overline{\mathbb{P}}$ is the product probability measure $\mathbb{P}\otimes\eta$.

□

**Remark 4.6** Similarly as noted in Remark 3.1, on the set $\{\int_E\beta_t(x)\,\eta_t(dx)=0\}$, one has $\int_E\mathbb{E}_{\mathbb{P}}[Y_T(x)\beta_T(x)|\mathcal{F}_t]\,\eta_t(dx)=0$, $\mathbb{P}$-a.s., and we omit the indicator $1\!\!1_{\{\int_E\beta_t(x)\,\eta_t(dx)>0\}}$ on the right-hand side of (19). The same rule will also be applied to what follows.

In the following, we apply the above proposition to several particular cases which were presented in Example 2.1.

**Case of one default:** We consider the case where $\chi=\tau$ and the filtration $\mathbb{G}$ is the standard progressive enlargement of $\mathbb{F}$ by $\tau$. Then the $\mathcal{N}_t$-conditional law of $\tau$ is given by

$$\eta_t(du)=\frac{1\!\!1_{\{t<u\}}\alpha_0(u)\nu(du)}{\int_t^\infty\alpha_0(u)\nu(du)}1\!\!1_{\{\tau>t\}}+\delta_\tau(du)1\!\!1_{\{\tau\le t\}}$$

For any non-negative $\mathcal{F}_T\otimes\mathcal{B}(\mathbb{R}_+)$-measurable function $Y_T(\cdot)$, one has (c.f. [15, Theorem 3.1])

$$\mathbb{E}_{\mathbb{P}}[Y_T(\tau)|\mathcal{G}_t]=\frac{\int_t^\infty\mathbb{E}_{\mathbb{P}}[Y_T(u)\alpha_T(u)|\mathcal{F}_t]\nu(du)}{\int_t^\infty\alpha_t(u)\nu(du)}1\!\!1_{\{\tau>t\}}+\frac{\mathbb{E}_{\mathbb{P}}[Y_T(u)\alpha_T(u)|\mathcal{F}_t]}{\alpha_t(u)}\Big|_{u=\tau}1\!\!1_{\{\tau\le t\}}. \tag{23}$$

**One default with insider information** Consider one default with insider's information as in Example 2.1 (3) where the filtration $\mathbb{G}$ is given by $(\mathcal{F}_t\vee\mathcal{N}_t)_{t\ge0}$ with $\mathcal{N}_t=\sigma(1\!\!1_{\{\tau\le s\}},\ s\le t\text{ or }s=t_0)$. Then the $\mathcal{N}_t$-conditional law of $\tau$ is given, according to $t<t_0$ or $t\ge t_0$, by

$$\eta_t(du)=\frac{1\!\!1_{\{t<u\le t_0\}}\alpha_0(u)\nu(du)}{\int_t^{t_0}\alpha_0(u)\nu(du)}1\!\!1_{\{t<\tau\le t_0\}}+\frac{1\!\!1_{\{t_0<u\}}\alpha_0(u)\nu(du)}{\int_{t_0}^\infty\alpha_0(u)\nu(du)}1\!\!1_{\{\tau>t_0\}}+\delta_\tau(du)1\!\!1_{\{\tau\le t\}},\ \text{if } t<t_0$$

$$\eta_t(du)=\frac{1\!\!1_{\{t<u\}}\alpha_0(u)\nu(du)}{\int_t^\infty\alpha_0(u)\nu(du)}1\!\!1_{\{\tau>t\}}+\delta_\tau(du)1\!\!1_{\{\tau\le t\}},\ \text{if } t\ge t_0.$$

For any non-negative $\mathcal{F}_T\otimes\mathcal{B}(\mathbb{R}_+)$-measurable function $Y_T(\cdot)$, one has respectively

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}[Y_T(\tau)|\mathcal{G}_t]=&\frac{\int_t^{t_0}\mathbb{E}_{\mathbb{P}}[Y_T(u)\alpha_T(u)|\mathcal{F}_t]\nu(du)}{\int_t^{t_0}\alpha_t(u)\nu(du)}1\!\!1_{\{t_0\ge\tau>t\}}\\&+\frac{\int_{t_0}^\infty\mathbb{E}_{\mathbb{P}}[Y_T(u)\alpha_T(u)|\mathcal{F}_t]\nu(du)}{\int_{t_0}^\infty\alpha_t(u)\nu(du)}1\!\!1_{\{\tau>t_0\}}+\frac{\mathbb{E}_{\mathbb{P}}[Y_T(u)\alpha_T(u)|\mathcal{F}_t]}{\alpha_t(u)}\Big|_{u=\tau}1\!\!1_{\{\tau\le t\}},\ \text{if } t<t_0.\end{aligned}$$

$$\mathbb{E}_{\mathbb{P}}[Y_T(\tau)|\mathcal{G}_t] = \frac{\int_t^\infty \mathbb{E}_{\mathbb{P}}[Y_T(u)\alpha_T(u)|\mathcal{F}_t]\nu(du)}{\int_t^\infty \alpha_t(u)\nu(du)} 1\!\!1_{\{\tau>t\}} + \frac{\mathbb{E}_{\mathbb{P}}[Y_T(u)\alpha_T(u)|\mathcal{F}_t]}{\alpha_t(u)}\Big|_{u=\tau} 1\!\!1_{\{\tau\leq t\}}, \text{ if } t \geq t_0.$$

We note that when $t \geq t_0$, the information flow becomes the same as for a standard investor, so that the conditional expectation formula in this case coincides with (23).

