Payroll Taxation, qualifications, wages and unemployment rates in a frictional labor market with productive interactions between segments

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Abstract

The present paper investigates the incidence of payroll taxation in a search and matching framework considering a production function with different type of workers. This allows understanding the productive interactions between segmented labor markets. General results are analytically demonstrated, and two kinds of reforms are numerically simulated: i) shifting the tax burden from low-skilled segments of the labor market to high-skilled segments, capital or consumption; ii) upgrading a share of low-skilled workers in high-skilled segments, which represents an educational policy. The tax reforms' efficiency increases with the substitutability between segments in the production function and with the constraints on the low-skilled wages (high minimum wages). The educational reform's efficiency increases with the complementarities between segments and is not much impacted by the constraints on low-skilled wages. The Malthusian effect of reducing low- skilled labor supply is reinforced by the demand increase due to the increase of high-skilled labor supply and the complementary between segments. The association of employment and productivity increases generates large output and tax revenue increases, which may inter-temporally finance educational reforms.

Keywords: Search and matching; segemented labor market; intra-firm bargaining; tax incidence
JEL: H22; J31; J38.

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1 Introduction

Consequences of taxation on labor market equilibrium are a central issue of applied public economics and more specifically of the understanding of public policies' impacts. Need for additional knowledge has been strengthened by the economic crisis, and even before in the polarization context (**?**). With the disappearance of routine jobs, some middle-skilled workers should be reallocated into manual jobs - at lower pay - which amplified inequalities or unemployment if high minimum wages prevent wage downwards adjustment. A large number of governments use the fiscal tool not only to levy resources but also to subsidy labor. Tax burden shifting from low wages to other tax bases is thought to allow maintaining net wages - and thus avoiding wage inequality increase - while decreasing low-skilled labor costs - and thus avoiding unemployment increase. France, for example set a new payroll tax rebate of 4% of the payroll bill for 2013 then 6% for years after 2014, targeted on low and medium wages. Such reforms generate payroll taxation differentiated by industrial sector or level of qualification. This differentiation may modify the structure of employment and unemployment as well as the structure of wages. However, this does not consider alternative educational reforms increasing the share of high-skilled workers, which altogether decreases low-skilled labor supply and increases low-skilled labor demand by complementarities to the increase of high-skilled employment.

The present paper aims at analyzing the impacts of differentiated payroll taxation in a model of search and matching taking into account the productive interaction between different inputs: employees of different qualifications and capital. This allows understanding the distortions generated on the labor markets as well as the distributive consequences. As shown by Dwenger et al. (2014); Kleven (2014), tax compliance and tax employment impact depend on public expenditure choices. Particularly, social investment may increase employment even if it necessitates additional public funds. The present paper also studies the impact of educational policies consisting to make workers change from a low-skilled segment to a higher- skilled segment of the labor market. Keeping tax rate constant, it leads to unemployment decrease and tax revenue increase.

In addition, incidence plays an major role: tax burden does not fall only onto the individuals officially taxed. The burden is shared among the agents interacting on markets. This also applies to payroll taxation: Gruber (1994, 1997); Anderson and Meyer (1997, 2000); Murphy (2007) demonstrated thanks to natural experiments in the United States and Chile that workers pay the major share of payroll taxes, whatever their official designation - employees' or employers' social security contributions. Furthermore, the sharing of the tax burden varies with the bargaining power of employees: the larger the employee's bargaining power, the higher the share of taxes borne by employees and the higher the share of exemptions that are eventually translated into net wage rises instead of labor cost reductions. Workers paid at the minimum wage have no bargaining power: their bargained wage would have been lower. Hence, tax exemptions at the level of the minimum wage are more fully converted into labor cost decreases than exemptions for higher wages. It is therefore of main importance to introduce bargaining power and minimum wage in the model.

The motive of differentiated payroll taxes is often employment, yet incidence of payroll taxes is a key parameter of the success of such policies. Due to incidence differences, the impact on employment of payroll tax cuts should be greater for low wages than for higher wages. There have been several empirical analyses of such policies in Europe, **?**Kramarz and Philippon (2001); Chéron et al. (2008) find significant impact for France when results of Bohm and Lind (1993); Bennmarker et al. (2009) for Suede and Korkeamäki and Uusitalo (2009) for Finland are more mitigated. The cause of the difference may lie in incidence and the fact that French payroll tax reductions were set very close to the minimum wage. Actually, Crépon and Deplatz (2001) show that the effect in France occurred through substitutions of low wage workers to higher wage workers. Nevertheless, Huttunen et al. (2013) consider payroll tax cuts on low wages and found also very weak impact. They use difference-in-difference methodology (per age categories) to assess the impact of a Finnish payroll tax cut targeting elder workers and low wages: they found no impact at the extensive margins and a small impact at the intensive margins.

This also meets the issue of optimal labor taxation as payroll taxation and labor income taxation probably have similar incidence even when labor taxation is not levied at source (excluding salience effects as highlighted by **?**). However, optimal labor taxation literature has first focused on the labor supply side and the adverse selection problem. Mirrlees (1971) considered a discrete distribution of workers, Saez (2001) generalized the approach with continuous productivity of workers and Kleven et al. (2009) generalized to couples and labor supply in the extensive margins. However, this literature does not consider any labor market as each unit of labor supplied finds an employer - there is no unemployment - and the wage is equal to the productivity of the worker.

The standard way of modeling labor markets has been developed by the search and matching literature (e.g. Pissarides (2000)). It provides a dynamic framework and reproduces the conditions of frictional unemployment, the rent of employment being shared between firms and workers. Stole and Zwiebel (1996b,a) renew the process of wage setting by the hypothesis that contract incompleteness does not enables neither firms nor workers to commit to future wages and employment decision, which leads to intra-firm bargaining engaged individually by workers. It results in lower wages and more employment than in standard model. None of these models take into account the structure of production and possible substitution between production factors.

Acemoglu (2001) built a matching model with two kinds of jobs (good jobs/bad jobs) and derives the impact of minimum wage on the structure of production. However, it does not fit the problematic of the present paper as there is only one type of workers and the two kinds of jobs are modeled as separate sectors of intermediate goods. Belan et al. (2010) introduced a model with frictional and classical unemployment and two kinds of workers. However, there are also two kinds of goods and this model does not allow understanding the interactions between the types of workers within the production process.

The choice of the model necessitates therefore the hiring of different kinds of workers for the same production process, taking into account the interaction effects through a multi-factor production function. Hence, the model developed below is based on Cahuc et al. (2008), including altogether matching, bargaining and multi-factor production function. The original paper was developed to understand over-employment in a normative point of view. However, over-employment in this model is directly linked to the wages being larger than the marginal productivity, which may be alternatively interpreted as an issue of value added sharing. The question of over-employment is not considered here as the present paper focuses on understanding the impact of taxation on the structure of wages and unemployment in positive point of view.

Intra-firm bargaining models has been criticized because they assumed permanent and individual bargaining when in most countries wages are bargained collectively and sequentially. The present paper answers this critic by considering different types of workers (with wages bargained collectively or individually) and by interpreting further the process of individual intra-firm bargaining and the bargaining power parameter itself.

Actually, the model is modified mainly in two ways. First, three kinds of inputs are considered. The factors representing the different kind of capital are included in the production function and in the decision of input demand by the firm; their allocation is not considered frictional, their remuneration is set internationally and exogenously. The constrained workers have no individual bargaining power. Their wages are determined collectively for each segment and these fixed wages are considered exogenous in the model. It may represent (exogenous) collective bargaining; it fits even more the case of workers whose qualification prevent them to access jobs paid over the minimum wage: in that case, the minimum wage is actually exogenous.

The last kind of input is constituted of workers with individual bargaining power. This does not come from their substitutability with other types of workers (inter-segment substitutability) but from their substitutability with other workers of the same type (intra-segment substitutability). The intra-segment substitutability does not necessitate that workers are heterogeneous within the segment but that the productivity of their job marginally increases with the personal investment of the worker. In that case, as presented by Goldin (2014), the hourly wages are convex with respect to the personal investment because it is costly for the employer to substitute one employee to another one with the same qualification and ability. For those kinds of jobs, the increase of productivity with personal investment lowers the substitutability with similar workers. This low intra-input substitutability allows those kinds of workers to extract surplus from the employer. This justifies their ability to bargain intra-firm and the modeling of their wage setting.

The second main modification is the introduction of taxation: capital income taxation, consumption taxation and taxation on wages which may represent either payroll taxes or labor income taxes. For the case of payroll taxation financing public social security systems, some countries distinguish between employers' and employees' social security contributions. This differentiation is not considered here because it is formal but has no economic reality except at the level of minimum wage. Formally, the model considers only employers' social security contributions and tax base is assumed to be the net wage. This modeling choice has no impact on unconstrained workers (above minimum wage), it matters only for constrained workers (at the minimum wage). Nevertheless, the model may be easily adapted to consider taxes officially on employees: only the case of constrained workers should be modified by considering the collectively bargained wage as the gross wage.

Furthermore, this model does not differentiate between contributive and non-contributive social security contributions. The type of policies studied consists in payroll tax cuts in order to decrease labor costs, with compensation if necessary to social security institutions in order not to decrease the benefits. In that way, social security contribution cuts actually have the same impact as tax cuts. It may be interesting to consider cuts both in social security contributions and benefits, but it is out of the scope of the present paper. It could be analyzed through the model presented here as it is equivalent for the constrained workers to a decrease of their exogenous wage (through the decrease of the in-kind part of this remuneration). For unconstrained workers, it would be only the decrease of a mandatory consumption of insurance.

The paper is organized as follows. Section 2 presents the general model: first the global setup, then the demand equation of firms, the wage bargaining process, finally the general equilibrium. Section 3 presents the formal solving of the model and general results, mainly the impact of the different parameters on employment and wages and the relative crossed effects of taxation.

Section 4 investigates numerically the case of the interactions between different types of workers. First, the case of two segments of unconstrained workers is considered with different values of the elasticity of substitution between low- and high-skilled workers. The tax reforms' efficiency increases with the substitutability between segments, but the magnitude of impacts are limited. The educational reform's efficiency increases with the complementarities between segments.

Then the case with three factors is considered: one type of unconstrained workers, one type of constrained workers and one type of capital. The tax reforms' efficiency is greater when the low-skilled are constrained by a minimum wage set relatively high. The educational reform's efficiency is not much impacted by the constraints on low-skilled wages. The Malthusian effect of reducing low-skilled labor supply is reinforced by the increase of demand due to the increase of high-skilled labor supply and the complementarities between segments. The association of employment and productivity increases generates large output and tax revenue increases, which may allow financing the educational policy. Section 5 concludes.

2 Theoretical framework

2.1 The general setup of the model

Let us consider an economy with a numeraire good produced thanks to $n \geq 1$ labor types ($i = 1, ..., n$) supplied by a continuum of infinitely lived workers of size $\vec{L} = (L_1, ..., L_n)$ (supplying each one unit of labor). The production function is $F(N_1, ..., N_n)$ where $N_i \geq 0$ is the level of employment of factor i ($\vec{N} = (N_1, ..., N_n)$). The inputs are of three kinds. The $m \leq n$ first factors ($(N_1, ..., N_m)$) are human input: different kinds of workers. The last $n-m$ factors ($\vec{K} = (N_{m+1}, ..., N_n)$) are capital. Their cost is constant at the internationally set interest rate r and can be acquired each period without friction.

Among the workers, some are unconstrained workers ($\vec{L}_u = (L_1, ..., L_l)$ among who $\vec{N}_u = (N_1, ..., N_l)$ are employed and $\vec{U}_u = (U_1, ..., U_l)$ are unemployed, with $U_i = L_i - N_i$). They negotiate individually their wages with their employers. They keep bargaining even when employed, which is the reason why the model of intra-firm bargaining has been chosen. Their remuneration $w_i(\vec{N})$ therefore depends on the quantity of each input.

