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ON CONTROL STRATEGIES FOR AVOIDING LOSS OF INFORMATION IN NON DESTRUCTIVE TESTING

PHILIPPE DESTUYNDER

Département d'ingénierie mathématique, laboratoire M2N
Conservatoire National des Arts et Métiers
292, rue saint Martin, 75003 Paris France

CAROLINE FABRE

Laboratoire de mathématiques d'Orsay, UMR 8628
Univ Paris-Sud, CNRS, Université Paris-Saclay
Orsay 91405 France

ABSTRACT. This study takes place in the framework of non destructive testing. Based on energy invariants, one can derive boundary quantities which could help to detect damages in a structure. Since only a small part of a structure is equipped with sensors, a lot of information may be lost through the boundary where there is no sensor. In this paper we study several control strategies in order to drive most of the information contained in the energy invariant used, to the sensors.

1. Introduction. Let us consider an open, nonempty, bounded and connected set $\Omega \subset \mathbb{R}^2$ with a boundary $\Gamma = \partial\Omega$ as shown on figure 1. There are at least two parts on the boundary Γ . One denoted by Γ_m , equipped with strain sensors and the complementary -say Γ_f - which is unequipped. On all the boundary, homogeneous Neumann boundary conditions are prescribed. The unit normal to Γ outward Ω is denoted by ν . We assume that there is a hole inside Ω occupying the open set T with boundary ∂T . The goal of the method that we describe in this paper, consists in detecting this hole from boundary measurements. The difficulty is that only a part of the external boundary of Ω is equipped with sensors. Therefore, a part of the information is lost. One can find such situation in [1] and [8]. One suggestion is to use an optimal control strategy in order to avoid if possible this loss of information through the unequipped boundary on $\Gamma \setminus \partial T$. A control is then set on a sub-open set of Ω say O_1 - denoted by f and is computed in order to minimize the loss. In fact, the cost of the optimal control is introduced with a small parameter -say ε - and when it tends to zero, we discuss the exact controllability of the quantity on $\Gamma \setminus \partial T$ which is the loss of information that we try to cancel. The paper is organized as follows.

1. Introduction;
2. The mathematical model;
3. The invariant used;
4. The criterion used and the optimality relations;
5. Robustness of the optimal control;

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6. Computation of approximate or exact control(s);
7. Conclusion.

Finally, we refer to [5], [7], [12], [13] for a state of the art on the NDT subject.

2. The mathematical model. Let us consider the open set Ω as shown on figure 1 and two open and nonempty sets O_1 and O_2 in Ω with $\overline{O_1} \cup \overline{O_2} \subset \Omega$ and $\overline{O_1} \cap \overline{O_2} = \emptyset$.

We introduce two functions f and g with compact supports respectively in O_1 and O_2 . Notice that in what follows g is a given data whereas f is a control.

We write $\Gamma = \Gamma_f \cup \Gamma_m \cup \partial T$. It is assumed that in neighborhoods of the connection points between Γ_f and Γ_m the unit normal ν satisfies $\nu \cdot e_1 = 0$ on Γ_f and $\nu \cdot e_2 = 0$ on Γ_m . This implies that all the angles at the corners between Γ_f and Γ_m are $\pm \frac{\pi}{2}$.

Furthermore, in all the paper, χ_O denotes the characteristic function of a set O .

FIGURE 1. The open set Ω

2.1. The Helmholtz equation used. Let us introduce the following mathematical model set in the open set Ω .

$$\left\{ \begin{array}{l} \text{Find } u \in H^1(\Omega) \text{ such that:} \\ -\omega^2 u - \operatorname{div}(c^2 \nabla u) = f + g \text{ in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma. \end{array} \right. \quad (1)$$

As soon as ω^2 is not solution of the following eigenvalue problem (concerning the values of λ , one can refer to [6], [17]):

$$\text{find } w \in H^1(\Omega), \lambda \in \mathbb{R}^+, \forall v \in H^1(\Omega), \int_{\Omega} c^2 \nabla w \cdot \nabla v = \lambda \int_{\Omega} w v, \quad (2)$$

Fredholm's theorem (see [2], [9]) enables one to ensure that there is a unique solution to (1) for any couple $(f, g) \in L^2(O_1) \times L^2(O_2)$. The set of eigenvalues λ_n solution of the previous model (2) is denoted by $\Lambda = \{\lambda_n, n \in \mathbb{N}\}$. Let us point out that $\lambda_0 = 0$. The wave velocity c can be discontinuous for instance in case of a heterogeneous material. For

sake of simplicity, it is assumed that it is constant in a neighbourhood of the boundary of Ω .

Remark 1. From classical results, one can prove some regularity results for u but the functional framework (space $H^1(\Omega)$) only enables one to define u on the boundary of Ω as a function of the space $H^{1/2}(\Gamma)$. Thus, the derivative of u along Γ belongs to the space $H^{-1/2}(\Gamma)$ (see [2], [17]). Nevertheless, using the domain derivative tool introduced by J. Hadamard [10], it is possible to derive estimates on this term in L^2 norm. Let us explain how in the next subsection.

2.2. Energy estimates on u . Let us introduce a vector field θ of $\mathbb{R}^2$ whose components are denoted by θ_α and are assumed to be at least in the space $W^{1,\infty}(\Omega)$. In all this section, we assume that the support of θ is contained in a close neighbourhood of the boundary Γ_f where there is no sensor (see figure 2). Furthermore, we assume that this neighbourhood doesn't meet the closure of the sets O_1 , O_2 and T .