**Multiple ordered defaults:** $\chi = \boldsymbol{\sigma} = (\sigma_1, \cdots, \sigma_n)$ where $\sigma_1 \leq \cdots \leq \sigma_n$ and $E = \{(u_1, \cdots, u_n) \in \mathbb{R}_+^n \,|\, u_1 \leq \cdots \leq u_n\}$. The filtration $(\mathcal{N}_t)_{t\geq 0}$ is generated by the process $(\sum_{i=1}^n 1\!\!1_{\{\sigma_i \leq t\}},\ t \geq 0)$. Assume that $\nu$ is the Lebesgue measure and that the $\mathcal{F}_t$-conditional law of $\chi$ has a density $\alpha_t(\cdot)$ with respect to $\nu(d\boldsymbol{u}) = d\boldsymbol{u}$. Then the $\mathcal{N}_t$-conditional law is given by

$$\eta_t(d\boldsymbol{u}) = \sum_{i=0}^n \frac{1\!\!1_{\{t<u_{i+1}\}}\alpha_0(\boldsymbol{\sigma}_{(i)}, \boldsymbol{u}_{(i+1:n)})\delta_{(\boldsymbol{\sigma}_{(i)})}(d\boldsymbol{u}_{(i)})d\boldsymbol{u}_{(i+1:n)}}{\int_t^\infty \alpha_0(\boldsymbol{\sigma}_{(i)}, \boldsymbol{u}_{(i+1:n)})d\boldsymbol{u}_{(i+1:n)}} 1\!\!1_{E_t^i}(\boldsymbol{\sigma}) \tag{24}$$

where

$$E_t^i := \{(u_1, \ldots, u_n) \in E \,|\, u_i \leq t < u_{i+1}\}. \tag{25}$$

Then by Proposition 4.5, we obtain

$$\mathbb{E}_{\mathbb{P}}[Y_T(\boldsymbol{\sigma})|\mathcal{G}_t] = \sum_{i=0}^n 1\!\!1_{\{\sigma_i\leq t<\sigma_{i+1}\}} \frac{\int_t^\infty \mathbb{E}_{\mathbb{P}}[Y_T(\boldsymbol{u})\alpha_T(\boldsymbol{u})|\mathcal{F}_t]\, d\boldsymbol{u}_{(i+1:n)}}{\int_t^\infty \alpha_t(\boldsymbol{u})\, d\boldsymbol{u}_{(i+1:n)}}\Big|_{\boldsymbol{u}_{(i)}=\boldsymbol{\sigma}_{(i)}}$$

which corresponds to [16, Proposition 2.2].

**Multiple non-ordered defaults:** $\chi = \boldsymbol{\tau} = (\tau_1, \cdots, \tau_n)$ and $E = \mathbb{R}_+^n$. The filtration $(\mathcal{N}_t)_{t\geq 0}$ is generated by the family of indicator processes $(1\!\!1_{\{\tau_i\leq t\}},\ t \geq 0)$, $i = 1, \cdots, n$. Assume in addition that the $\mathcal{F}_t$-conditional law of $\chi$ has a density $\alpha_t(\cdot)$ with respect to the Lebesgue measure $d\boldsymbol{u}$. Then the $\mathcal{N}_t$-conditional law of $\eta$ can be written in the form

$$\eta_t(d\boldsymbol{u}) = \sum_{I\subset\{1,\cdots,n\}} \frac{1\!\!1_{\{\boldsymbol{u}_{I^c}>t\}}\alpha_0(\cdot, \boldsymbol{u}_{I^c})\delta_{(\cdot)}(d\boldsymbol{u}_I)d\boldsymbol{u}_{I^c}}{\int_t^\infty \alpha_0(\cdot, \boldsymbol{u}_{I^c})d\boldsymbol{u}_{I^c}} 1\!\!1_{E_t^I}(\boldsymbol{\tau}), \tag{26}$$

where for $I \subset \{1, \ldots, n\}$, $\delta_{(\cdot)}(d\boldsymbol{u}_I)$ denotes the Dirac measure on the coordinates with indices in $I$, $\boldsymbol{u}_{I^c}$ denotes the vector $(u_j)_{j\in I^c}$, the event $\{\boldsymbol{u}_{I^c} > t\}$ denotes $\bigcap_{j\in I^c}\{\boldsymbol{u}_j > t\}$, and

$$E_t^I := \{(u_1, \ldots, u_n) \in E \,|\, \forall\, i \in I,\ u_i \in [0, t],\ \forall\, j \in I^c,\ u_j > t\}.$$

By Proposition 4.5, we obtain

$$\mathbb{E}_{\mathbb{P}}[Y_T(\boldsymbol{\tau})|\mathcal{G}_t] = \sum_{I\in\{1,\cdots,n\}} 1\!\!1_{\{\boldsymbol{\tau}_I\leq t,\, \boldsymbol{\tau}_{I^c}>t\}} \frac{\int_t^\infty \mathbb{E}_{\mathbb{P}}[Y_T(\boldsymbol{u})\alpha_T(\boldsymbol{u})|\mathcal{F}_t]d\boldsymbol{u}_{I^c}}{\int_t^\infty \alpha_t(\boldsymbol{u})d\boldsymbol{u}_{I^c}}\Big|_{\boldsymbol{u}_I=\boldsymbol{\tau}_I}$$

where $\boldsymbol{\tau}_I := (\tau_i)_{i\in I}$ and $1\!\!1_{\{\boldsymbol{\tau}_I\leq t,\, \boldsymbol{\tau}_{I^c}>t\}}$ corresponds to $1\!\!1_{E_t^I}(\boldsymbol{\tau})$.