Last, the constrained workers ($\vec{L}_c = (L_{l+1}, ..., L_m)$ among who $\vec{N}_c = (N_{l+1}, ..., N_m)$ are employed and $\vec{U}_c = (U_{l+1}, ..., U_m)$ are unemployed, with $U_i = L_i - N_i$) cannot negotiate their wage individually. They are employed at wage $\underline{w}_i$, collectively bargained, applying to all worker of their type. Depending of the use of the model, it can be considered as the collective bargaining by unions for each type of job or as the legal minimum wage. The model does not endogenize this collective bargaining and the wages $\underline{w}_i$ for $i = l+1, ..., m$ are considered exogenous. Those workers are subject to classical unemployment in addition to frictional unemployment.

To hire workers, firms post vacancies separately on each segments (with a segment specific hiring cost γ_i per unit of time and per vacancy posted). These vacancies meet the pool of unemployed workers of the type. Matching functions $h_i(U_i, V_i)$ give for each segment of the labor market the mass of aggregate contacts depending on the mass of unemployed U_i and the mass of vacancies V_i for the type of workers. With $\theta_i = V_i/U_i$ the tightness of segment i, the probability per unit of time to fill a vacant job is $q_i(\theta_i) = h_i(U_i, V_i)/V_i$ ($q_i'(\theta_i) < 0$ and $q_i(0) = +\infty$) and the probability per unit of time to find a job is $p_i = h_i(U_i, V_i)/U_i = \theta_i q(\theta_i)$ (with $d[\theta_i q(\theta_i)]/d\theta_i > 0$). The segment-specific exogenous probability of job destruction by unit of time is s_i.

Furthermore, a tax function T_i is considered such that the gross wage is $T_i[w_i(\vec{N})]$ when the net wage is $w_i(\vec{N})$. This tax function may represent most tax schedules around the globe, whatever social security contribution - often linear - or labor income tax schedule - often piecewise linear. For the capital factors, this tax function gives the level of capital income tax. Considering numerical application, it should be kept in mind that tax rates apply to the net remuneration of inputs: a given rate corresponds to a much lower nominal tax rate when applied to the gross wage instead of the net wage. For example, a tax rate of 25% on the gross wage is equivalent to a tax rate of 33.3% on the net wage and a tax rate of 50% on the gross wage is equivalent to a tax rate of 100% on the net wage. Hence, numerical analyses can consider tax rates on net remuneration as large as 100%. Last, a specification of consumption tax of rate t may be easily introduced by considering a net firm income $F(\vec{N}) = (1-t)G(\vec{N})$ where $G(\vec{N})$ is the actual production function. On contrary to other taxes, this consumption tax is specified with a rate t applying to gross sales. To fit usual consumption taxes applying on net prices, one should just consider the net rate $u = t/(1-t)$.

The equilibrium on the market is reach through the confrontation of a labor demand curves and wage bargaining curves on each segment of the labor market - depending on equilibria on the other labor markets. The demand for each level of labor is determined ex ante by the quantity of vacancies posted on each segments of the labor market. It depends on the anticipation of the ex post wage bargaining, itself depending on the level of unemployment, the unemployment benefits and the marginal productivity of each type of input. The overall model is dynamic and time is continuous. The equilibrium is calculated through the use of Bellman equations for the value of profit flows for firms, and the value of employment and unemployment for workers.

2.2 Labor demand

Demand on each segment of the labor market is determined by the maximization by the firm of the value of its profit flows. The Bellman equation of the value of the firm between time t and $t + dt$ is given by equation 1, subject to equation 2 giving the evolution of the number of each input depending on the rate of job destruction, the number of vacancies and the matching function itself depending on the tightness of the segment.

$$\Pi(\vec{N}) = \max_{\vec{V}} \frac{1}{1+rdt} \left\{ \left[F(\vec{N}) - \sum_{j=1}^{n} \Big(T[w_j(\vec{N})]N_j + \gamma_j V_j \Big) \right] dt + \Pi(\vec{N}^{t+dt}) \right\} \tag{1}$$

$$N_i^{t+dt} = N_i(1 - s_i dt) + V_i q_i(\theta_i) dt \tag{2}$$

At this stage, no distinction between constrained and unconstrained factors should be made. The only difference between the two kinds of factors is that the remuneration $w_i(\vec{N})$ of constrained factors is constant (equal to r for capital and to $\underline{w}_i$ for low-skilled workers). The solution to the firm maximization problem is found by calculating with two different methods the marginal profits with respect to each type of workers, noted $J_i(\vec{N}) = \partial\Pi(\vec{N})/\partial N_i$. The first method uses the first order condition with respect to the number of vacancies V_i posted by firms, leading to equation 3 at steady state. The second method is derived from the envelop theorem and results in equation 4.

$$J_i(\vec{N}) = \frac{\gamma_i}{q_i} \tag{3}$$

$$J_i(\vec{N}) = \frac{\frac{\partial F(\vec{N})}{\partial N_i} - T[w_i(\vec{N})] - \sum_{j=1}^{n} N_j \frac{\partial T[w_j(\vec{N})]}{\partial N_i}}{r + s_i} \tag{4}$$

Indeed, first order condition with respect to V_i is $-\gamma_i dt + J_i(\vec{N}^{t+dt}) dN_i^{t+dt}/dV_i = 0$ where $dN_i^{t+dt}/dV_i = q_i dt$ from equation 2. At steady state, $\vec{N}^{t+dt} = \vec{N}$ which gives equation 3. In addition, the envelop theorem applied by differentiating equation 1 with respect to N_i gives:

$$\left[\frac{\partial F(\vec{N})}{\partial N_i} - \sum_{j=1}^{n} N_j \frac{\partial T[w_j(\vec{N})]}{\partial N_i} - T[w_i(\vec{N})] \right] dt + \frac{\partial N_i^{t+dt}}{\partial N_i} J_i(\vec{N}^{t+dt}) = J_i(\vec{N})(1 + rdt)$$

With $\frac{\partial N_i^{t+dt}}{\partial N_i} = (1 - s_i dt)$ from equation 2, which gives equation 4 at steady state. Combining equation 3 and 4 gives the decomposition of the marginal productivity with respect to the workers of type i in equation 5.

$$\frac{\partial F(\vec{N})}{\partial N_i} = T[w_i(\vec{N})] + \frac{\gamma_i(r+s_i)}{q_i(\theta_i)} + \sum_{j=1}^{l} N_j \frac{\partial T[w_j(\vec{N})]}{\partial N_i} \tag{5}$$

Where $\partial F(\vec{N})/\partial N_i$ is the marginal productivity of workers of type i; $T[w_i(\vec{N})]$ their gross wage; $\gamma_i(r+s_i)/q_i(\theta_i)$ the hiring costs increasing with the vacancy posting cost γ_i and with the rate of job destruction s_i and decreasing with the probability $q_i(\theta_i)$ that a vacancy meets an unemployed worker; $N_j \partial T[w_j(\vec{N})]/\partial N_i$ the change in the wage bill for workers of type j due to the change in the level of employment of workers of type i through the intra-firm bargaining process. As only unconstrained workers may negotiate their wages, the sum of the wage bill effects are calculated only over factors $j \in [1, l]$.

Equation 5 gives a relation between the wage bargaining function as anticipated by firms and the level of employment targeted by firms through their vacancies' posting. It corresponds to labor demand curves. This demand is not such that overall marginal labor costs - gross wages plus hiring costs - equals the marginal productivity of workers. It depends also on the variations of the overall wage bill due to the change in the employment level because changing the level of employment (and therefore of unemployment) changes the wages through changes in the outside options of workers and firms. As shown by Stole and Zwiebel (1996b) and Stole and Zwiebel (1996a) and confirmed by Cahuc et al. (2008), labor demand may be such that the marginal productivity of a type of worker is lower than the overall marginal cost of such a type of labor.

2.3 Wage determination

Labor demand equation 5 gives a first relation between the number of employees of each segment and their wages. The actual wages and employment levels for each type of worker need another relation to be fully determined: this second relation comes from the intra-firm bargaining determining the wage function $w_j(\vec{N})$ for unconstrained workers. Constrained factors are remunerated at fixed level r for capital and $\underline{w}_i$ for constrained workers. Consequently, the present section concerns only unconstrained factors N_i for $i \in [1, l]$. From Bellman equation 6 of the value of being in employment E_i for worker of type i may be derived directly equation 7.

$$rE_i = w_i(\vec{N}) + s_i(U_i - E_i) \tag{6}$$

$$E_i - U_i = \frac{w_i(\vec{N}) - rU_i}{r + s_i} \tag{7}$$

Given the type specific bargaining power β_i of workers of type i, according to the fact that the rent of employement for workers is the difference of values $E_i - U_i$ between employement and unemployement and the rent of employement for the firm net of vacancy costs is the marginal productivity $J_i(\vec{N})$ of workers of type i, Mortensen and Pissarides (2001) have shown that the bargaining process results in equation 8 in the presence of tax rate τ_i on wages.

$$\beta_i J_i(\vec{N}) = (1 - \beta_i)(1 + \tau_i)(E_i - U_i) \tag{8}$$

The bargaining power is a central in those kinds of models (even if numerical analyses in section 3 show it has limited impact on equilibria) but difficulties remain to rightly interpret the economic reality behind this parameter. It does not come neither from rarity of the type of workers nor from their productivity. In the one hand, rarity is taken into account through the intensity of use of the factor type: the endogenous tightness of labor markets in search and matching models. In the other hand, the production function defines workers' productivities independently from their bargaining power parameter. The thesis of the present article is that individual bargaining power does not represent any form of substitutability of workers between worker types (which would be implicitly assumed by considering productivity or rarity of worker types) but substitutability within worker types.

This thesis may be further understood considering the wage analysis of Goldin (2014). She focused on gender pay gap and draws very general results on wage variations within jobs and qualifications. She found some jobs with wages proportional to the personal implication of the workers (working time, acceptance of unusual periods of work, any-time availability). These are jobs where tasks may be easily shared between different workers, where substituting one worker to another does not decrease the productivity. Her example is pharmacists: each task is independent from the preceding one and all the needed information appears on the computer screen when loading the patient fill. Workers in such industries, whatever their qualification and inter-segment substitutability, are very substitutable the ones to the others and have a low bargaining power.

Other kinds of job present a wage function convex with respect to the employee's personal investment. Goldin (2014) and previously Goldin and Katz (2008) found that business and law jobs present such schemes. This comes from the need to fully follow contracts or clients and know all their details and specificities, which cannot be transferred without huge costs to a substitute worker. This creates a low intra-segment substitutability allowing the employee to extract a larger share of the surplus than more substitutable types of workers with the same rarity and skills. This is exactly what is reflected by the bargaining power parameter. Furthermore, it also justifies the intra-firm bargaining process as it is actually the position of insider and the knowing of all the specificities of the very position that allows such non-substitutable worker to extract a share of the profit.

No straightforward monotonous relation should exist between qualification and bargaining power - pharmacists being an example of high-skilled workers with low bargaining power. Nevertheless, from a broad perspective, a positive correlation between qualification and bargaining power is highly plausible. Hence, we use a positive link between productivity and bargaining power in numerical simulations, even if the bargaining power does not come from productivity itself.

According to equation 7 of the difference of value between employment and unemployment and equation 4 of the marginal productivity of workers of type i, equation 8 may be rewritten as the differential equation 9 of the wage as a function of employment levels in each segments.

$$T[w_i(\vec{N})] = (1 - \beta_i)(1 + \tau_i)rU_i + \beta_i \left[\frac{\partial F(\vec{N})}{\partial N_i} - \sum_{j=1}^{l} N_j \frac{\partial T[w_j(\vec{N})]}{\partial N_i} \right] \tag{9}$$

As intra-firm bargaining take place individually for each worker already employed, it does not anticipate the possible change in the overall employment rate resulting for the new wage, which means that rU_i is considered as constant in that differential equation. The condition at limit necessary to solve this differential equation is that the overall gross wage bill $N_i T[w_i(\vec{N_i})]$ for workers of type i tends towards zero when employment N_i on this segment tends towards zero.