FIGURE 2. The support of the vector field θ

Let us introduce the bilinear continuous symmetrical form ($D_s\theta$ is the symmetrical part of the jacobian matrix $D\theta$):

$$(u, v) \in H^1(\Omega)^2 \rightarrow d(u, v) = \omega^2 \left[\int_{\Gamma} uv \theta \cdot \nu - \int_{\Omega} uv \operatorname{div}(\theta) \right] + \int_{\Omega} c^2 [(\nabla u \cdot \nabla v) \operatorname{div}(\theta) - 2(D_s\theta \nabla u \cdot \nabla v)]. \quad (3)$$

We prove the following lemma:

Lemma 2.1. (i) For every $\theta \in W^{1,\infty}(\Omega)$, every $(f, g) \in L^2(\Omega)^2$ and u solution of the Helmholtz's equation (1), one has

$$d(u, u) = \int_{\Gamma} c^2 \left| \frac{\partial u}{\partial s} \right|^2 \theta \cdot \nu. \quad (4)$$

(ii) There exists $\theta \in W^{1,\infty}(\Omega)$, such that for every $(f, g) \in L^2(\Omega)^2$ and u solution of the Helmholtz equation (1), one has

$$d(u, u) = \int_{\Gamma_f} c^2 \left| \frac{\partial u}{\partial s} \right|^2 \nu_1^2. \quad (5)$$

(iii) There exists a positive constant c_0 such that for every $(f, g) \in L^2(\Omega)^2$ and u solution of the Helmholtz equation (1), one has

$$\int_{\Gamma} c^2 \left| \frac{\partial u}{\partial s} \right|^2 \leq c_0 \|u\|_{1,2,\Omega}^2. \quad (6)$$

Proof - (i) Assuming in a first step, a regularity of u ($u \in H^2(\Omega)$ is sufficient), one obtains from (1) the following identities where $D\theta$ is the Jacobian matrix associated to the vector field θ :

$$\begin{aligned} 0 &= -\omega^2 \int_{\Omega} u \nabla u \cdot \theta - \int_{\Omega} \operatorname{div}(c^2 \nabla u) \nabla u \cdot \theta = \\ &= -\omega^2 \int_{\Gamma} \frac{u^2}{2} \theta \cdot \nu + \frac{\omega^2}{2} \int_{\Omega} u^2 \operatorname{div}(\theta) - \int_{\Gamma} c^2 \frac{\partial u}{\partial \nu} \nabla u \cdot \theta + \int_{\Gamma} \frac{c^2}{2} |\nabla u|^2 \theta \cdot \nu \\ &\quad - \int_{\Omega} \frac{c^2}{2} |\nabla u|^2 \operatorname{div}(\theta) + \int_{\Omega} c^2 (D\theta \nabla u \cdot \nabla u). \end{aligned}$$

From the boundary conditions satisfied by u , the previous expression leads to the next one:

$$\int_{\Gamma} c^2 \left| \frac{\partial u}{\partial s} \right|^2 \theta \cdot \nu = \omega^2 \left[\int_{\Gamma} u^2 \theta \cdot \nu - \int_{\Omega} u^2 \operatorname{div}(\theta) \right] + \int_{\Omega} c^2 [|\nabla u|^2 \operatorname{div}(\theta) - 2(D\theta \nabla u \cdot \nabla u)]. \quad (7)$$

Equation (7) ends the proof of assertion (i).

Let us now turn to the second point (ii). Since the term on the lefthandside is an integral over all the boundary of the open set Ω , it could include a contribution from the boundary ∂T of the hole T . But θ is zero far from the neighbourhood of Γ_f which doesn't meet the hole T . Therefore this boundary integral is restricted to $\Gamma_f \cup \Gamma_m$. Let us choose θ as follows, using the coordinates shown on figure 2, denoting by O the neighbourhood Γ_f and by $x = (x_1, x_2)$ a generic point in Ω :

$$\begin{aligned} \eta &\in W^{1,\infty}(O) \quad \text{with} \quad \eta(x) = 0 \text{ if } x \notin O, \\ \eta(x) &= 1 \text{ if } x \in \Gamma_f \text{ and } \nu_1(x) \neq 0, \\ \eta(x) &= 0 \text{ if } x \in \Gamma_m, \\ \text{and } \theta &= \begin{pmatrix} \nu_1 \eta(x) \\ 0 \end{pmatrix}. \end{aligned} \quad (8)$$

On Γ_f , the function $\nu_1 = (e_1, \nu)$ is assumed to be smooth enough (for instance $C^1(\overline{\Gamma_f})$) and zero in a close vicinity of the junctions between Γ_f and Γ_m . The function θ is then vanishing at the extremities of Γ_f and on Γ_m . Therefore one has from (7):

$$\int_{\Gamma_f} c^2 \left| \frac{\partial u}{\partial s} \right|^2 \nu_1^2 = \omega^2 \left[\int_{\Gamma_f} u^2 \theta \cdot \nu - \int_{\Omega} u^2 \operatorname{div}(\theta) \right] + \int_{\Omega} c^2 [|\nabla u|^2 \operatorname{div}(\theta) - 2(D\theta \nabla u \cdot \nabla u)], \quad (9)$$

which is the point (ii) (5). Let us notice that one can easily obtain that there exists a positive constant say c_0 such that:

$$\int_{\Gamma_f} c^2 \left| \frac{\partial u}{\partial s} \right|^2 \nu_1^2 \leq c_0 \|u\|_{1,2,\Omega}^2. \quad (10)$$

The general case of solution with finite energy can be easily obtained with a density argument.

Let us now turn to the proof of the third point (iii). Let us notice that u is not globally $H^2(\Omega)$ since the velocity is not constant. Its proof is based on suitable choices of vector fields θ in assertion (7). Let $\mathbb{T}$ be an open set with $\bar{T} \subset \mathbb{T}$. With $\theta \in C_0^\infty(\mathbb{T})$ with $\theta = \nu$ on ∂T , we get (5) on ∂T .

Let $K \subset \Gamma_f$ be a compact set with $K \cap \partial\Gamma_f = \emptyset$. With $\theta = \eta\nu$ on Γ_f , $\eta = 1$ on K and $\eta = 0$ on Γ_m , (7) leads to

$$\frac{\partial u}{\partial s} \in L_{loc}^2(\Gamma_f)$$

and $u \in H^1(\Omega) \rightarrow \frac{\partial u}{\partial s} \in L_{loc}^2(\Gamma_f)$ is linear and continuous. It is easy to obtain an analogous result on Γ_m thus $\frac{\partial u}{\partial s} \in L_{loc}^2(\Gamma_m)$. The last point concerns the corners. Thanks to our hypothesis on the corners between Γ_f and Γ_m (the angles are $\pm\pi/2$) and du to the fact that c is constant there, it is known that the function u is a H^2 -one near the corner. This easily ends Lemma's proof.

Remark 2. If u is not solution of the Helmholtz equation (1), the previous equality is not necessarily true.

Remark 3. Lemma 2.1(ii) proves that d is a positive bilinear form (with the suitable choice of θ). One can ask if it is definite. We give a positive answer under some geometric conditions. If u is regular (say $C^2(\Omega)$) and if $d(u, u) = 0$ then $\frac{\partial u}{\partial s} = 0$ on Γ_f where $\nu_1 \neq 0$. But u is locally (in the neighbourhood O_f of Γ_f) solution of:

$$-\omega^2 u - \operatorname{div}(c^2 \nabla u) = 0 \text{ in } O_f \text{ and } \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma_f.$$

The solution of this model can be computed using local coordinates (s, ξ) (see figure 3) assuming that the boundary Γ_f is smooth enough.