If the ordered defaults $\boldsymbol{\sigma}$ is defined as the increasing permutation of $\boldsymbol{\tau}$, then there exists an explicit relation between the density processes $\alpha^{\boldsymbol{\sigma}}(\cdot)$ of $\boldsymbol{\sigma}$ and $\alpha^{\boldsymbol{\tau}}(\cdot)$ of $\boldsymbol{\tau}$ by using the order statistics. For any $\boldsymbol{u} \in \mathbb{R}_+^n$ such that $u_1 < \cdots < u_n$, one has for any $t \geq 0$,

$$\alpha_t^{\boldsymbol{\sigma}}(u_1, \cdots, u_n) = 1\!\!1_{\{u_1<\cdots<u_n\}} \sum_{\Pi} \alpha_t^{\boldsymbol{\tau}}(u_{\Pi(1)}, \cdots, u_{\Pi(n)})$$

where $(\Pi(1), \cdots, \Pi(n)$ is a permutation of $(1, \cdots, n)$. If in addition $\boldsymbol{\tau}$ is exchangeable (see e.g. [18]), then for any permutation, $(\tau_{\Pi(1)}, \cdots, \tau_{\Pi(n)})$ has the same distribution as $(\tau_1, \cdots, \tau_n)$ so that

$$\alpha_t^{\boldsymbol{\sigma}}(u_1, \cdots, u_n) = \mathbb{1}_{\{u_1 < \cdots < u_n\}} n! \, \alpha_t^{\boldsymbol{\tau}}(u_1, \cdots, u_n).$$

In this case we say that the default portfolio is homogeneous.

# 5 Martingale characterization

It is important to study martingale properties for financial applications such as pricing of credit sensitive contingent claims. In this section, we are interested in the characterization of martingale processes in different enlarged filtrations, notably in the observation information filtration $\mathbb{G}$.

We first recall a martingale criterion in the initial enlargement of filtration in Amendinger [3] (see also Callegaro, Jeanblanc and Zargari [7]). It corresponds in our setting to the total information filtration $\mathbb{H} = (\mathcal{H}_t)_{t \geq 0}$, $\mathcal{H}_t = \cap_{s>t}(\mathcal{F}_s \vee \sigma(\chi))$. For the ease of readers, we also give the proof below.

**Proposition 5.1** *An $(\mathcal{F}_t \otimes \mathcal{E})_{t \geq 0}$-adapted process $(M_t(\cdot), t \geq 0)$ is an $(\mathcal{F}_t \otimes \mathcal{E})_{t \geq 0}$-martingale under the probability measure $\mathbb{P}'$ defined in (17) if and only if $(\alpha_t(x) M_t(x), t \geq 0)$ is a parametrized $(\mathbb{F}, \mathbb{P})$-martingale depending on $x \in E$. Moreover, if this condition is satisfied, then $(M_t(\chi), t \geq 0)$ is an $(\mathbb{H}, \mathbb{P})$-martingale on $(\Omega, \mathcal{A})$.*

Proof: For any $\mathcal{F}_T \otimes \mathcal{E}$-measurable random variable $M_T(\cdot)$ and $t \leq T$, since the Radon-Nikodym derivative of $\mathbb{P}'$ with respect to $\overline{\mathbb{P}}$ is $\beta_T(\cdot)$ on $\mathcal{F}_T \otimes \mathcal{E}$, we have that

$$\alpha_t(\cdot) \mathbb{E}_{\mathbb{P}'}[M_T(\cdot) | \mathcal{F}_t \otimes \mathcal{E}] = \mathbb{E}_{\overline{\mathbb{P}}}\big[M_T(\cdot) \alpha_T(\cdot) \,\big|\, \mathcal{F}_t \otimes \mathcal{E}\big] = \mathbb{E}_{\mathbb{P}}[M_T(\cdot) \alpha_T(\cdot) | \mathcal{F}_t].$$

Note that $\alpha_t(\cdot) > 0$ almost surely under $\mathbb{P}'$. In fact, by (17) one has

$$\mathbb{E}_{\mathbb{P}'}[\mathbb{1}_{\{\alpha_t(\cdot)=0\}}] = \mathbb{E}_{\mathbb{P}}[\mathbb{1}_{\{\alpha_t(\chi)=0\}}] = \mathbb{E}_{\mathbb{P}}\left[\int_E \mathbb{1}_{\{\alpha_t(x)=0\}} \alpha_t(x) \nu(dx)\right] = 0.$$

Therefore, the process $M(\cdot)$ is an $(\mathbb{P}', \mathcal{F}_t \otimes \mathcal{E})_{t \geq 0}$-martingale if and only if $(\alpha_t(x) M_t(x), t \geq 0)_{x \in E}$ is a parametrized $(\mathbb{F}, \mathbb{P})$-martingale depending on $x \in E$. Finally, (21) implies that

$$\mathbb{E}_{\mathbb{P}}[M_T(\chi) | \mathcal{H}_t] = \mathbb{E}_{\mathbb{P}'}[M_T(\cdot) | \mathcal{F}_t \otimes \mathcal{E}](\chi).$$

Therefore we obtain the second assertion of the proposition. □

### 5.1 Martingales in the filtration $\mathbb{G}$

We now deduce from Proposition 4.5 the following martingale criterion for the accessible information filtration $\mathbb{G}$.