2.4 Labor market equilibrium

The differential equations for all segments are solved given the situation of the other segments (the reservation wages rU_i and the labor market tightness θ_i) and therefore give solutions in partial equilibrium. Actually, solution to the differential equation system 9 gives the wage function depending on the employment structure $\vec{N}$ and the values of unemployment. Meeting the demand functions 5 linking the level of employment to the wages anticipated by firms allows determining the general equilibrium. It defines two sets of n equations linking directly N_i and θ_i. The first set of equations comes from the labor market allocation process. Equation 2 gives $N_i s_i = V_i q_i(\theta_i)$ and consequently the first set of equations linking N_i to θ_i is equation 10.

$$\theta_i q_i(\theta_i) = \frac{s_i N_i}{L_i - N_i} \tag{10}$$

The second set of equations comes from the labor demand equation 5 knowing the remuneration of constrained factors and the wage functions of unconstrained factors (results of differential equations 9). However, these last functions depend on the value of unemployment rU_i for unconstrained workers, which is determined at general equilibrium. The bellman equation for the value of unemployment is:

$$rU_i = b_i + \theta_i q_i(\theta_i)(E_i - U_i)$$

Where b_i is the income flow at unemployment. As equation 8 gives $E_i - U_i = \beta_i/[(1-\beta_i)(1+\tau_i)]J_i(\vec{N}) = \beta_i/[(1-\beta_i)(1+\tau_i)]\gamma_i/q_i(\theta_i)$ because of equation 3, the value of unemployment is given by equation 11.

$$rU_i = b_i + \frac{\beta_i}{1-\beta_i}\frac{\gamma_i \theta_i}{1+\tau_i} \tag{11}$$

Hence, it is possible to find equation 12 giving the high-skill workers' wage at equilibrium by including equations 5 and 11 in equation 9.

$$T[w_i(\vec{N})] = (1+\tau_i)b_i + \frac{\gamma_i \beta_i}{1-\beta_i}\left(\theta_i + \frac{r_i + s_i}{q_i(\theta_i)}\right) \tag{12}$$

To calculate the structure of employment $\vec{N}$ and wages $\vec{w}(\vec{N})$, solutions of differential equations 9 should be incorporated in this system, which gives the second set of relations between wages and employment and eventually the equilibrium wages and employment levels. If the bargaining power is fully owned by the employer, that is if $\beta_i = 0$, differential equation 9 become equation 13 giving directly the bargained net wage.

$$w_i(\vec{N}) = rU_i = b_i \tag{13}$$

The employee without any bargaining power should accept its reservation wage and nothing more. In that case, the net wage is independent from the payroll tax which is fully borne by the employer. It is the case of constrained workers. At the opposite, if the full bargaining power is owned by the employee, differential equation 9 become:

$$T[w_i(\vec{N})] = \frac{\partial F(\vec{N})}{\partial N_i} - \sum_{j=1}^{n} N_j \frac{\partial T[w_j(\vec{N})]}{\partial N_i}$$

Yet:

$$\frac{\partial \sum_{j=1}^{n} N_j T[w_j(\vec{N})]}{\partial N_i} = T[w_j(\vec{N})] + \sum_{j=1}^{n} N_j \frac{\partial T[w_j(\vec{N})]}{\partial N_i}$$

And therefore differential equation 9 is equivalent to equation 14

$$\frac{\partial\left(\sum_{j=1}^{n} N_j T[w_j(\vec{N})] - F(\vec{N})\right)}{\partial N_i} = 0 \tag{14}$$

And consequently $\sum_{j=1}^{n} N_j T[w_j(\vec{N})] - F(\vec{N})$ is constant with respect to $\vec{N}$. Yet it is zero when $\vec{N} = \vec{0}$. Hence, $\sum_{j=1}^{n} N_j T[w_j(\vec{N})] = F(\vec{N})$ and there is no equilibrium because the full output is paid in wage and nothing remains for hiring costs. However, this hypothesis of full bargaining power of the employees is very unlikely and in the following the bargaining power of unconstrained workers is assumed to be strictly between 0 and 1.

3 Formal solving when taxes are piecewise linear

3.1 The wage functions when taxes are piecewise linear

Given the present knowledge on differential equations, it is not possible to solve formally such a differential equation system for a general tax function T. Basically, it is possible mainly in the linear case. However, the linear case is indeed the most probable as the tax schedules actually set in most countries are flat or piecewise linear. Hence, the more general case is to consider a piecewise linear income tax schedule where the marginal tax rate at the wage level of workers of type i is τ_i: $T_i[w_i(\vec{N})] = (1+\tau_i)w_i(\vec{N})$. Similarly, τ_i for $i \in [m+1, n]$ is directly interpreted as the marginal tax rate on capital income. Some continuously progressive tax schedules are also possible, even if less likely. It is the case for payroll tax in France where a payroll tax rebate at the level of the minimum wage is continuously reduced giving birth to actually continuously progressive marginal tax rates on labor income. With a piecewise linear specification, the system of differential equations become as presented by equation 15 for factors $i \in [1, l]$.

$$w_i(\vec{N}) = (1-\beta_i)rU_i + \frac{\beta_i}{1+\tau_i}\left[\frac{\partial F(\vec{N})}{\partial N_i} - \sum_{j=1}^{l}(1+\tau_j)N_j\frac{\partial w_j(\vec{N})}{\partial N_i}\right] \tag{15}$$

The differential equations cannot be solved independently the ones from the others because each function $w_i(\vec{N})$ depends on the derivatives of the wage functions on other segments. The first stage for solving this differential equations consists in disentangling partially this system. Appendix A.1 shows how and demonstrates that the system is equivalent to those of equations 16.

$$w_i(\vec{N}) = (1-\beta_i)rU_i + \frac{\beta_i}{1+\tau_i}\left[\frac{\partial F(\vec{N})}{\partial N_i} - \sum_{j=1}^{l}(1+\tau_i)\chi_{ij}N_j\frac{\partial w_i(\vec{N})}{\partial N_j}\right] \tag{16}$$

Where the parameter $\chi_{ij} = \frac{\beta_j}{1-\beta_j}\frac{1-\beta_i}{\beta_i}$ gives the comparison between the bargaining powers of workers of types i and j. There is no issue of dividing by zero because the only factors whose bargaining power is considered in the previous equation are those for $i \in [1, l]$ whose bargaining power is strictly positive (the other are indeed constrained factors because their unemployment benefit is lower than the minimum wage).

The second stage is the actual resolution of the differential equations. It consists in several changes of variables, the most important being the change in polar coordinates allowing to actually resolve the differential equations, and some integration per part. It is quite technical and has no economic meaning by itself; it is therefore presented in the appendix section (appendix A.2), and allows to demonstrate lemma 1.

Lemma 1. *The solution of the system of wage bargaining differential equations 15 - with condition at limit being that the payroll bill of each segment tends towards zero when the employment on that segment tends toward zero - is given by equation 17* for all unconstrained workers (when $i \in [1, l]$).

$$w_i(\vec{N}) = (1 - \beta_i)rU_i + \int_0^1 u^{\frac{1+\tau_i}{\beta_i}-1} \frac{\partial F(\vec{N}_u A_i(u), \vec{N}_c, \vec{K})}{\partial N_i} du \tag{17}$$

Where matrix $A_i(u)$ is given by equation 18.

$$A_i(u) = \begin{pmatrix} u^{(1+\tau_i)\frac{\beta_1}{1-\beta_1}\frac{1-\beta_i}{\beta_i}} & 0 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 & 0 \\ 0 & 0 & u^{(1+\tau_i)\frac{\beta_j}{1-\beta_j}\frac{1-\beta_i}{\beta_i}} & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & u^{(1+\tau_i)\frac{\beta_l}{1-\beta_l}\frac{1-\beta_i}{\beta_i}} \end{pmatrix} \tag{18}$$

Proof. See appendix A.2.

Equation 17 provides a decreasing relationship between employment N_i and net wage w_i as soon as factorial marginal productivity decreases. As equation 10 provides an increasing relationship between these two variables, it allows to define a general equilibrium as in the following subsection. Furthermore, an increase of taxes or bargaining powers for one kind of workers generates a net wage increase for types of workers who are complement (the marginal productivity of one type of workers increases with the employment in other segments) and a net wage decrease for types of workers who are substitutes (the marginal productivity of one type of workers decreases with the employment in other segments).

3.2 General equilibrium when taxes are piecewise linear

The two sets of n equations 17 and 10 provides n labor demand equations and n wage setting equations with $2n$ variables: n input quantities N_i and n tightness θ_i. Incorporating the wage functions from equation 17 into the general equilibrium wage equations 12 and replacing the value of unemployment thanks to equation 11 gives the general equilibrium system 19. The equations for constrained inputs come from the demand equations 5 and the derivatives of the wage functions from equations 17.17.

$$\begin{cases} \theta_i q_i(\theta_i) = \frac{s_i N_i}{L_i - N_i} & (a) \\ \left[\begin{array}{ll} if\ i \in [1, l] & \\ \int_0^1 u^{\frac{1+\tau_i}{\beta_i}-1} \frac{\partial F(\vec{N} A_i(u))}{\partial N_i} du = \beta_i b_i + \frac{\gamma_i}{1+\tau_i} \frac{\beta_i}{1-\beta_i} \left(\beta_i \theta_i + \frac{r+s_i}{q_i(\theta_i)}\right) & (b_u) \\ & \\ if\ i \in [l+1, n] & \\ \frac{\partial F(\vec{N})}{\partial N_i} - \sum_{j=1}^{l} (1+\tau_j) N_j \int_0^1 u^{\frac{1+\tau_j}{\beta_j}-1} \frac{\partial^2 F(\vec{N} A_j(u))}{\partial N_j \partial N_i} du = (1+\tau_i) w_i + \frac{\gamma_i (r+s_i)}{q_i(\theta_i)} & (b_c) \end{array} \right. \end{cases} \tag{19}$$

An additional slight assumption should be made to demonstrate the existence of equilibrium, as it is stated by proposition 1: no couple of inputs are strictly substitutes. This means that no input's marginal productivity strictly increases when the quantity of another input decreases.

Proposition 1: The existence of general equilibrium. *Under likely hypotheses on the production function - decreasing factorial productivity and imperfect substitution of factors - general equilibrium exists on the segmented labor market, which is solution of the system of equations 19.*

Proof.: Equation 19a provides a strictly increasing relation between θ_i and N_i. Considering the implicit increasing function $\theta_i(N_i)$, the problem may be reduced to the n equations 19b for the n unknown N_i. These equations are of the type $lht_i(\vec{N}) = rht_i(N_i)$. Right hand term functions rht_i stricly increase - from $\beta_i b_i$ if $i \in [1, l]$ and from $(1+\tau_i)w_i$ if $i \in [l+1, n]$ - to infinity when N_i goes from 0 to L_i. Hypotheses about the production function induce that the left hand term functions lht_i decrease with respect to N_i and increase with respect to N_j $j \neq i$.

In addition, let us assume that for any i, $lht_i(\vec{N}^{-i}, 0, \vec{N}^{+i})$ is larger than $rht_i(0)$ (where $\vec{N}^{-i} = (N_1, ..., N_{i-1})$ and $\vec{N}^{+i} = (N_{i+1}, ..., N_n)$). If it is not the case, N_i is zero at equilibrium and let us consider the labor market without this fictive segment (let us call this the no-fictive segment assumption). It means that for any values of $\vec{N}^{-i}$ and $\vec{N}^{+i}$, the equation $lht_i(\vec{N}^{-i}, N_i, \vec{N}^{+i}) = rht_i(N_i)$ has unique solution strictly between zero and L_i. This solution $N_i^*(\vec{N}^{-i}, \vec{N}^{+i})$ increases with respect to each N_j $j \neq i$, because $rht_i(N_i)$ does not depend on any N_j $j \neq i$ and $lht_i(\vec{N})$ increases with respect each N_j $j \neq i$. This partial equilibrium on segment i is shown by Figure 1.