Let us set:

$$q = \frac{\partial u}{\partial s} \text{ in } O_f.$$

It is solution of a partial differential equation on O_f and satisfies both:

$$q = \frac{\partial q}{\partial \nu} = 0 \text{ on } \Gamma_f.$$

Then, one can conclude from Holmgren theorem that $q = 0$ in O_f and thus u is a function of the only coordinate ξ (see figure 3) in O_f . Because u is therefore constant on Γ_f , one has (R is the radius of curvature of Γ_f and the velocity c is assumed to be constant for sake of simplicity) along the the line normal to Γ_f

$$-c^2 \frac{\partial^2 u}{\partial \xi^2} - \frac{c^2}{R + \xi} \frac{\partial u}{\partial \xi} - \omega^2 u = 0.$$

Let us consider for instance a point M_0 of Γ_f where $R = \infty$ (inflection point). The equation satisfied by u becomes along the normal to Γ_f at this point:

$$-c^2 \frac{\partial^2 u}{\partial \xi^2} - \omega^2 u = 0,$$

FIGURE 3. Local coordinates near Γ_f

which implies $(\frac{\partial u}{\partial \xi}(0) = 0)$:

$$u(\xi) = A \cos\left(\frac{\omega \xi}{c}\right).$$

Assuming that O_f can be prolongedated up to this next boundary of Ω , the line considered crosses necessarily another point -say M_1 - of the boundary of the open set Ω and one should have at this new point $(\frac{\partial u}{\partial s} = 0$ along the line):

$$\frac{\partial u}{\partial \nu}(M_1) = \frac{\partial u}{\partial \xi}(M_1) \nu(M_0) \cdot \nu(M_1) = 0.$$

Assuming that $\nu(M_0) \cdot \nu(M_1) \neq 0$, and that (D is the lenght between M_0 and M_1):

$$\omega \notin \frac{\pi c}{D} \mathbb{Z},$$

if the line doesn't meet the open set $O_1 \cup O_2$, and one can conclude that $u = 0$ along this line and therefore, from Holmgren's theorem again, $u = 0$ on $\Omega - (O_1 \cup O_2)$. Of course, any regular functions with compact supports in O_1 and/or O_2 satisfies $d(u, u) = 0$ thus our remark seems to be optimal.

3. The invariant used. Let us come back to the analogous formula (7) but derived with now $\theta = e_1$ (there is no more truncation η). Therefore one should add the terms on O_1 and O_2 which contain respectively the support of the two functions f and g . This leads to the

following expression:

$$\int_{\Gamma} c^2 \left| \frac{\partial u}{\partial s} \right|^2 \nu_1 - \omega^2 \int_{\Gamma} u^2 \nu_1 - \int_{O_1} f \frac{\partial u}{\partial x_1} - \int_{O_2} g \frac{\partial u}{\partial x_1} = 0. \quad (11)$$

But the integral over Γ_f includes two contributions: one where $\nu_1 = 0$, and a second one denoted by Γ_f^1 , where $\nu_1 \neq 0$. The contributions on Γ_m and on the boundary of the defect (the hole) *ie* ∂T , are taken into account in this expression. For any ω, f and g given, we define the functional G by:

$$G(\omega, f, g) = \int_{\Gamma_m} c^2 \left| \frac{\partial u}{\partial s} \right|^2 \nu_1 - \omega^2 \int_{\Gamma_m} u^2 \nu_1 - \int_{O_1} f \frac{\partial u}{\partial x_1} - \int_{O_2} g \frac{\partial u}{\partial x_1}. \quad (12)$$

This quantity can be measured as far as u is measured on Γ_m but also on O_1 and O_2 . In fact from (11) one can state that the term representing the loss of information in the global expression of G is:

$$\Delta(\omega, f, g) = - \int_{\Gamma_f \cup \Gamma_{\partial T}} c^2 \left| \frac{\partial u}{\partial s} \right|^2 \nu_1 + \omega^2 \int_{\Gamma_f \cup \Gamma_{\partial T}} u^2 \nu_1. \quad (13)$$

If one can ensure by a judicious choice of f that the integral over Γ_f are zero, the functional G becomes an indicator of the presence of the hole. Indeed, let us notice that if there is no hole one can claim that $G = 0$.

Conversely, if for every $\omega \in [\omega_1, \omega_2]$, ($\omega_1 < \omega_2$) and for every $g \in L^2(O_2)$, one has $G = 0$ then it is possible to discuss the non existence of a defect as in [3] and [4]. For instance, if Γ_f is flat, there would'nt be any contribution from the boundary Γ_f in the expression of G . The terms on Γ_f can be viewed as a loss of information and more precisely as a perturbator with respect to the meaning of G . This is the reason why it is suggested in this paper to use the function f as a control variable in order to cancel this parasite term on Γ_f . This control should be chosen such that the information measured on Γ_m is still meaningful. In fact, one can observe that the term is vanishing if

$$[-\omega u \pm c \frac{\partial u}{\partial s}] \nu_1 = 0 \quad \text{on } \Gamma_f.$$

Hence, we suggest to define a control f such that -for instance-

$$[-\omega u + c \frac{\partial u}{\partial s}] \nu_1 = 0 \quad \text{on } \Gamma_f.$$

4. The criterion used and the optimality relations. For any $f \in L^2(O_1)$ we define the criterion:

$$f \in L^2(O_1) \rightarrow J^\varepsilon(f) = \frac{1}{2} \int_{\Gamma_f} (-\omega u + c \frac{\partial u}{\partial s})^2 \nu_1^2 ds + \frac{\varepsilon}{2} \int_{O_1} f^2 dx. \quad (14)$$