**Theorem 5.2** *Let $(M_t(\cdot), t \geq 0)$ be $(\mathcal{F}_t \otimes \mathcal{N}_t^E)_{t\geq 0}$-adapted processes. If the process*

$$\widetilde{M}_t(\chi) = M_t(\chi) \int_E \beta_t(x)\eta_t(dx), \quad t \geq 0$$

*verifies*

$$\forall T \geq t \geq 0, \quad \int_E \mathbb{E}_{\mathbb{P}}[\widetilde{M}_T(x)|\mathcal{F}_t]\,\eta_t(dx) = \widetilde{M}_t(\chi), \tag{27}$$

*then $(M_t(\chi), t \geq 0)$ is a $(\mathbb{G}, \mathbb{P})$-martingale.*

Proof: For any $t \geq 0$, let $\eta_t^E$ be the conditional law of $\chi$ given $\mathcal{N}_t^E$ on $(E, \mathcal{E})$, i.e., for any bounded or non-negative Borel function $f$ on $E$,

$$\int_E f(x)\eta_t(dx) = \Big(\int_E f(x)\eta_t^E(dx)\Big)(\chi). \tag{28}$$

Then, by Proposition 4.5, for $T \geq t \geq 0$,

$$\mathbb{E}_{\mathbb{P}}[M_T(\chi)|\mathcal{G}_t] = \frac{\int_E \mathbb{E}_{\mathbb{P}}[M_T(x)\beta_T(x)|\mathcal{F}_t]\eta_t^E(dx)}{\int_E \beta_t(x)\eta_t^E(dx)}(\chi).$$

By Fubini's theorem for conditional expectations,

$$\int_E \mathbb{E}_{\mathbb{P}}\Big[M_T(x)\beta_T(x)\,\Big|\,\mathcal{F}_t\Big]\eta_t^E(dx) = \mathbb{E}_{\overline{\mathbb{P}}}\Big[M_T(\cdot)\beta_T(\cdot)\,\Big|\,\mathcal{F}_t \otimes \mathcal{N}_t^E\Big] \tag{29}$$

Note that by (20), $d\mathbb{P}' = \beta_T(\cdot)d\overline{\mathbb{P}}$ on $\mathcal{F}_T \otimes \mathcal{E}$. Since $\mathbb{P}'$ is induced by the graph map $\Gamma_\chi$, for any bounded or non-negative $\mathcal{F}_T \otimes \mathcal{E}$-measurable random variable $\varphi_T(\cdot)$, one has

$$\mathbb{E}_{\overline{\mathbb{P}}}\Big[\varphi_T(\cdot)\beta_T(\cdot)\Big] = \mathbb{E}_{\mathbb{P}'}[\varphi_T(\cdot)] = \mathbb{E}_{\mathbb{P}}[\varphi_T(\chi)],$$

which only depends on the value of the random variable $\varphi_T(\chi)$ on $\Omega$. Hence the random variable $\mathbb{E}_{\overline{\mathbb{P}}}[M_T(\cdot)\beta_T(\cdot)\,|\,\mathcal{F}_t \otimes \mathcal{N}_t^E]$, which is the right-hand side of the equality (29), only depends on the value of $M_T(\chi)$. Therefore in the computation of (29), we may assume without loss of generality that

$$\widetilde{M}_T(\cdot) = M_T(\cdot) \int_E \beta_T(x)\eta_T^E(dx). \tag{30}$$

By Bayes' formula, one has

$$\int_E \mathbb{E}_{\mathbb{P}}\Big[M_T(x)\beta_T(x)\,\Big|\,\mathcal{F}_t\Big]\eta_t^E(dx) = \mathbb{E}_{\overline{\mathbb{P}}}\Big[M_T(\cdot)\int_E \beta_T(x)\eta_T^E(dx)\,\Big|\,\mathcal{F}_t \otimes \mathcal{N}_t^E\Big], \tag{31}$$

which by (30) leads to

$$\int_E \mathbb{E}_{\mathbb{P}}\Big[M_T(x)\beta_T(x)\,\Big|\,\mathcal{F}_t\Big]\eta_t^E(dx) = \int_E \mathbb{E}_{\mathbb{P}}[\widetilde{M}_T(x)|\mathcal{F}_t]\eta_t^E(dx).$$

Finally the condition (27) implies

$$\mathbb{E}_{\mathbb{P}}[M_T(\chi)|\mathcal{G}_t] = \frac{\widetilde{M}_t(\chi)}{\big(\int_E \beta_t(x)\eta_t(dx)\big)(\chi)} = M_t(\chi).$$

The theorem is thus proved. □

**Remark 5.3** The condition (27) is satisfied notably when

$$\int_E \mathbb{E}_{\mathbb{P}}[\widetilde{M}_T(x)|\mathcal{F}_t]\eta_t^E(dx) = \widetilde{M}_t(x)$$

for any $x \in E$, which means that $(\widetilde{M}_t(\cdot))_{t\geq 0}$ is an $(\mathcal{F}_t \otimes \mathcal{N}_t^E)_{t\geq 0}$-martingale under the product measure $\overline{\mathbb{P}}$. This observation allows to construct $(\mathbb{G},\mathbb{P})$-martingales. We begin with a $((\mathcal{F}_t \otimes \mathcal{N}_t^E)_{t\geq 0}, \overline{\mathbb{P}})$-martingale $\widetilde{M}(\cdot)$ (which could be chosen easily as conditional expectations on the product probability space $\Omega \times E$ since $\overline{\mathbb{P}}$ is the product measure). Then the process

$$M_t(\chi) := \widetilde{M}_t(\chi)\Big(\int_E \beta_t(x)\eta_t(dx)\Big)^{-1}, \quad t \geq 0$$

is a $(\mathbb{G},\mathbb{P})$-martingale.