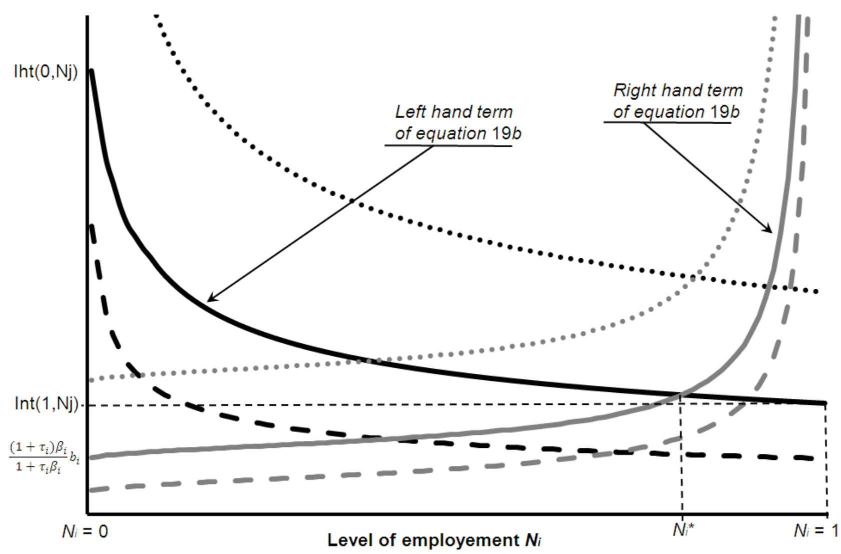


Figure 1: Equilibrium on a segemented labor market

Now let us build an infinite sequence of vectors $\vec{N}(\nu)$. Let us assume that the first terms of the series are $(1/2^\nu, ...1/2^\nu)$ until first rank μ where for each i $lht_i(\vec{N}(\nu)) > rht_i(N_i(\nu))$. The rank μ exists due to the no-fictive segment assumption. After this rank μ, let us define $N_i(\nu+1)$ as the partial equilibrium on segment i given $N_j = N_j(\nu+1)$ if $j < i$ and $N_j = N_j(\nu)$ if $j > i$. Each sequence $N_i(\nu)$ increases after rank μ, because $N_i(\mu)$ is under partial equilibrium on segment i, then $N_i(\nu+1)$ is the new equilibrium with increased N_j $j \neq i$. The sequence of vectors $\vec{N}(\nu)$ increases and is bounded (by $\vec{L}$), so it converges between zero and $\vec{L}$. The algebraic limit theorem induces that the limit $\vec{N}(\infty)$ of this sequence verifies the equation

$lht_i(\vec{N}(\infty)) = rht_i(N_i(\infty))$ for each i and is therefore solution of the problem 19.***Q.E.D.***

Uniqueness of this equilibrium is not directly demonstrable in the general case. However, it is very likely as soon as there are no increasing returns to scale. If there actually are increasing returns to scale, the multiple equilibria result is usual. Furthermore, the impact on employment equilibrium of various parameters may be easily understood thanks to Figure 1. A parameter increasing lht_i (pushing the black solid line onto the black dotted line) leads to an increase of the level of employment. Reciprocally, a parameter decreasing lht_i (pushing the black solid line onto the black dashed line) leads to a decrease of the level of employment. A parameter increasing rht_i (pushing the grey solid line onto the grey dotted line) leads to a decrease of the level of employment. Reciprocally, a parameter decreasing rht_i (pushing the grey solid line onto the grey dashed line) leads to a decrease of the level of employment. In that way, all parameters but the bargaining power have unambiguous impact on the equilibrium, as presented in table 1.

Table 1: Impact of model parameters on the level of employement

Parameter	Variations in Eq. 19	Employement variation
Total factor produvitity	$lht \nearrow$	Increase
Matching function efficiency $q(.)$	$rht \searrow$	Increase
Segment size L	$rht \searrow$	Increase
Unemployement benefits b	$rht \nearrow$	Decrease
Vacancy posting cost γ	$rht \nearrow$	Decrease
Job destruction rate s	$rht \nearrow$	Decrease
Interest rate r	$rht \nearrow$	Decrease
Own payroll tax rate τ_i	$lht \searrow$ and $rht \nearrow$	Decrease
Crossed payroll tax rate τ_j $(j \neq i)$	$lht \searrow$	Decrease

Particularly, taxation of a given segment has a negative impact on every segments and shifting taxation from one segment to another has ambiguous impact, which is assessed numerically in the following section, on particular cases of the present model.

4 Interactions between multiple worker types

Numerical analysis aims at understanding the interactions between different types of workers subject to different taxes. The focus is made on two different kinds of interactions. The first one is the interaction between two types of unconstrained workers. The second one is the interaction between a constrained and an unconstrained segment. For that second case, a third factor (capital) is also added. This allows analyzing the impact of shifting taxes from low-skilled not only onto high-skilled, but also onto capital income and sales taxes.

Furthermore, to go into numbers and provides numerical simulations, additional hypotheses should be made. Functional forms should be chosen for the production function and the matching function. This last function is assumed to be $h_i(u_i, V_i) = a_i u_i^{1-\eta_i} V_i^{\eta_i}$. Consequently, $q_i(\theta_i) = a_i \theta_i^{\eta_i - 1}$ and $\theta_i q_i(\theta_i) = a_i \theta_i^{\eta_i} =$

$s_i N_i/(L_i - N_i)$. Hence, equation 10 become 20.

$$\theta_i = \left(\frac{s_i}{a_i}\right)^{\frac{1}{\eta_i}} \left(\frac{N_i}{L_i - N_i}\right)^{\frac{1}{\eta_i}} \tag{20}$$

The matching function parameters are calibrated according to Petrongolo and Pissarides (2000) survey of the empirical literature on the matching function and Borowczyk Martins et al. (2011) who corrects for an estimation bias due to endogenous search behavior from each side of the market. The parameter are actually set at $s = 0.1$, $a = 1$ and $\eta = 0.68$. These parameters' impacts are straightforward and do not change the qualitative results of the various simulated reforms.

4.1 Two unconstrained worker types

The first kind of numerical analysis restrains the labor market to two segments. This relative simplicity allows keeping quite complex functional form for the production function in order to catch the impact of the level of substitutability between segments. The production is processed with two types of labor, one more qualified than the other, in a constant elasticity of substitution framework. The functional form is $F(N_1, N_2) = A\left(\alpha_1 N_1^\delta + \alpha_2 N_2^\delta\right)^{\frac{\alpha}{\delta}}$, with $\alpha_2 = 1 - \alpha_1$ and $\delta = (\sigma - 1)/\sigma$ where σ is the inter-factorial substitution elasticity. The difference of qualification is obtained by using different parameters α_1 of productivity in the CES production function. It is also assumed that the higher qualified workers have more bargaining power than the less qualified one.

For each value of the elasticity of substitution (0.2, 0.5, 2 and 5), the model is calibrated with taxes of 40% on each factor and α_1 such that the unemployment rate of low-skilled workers is 15%. From this situation, two reforms are simulated: a tax reform and an educational reform. The tax reform consists in shifting the tax burden from the low-skilled to the high-skilled workers. For each decrease of the tax rate for low-skilled - per 2 percentage points increments - the tax rate for high-skilled is adjusted in order to get the overall tax revenue constant. The impact of this reform on the unemployment rates of each segment of the labor market is presented in figure 2. Incidence on high-skilled wages and impacts on outputs and profits are presented in figure 3.

The relatively high level of high-skilled unemployment when the two segments are highly substitutes is due to the assumption of the simulation, e.g. initial unemployment rate of low-skilled at 15%. Indeed, factor substitutability with wage bargained imposes relatively close rates of unemployment between segments. Whatever the level, the unemployment rate of high-skilled is not impacted when shifting the tax burden on this segment of the labor market. At the opposite, reducing the tax burden on low-skilled reduces their unemployment; the unemployment reduction increases with the substitutability between low-skilled and high-skilled workers.

However, the magnitude of the unemployment reduction remains modest, partly because of the incidence effect: the low-skilled labor cost decrease is lower than the low-skilled tax decrease. The tax cut beneficiates also to low-skilled in form of net wages' increase, from 15% in the most complementary case to 40% in the most substitutable case. Hence, the overall output increase is also modest. Tax increase incidence on high-skilled wages is even higher, up to full shifting on net wages (and no labor cost impact) when the two labor market segments are very substitutes.

The second type of reform is educational. The present paper cannot determine the actual content and cost of such a reform, but only considers that if it succeeds it shifts workers from one segment to the other.

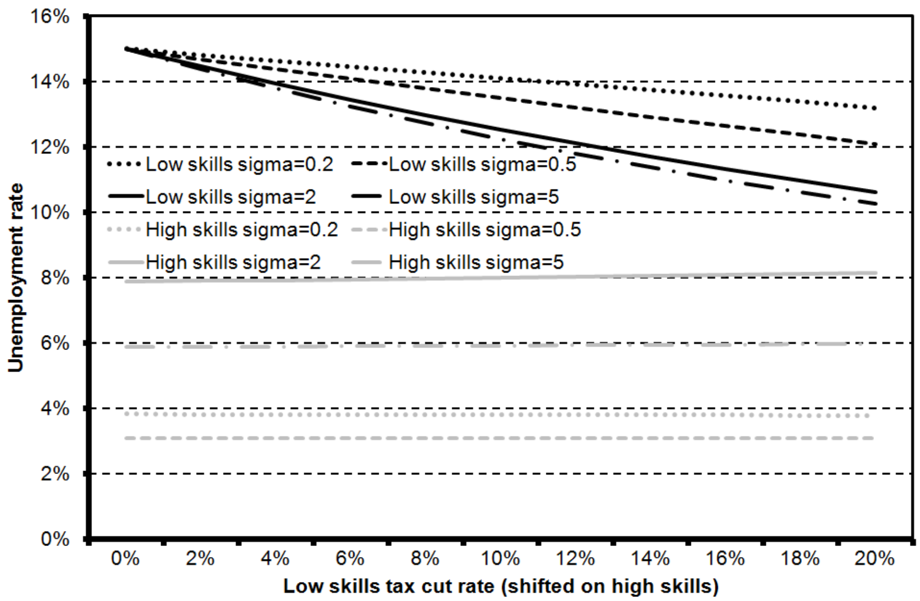


Figure 2: Impact of tax shifting on high and low qualification workers' unemployment

The simulation is calculated for a high qualification worker with twice the productivity and the bargaining power as the low qualification worker

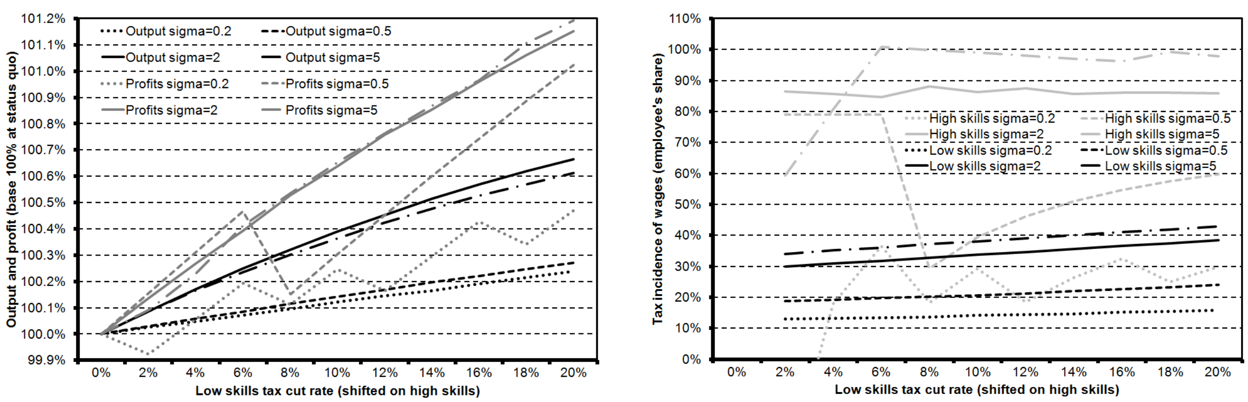


Figure 3: Impact of tax shifting on ouput, profits and wages

The simulation is calculated for a high qualification worker with twice the productivity and the bargaining power as the low qualification worker

This shift between segments is done keeping the tax rates constant, thus leading to changes in overall tax revenue because tax bases are modified. The level of the tax revenue variation should be analyzed in order to verify if it is sufficient to finance the educational reform. The impact on unemployment is presented in figure 4. Impacts on tax revenue, output and profits are presented in figure 5.