It is a positive and strictly convex function of f . Furthermore, it is continuous from Lemma 2.1. In fact, for the numerical implementation and from (12) it is more accurate to use the following expression for J^ε even if for the continous formulation it is strictly the same. Let

us make it explicit ($\tau_1 = (e_2, \nu)$ and R is the radius of curvature of Γ_f):

$$\begin{aligned} J^\varepsilon(f) &= \frac{1}{2} \int_{\Gamma_f} (\omega^2 u^2 \nu_1^2 - 2\omega c u \frac{\partial u}{\partial s} \nu_1^2 + c^2 (\frac{\partial u}{\partial s})^2 \nu_1^2) ds + \frac{\varepsilon}{2} \int_{O_1} f^2 dx \\ &= \int_{\Gamma_f} \omega \nu_1 (\omega \nu_1 + \frac{c \tau_1}{R}) u^2 + \frac{1}{2} \int_{\Omega} (c^2 |\nabla u|^2 - \omega^2 u^2) \operatorname{div}(\theta) - \int_{\Omega} c^2 (D\theta \nabla u \cdot \nabla u) \\ &\quad + \frac{\varepsilon}{2} \int_{O_1} f^2 dx. \end{aligned} \quad (15)$$

The advantage of this last expression is that it is stable in the space $H^1(\Omega)$ which is the right one in a numerical approximation of the initial model (1). Let us now introduce the model which can be used in order to increase the efficiency of the strategy for detecting defects.

$$\min_{f \in L^2(O_1)} J^\varepsilon(f). \quad (16)$$

The classical optimization results (see [11], [16]) can be applied to the previous model and the solution is denoted by f^ε . The corresponding solution is u^ε . This leads to the result that we summarize hereafter:

Theorem 4.1. *Let $g \in L^2(O_2)$ and $\omega^2 \notin \Lambda$. For sake of simplicity, it is also assumed that the wave velocity c , is constant on the boundary Γ_f . For any $\varepsilon > 0$ there is a unique solution to the problem (16). Furthermore, we introduce the adjoint state p which is solution of:*

$$\begin{cases} -\omega^2 p^\varepsilon - \operatorname{div}(c^2 \nabla p^\varepsilon) = 0 \text{ in } \Omega, \\ \frac{\partial p^\varepsilon}{\partial \nu} = 0 \text{ on } \Gamma_m \cup \Gamma_{\partial T}, \\ c^2 \frac{\partial p^\varepsilon}{\partial \nu} = (\omega^2 \nu_1^2 + 2 \frac{\tau_1 \nu_1 \omega c}{R}) u^\varepsilon - 2 \frac{\tau_1 \nu_1 c^2}{R} \frac{\partial u^\varepsilon}{\partial s} - c^2 \nu_1^2 \frac{\partial^2 u^\varepsilon}{\partial s^2} \text{ on } \Gamma_f, \end{cases} \quad (17)$$

where $\tau_1 = e_2 \cdot \nu$ and R is the radius of curvature of the boundary Γ_f . The optimality condition for the solution f of (16) is:

$$p^\varepsilon + \varepsilon f^\varepsilon = 0 \text{ in } O_1. \quad (18)$$

Thus the solution of (16) is equivalent to solve (1)-(17) and (18).

Remark 4. In practical implementation it is easier and more stable to use the expression of J^ε given at (15). But this expression is equivalent to (14) for the continuous model. The solution method which can be used can be a direct solving of the linear system (1)-(17) and (18). Another possibility is to use the preconditioned (by the diagonal of the quadratic form with respect to f representing J^ε) conjugate gradient algorithm which is a very popular and efficient method.

5. Robustness of the optimal control. The question that we discuss in this section concerns the asymptotic behaviour of the solution f when $\varepsilon \rightarrow 0$. Let us assume that there exists at least one control function $f^e \in L^2(O_1)$ such that if u^e is the solution of (1) with this choice for f , one has:

$$(-\omega u^e + c \frac{\partial u^e}{\partial s}) \nu_1 = 0 \text{ on } \Gamma_f.$$

We say that f^e is an exact control for the criterion chosen. In this case, one has the following result.

Theorem 5.1. *Let f^ε be the unique solution of problem (16). The corresponding solution to equation (1) is denoted by u^ε . There exists a unique function-say $f^0 \in L^2(O_1)$ - and the corresponding solution u^0 of (1) which satisfy the following requirements:*

$$\begin{cases} 1. (-\omega u^0 + c \frac{\partial u^0}{\partial s}) \nu_1 = 0 \text{ on } \Gamma_f; \\ 2. \lim_{\varepsilon \rightarrow 0} \|f^\varepsilon - f^0\|_{0,O_1} = 0 \text{ and } \lim_{\varepsilon \rightarrow 0} \|u^\varepsilon - u^0\|_{1,\Omega} = 0; \\ 3. f^0 \text{ is the minimal exact control in } L^2(O_1). \end{cases} \quad (19)$$

Proof Let us first observe that:

$$J^\varepsilon(f^\varepsilon) \leq J^\varepsilon(f^e).$$

Hence:

$$\begin{cases} \|f^\varepsilon\|_{0,O_1} \leq \|f^e\|_{0,O_1}, \\ \int_{\Gamma_f} |(-\omega u^\varepsilon + c \frac{\partial u^\varepsilon}{\partial s}) \nu_1|^2 ds \leq \varepsilon \|f^e\|_{0,O_1}^2. \end{cases}$$

This enables one to deduce that there is a subsequence $f^{\varepsilon'}$ and an element $f^0 \in L^2(O_1)$ such that:

$$f^{\varepsilon'} \rightharpoonup f^0 \text{ in } L^2(O_1)\text{-weak.}$$

Classically, $u^{\varepsilon'}$ is weakly convergent to u^0 solution of (1) associated to f^0 , in the space $H^1(\Omega)$. From the semi-continuity of convex functions, one can deduce that:

$$(-\omega u^0 + c \frac{\partial u^0}{\partial s}) \nu_1 = 0.$$

This proves that f^0 is an exact control.

From the property that f^0 is an exact control one deduces that on the one hand:

$$\int_{O_1} |f^{\varepsilon'}|^2 \leq \int_{O_1} |f^0|^2,$$

and on the other hand:

$$\begin{aligned} \int_{O_1} |f^{\varepsilon'} - f^0|^2 &= \int_{O_1} \int_{O_1} |f^{\varepsilon'}|^2 - 2 \int_{O_1} f^{\varepsilon'} f^0 + \int_{O_1} |f^0|^2 \\ &\leq \int_{O_1} |f^0|^2 - 2 \int_{O_1} f^{\varepsilon'} f^0 + \int_{O_1} |f^0|^2 = 2 \int_{O_1} f^0 (f^0 - f^{\varepsilon'}). \end{aligned}$$