**Corollary 5.4** *Given a strictly positive $(\mathbb{G},\mathbb{P})$-martingale $(M_t(\chi), t \geq 0)$ with $\mathbb{P}$-expectation 1, let $\mathbb{Q}$ be an equivalent probability measure of $\mathbb{P}$ defined by*

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\,|_{\mathcal{G}_t} = M_t(\chi).$$

*Then, for any non-negative Borel function $f(\cdot)$ on $E$, one has*

$$\mathbb{E}_{\mathbb{Q}}[f(\chi)|\mathcal{F}_t] = \frac{\mathbb{E}_{\mathbb{P}}[f(\chi)M_t(\chi)|\mathcal{F}_t]}{\mathbb{E}_{\mathbb{P}}[M_t(\chi)|\mathcal{F}_t]} = \frac{\int_E f(x)M_t(x)\alpha_t(x)\nu(dx)}{\int_E M_t(x)\alpha_t(x)\nu(dx)}.$$

*In other words, the $\mathcal{F}_t$-conditional density of $\chi$ under $\mathbb{Q}$ is*

$$\alpha_t^{\mathbb{Q}}(x) = \frac{M_t(x)\alpha_t(x)}{\int_E M_t(x)\alpha_t(x)\nu(dx)}.$$

The immersion property between filtrations $\mathbb{F}$ and $\mathbb{G}$ asserts that any $\mathbb{F}$-martingale remains a $\mathbb{G}$-martingale. We present below a direct consequence of Theorem 5.2 concerning the immersion property.

**Corollary 5.5** *Assume that, for $0 \leq t \leq T$, one has*

$$\int_E \beta_t(x)\eta_t^E(dx) = \int_E \beta_T(x)\eta_t^E(dx), \tag{32}$$

*then $(\mathbb{F},\mathbb{G})$ satisfies the immersion property.*

PROOF: Let $M$ be an $(\mathbb{F},\mathbb{P})$-martingale. For $t \geq 0$ and $x \in E$, let

$$\widetilde{M}_t(x) = M_t \int_E \beta_t(x)\eta_t^E(dx).$$

One has

$$\mathbb{E}_{\mathbb{P}}[\widetilde{M}_T(x)|\mathcal{F}_t]\eta_t^E(dx) = \mathbb{E}_{\mathbb{P}}\left[M_T \int_E \beta_T(x)\eta_t^E(dx) \,\middle|\, \mathcal{F}_t\right] = M_t \int_E \beta_t(x)\eta_t^E(dx),$$

where the last equality comes from (32). By Theorem 5.2, we obtain that $M$ is a $(\mathbb{G},\mathbb{P})$-martingale. $\square$

## 5.2 Special cases of ordered and non-ordered multi-defaults

We apply the martingale characterization result to several particular cases. In the case of single default when $\chi = \tau$ and $E = \mathbb{R}_+$, Theorem 5.2 leads to [15, Theorem 5.7]. In the following, we give some extensions for multi-default cases by using Theorem 5.2.

**Case of ordered defaults:** This case can be viewed as a generalization of the single default case. By (24), we have for any $\boldsymbol{v} \in E = \{(v_1, \cdots, v_n) \in \mathbb{R}_+^n \mid v_1 \leq \cdots \leq v_n\}$,

$$\Big(\int_E \beta_t(\boldsymbol{u})\eta_t(d\boldsymbol{u})\Big)(\boldsymbol{v}) = \sum_{i=0}^n \frac{\int_t^\infty \alpha_t(\boldsymbol{v}_{(i)}, \boldsymbol{u}_{(i+1:n)})d\boldsymbol{u}_{(i+1:n)}}{\int_t^\infty \alpha_0(\boldsymbol{v}_{(i)}, \boldsymbol{u}_{(i+1:n)})d\boldsymbol{u}_{(i+1:n)}} 1\!\!1_{E_t^i}(\boldsymbol{v}).$$

Since $(\mathcal{N}_t^E)_{t\geq 0}$ is generated by the process $(N_t = \sum_{i=1}^n 1\!\!1_{\{u_i \leq t\}},\ t \geq 0)$, the $\mathbb{F} \otimes \mathcal{N}^E$-adapted process $M(\boldsymbol{u})$ can be written in the form

$$M_t(\boldsymbol{u}) = \sum_{i=0}^n M_t^i(\boldsymbol{u}_{(i)})1\!\!1_{E_t^i}(\boldsymbol{u}), \quad t \geq 0$$

where $M^i(\cdot)$ is $\mathbb{F} \otimes \mathcal{B}(\mathbb{R}_+^i)$-adapted and $E_t^i$ is defined in (25), then one has

$$\widetilde{M}_t(\boldsymbol{u}) = \sum_{i=0}^n \left(M_t^i(\boldsymbol{u}_{(i)}) \frac{\int_t^\infty \alpha_t(\boldsymbol{u})d\boldsymbol{u}_{(i+1:n)}}{\int_t^\infty \alpha_0(\boldsymbol{u})d\boldsymbol{u}_{(i+1:n)}}\right) 1\!\!1_{E_t^i}(\boldsymbol{u}).$$