As previously, unemployment rate on high-skilled segment of the labor market is not impacted by the reform. Actually, the increase of the high-skilled labor supply leads to wage decreases and not unemployment because the wages is bargained and sufficiently high to be reduced. The impact on low-skilled is substantial;

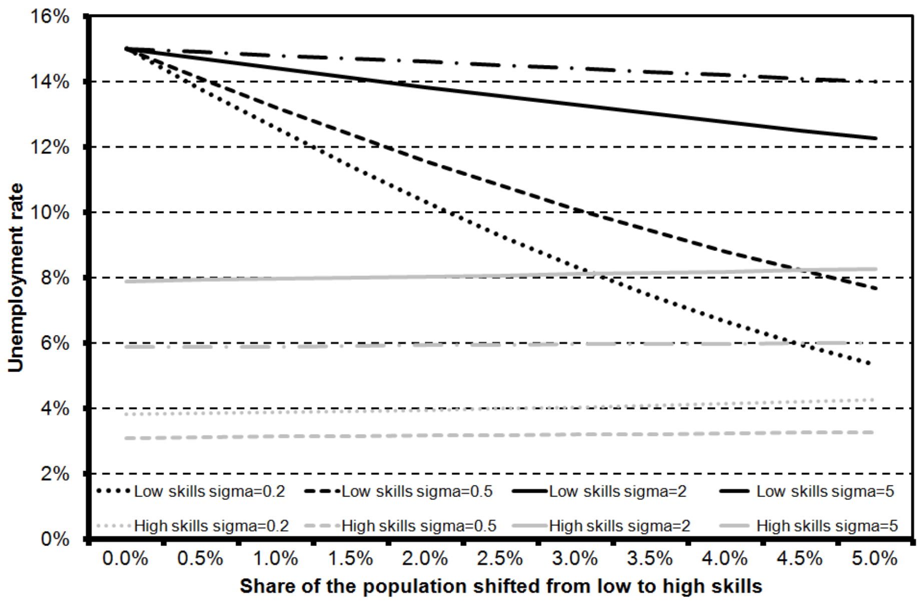


Figure 4: Impact of tax shifting on high and low qualification workers' unemployment

The simulation is calculated for a high qualification worker with twice the productivity and the bargaining power as the low qualification worker

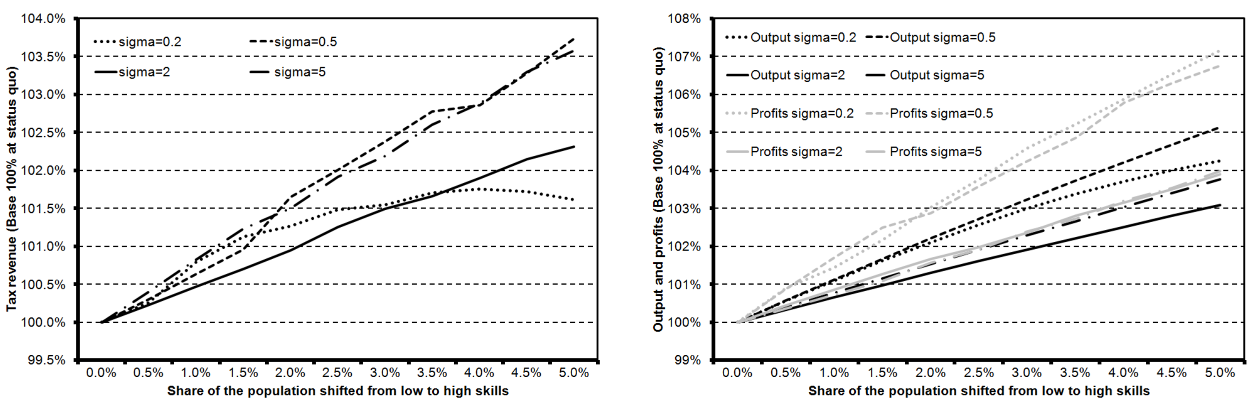


Figure 5: Impact of tax shifting on ouput, profits and wages

The simulation is calculated for a high qualification worker with twice the productivity and the bargaining power as the low qualification worker

it is very sensible to substitutability, with very modest unemployment decrease when segments are substitutes but very large unemployment decrease when segments are complements.

Actually, the upgrading of low-skilled workers into high-skilled workers creates a Malthusian impact: the decrease of low-skilled labor supply leading to unemployment decrease and wage increase. However, the change in high-skilled absolute employment level generates an equilibrium impact which can be of first order. Indeed, the high-skilled unemployment rate does not vary although the high-skilled labor supply increases, meaning that high-skilled absolute employment increases in the proportion of the worker shift between segments. If the segments are complements, this activity increase generates labor demand increase on the low-skilled segment.

This activity increase, coupled with the increase in the mean productivity (relatively more high-skilled employed) leads to large output increases and even larger profit increases. Indeed, the high-skilled labor supply creates both mean productivity increase and high-skilled wage decrease, increasing even more profits. This reform, due to its overall employment impact, allows levying more taxes with constant tax rates in a large magnitude: between half and one percent increase of public financing for one percent workers shifted, up to 3.5% tax revenue increase for 5% workers shift in the most favorable cases.

4.2 The case of three factors, low- and high-skilled workers and capital

The second simulation considers only one middle range elasticity of substitution between production factors: an elasticity of one and a Cobb-Douglas production function. This simple functional form allows extending the number of factors to three: high-skilled and low-skilled labor and capital. Furthermore, to better fit the issue of low-skilled labor costs, this segment is considered constrained, thus the low-skilled net wage is exogenous (at minimum wage level $\underline{w}$), giving birth to classical unemployment in addition to frictional unemployment. Furthermore, the incidence of tax decrease on this segment results fully in labor cost decrease. The model considers one of each kind of production factor: unconstrained workers (indexed by u), constrained workers (indexed by c) and capital (indexed by k).

The production function is $F(L_u, L_c, K) = (1-t)AN_u^{\alpha_u}N_c^{\alpha_c}K^{\alpha_k}$. Parameter t is the sales tax rate, which allow to shift the tax burden from the constrained workers (tax rate τ_c) not only onto unconstrained workers (tax rate τ_u) but also onto sales taxes (tax rate t) or capital income taxes (tax rate τ_k). The problem of general equilibrium 19 become the equation system 21.

$$\left\{\begin{array}{ll}\frac{(1-\beta)(1-t)A\alpha_u N_u^{\alpha_u-1}N_c^{\alpha_c}K^{\alpha_k}}{(1-\beta+\alpha_u\beta)} = (1+\tau_u)(1-\beta)b_u + \gamma_u\left[\beta\left(\frac{s_u}{a_u}\right)^{\frac{1}{\eta_u}}\left(\frac{N_u}{L_u-N_u}\right)^{\frac{1}{\eta_u}} + \frac{r+s_u}{a_u}\left(\frac{s_u}{a_u}\right)^{\frac{1-\eta_u}{\eta_u}}\left(\frac{N_u}{L_u-N_u}\right)^{\frac{1-\eta_u}{\eta_u}}\right] & (u)\\ \frac{(1-\beta)(1-t)A\alpha_c N_u^{\alpha_u}N_c^{\alpha_c-1}K^{\alpha_k}}{1-\beta+\alpha_u\beta} = (1+\tau_c)\underline{w} + \gamma_c\frac{r+s_c}{a_c}\left(\frac{s_c}{a_c}\right)^{\frac{1-\eta_c}{\eta_c}}\left(\frac{N_c}{L_c-N_c}\right)^{\frac{1-\eta_c}{\eta_c}} & (c)\\ \frac{(1-\beta)(1-t)A\alpha_k N_u^{\alpha_u}N_c^{\alpha_c}K^{\alpha_k-1}}{1-\beta+\alpha_u\beta} = (1+\tau_k)r & (k)\end{array}\right. \tag{21}$$

It is possible to write K as a function of N_c and N_u according to equation 21k, then to incorporate it in other equations. Following, it is possible to rewrite N_u as a function of N_c according to equation 21c and to incorporate it in equation 21u which become an equation of the unique variable N_c. Adding the constant returns to scale hypothesis, the system become as presented by equation 22.

$$
\begin{cases}
K = \left[\frac{\alpha_k(1-\beta)(1-t)A}{(1-\beta+\alpha_u\beta)(1+\tau_k)r}\right]^{\frac{1}{1-\alpha_k}} N_u^{\frac{\alpha_u}{1-\alpha_k}} N_c^{\frac{\alpha_c}{1-\alpha_k}} & (k) \\
N_u = \alpha_c^{-\frac{1-\alpha_k}{\alpha_u}} \left[\frac{(1-\beta)(1-t)A}{1-\beta+\alpha_u\beta}\right]^{-\frac{1}{\alpha_u}} \left[\frac{\alpha_k}{(1+\tau_k)r}\right]^{-\frac{\alpha_k}{\alpha_u}} N_c & \\
\quad * \left[(1+\tau_c)\underline{w}\gamma_c \frac{r+s_c}{a_c}\left(\frac{s_c}{a_c}\right)^{\frac{1-\eta_c}{\eta_c}}\left(\frac{N_c}{L_c-N_c}\right)^{\frac{1-\eta_c}{\eta_c}}\right]^{\frac{1-\alpha_k}{\alpha_u}} & (c) \\
\alpha_u\alpha_c^{\frac{\alpha_c}{\alpha_u}} \left[\frac{(1-\beta)(1-t)A}{1-\beta+\alpha_u\beta}\right]^{\frac{1}{\alpha_u}} \left[\frac{\alpha_k}{(1+\tau_k)r}\right]^{\frac{\alpha_k}{\alpha_u}} & \\
\quad * \left[(1+\tau_c)\underline{w} + \gamma_c \frac{r+s_c}{a_c}\left(\frac{s_c}{a_c}\right)^{\frac{1-\eta_c}{\eta_c}}\left(\frac{N_c}{L_c-N_c}\right)^{\frac{1-\eta_c}{\eta_c}}\right]^{-\frac{\alpha_c}{\alpha_u}} = (1+\tau_u)(1-\beta)b_u & (u) \\
\quad\quad +\gamma_u \left[\beta\left(\frac{s_u}{a_u}\right)^{\frac{1}{\eta_u}}\left(\frac{N_u}{L_u-N_u}\right)^{\frac{1}{\eta_u}} + \frac{r+s_u}{a_u}\left(\frac{s_u}{a_u}\right)^{\frac{1-\eta_u}{\eta_u}}\left(\frac{N_u}{L_u-N_u}\right)^{\frac{1-\eta_u}{\eta_u}}\right] &
\end{cases}
\tag{22}
$$

Resolution only consists in solving equation 22u with unique unknown N_c. The left-end term strictly decreases from $\alpha_u\alpha_c^{\frac{\alpha_c}{\alpha_u}} \left[\frac{(1-\beta)(1-t)A}{1-\beta+\alpha_u\beta}\right]^{\frac{1}{\alpha_u}} \left[\frac{\alpha_k}{(1+\tau_k)r}\right]^{\frac{\alpha_k}{\alpha_u}} (1+\tau_c)\underline{w}$ to 0 when N_c goes from 0 to L_c. As N_u strictly increases in N_c according to equation 22c, the right-hand term strictly increases from $(1-\beta)(1+\tau_u)b_u$ to infinity. This equation has therefore a unique strictly positive solution as long as $\frac{(1-\beta)(1+\tau_u)b_u}{(1+\tau_c)\underline{w}} < \alpha_u\alpha_c^{\frac{\alpha_c}{\alpha_u}} \left[\frac{(1-\beta)(1-t)A}{1-\beta+\alpha_u\beta}\right]^{\frac{1}{\alpha_u}} \left[\frac{\alpha_k}{(1+\tau_k)r}\right]^{\frac{\alpha_k}{\alpha_u}}$. Otherwise, the unique equilibrium is the absence of production.