Therefore:

$$\lim_{\varepsilon' \rightarrow 0} \|f^{\varepsilon'} - f^0\|_{0,O_1} = 0.$$

Let us imagine that there are two accumulation points to the sequence f^ε -say f^0 and f^1 -corresponding respectively to two subsequences of f^ε denoted by f^{ε^0} and f^{ε^1} . Both are exact control and this leads to:

$$\|f^0\|_{0,O_1} \leq \liminf_{\varepsilon^0 \rightarrow 0} \|f^{\varepsilon^0}\|_{0,O_1} \leq \|f^1\|_{0,O_1} \leq \liminf_{\varepsilon^1 \rightarrow 0} \|f^{\varepsilon^1}\|_{0,O_1} \leq \|f^0\|_{0,O_1}.$$

Or else:

$$\|f^0\|_{0,O_1} = \|f^1\|_{0,O_1}.$$

In other words, because the functions f^0 and f^1 are both exact controls, one has:

$$J^\varepsilon(f^0) = J^\varepsilon(f^1).$$

But J^ε is strictly convex, thus $f^0 = f^1$. Finally we proved the uniqueness of the accumulation point and all the sequence f^ε converges to this single point f^0 . From the inequality:

$$\forall f^\varepsilon \in L^2(O_1), \text{ (exact control), } \|f^0\|_{0,O_1} \leq \liminf_{\varepsilon \rightarrow 0} \|f^\varepsilon\|_{0,O_1} \leq \|f^\varepsilon\|_{0,O_1}$$

we also deduce that f^0 is the exact control with the minimal $L^2(O_1)$ -norm. $\square$

6. Computation of approximate or exact control(s). A first possibility consists in minimizing the functional J^ε . But the result can't be stable if there is no exact control, at least as far as one is concerned with the continuous model. Let us discuss two direct methods for computing an approximation of an exact control (if it exists).

6.1. The asymptotic strategy. The first idea is to look for an asymptotic expansion of u^ε , p^ε , f^ε for ε small enough. Let us set:

$$\begin{cases} u^\varepsilon = u^0 + \varepsilon u^1 + \dots, \\ p^\varepsilon = p^0 + \varepsilon p^1 + \dots, \\ f^\varepsilon = f^0 + \varepsilon f^1 + \dots \end{cases} \quad (20)$$

Remark 5. At this step, nothing guaranties that f^0 exists but if it is the case, it is natural to use the same notation as the one used for denoting the limit of f^ε discussed in the previous section. In fact, in the following of this section, we proceed by necessary conditions which should be satisfied by f^0 .

By introducing the assumed expressions (20) into the equations satisfied by $(u^\varepsilon, p^\varepsilon, f^\varepsilon)$ and by equating the term of same power in the resulting expressions, one obtains:

1. *Order 0:*

$$\left\{ \begin{array}{l} -\omega^2 u^0 - \operatorname{div}(c^2 \nabla u^0) = f^0 \chi_{O_1} + g \chi_{O_2} \text{ in } \Omega \\ \frac{\partial u^0}{\partial \nu} = 0 \text{ on } \partial\Omega, \\ -\omega^2 p^0 - \operatorname{div}(c^2 \nabla p^0) = 0 \text{ in } \Omega \\ \frac{\partial p^0}{\partial \nu} = 0 \text{ on } \Gamma_m \cup \partial T; \\ c^2 \frac{\partial p^0}{\partial \nu} = (\omega^2 \nu_1^2 + 2 \frac{\tau_1 \nu_1 \omega c}{R}) u^0 - 2 \frac{\tau_1 \nu_1 c^2}{R} \frac{\partial u^0}{\partial s} - c^2 \nu_1^2 \frac{\partial^2 u^0}{\partial s^2} \text{ on } \Gamma_f, \\ p^0 = 0 \text{ in } O_1. \end{array} \right. \quad (21)$$

2. *Order 1:*

$$\left\{ \begin{array}{l} -\omega^2 u^1 - \operatorname{div}(c^2 \nabla u^1) = f^1 \chi_{O_1} \text{ in } \Omega \\ \frac{\partial u^1}{\partial \nu} = 0 \text{ on } \partial\Omega, \\ -\omega^2 p^1 - \operatorname{div}(c^2 \nabla p^1) = 0 \text{ in } \Omega \\ \frac{\partial p^1}{\partial \nu} = 0 \text{ on } \Gamma_m \cup \partial T, \\ c^2 \frac{\partial p^1}{\partial \nu} = (\omega^2 \nu_1^2 + 2 \frac{\tau_1 \nu_1 \omega c}{R}) u^1 - 2 \frac{\tau_1 \nu_1 c^2}{R} \frac{\partial u^1}{\partial s} - c^2 \nu_1^2 \frac{\partial^2 u^1}{\partial s^2} \text{ on } \Gamma_f, \\ p^1 + f^0 = 0 \text{ in } O_1. \end{array} \right. \quad (22)$$

3. *and so on ...*

6.1.1. *Treatment of the order 0.* Assuming that the boundary Γ_f is smooth enough, the first point is that from Holmgren theorem one can state that p^0 is zero. This implies that:

$$(\omega^2 \nu_1^2 + 2 \frac{\tau_1 \nu_1 \omega c}{R}) u^0 - 2 \frac{\tau_1 \nu_1 c^2}{R} \frac{\partial u^0}{\partial s} - c^2 \nu_1^2 \frac{\partial^2 u^0}{\partial s^2} = 0 \text{ on } \Gamma_f.$$

Or else, using the variational formulation for p^0 :

$$(-\omega u^0 + c \frac{\partial u^0}{\partial s}) \nu_1 = 0 \text{ on } \Gamma_f.$$

6.1.2. *Treatment of the order 1.* From $f^0 = -p^1$ in O_1 , one deduces that u^0 should be solution of:

$$\left\{ \begin{array}{l} -\omega^2 u^0 - \operatorname{div}(c^2 \nabla u^0) = -p^1 \chi_{O_1} + g \chi_{O_2} \text{ in } \Omega, \\ \frac{\partial u^0}{\partial \nu} = 0 \text{ on } \partial\Omega. \end{array} \right.$$

Let us define the sub-open set of Γ_f denoted by Γ_f^1 and such that one has on this subset $\nu_1 \neq 0$ (see figure (4)).

Let us point out that u^1 is unknown. The function p^1 will be defined as soon as $\frac{\partial p^1}{\partial \nu}$ on Γ_f^1 will be known. Let us construct a relation concerning this term which will ensure that the condition on Γ_f^1 concerning u^0 will be satisfied.