Therefore, for $T \geq t \geq 0$,

$$\mathbb{E}_{\mathbb{P}}[\widetilde{M}_T(\boldsymbol{u})|\mathcal{F}_t] = \sum_{i=0}^n \frac{\mathbb{E}_{\mathbb{P}}[M_T^i(\boldsymbol{u}_{(i)}) \int_T^\infty \alpha_T(\boldsymbol{u})d\boldsymbol{u}_{(i+1:n)}|\mathcal{F}_t]}{\int_T^\infty \alpha_0(\boldsymbol{u})d\boldsymbol{u}_{(i+1:n)}} 1\!\!1_{E_T^i}(\boldsymbol{u})$$

and

$$\begin{aligned}
&\Big(\int_E \mathbb{E}_{\mathbb{P}}[\widetilde{M}_T(\boldsymbol{u})|\mathcal{F}_t]\eta_t(d\boldsymbol{u})\Big)(\boldsymbol{v}) \\
&= \sum_{j=0}^n \Big(\sum_{i\geq j} \frac{\int_t^T \mathbb{E}_{\mathbb{P}}[M_T^i(\boldsymbol{v}_{(i)}) \int_T^\infty \alpha_T(\boldsymbol{v}_{(i)}, \boldsymbol{u}_{(i+1:n)})d\boldsymbol{u}_{(i+1:n)}|\mathcal{F}_t]d\boldsymbol{v}_{(j+1:i)}}{\int_t^\infty \alpha_0(\boldsymbol{v})d\boldsymbol{v}_{(j+1:n)}}\Big) 1\!\!1_{E_t^j}(\boldsymbol{v}).
\end{aligned}$$

So the condition (27) is equivalent to, for any $j \in \{0, \dots, n\}$,

$$\sum_{i \geq j} \int_t^T \mathbb{E}_{\mathbb{P}}\Big[M_T^i(\boldsymbol{u}_{(i)}) \int_T^\infty \alpha_T(\boldsymbol{u}) d\boldsymbol{u}_{(i+1:n)} \Big| \mathcal{F}_t\Big] d\boldsymbol{u}_{(j+1:i)} = M_t^j(\boldsymbol{u}_{(j)}) \int_t^\infty \alpha_t(\boldsymbol{u}) d\boldsymbol{u}_{(j+1:n)},\ t \geq u_j \tag{33}$$

which implies the following characterization result.

**Proposition 5.6** *The condition* (27) *is equivalent to the following: for any $j \in \{0, \dots, n\}$ and any $\boldsymbol{u}_{(j)} \in \mathbb{R}_+^j$, $u_1 \leq \dots \leq u_j$,*

$$M_t^j(\boldsymbol{u}_{(j)}) \int_t^\infty \alpha_t(\boldsymbol{u}) d\boldsymbol{u}_{(j+1:n)} - \int_0^t M_{u_{j+1}}^{j+1}(\boldsymbol{u}_{(j+1)}) \int_{u_{j+1}}^\infty \alpha_{u_{j+1}}(\boldsymbol{u}) d\boldsymbol{u}_{(j+2:n)} du_{j+1},\ \ t \geq u_j \tag{34}$$

*is an $(\mathbb{F}, \mathbb{P})$-martingale.*

Proof: For any $j \in \{0, \cdots, n\}$, let $(A_j)$ be the equality (33) for $T \geq t \geq 0$ and $\boldsymbol{u}_{(j)} = (u_1, \cdots, u_j) \in \mathbb{R}_+^j$ such that $u_1 \leq \dots \leq u_j \leq t$ and let $(B_j)$ be the martingale property of (34). We will prove by reverse induction on $j$ that

$$(\forall\, i \geq j,\ (A_i)) \Longleftrightarrow (\forall\, i \geq j,\ (B_i)). \tag{35}$$

Note that the conditions $(A_n)$ and $(B_n)$ are acutally the same. Assume that the equivalence (35) has been proved for $j' \geq j$. We will prove the equivalence for $j$. By the induction assumption it suffice to prove $(A_j) \Leftrightarrow (B_j)$ given that $(A_i)$ and $(B_i)$ are satisfied for all $i > j$. Thus for $i \geq j+1$ and $t < u_{j+1}$ one has

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}}\Big[M_T^i(\boldsymbol{u}_{(i)}) \int_T^\infty \alpha_T(\boldsymbol{u}) d\boldsymbol{u}_{(i+1:n)} \Big| \mathcal{F}_t\Big] = \mathbb{E}_{\mathbb{P}}\Big[\mathbb{E}\Big[M_T^i(\boldsymbol{u}_{(i)}) \int_T^\infty \alpha_T(\boldsymbol{u}) d\boldsymbol{u}_{(i+1:n)} \Big| \mathcal{F}_{u_i}\Big] \mathcal{F}_t\Big] \\
&= \mathbb{E}_{\mathbb{P}}\Big[M_{u_i}^i(\boldsymbol{u}_{(i)}) \int_{u_i}^\infty \alpha_{u_i}(\boldsymbol{u}) d\boldsymbol{u}_{(i+1:n)} \Big| \mathcal{F}_t\Big] - \int_{u_i}^T \mathbb{E}_{\mathbb{P}}\Big[M_{u_{i+1}}^{i+1}(\boldsymbol{u}_{(i+1)}) \int_{u_{i+1}}^\infty \alpha_{u_{i+1}}(\boldsymbol{u}) d\boldsymbol{u}_{(i+2:n)} \Big| \mathcal{F}_t\Big] du_{i+1}.
\end{aligned}$$