For each level of bargaining power, the model is calibrated with taxes of 40% on each labor segment, 25% on capital income and 10% on sales; parameter α_1 is set such that the unemployment rate of low-skilled is 15%. From this reference situation, four reforms are simulated: three fiscal reforms and an educational reform. The fiscal reforms consist in shifting the tax burden from the low-skilled to another base: i) high-skilled workers; ii) capital income; iii) sales. For each decrease of the low-skilled tax rate, the tax rate on the base to which taxation is shifted is adjusted in order to get the overall tax revenue constant. The impact of these fiscal reforms on the unemployment rates of each segment of the labor market is presented in figure 6. The impacts on output and profits are presented in figure 7.

Shifting tax burden from low-skilled to capital income succeed in decreasing low-skilled unemployment if high-skilled bargaining power is not to large. Otherwise, low-skilled tax decrease possibilities are limited. Even if possible, the efficiency of such a tax shift is lower than when tax burden is transferred onto high-skilled workers or sales. For the shift on these two bases, the impact of the high-skilled bargaining power on the efficiency of the reforms in terms of unemployment is negligible: the reforms are always efficient. As previously, high-skilled unemployment rate is not impacted by the reforms because the bargained wages absorbs the demand shocks.

It appears that the low-skilled unemployment reduction is even greater when the tax burden is transferred onto high-skilled workers instead of sales. Indeed, high-skilled segment is the most flexible production input in terms of price. Taxing sales is equivalent to taxing the whole production, and therefore taxing constrained labor segment and capital in addition to unconstrained labor segment. Hence, the tax increase to compensate low-skilled tax revenue decrease is less fully absorbed by input prices, leading to a lower activity increase.

Apart from the reforms shifting the tax burden on capital, which has a negative impact on overall output, the other two tax reforms allow increasing substantially the overall output. As for unemployment, the output increase is even larger when the tax burden is shifted onto the flexible unconstrained labor segment instead of

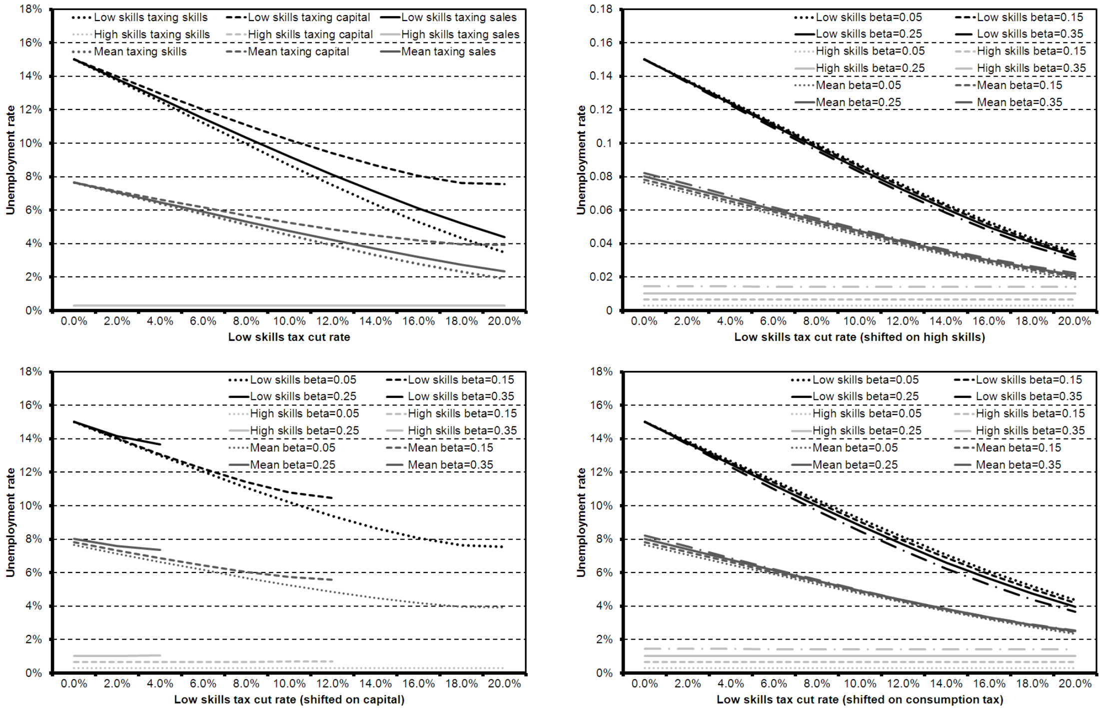


Figure 6: Impact of taxes on unemployment rates of high and low qualification workers

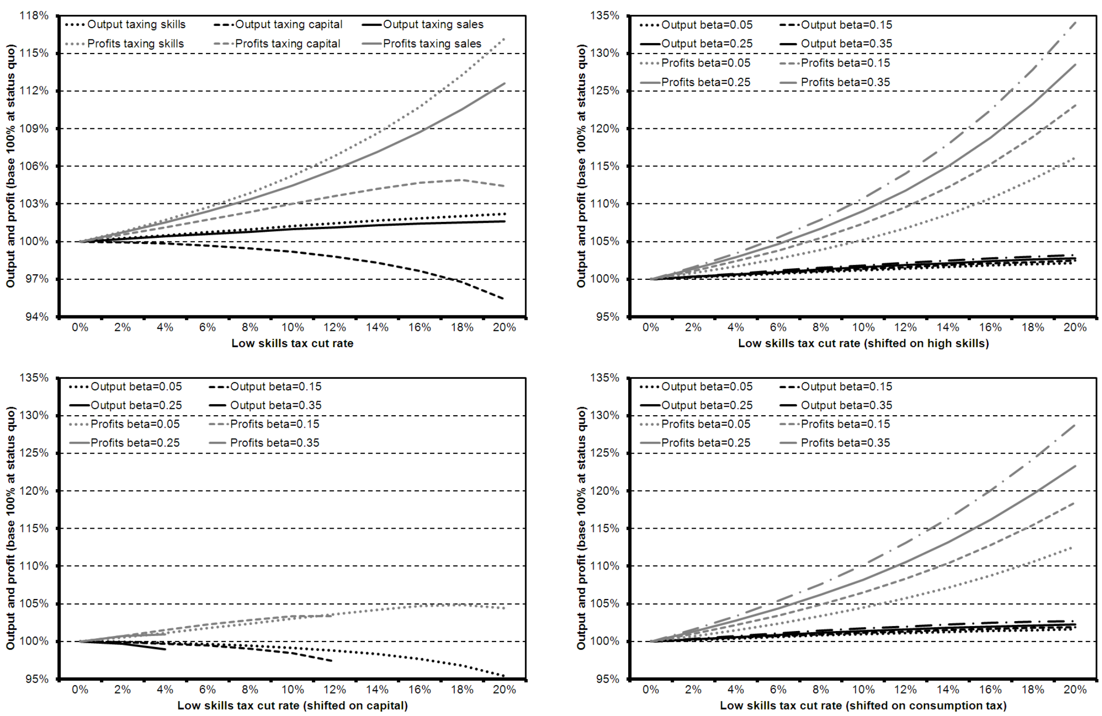


Figure 7: Impact of taxes on output and profits

sales. Profits always increase - even when taxing capital - and the increase is larger with taxing sales instead of capital, and high-skilled instead of sales.

The last reform simulated is the same educational reform as in the previous subsection: upgrading a share of constrained workers into the unconstrained labor segment. The impacts on unemployment are presented in 8. Impacts on output, profits and tax revenue are presented in figure 9.

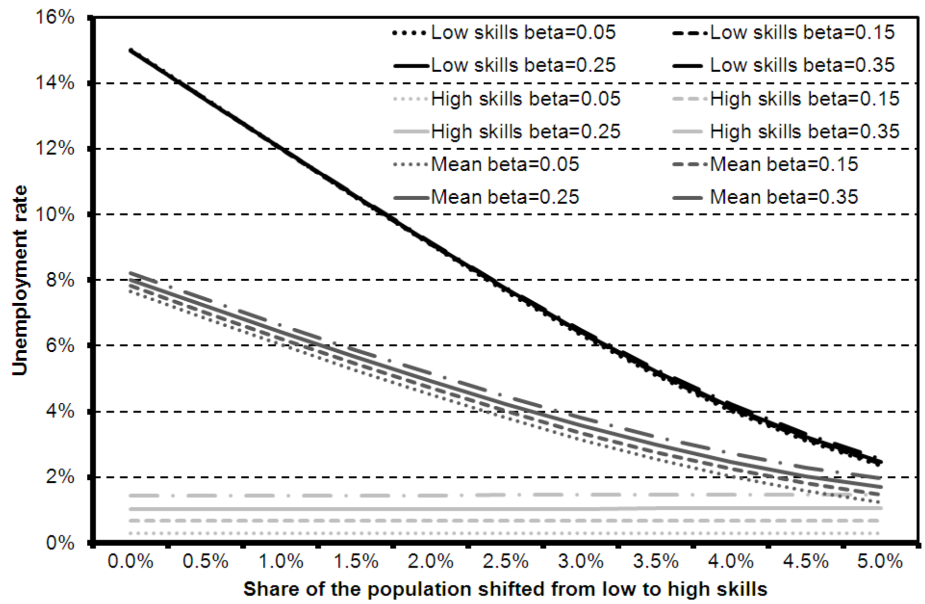
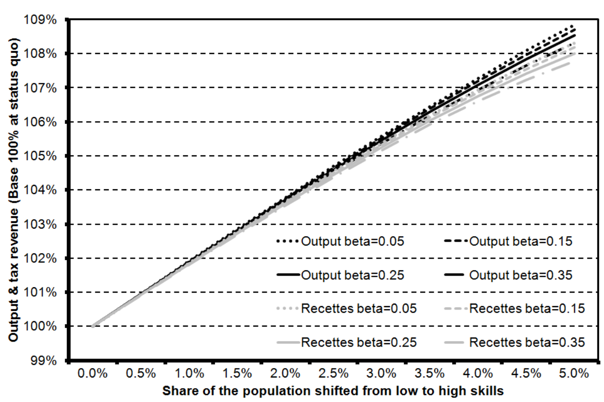


Figure 8: Impact of tax shifting on high and low qualification workers' unemployment

The simulation is calculated for a high qualification worker with twice the productivity and the bargaining power as the low qualification worker



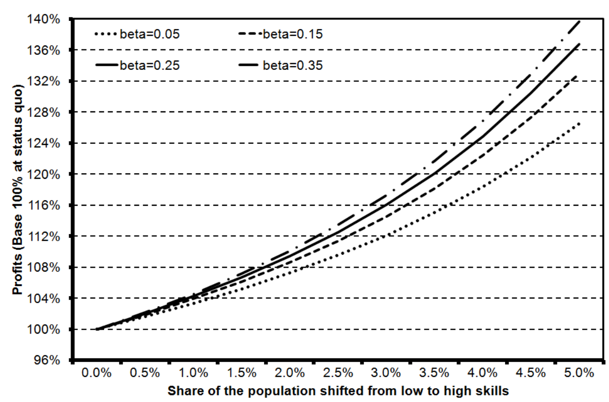


Figure 9: Impact of taxes on output and profits

As for all the different simulations and for the same reason of flexibility, unemployment on the high-skilled segment is both very low and not impacted by educational reform. Concerning specifically the three factors case, the constraint on wages due to minimum wage makes educational reforms even more efficient. The

Malthusian effect is strongly reinforced by the demand effect. With fixed low-skilled wages, the demand increase due to increase of high-skilled labor supply - without additional unemployment in that segment - push upwards the virtual bargained wage on the low-skilled segment, getting it closer to the minimum wage and consequently strongly lowering the constraint. This leads to strong overall employment increase with a strongly increased mean productivity, hence a very strong output, profit and tax revenue increase.