From the equation satisfied by u^0 one should have for any smooth enough function q :

$$-\omega^2 \int_{\Omega} u^0 q - \int_{\Omega} \operatorname{div}(c^2 \nabla u^0) q = - \int_{O_1} p^1 q + \int_{O_2} g q. \quad (23)$$

FIGURE 4. Definition of Γ_f^1

From two integrations by parts and assuming that for a given smooth enough function μ , (for instance in the space $H^1(\Gamma_f^1)$), the function q is solution of:

$$-\omega^2 q - \operatorname{div}(c^2 \nabla q) = 0 \text{ in } \Omega, \quad \frac{\partial q}{\partial \nu} = -(c \frac{\partial \mu}{\partial s} + \omega \mu) \text{ on } \Gamma_f^1, \quad \frac{\partial q}{\partial \nu} = 0 \text{ on } \partial\Omega \setminus \Gamma_f^1,$$

one obtains the following identity which should be satisfied by any function μ smooth enough in order to make sense to the expressions used:

$$\forall \mu \in H^1(\Gamma_f^1), \quad - \int_{\Gamma_f^1} c^2 u^0 (c \frac{\partial \mu}{\partial s} + \omega \mu) + \int_{O_1} p^1 q = \int_{O_2} g q. \quad (24)$$

But, at this step, nothing is done in order to prescribe the condition which should be satisfied by u^0 on Γ_f^1 (where $\nu_1 \neq 0$).

Assuming for instance that $\mu \in H_0^1(\Gamma_f^1)$, one can rewrite the relation (24) as follows:

$$\forall \mu \in H_0^1(\Gamma_f^1), \quad \int_{\Gamma_f^1} c^2 (c \frac{\partial u^0}{\partial s} - \omega u^0) \mu + \int_{O_1} p^1 q = \int_{O_2} g q. \quad (25)$$

Now we apply the condition that we are wishing for u^0 . This implies that:

$$\forall \mu \in H_0^1(\Gamma_f^1), \quad \int_{O_1} p^1 q = \int_{O_2} g q. \quad (26)$$

6.1.3. Characterization of an exact control. We are now able to characterize the exact control with $L^2(O_1)$ -minimum norm. Let us introduce the linear form $l(\cdot)$ and the bilinear

and continuous form $\beta(.,.)$ defined on the space $H_0^1(\Gamma_f^1)$ by:

$$\left\{ \begin{array}{l} \mu \in H_0^1(\Gamma_f^1), \quad l(\mu) = \int_{O_2} gq(\mu), \\ (\xi, \mu) \in H_0^1(\Gamma_f^1) \rightarrow \beta(\xi, \mu) = \int_{O_1} p^1(\xi)q(\mu), \\ \text{where } p^1(\xi) \text{ and } q(\mu) \text{ are respectively solutions of:} \\ -\omega^2 p^1(\xi) - \operatorname{div}(c^2 \nabla p^1(\xi)) = 0, \quad -\omega^2 q(\mu) - \operatorname{div}(c^2 \nabla q(\mu)) = 0 \text{ in } \Omega, \\ \frac{\partial p^1(\xi)}{\partial \nu} = 0, \quad \frac{\partial q(\mu)}{\partial \nu} = 0 \text{ on } \Gamma_m \cup \partial T \cup \{\Gamma_f \setminus \Gamma_f^1\}, \\ \frac{\partial p^1}{\partial \nu} = -(c \frac{\partial \xi}{\partial s} + \omega \xi), \quad \frac{\partial q}{\partial \nu} = -(c \frac{\partial \mu}{\partial s} + \omega \mu) \text{ on } \Gamma_f^1. \end{array} \right. \quad (27)$$

We then consider the following problem:

$$\left\{ \begin{array}{l} \text{find } \xi \in H_0^1(\Gamma_f^1) \text{ such that:} \\ \forall \mu \in H_0^1(\Gamma_f^1), \quad \beta(\xi, \mu) = l(\mu). \end{array} \right. \quad (28)$$

Let us assume that there is a solution -say ξ - to equation (28). By comparing (28) and (25), one deduces that:

$$\forall \mu \in H_0^1(\Gamma_f^1), \quad \int_{\Gamma_f^1} (c \frac{\partial u^0}{\partial s} - \omega u^0) \mu = 0, \text{ or else } c \frac{\partial u^0}{\partial s} - \omega u^0 = 0. \quad (29)$$

This proves that $-p^1 \chi_{O_1}$ is an exact control for u^0 . In fact, u^0 would satisfy the required condition on Γ_f^1 . The main question is now to study the variational model (28). Unfortunately, this is a difficult problem that we are still not able to solve completely. On the one hand, the bilinear form $\beta(.,.)$ is symmetrical, positive and clearly continuous on $H_0^1(\Gamma_f^1)$. On the other hand, $\beta(\xi, \xi) = 0$ implies that $p^1(\xi) = 0$ on the open set O_1 .

But, p^1 is solution of the equation (22). Thus -assuming that all the necessary hypotheses are satisfied, one can claim from Holmgren's theorem that $p^1 = 0$ on Ω and finally that:

$$c \frac{\partial \xi}{\partial s} + \omega \xi = 0 \text{ on } \Gamma_f^1.$$

By solving this ordinary differential equation on each connected component of Γ_f^1 , one obtains:

$$\xi = C e^{-\frac{\omega}{c}s},$$

and from the boundary condition satisfied by ξ , one finally gets that $\xi = 0$. Hence, the kernel of $\beta(.,.)$ is reduced to $\{0\}$. Because the mapping:

$$\xi \in H_0^1(\Gamma_f^1), \rightarrow \sqrt{\beta(\xi, \xi)}$$

is a continuous norm on the space $H_0^1(\Gamma_f^1)$, one can define (see [14]) the completed space V^β of $H_0^1(\Gamma_f^1)$ with respect to this norm. As soon as the function g belongs to the dual space $V^{\beta'}$, the Lax-Milgram theorem enables to claim that (28) has a unique solution in V^β . The point is to characterize this completed space (V^β and its dual $V^{\beta'}$). This could be derived

from an inverse inequality but it is not established excepted in very particular geometry (and will be the study of forthcoming paper). Nevertheless in practical applications one can use the fact that the solution of the acoustic model is close to an eigenmode as soon as the frequency used ($\frac{\omega}{2\pi}$) is close to an eigenfrequency of the structure. This is the discussion suggested in the next subsection.