Therefore the equality (33) is equivalent to

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}}\Big[M_T^j(\boldsymbol{u}_{(j)}) \int_T^\infty \alpha_T(\boldsymbol{u}) d\boldsymbol{u}_{(i+1:n)} \Big| \mathcal{F}_t\Big] - M_t^j(\boldsymbol{u}_{(j)}) \int_t^\infty \alpha_t(\boldsymbol{u}) d\boldsymbol{u}_{(j+1:n)} \\
&= \sum_{i \geqslant j+1} \bigg( \int_t^T \mathbb{E}_{\mathbb{P}}\Big[M_{u_i}^i(\boldsymbol{u}_{(i)}) \int_{u_i}^\infty \alpha_{u_i}(\boldsymbol{u}) d\boldsymbol{u}_{(i+1:n)} \Big| \mathcal{F}_t\Big] d\boldsymbol{u}_{(j+1:i)} \\
&\qquad - \int_t^T \mathbb{E}_{\mathbb{P}}\Big[M_{u_{i+1}}^{i+1}(\boldsymbol{u}_{(i+1)}) \int_{u_{i+1}}^\infty \alpha_{u_{i+1}}(\boldsymbol{u}) d\boldsymbol{u}_{(i+2:n)} \Big| \mathcal{F}_t\Big] d\boldsymbol{u}_{(i+2:n)} \bigg) \\
&= \int_t^T \mathbb{E}_{\mathbb{P}}\Big[M_{u_{j+1}}^{j+1}(\boldsymbol{u}_{(j+1)}) \int_{u_{j+1}}^\infty \alpha_{u_{j+1}}(\boldsymbol{u}) d\boldsymbol{u}_{(j+2:n)} \Big| \mathcal{F}_t\Big] du_{j+1}.
\end{aligned}$$

Hence we obtain the equivalence of $(A_j)$ and $(B_j)$. $\square$

**Case of non-ordered defaults:** The case of non-ordered defaults and ordered ones can be treated in similar way. The only difference is to make precise the corresponding prediction process. For $\boldsymbol{v} \in \mathbb{R}_+^n$, one has by (26) that

$$\Big(\int_E \beta_t(\boldsymbol{u})\eta_t(d\boldsymbol{u})\Big)(\boldsymbol{v}) = \sum_{I\subset\{1,\cdots,n\}} \frac{\int_t^\infty \alpha_t(\boldsymbol{v}_I, \boldsymbol{u}_{I^c})d\boldsymbol{u}_{I^c}}{\int_t^\infty \alpha_0(\boldsymbol{v}_I, \boldsymbol{u}_{I^c})d\boldsymbol{u}_{I^c}} \mathbb{1}_{E_t^I}(\boldsymbol{v}).$$

The $\mathbb{F}\otimes\mathcal{N}^E$-adapted process $M(\boldsymbol{u})$ can be written in the form

$$M_t(\boldsymbol{u}) = \sum_{I\subset\{1,\ldots,n\}} M_t^I(\boldsymbol{u}_I)\mathbb{1}_{E_t^I}(\boldsymbol{u}), \quad t \geq 0$$

where $M^I(\cdot)$ is $\mathbb{F}\otimes\mathcal{B}(\mathbb{R}_+^I)$-adapted, then one has

$$\widetilde{M}_t(\boldsymbol{u}) = \sum_{I\subset\{1,\ldots,n\}} \Big(M_t^I(\boldsymbol{u}_I)\frac{\int_t^\infty \alpha_t(\boldsymbol{u})d\boldsymbol{u}_{I^c}}{\int_t^\infty \alpha_0(\boldsymbol{u})d\boldsymbol{u}_{I^c}}\Big)\mathbb{1}_{E_t^I}(\boldsymbol{u})$$

and for $T \geq t \geq 0$,

$$\mathbb{E}_{\mathbb{P}}[\widetilde{M}_T(\boldsymbol{u})|\mathcal{F}_t] = \sum_{I\subset\{1,\ldots,n\}} \frac{\mathbb{E}_{\mathbb{P}}[M_T^I(\boldsymbol{u}_I)\int_T^\infty \alpha_T(\boldsymbol{u})d\boldsymbol{u}_{I^c}|\mathcal{F}_t]}{\int_T^\infty \alpha_0(\boldsymbol{u})d\boldsymbol{u}_{I^c}}\mathbb{1}_{E_t^I}(\boldsymbol{u}).$$

Therefore

$$\begin{aligned}
&\Big(\int_E \mathbb{E}_{\mathbb{P}}[\widetilde{M}_T(\boldsymbol{u})|\mathcal{F}_t]\eta_t(d\boldsymbol{u})\Big)(\boldsymbol{v}) \\
&= \sum_{J\subset\{1,\ldots n\}} \Big(\sum_{I\supset J}\int_t^\infty \frac{\mathbb{E}_{\mathbb{P}}[M_T^I(\boldsymbol{v}_I)\int_T^\infty \alpha_T(\boldsymbol{v}_I, \boldsymbol{u}_{I^c})d\boldsymbol{u}_{I^c}|\mathcal{F}_t]}{\int_T^\infty \alpha_0(\boldsymbol{v}_I, \boldsymbol{u}_{I^c})d\boldsymbol{u}_{I^c}}\mathbb{1}_{E_T^I}(\boldsymbol{v})\alpha_0(\boldsymbol{v})d\boldsymbol{v}_{J^c}\Big)\frac{\mathbb{1}_{E_t^J}(\boldsymbol{v})}{\int_t^\infty \alpha_0(\boldsymbol{v})d\boldsymbol{v}_{J^c}} \\
&= \sum_{J\subset\{1,\ldots,n\}} \Big(\sum_{I\supset J} \frac{\int_t^T \mathbb{E}_{\mathbb{P}}[M_T^I(\boldsymbol{v}_I)\int_T^\infty \alpha_T(\boldsymbol{v}_I, \boldsymbol{u}_{I^c})d\boldsymbol{u}_{I^c}|\mathcal{F}_t]d\boldsymbol{v}_{I\setminus J}}{\int_t^\infty \alpha_0(\boldsymbol{v})d\boldsymbol{v}_{J^c}}\Big)\mathbb{1}_{E_t^J}(\boldsymbol{v})
\end{aligned}$$