Nevertheless, another variable may have an impact on the efficiency of the tax and educational reforms: the initial proportion of low- and high-skilled workers. To measure this impact, the previous simulations are run with alternative calibration of the initial proportion of workers on the two segments of the labor market: two thirds of low-skilled instead of one half. Differences of impacts of tax reforms are presented in figure 10. Differences of impacts of educational reform are presented in figure 11.

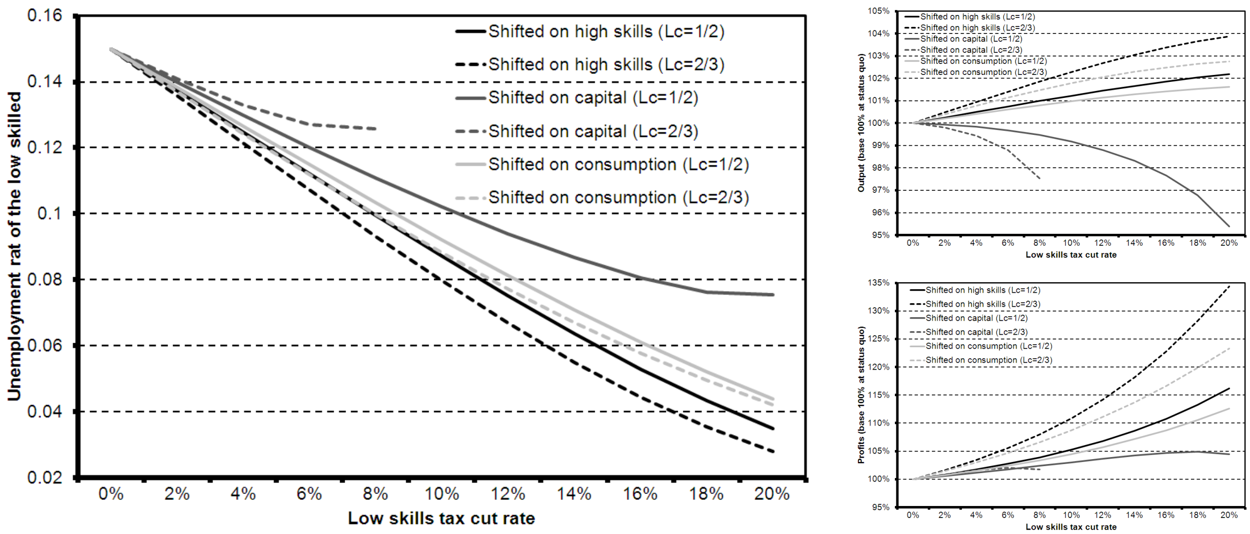


Figure 10: Impact of taxes depending of proportion of low skills

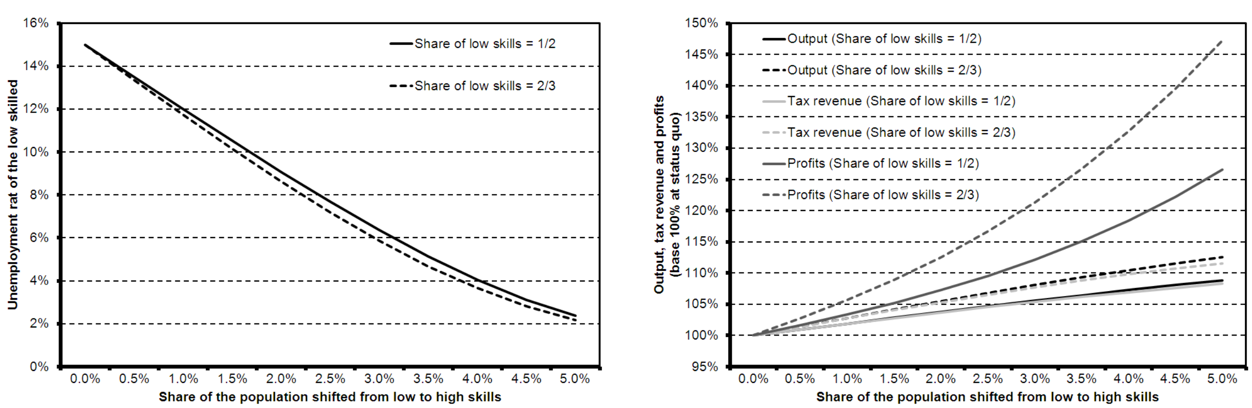


Figure 11: Impact of educational reform depending of proportion of low skills

When the initial rate of low-skilled is higher, the impact of reforms - whatever tax reform or educational reform - is stronger. The differences are relatively weak for unemployment, output and tax revenue, but are substantial concerning profits. This means that the educational reforms are at decreasing marginal efficiency. However, the difference of unemployment impact is small although the difference of low-skilled workers proportion is very large - two thirds versus one half. Consequently, the rate of decrease of the marginal efficiency of educational reforms is itself very small. Similarly, the interaction between the two kinds of reforms is negative: a lower proportion of low-skilled - which can be due to successful educational reforms - lowers the efficiency of tax reforms, but also in very small magnitude.

5 Conclusion and comments

The present article models a labor market with heterogeneous workers in a search and matching framework. A global production function is considered to take into account the interactions between segments. The different interacting inputs may be capital, unconstrained or constrained workers. Two kinds of substitutability between workers are considered. The usual one comes from the inter-factorial substitutability, depending on the productivity and rarity of the workers of the given type. The productivity and rarity are determined endogenously from the production function, the size of the segment and the matching process (leading to a level of employment, a marginal productivity and a tightness of the segment of the labor market). An additional substitutability comes from the bargaining power parameter and is interpreted as the intra-segment substitutability. Workers without bargaining power - or with very large intra-segment substitutability - rely on collective bargaining for their wage setting and are considered as constrained workers. Particularly, workers at minimum wage are of this category. Workers with bargaining power individually bargain their wage continuously intra-firm.

The present paper solves formally the model with an arbitrarily large number of segments, and presents the impact of the different parameters of the model on the equilibrium. The employment on each segment increases with total factor productivity, matching function efficiency and segment size. It decreases with unemployment benefit, vacancy posting cost, job destruction rate, interest rate and payroll taxes (own segment tax and tax on other segments). The crossed impact of taxation is greater (more negative) from a large bargaining power segment to a low bargaining power segment than the opposite.

In addition, two different simplified versions of the model are numerically analyzed. First, a version with two segments of unconstrained workers and a CES production function is calibrated in order to test the impact of the inter-factorial elasticity of substitution. Second, a model with three factors - a constrained workers' segment, an unconstrained workers' segment and capital - tests tax burden shifting in greater details when one segment is constrained by minimum wage. Two kinds of reforms are tested in the two frameworks: a tax reform consisting in shifting the burden from the segment with largest rate of unemployment to other bases; and an educational reform consisting in shifting some workers from a low productivity segment to a larger productivity segment.

The tax reforms' efficiency increases with the substitutability between segments in the production function and with the constraints on the low-skilled wages (high minimum wages). The educational reform's efficiency increases with the complementarities between segments and is not much impacted by the constraints on low-skilled wages. The Malthusian effect of reducing low-skilled labor supply is reinforced by the increase of demand due to the increase of high-skilled labor supply and the complementary between segments. The association of employment and productivity increases generates large output and tax revenue increases, which can finance the educational policies. The educational reforms' efficiency marginally decreases, but the rate of decrease is very small. Similarly, the interaction between the two kinds of reforms is negative: a lower proportion of low-skilled - which can be due to successful educational reforms - lowers the efficiency of tax reforms, but also in very small extend.

It appears that tax reforms may be efficient to lower unemployment rate only of segment fully constrained by the minimum wages. The impact on segments with bargained wage above this level - and hence with at least little possibility of wage bargaining - is much lower, especially when the segments are complements. Furthermore, the shift in the tax burden from low-skilled to high-skilled segments may prove hardly sustainable from a political economy perspective. In main countries experiencing these kinds of reforms, the shift concerns mainly payroll taxes financing social protection - often in Bismarckian countries - and the reforms tend to set a redistributive profile of social protection funding. This redistributive financing profile lowers the political support of middle class to social protection and negative correlation appears between redistributive profile of financing and the generosity of welfare states (Prasad and Deng (2009)). Yet, a fraction of social protection constitutes social investment, which may be a crucial parameter strengthening labor force participation and hence employment (Dwenger et al. (2014); Kleven (2014)). More broadly, in the curse of the biased technological change with decrease of routine jobs, tax burden shifts may prove to be insufficient or very costly in a dynamic framework. Another possibility is to allow labor cost decrease at the bottom of the distribution not by tax cuts but by minimum wage decreases. However, this could generate strong in job inequalities and in job poverty.

It appears out of the different simulations that the alternative educational reform keeps effective for low productivity segments even if wages are not fully constrained. Such reforms may be even more at stake in the context of polarization. It allows increasing upwards shifts from routine to cognitive jobs, which induces a labor supply relative decrease at the bottom of the distribution and in addition a demand increase for manual (or contact) jobs complements to new cognitive jobs. These manual jobs may be complements to production - services to firms - or consumed by the increased proportion of affluent people - personal services. Furthermore, education for the bottom manual workers, as much as better job conditions (social security and in the job training) allows increasing the quality of these manual services in complement to technological progress.

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A Formal resolution of the differential equations

A.1 Disentengling the system of differential equations

The partial derivative of equation 15 with respect to N_k, $k \in [1,l]\backslash i$, is:

$$\frac{\partial w_i(\vec{N})}{\partial N_k} = \frac{\beta_i}{1+\tau_i}\left[\frac{\partial^2 F(\vec{N})}{\partial N_i N_k} - \sum_{j=1}^{l}(1+\tau_j)N_j\frac{\partial^2 w_j(\vec{N})}{\partial N_i N_k} - (1+\tau_k)\frac{\partial w_l(\vec{N})}{\partial N_i}\right]$$

Yet, when $i, k \in [1,l]$ and $i \neq k$:

$$\frac{\partial^2}{\partial N_i \partial N_k}\sum_{j=1}^{l}(1+\tau_j)N_j w_j(\vec{N}) = \sum_{j=1}^{l}(1+\tau_j)N_j\frac{\partial^2 w_j(\vec{N})}{\partial N_k \partial N_i} + (1+\tau_i)\frac{\partial w_i(\vec{N})}{\partial N_k} + (1+\tau_k)\frac{\partial w_k(\vec{N})}{\partial N_i}$$

And therefore the derivative with respect to N_k of differential equation 15 for $k \in [1,l]\backslash i$ is:

$$(1-\beta_i)\frac{\partial w_i(\vec{N})}{\partial N_k} = \frac{\beta_i}{1+\tau_i}\frac{\partial^2}{\partial N_i \partial N_k}\left[F(\vec{N}) - \sum_{j=1}^{l}(1+\tau_j)N_j w_j(\vec{N})\right]$$

Comparing derivative with respect to N_k of differential equation 15 for i and derivative with respect to N_i of differential equation 15 for k with $i, k \in [1,l]$ and $i \neq k$ gives equation 23.

$$(1+\tau_k)\frac{\partial w_k(\vec{N})}{\partial N_i} = \frac{1-\beta_i}{\beta_i}\frac{\beta_k}{1-\beta_k}(1+\tau_i)\frac{\partial w_i(\vec{N})}{\partial N_k} = \chi_{ik}(1+\tau_i)\frac{\partial w_i(\vec{N})}{\partial N_k} \tag{23}$$

Which implies that:

$$\sum_{j=1}^{l}(1+\tau_j)N_j\frac{\partial w_j(\vec{N})}{\partial N_i} = \sum_{j=1}^{l}(1+\tau_i)\chi_{ij}N_j\frac{\partial w_i(\vec{N})}{\partial N_j}$$

And differential equation 15 may be rewritten as differential equation 16.