6.1.4. What happens when ω^2 is close to an eigenvalue of the model. Let us consider a particular case which is similar to a finite dimensional situation. This is the case when the pulsation ω is very close to the square root of an eigenvalue of the acoustic operator (see equation 2). This is the resonance case. The solution would be close to an eigenfunction of the Helmotz's operator and this is the property used on order to construct an approximate control. Let us assume for sake of brevity, that the eigenvalue λ_{i_0} is single and let us set:

$$\omega^2 = \lambda_{i_0} - v. \quad (30)$$

For any $\xi \in H_0^1(\Gamma_f^1)$, (in fact one could extend the expression to $\xi \in L^2(\Gamma_f^1)$), the solution p^1 is :

$$\begin{cases} p^1 = \sum_{i \geq 1} -\frac{\int_{\Gamma_f^1} (c \frac{\partial \xi}{\partial s} + \omega \xi) w_i}{\lambda_i - \omega^2} w_i \\ = -\frac{\int_{\Gamma_f^1} (c \frac{\partial \xi}{\partial s} + \omega \xi) w_{i_0}}{v} w_{i_0} - \sum_{i \geq 1, i \neq i_0} \frac{\int_{\Gamma_f^1} (c \frac{\partial \xi}{\partial s} + \omega \xi) w_i}{v + \lambda_i - \lambda_{i_0}} w_i. \end{cases} \quad (31)$$

Hence assuming that $|v| < \delta = \min_{i \geq 1, i \neq i_0} |\lambda_i - \lambda_{i_0}|$, one has:

$$\begin{cases} \|p^1 + \frac{\int_{\Gamma_f^1} (c \frac{\partial \xi}{\partial s} + \omega \xi) w_{i_0}}{v} w_{i_0}\|_{0, \Omega} \leq \|p^1 + \frac{\int_{\Gamma_f^1} (c \frac{\partial \xi}{\partial s} + \omega \xi) w_{i_0}}{v} w_{i_0}\|_{0, \Omega} \\ = [\sum_{i \neq i_0} |\frac{\int_{\Gamma_f^1} (c \frac{\partial \xi}{\partial s} + \omega \xi) w_i}{v + \lambda_i - \lambda_{i_0}}|^2]^{1/2} \leq \frac{1}{\delta - |v|} \|p_c^1\|_{0, \Omega}, \end{cases} \quad (32)$$

where p_0^1 is defined by:

$$p_c^1 = p^1 - p_{i_0}^1, \quad \text{with } p_{i_0}^1 = -\frac{\int_{\Gamma_f^1} (c \frac{\partial \xi}{\partial s} + \omega \xi) w_{i_0}}{v} w_{i_0}. \quad (33)$$

Let us choose a function $\xi = \alpha w_{i_0} \nu_1 \in H_0^1(\Gamma_f^1)$ where α is constant. The coefficient ν_1 ensures that this choice for ξ is effectively zero on the boundary of Γ_f^1 as soon as the boundary Γ_f is at least C^1 in order to avoid discontinuity of ν_1 . Then one has (R is the radius of curvature of the boundary Γ_f^1 ; its is assumed to be zero at all the extremities of connected components of Γ_f^1):

$$\int_{\Gamma_f^1} (c \frac{\partial \xi}{\partial s} + \omega \xi) w_{i_0} = \alpha \int_{\Gamma_f^1} (\omega \nu_1 + \frac{c \tau_1}{2R}) w_{i_0}^2.$$

Hence, assuming that $\omega\nu_1 + \frac{c\tau_1}{2R} \neq 0$ (which is compatible with the fact that we choose ω for a given structure), one can claim that $\|p_{i_0}^1\|_{0,1} \simeq 1/\nu$. Finally, there is a constant c_1 such that with this choice of ξ , one has:

$$\|p^1 - p_{i_0}^1\|_{0,O_1} \leq c_1 |\nu| \|p_{i_0}^1\|_{0,O_1} \quad (34)$$

The coefficient α is defined from (28) by:

$$\alpha = \frac{\int_{O_2} g p_{i_0}^1}{\int_{O_1} (p_{i_0}^1)^2}, \quad (35)$$

and the approximation of the control f^0 is given by:

$$f_{i_0}^0 = -\alpha p_{i_0}^1 \chi_{O_1}, \quad (36)$$

where $p_{i_0}^1$ is defined by solving (22), and introducing $\xi = \xi_0 = w_{i_0} \nu_1$ in the expression of $p_{i_0}^1$ given at (33).

Remark 6. Clearly the choice, made for ξ is restrictive. But if the frequency $\omega/2\pi$ is close to an eigenfrequency of the structure, it is justified as far as the the solution of the Helmholtz's equation of our model is also close to the corresponding eigenmode. It is possible, without any new difficulty, to expand the method with a finite number of eigenmodes, instead of only one has we did. But in such a case the choice of ξ is done in the finite dimensional space spanned by a choice of vectors $w_i \nu_1$ restricted to the boundary Γ_f^1 (see [15]). In this case, the coefficient α is replaced by a vector and defined by solving a linear system deduced from (28) as for (35).

Remark 7. The control can be one of the two methods described in this paper. The first one consists in finding the optimal control defined in section (4) at equation (16) with a particular choice of the parameter ε . In fact, the Helmholtz's equation will be replaced in a numerical approximation by a finite element model or by a modal model using the eigenmodes computed from a finite element method. In fact, a splitting of the spectrum will be used in order to focus on the eigenmodes which will be concerned by the non destructive strategy suggested. From the analysis performed in section 5, the solution of the optimal control model (in the reduced basis spanned by the selected eigenvectors), one can claim that the solution converges as $\varepsilon \rightarrow 0$ to the one of the limit model that has been studied in this subsection.

6.2. The simplest method. The goal of the control method that has been introduced in section 4, is to reduce the loss of information on the boundary Γ_f^1 . The criterion J^ε which has been introduced at (15) contains a regulation term on the control which represents the cost of the control. We studied in section 6 the asymptotic behaviour of the solution (optimal control) when the marginal cost ε tends to zero. In this section, we discuss another strategy which is not yet mathematically founded but which can be used as a simple -and even simplest- method. In the following we shall use the notation $u^0 = u_g + z$ where u_g is solution of:

$$\begin{cases} u_g \in H^1(\Omega), \text{ such that:} \\ \forall v \in H^1(\Omega), -\omega^2 \int_{\Omega} u_g v + \int_{\Omega} c^2 \nabla u_g \cdot \nabla v = \int_{O_2} g v, \end{cases} \quad (37)$$

and our goal is still to define z such that $(-\omega u^0 + c \frac{\partial u^0}{\partial s})\nu_1$ would be as smallest as possible on Γ_f^1 . We try in this section to avoid the difficulty connected to the small parameter ε . The basic idea is to try to solve the terms of order one for p^1 in the assumed asymptotic expansion (27).