Therefore the conditon (27) is actually equivalent to, for any $J \subset \{1,\ldots,n\}$, and $\boldsymbol{u}_J \in \mathbb{R}_+^J$ such that $\boldsymbol{u}_J^{\max} := \max_{j\in J} u_j \leq t$,

$$\sum_{I\supset J}\int_t^T \mathbb{E}_{\mathbb{P}}\Big[M_T^I(\boldsymbol{u}_I)\int_T^\infty \alpha_T(\boldsymbol{u})d\boldsymbol{u}_{I^c}\Big|\mathcal{F}_t\Big]d\boldsymbol{u}_{I\setminus J} = M_t^J(\boldsymbol{u}_J)\int_t^\infty \alpha_t(\boldsymbol{u})d\boldsymbol{u}_{J^c}. \tag{36}$$

Similar as in the ordered case, we have the following characterization result for non-ordered defaults. The proof is analogous to that of Proposition 5.6. We therefore omit it.

**Proposition 5.7** *The condition* (27) *is equivalent to the following: for any* $J \subset \{0,\cdots,n\}$ *and any* $\boldsymbol{u}_J \in \mathbb{R}_+^J$*, the process*

$$M_t^J(\boldsymbol{u}_J)\int_t^\infty \alpha_t(\boldsymbol{u})d\boldsymbol{u}_{J^c} - \sum_{k\in J^c}\int_{\boldsymbol{u}_J^{\max}}^t M_{u_k}^{J\cup\{k\}}(\boldsymbol{u}_{J\cup\{k\}})\Big(\int_{u_k}^\infty \alpha_{u_k}(\boldsymbol{u})d\boldsymbol{u}_{J^c\setminus\{k\}}\Big)du_k, \ \ \boldsymbol{u}_J^{\max} \leq t \tag{37}$$

*is an* $(\mathbb{F},\mathbb{P})$*-martingale.*

We finally give an example to show how to construct a $\mathbb{G}$-martingale when $n = 2$ and $\chi = (\tau_1, \tau_2)$ by using the above proposition.

**Example 5.8** We define $M(\cdot,\cdot)$ to be an $(\mathcal{F}_t \otimes \mathcal{N}_t^E)_{t\geq 0}$-adapted process which is written in the form

$$M_t(u_1, u_2) = \mathbb{1}_{\{u_1>t, u_2>t\}} M_t^0 + \sum_{\{i,j\}=\{1,2\}} \mathbb{1}_{\{u_i\leq t, u_j>t\}} M_t^i(u_i) + \mathbb{1}_{\{u_1\leq t, u_2\leq t\}} M_t^{1,2}(u_1, u_2), \quad (38)$$

where $M^0$ is $\mathbb{F}$-adapted, $M^1(\cdot)$ and $M^2(\cdot)$ are $\mathbb{F}\otimes\mathcal{B}(\mathbb{R}_+)$-adapted, and $M^{1,2}$ is $\mathbb{F}\otimes\mathcal{B}(\mathbb{R}_+^2)$-adapted processes. Then $(M_t(\tau_1, \tau_2), t \geq 0)$ is a $\mathbb{G}$-adapted process. Let $(\alpha_t(u_1, u_2), t \geq 0)$ be the $\mathbb{F}$-conditional density process of the couple of non-ordered default times $(\tau_1, \tau_2)$, as defined in Hypothesis 4.1. Let $(L_t^{1,2}(u_1, u_2), t \geq \max(u_1, u_2))$ be a family of $(\mathbb{F}, \mathbb{P})$-martingales and

$$M_t^{1,2}(u_1, u_2) = \frac{L_t^{1,2}(u_1, u_2)}{\alpha_t(u_1, u_2)}, \quad t \geq \max(u_1, u_2).$$

In a recursive way, for $\{i, j\} = \{1, 2\}$, let $(L_t^i(u_i), t \geq u_i)$ be $(\mathbb{F}, \mathbb{P})$-martingales and

$$M_t^i(u_i) = \frac{L_t^i(u_i) + \int_{u_i}^t M_{u_j}^{1,2}(u_1, u_2)\alpha_{u_j}(u_1, u_2)du_j}{\int_t^\infty \alpha_t(u_1, u_2)du_j}, \quad t \geq u_i.$$

Finally, let $(L_t^0, t \geq 0)$ be a $(\mathbb{F}, \mathbb{P})$-martingale and

$$M_t^0 = \frac{L_t^0 + \sum_{\{i,j\}=\{1,2\}} \int_0^t M_{u_i}^i(u_i) \int_{u_i}^\infty \alpha_{u_i}(u_1, u_2)du_j du_i}{\int_t^\infty \alpha_t(u_1, u_2)du_1 du_2}, \quad t \geq 0.$$

Then the process $(M_t(\tau_1, \tau_2), t \geq 0)$ constructed as above is a $\mathbb{G}$-martingale.

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