A.2 Differential equations for multiple worker types

This differential equation must be solved in several stage: two successive change of coordinate, the actual resolution of the differential equation and the return to the original set of coordinates. For all the demonstration, $i \in [1,l]$ as there is no bargaining for other factors and w_i for $i \in [l+1,n]$ is constant and equal to $\underline{w}$ for constrained workers and r for capital.

A.2.1 First change of coordinates

Let consider a first change of coordinate such that $\vec{M}_i = (M_{i1}, ..., M_{in})$, $v_i(\vec{M}_i) = w_i(\vec{N})$ and:

$$\sum_{j=1}^{l} M_{ij}\frac{\partial v_i(\vec{M}_i)}{\partial M_{ij}} = \sum_{j=1}^{l}(1+\tau_i)\chi_{ij}N_j\frac{\partial w_i(\vec{N})}{\partial N_j}$$

It works in particular if for all $j \in [1,l]$:

$$M_{ij}\frac{\partial v_i(\vec{M}_i)}{\partial M_{ij}} = (1+\tau_i)\chi_{ij}N_j\frac{\partial w_i(\vec{N})}{\partial N_j} \tag{24}$$

And $M_{ij} = N_j$ for all $j \in [l+1, n]$. Yet by definition, for $j \in [1, l]$:

$$\frac{\partial w_i(\vec{N})}{\partial N_j} = \frac{dM_{ij}}{dN_j}\frac{\partial v_i(\vec{M}_i)}{\partial M_{ij}}$$

And therefore equation 24 become:

$$M_{ij}\frac{\partial v_i(\vec{M}_i)}{\partial M_{ij}} = (1+\tau_i)\chi_{ij}N_j\frac{dM_{ij}}{dN_j}\frac{\partial v_i(\vec{M}_i)}{\partial M_{ij}}$$

Allowing to define the differential equation for the functions $M_{ij}(N_j)$, which are:

$$M_{ij} = (1+\tau_i)\chi_{ij}N_j dM_{ij}/dN_j$$

One solution is $M_{ij} = N_j^{\chi_{ji}/(1+\tau_i)}$ as $1/\chi_{ij} = \chi_{ji}$. Furthermore, we call $G(\vec{M}_i) = F(\vec{N})$. Hence, for $j \in [1, l]$, $\partial F(\vec{N})/\partial N_j = (\partial G(\vec{M}_i)/\partial M_{ij})(dM_{ij}/dN_j)$, that is $\partial F(\vec{N})/\partial N_j = [\chi_{ji}/(1+\tau_i)]N_j^{\chi_{ji}/(1+\tau_i)-1}(\partial G(\vec{M}_i)/\partial M_{ij})$. And in particular, as $i \in [1, l]$, $\partial F(\vec{N})/\partial N_i = [N_i^{-\tau_i/(1+\tau_i)}/(1+\tau_i)](\partial G(\vec{M}_i)/\partial M_{ii}) = [M_{ii}^{-\tau_i}/(1+\tau_i)](\partial G(\vec{M}_i)/\partial M_{ii})$.The differential equation is the new set of coordinates is consequently given by equation 25.

$$v_i(\vec{M}_i) = (1-\beta_i)rU_i + \frac{\beta_i}{1+\tau_i}\left[\frac{M_{ii}^{-\tau_i}}{1+\tau_i}\frac{\partial G(\vec{M}_i)}{\partial M_{ii}} - \sum_{j=1}^{l} M_{ij}\frac{\partial v_i(\vec{M}_i)}{\partial M_{ij}}\right] \tag{25}$$

A.2.2 Second change: spherical coordinates

Another change of coordinate should now be made with spherical coordinates $(\rho_i, \phi_{i1}, ..., \phi_{i,l-1})$ where ρ_i is the canonical norm of the vector $\vec{M}_i^u = (M_{i1}, ..., M_{il})$ (eg: $\rho_i^2 = \sum_{j=1}^{l} M_{ij}^2$) and $\vec{\phi}_i = (\phi_{i1}, ..., \phi_{i,l-1})$ the angles. Let determine the angles as in equation 26.

$$\begin{cases} \phi_{i,1} & \textit{such that} & M_{i,l} = \rho_i \sin\phi_{i1} \\ \phi_{i,2} & \textit{such that} & M_{i,l-1} = \rho_i \cos\phi_{i1}\sin\phi_{i2} \\ & & ... \\ \phi_{i,j} & \textit{such that} & M_{i,l+1-j} = \rho_i \cos\phi_{i1}...\cos\phi_{i,j-1}\sin\phi_{ij} \\ & & ... \\ \phi_{i,l-1} & \textit{such that} & M_{i,2} = \rho_i \cos\phi_{i1}...\cos\phi_{i,l-2}\sin\phi_{i,l-1} \end{cases} \tag{26}$$

It follows that $M_{i,1}^2 = \rho^2 - \sum_{j=2}^{l} M_{i,j}^2$. Yet:

$$\begin{cases} M_{i,l}^2 + M_{i,l-1}^2 = \rho^2(1-\cos^2\phi_{i1} + \cos^2\phi_{i,1}\sin^2\phi_{i,2}) & = \rho^2[1-\cos^2\phi_{i1}(1-\sin^2\phi_{i,2})] \\ & = \rho^2[1-\cos^2\phi_{i1}\cos^2\phi_{i,2}] \\ ... + M_{i,l-2}^2 = \rho^2[1-\cos^2\phi_{i1}\cos^2\phi_{i,2}(1-\sin^2\phi_{i,3})] = \rho^2[1-\cos^2\phi_{i1}\cos^2\phi_{i,2}\cos^2\phi_{i,3}] \\ ... \\ ... + M_{i,2}^2 = \rho^2[1-\cos^2\phi_{i1}...\cos^2\phi_{i,n-2}(1-\sin^2\phi_{i,l-1})] = \rho^2[1-\cos^2\phi_{i1}...\cos^2\phi_{i,n-1}] \end{cases}$$

And therefore $M_{i,1}$ is given by equation 27.

$$M_{i,1} = \rho\cos\phi_{i1}...\cos\phi_{i,n-2}\cos\phi_{i,l-1} \tag{27}$$

Equations 26 and 27 imply that $\rho_i \partial M_{ij}/\partial\rho_i = M_{ij}$ and therefore:

$$\rho_i\partial v_i(\vec{M}_i)/\partial\rho_i = \rho_i\sum_{j=1}^{l}(\partial v_i(\vec{M}_i)/\partial M_{ij})(\partial M_{ij}/\partial\rho_i) = \sum_{j=1}^{l} M_{ij}\partial v_i(\vec{M}_i)/\partial M_{ij}$$

Hence, the differential equation in spherical coordinate is given by equation 28.

$$v_i(\rho_i,\vec{\phi}_i,\vec{N}_c,\vec{K}) = (1-\beta_i)rU_i + \frac{\beta_i}{1+\tau_i}\left[\frac{M_{ii}^{-\tau_i}}{1+\tau_i}\frac{\partial G(\rho_i,\vec{\phi}_i,\vec{N}_c,\vec{K})}{\partial M_{ii}} - \rho_i\frac{\partial v_i(\rho_i,\vec{\phi}_i,\vec{N}_c,\vec{K})}{\partial \rho_i}\right] \tag{28}$$

A.2.3 Differential equations without crossed derivative

The homogenous equation is $v_i(\rho_i,\vec{\phi}_i,\vec{N}_c,\vec{K}) + (\beta_i/[1+\tau_i])\rho_i\partial v_i(\rho_i,\vec{\phi}_i,\vec{N}_c,\vec{K})/\partial\rho_i = 0$ whose solution is $v_i(\rho_i,\vec{\phi}_i,\vec{N}_c,\vec{K}) = C\rho_i^{-\frac{1+\tau_i}{\beta_i}}$. The method of variation of the constant gives:

$$C(\rho_i,\vec{\phi}_i,\vec{N}_c,\vec{K})\rho_i^{-\frac{1+\tau_i}{\beta_i}} + \tfrac{\beta_i}{1+\tau_i}\rho_i\left(\tfrac{\partial C(\rho_i,\vec{\phi}_i,\vec{N}_c,\vec{K})}{\partial\rho_i}\rho_i^{-\frac{1+\tau_i}{\beta_i}} - \tfrac{1+\tau_i}{\beta_i}C(\rho_i,\vec{\phi}_i,\vec{N}_c,\vec{K})\rho_i^{-\frac{1+\tau_i}{\beta_i}-1}\right) = \tfrac{\beta_i}{1+\tau_i}\tfrac{M_{ii}^{-\tau_i}}{1+\tau_i}\tfrac{\partial G(\vec{M}_i)}{\partial M_{ii}}$$

And consequently the derivative of the constant is:

$$\rho_i^{-\frac{1+\tau_i}{\beta_i}+1}\frac{\partial C(\rho_i,\vec{\phi}_i,\vec{N}_c,\vec{K})}{\partial\rho_i} = \frac{M_{ii}^{-\tau_i}}{1+\tau_i}\frac{\partial G(\vec{M}_i)}{\partial M_{ii}}$$

Hence, the result of the differential equation with the spherical coordinates is:

$$v_i(\rho_i,\vec{\phi}_i,\vec{N}_c,\vec{K}) = (1-\beta_i)rU_i + \rho_i^{-\frac{1+\tau_i}{\beta_i}}\left(\kappa_i(\vec{\phi}_i,\vec{N}_c,\vec{K}) + \int_0^{\rho_i}\frac{M_{ii}^{-\tau_i}}{1+\tau_i}z^{\frac{1+\tau_i}{\beta_i}-1}\frac{\partial G(\vec{M}_i)}{\partial M_{ii}}dz\right)$$

Where M_{ii} is indeed a function of z. In the present case, the condition at limit is that $\rho_i v_i$ tends towards zero when ρ_i tends towards zero, which means that $\kappa_i = 0$. Furthermore, it appears that $(u\rho_i,\vec{\phi}_i) = (uM_{i1},...,uM_{il})$ so doing the change of variable $z = u\rho_i$ (u from 0 to 1, $dz = \rho_i du$, $\vec{M}_i(z) = (u\vec{M}_i^u,\vec{N}_c,\vec{K})$). The integral become:

$$\int_0^1\frac{u^{-\tau_i}M_{ii}^{-\tau_i}}{1+\tau_i}u^{\frac{1+\tau_i}{\beta_i}-1}\rho_i^{\frac{1+\tau_i}{\beta_i}-1}\frac{\partial G(u\vec{M}_i^u,\vec{N}_c,\vec{K})}{\partial M_{ii}}\rho_i du = \frac{M_{ii}^{-\tau_i}}{1+\tau_i}\rho_i^{\frac{1+\tau_i}{\beta_i}}\int_0^1 u^{\frac{1+\tau_i}{\beta_i}-1-\tau_i}\frac{\partial G(u\vec{M}_i^u,\vec{N}_c,\vec{K})}{\partial M_{ii}}du$$

In addition:

$$\frac{\partial G(u\vec{M}_i^u,\vec{N}_c,\vec{K})}{\partial M_{ii}} = \frac{1+\tau_i}{u^{-\tau_i}M_{ii}^{-\tau_i}}\frac{\partial F(u\vec{M}_i^u,\vec{N}_c,\vec{K})}{\partial N_i}$$

Let call $\mu_{ij} = uM_{ij}$ for $j\in[1,l]$, it is equal to $\nu_j^{\chi_{ji}/(1+\tau_i)}$ in the initial coordinates, yet $M_{ij} = N_j^{\chi_{ji}/(1+\tau_i)}$, so $\nu_j^{\chi_{ji}/(1+\tau_i)} = uN_j^{\chi_{ji}/(1+\tau_i)}$ and $\nu_j = u^{(1+\tau_i)\chi_{ij}}N_j$ and the net wage is given by equation 17.