For any $\xi \in L^2(\Gamma_f)$, let us set $p^1 \in H^1(\Omega)$ solution of:

$$\begin{cases} -\omega^2 p^1 - \operatorname{div}(c^2 \nabla p^1) = 0 \text{ in } \Omega, \\ \frac{\partial p^1}{\partial \nu} = 0 \text{ on } \Gamma_m \cup \partial T \cup \{\Gamma_f \setminus \Gamma_f^1\}, \\ \frac{\partial p^1}{\partial \nu} = \xi \text{ on } \Gamma_f^1. \end{cases} \quad (38)$$

Then, let us introduce $z \in H^1(\Omega)$ solution of:

$$\begin{cases} -\omega^2 z - \operatorname{div}(c^2 \nabla z) = -p^1 \chi_{O_1} \text{ in } \Omega, \\ \frac{\partial z}{\partial \nu} = 0 \text{ on } \partial \Omega. \end{cases}$$

We introduce the mapping L from $L^2(\Gamma_f)$ into itself by (see 5 for the regularity property of $\frac{\partial z}{\partial s}$ on Γ_f):

$$\xi \in L^2(\Gamma_f^1) \rightarrow L(\xi) = \nu_1(-\omega z + c \frac{\partial z}{\partial s}).$$

For another function $\mu \in L^2(\Gamma_f^1)$, we define the function y defined similarly to z with ξ . Let us now consider two functions $(\xi, \mu) \in [L^2(\Gamma_f^1)]^2$ and let us define the symmetrical and positive bilinear form:

$$a(\xi, \mu) = \int_{\Gamma_f^1} (-\omega z + c \frac{\partial z}{\partial s})(-\omega y + c \frac{\partial y}{\partial s}) \nu_1^2,$$

and the linear form $l(\cdot)$:

$$\mu \in L^2(\Gamma_f^1) \rightarrow l(\mu) = - \int_{\Gamma_f^1} (-\omega u_g + c \frac{\partial u_g}{\partial s})(-\omega y + c \frac{\partial y}{\partial s}) \nu_1^2.$$

There is a kernel K to $a(\cdot, \cdot)$ which is such that:

$$K = \{\mu \in L^2(\Gamma_f^1), (-\omega y + c \frac{\partial y}{\partial s})\nu_1 = 0 \text{ on } \Gamma_f^1\}. \quad (39)$$

From the linearity and continuity of the mapping:

$$\mu \in L^2(\Gamma_f^1) \rightarrow (-\omega y + c \frac{\partial y}{\partial s})\nu_1 \in L^2(\Gamma_f^1),$$

one can claim that K is a closed sub-set of $L^2(\Gamma_f^1)$. More precisely, K is a finite dimensional sub-space of $L^2(\Gamma_f^1)$ because it corresponds to functions such that: $y = A_k e^{\frac{\omega}{c}s}$ on each connected component of Γ_f^1 (and we assume that there is a finite number of such components).

Let us denote by:

$$\mathbb{V} = \overline{L^2(\Gamma_f^1)/K}^c \quad (40)$$

the completed space of $L^2(\Gamma_f^1)/K$ equipped with the norm (which is a semi-norm on $L^2(\Gamma_f^1)$) defined by:

$$\mu \in L^2(\Gamma_f^1) \rightarrow \sqrt{a(\mu, \mu)}.$$

One can observe that:

$$\forall \mu \in K, l(\mu) = 0.$$

Therefore, it makes sense to try to solve the following variational problem:

$$\begin{cases} \text{find } \xi \in \mathbb{V} \text{ such that:} \\ \forall \mu \in \mathbb{V}, a(\xi, \mu) = l(\mu). \end{cases} \quad (41)$$

It is equivalent to minimize over the space $L^2(\Gamma_f^1)$ the functional:

$$G(\xi) = \frac{1}{2}a(\xi, \xi) - l(\xi),$$

or else, because z is a function of ξ :

$$\min_{\xi \in L^2(\Gamma_f^1)} \frac{1}{2} \int_{\Gamma_f^1} \left| -\omega(z + u_g) + c \frac{\partial(z + u_g)}{\partial s} \right|^2 \nu_1^2.$$

If the minimum is zero, then the control $f = -p(\xi)\chi_{O_1}$ is exact. Let us explain why. In this case, once ξ defined, we observe that the function $h = u_g + z$ satisfies the condition:

$$(-\omega h + c \frac{\partial h}{\partial s})\nu_1 = 0 \text{ on } \Gamma_f.$$

Hence the function p^1 obtained by solving (38) with the value of ξ found by solving (41) is such that $f^e = -p^1\chi_{O_1}$ is an exact control. One could object that it is necessary to characterize the completed of the quotient space $\overline{V/K}^c$ in order to ensure that we are able to compute practically f^e as an element of the space $L^2(O_1)$. Obviously the question is simplified as far as one restricts the space $L^2(\Gamma_f^1)$ to a finite dimensional subspace which is the case of a finite element approximation of the solution of (41).

The goal of this section is to start a discussion and to underline the difference with the asymptotic method described in subsection 6.1

7. Conclusion. In this paper we have discussed the possibility to extend the strategy introduced in [3] for detecting defects in a rectangular structure. In the present, case the geometry can be more general but a control function has been introduced in order to restrict the loss of information through the lateral boundary of the open set Ω which is not equipped with sensors and which is not parallel to the axis x_1 . This control function is solution of an optimal control model and can be computed for any given excitation (named g in this paper) and for any -also given- frequency $\frac{\omega}{2\pi}$.

The main result is that the boundary criterion which permits this reduction, is stable because of an hidden regularity of the Helmholtz's equation. Furthermore, it has been proved that the control method is robust (stable) when the cost of the control (the parameter ε) tends to zero as far as the frequency is chosen close to an eigenfrequency of the structure.

Another step would be to prove an inverse inequality in order to characterize the functional space V^β (see section 6) in which the function ξ of section 5 should be. But let us underline that this choice of frequency -quite close to an eigenfrequency of the structure- enables one to get an almost finite dimensional control model for which the question is solved.

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E-mail address: philippe.destuynder@cnam.fr

E-mail address: caroline.fabre@u-psud.fr