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Generic regularity of conservative solutions to Camassa-Holm type equations

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Abstract

This paper mainly proves the generic properties of the Camassa-Holm equation and the two-component Camassa-Holm equation by Thom’s transversality Lemma. We reveal their differences in generic regularity and singular behavior.

1 Introduction

The Camassa-Holm equation and the two-component Camassa-Holm equations have been studied by many people in different aspects [3, 4, 5, 7, 9, 10, 13, 14, 15, 16, 18, 19]. These two equations mainly describe the surface wave phenomenon of shallow water. The Cauchy problem for the two-component Camassa-Holm system (2CH) can be written in the form

\[
\begin{cases}
    u_t + (u^2/2)_x + P_x = 0, \\
    \rho_t + (u\rho)_x = 0,
\end{cases}
\]

where the nonlocal source term \(P\) is defined as a convolution:

\[
P = \frac{1}{2} e^{-|x|} \ast \left( u^2 + \frac{(u^2 + \rho^2)}{2} \right).
\]

Here \(u\) is the averaged horizontal velocity and \(\rho\) is related to the deviation of the free surface. The initial data is specified as

\[
    u(0, x) = u_0(x), \quad \rho(0, x) = \rho_0(x).
\]

When \(\rho \equiv 0\), the above system becomes the one-component Camassa-Holm equation (CH)

\[
u_t + (u^2/2)_x + P_x = 0,
\]
where the nonlocal source term $P$ is defined as a convolution:

$$P = \frac{1}{2} e^{-|x|} * \left( u^2 + \frac{u_x^2}{2} \right),$$  \hspace{1cm} (1.5)

or written as

$$u_t - u_{txtt} + 3uu_x = 2u_xu_{xx} + uu_{xxx}. \hspace{1cm} (1.6)$$

The Camassa-Holm equation is proposed by Camassa and Holm in [7]. The existence of weak solution is proved in [23, 24] by a compactness argument, and in [4, 5, 19] by a characteristic method, while the uniqueness of energy conservative weak solution is proved in [3]. For the two-component Camassa-Holm system, the existence of solution is proved by different groups using different methods [14, 15] [16, 17] [18]. [16, 17, 14, 18] mainly deal with the case of positive $\rho$. They proved various results on singularity formation, global existence of solution and regularity. In the paper [15], the authors proposed a new kind of weak solution, the $\alpha$-dissipative solution, without the restriction of $\rho > 0$ and proved the global existence. In [20], we proved, for the general $\rho$, the uniqueness of energy conservative weak solution.

In this paper, we mainly study the generic regularity of the conservative solution of (CH) and (2CH). The problem of generic regularity of a solution dates back to Schaeffer [21], who showed that the generic solutions of one-dimensional scalar conservation law are piecewise smooth, with finitely many shocks in a bounded domain in $(t, x)$ plane. His proof relies on the explicit expression of solution by Lax formula. Later Dafermos and Geng [11] proved the similar result for $2 \times 2$ Temple class system. Very recently, Caravenna and Spinolo [8] constructed a $3 \times 3$ system showing that Schaeffer’s regularity theorem fails for general $n \times n$ system with $n \geq 3$.

Recently, Bressan and his collaborators revive this field by considering generic properties for variational wave equation. In [2], the authors mainly prove that for an open dense set of $C^3$ initial data, the solution is piecewise smooth in the $t - x$ plane, while the gradient $u_x$ can blow up along finitely many characteristic curves. Later in [6], the authors provides a detailed asymptotic description of the solution in a neighborhood of each generic singular point. They mainly rely on the equivalent ‘ODE-type’ expression of the weak solution.

In this paper, we propose the same question to Camassa-Holm type equation. Compared with the variational wave equation, the main difference lies in the nonlocal term in Camassa-Holm type equation. In variational wave equation, since the wave speed is finite, the behavior of the singularity mainly relies on the information in its domain of dependence. While in the Camassa-Holm type equations, since $P$ is nonlocal, there does not exist an explicit domain of dependence. In the following, we will show that this difficulty can be solved by detailed estimate of the source term (see (3.42)-(3.43)).

Another highlight point in this paper is the difference between the generic singularity behavior of Camassa-Holm equation and two-component Camassa-Holm equation. As we mentioned before, (CH) is a special case of (2CH). From this view, (2CH) should inherit all the singular behavior from (CH). However, we find that they have totally different types of the generic singular behaviors. The main point here is that when the density $\rho$ is positive, we have the upper bound of $|u_x|$, which means there is no blow-up phenomenon in the domain of $\rho > 0$. This shows that we can improve the regularity of the solution dramatically by considering the generic solutions. Compared with the fact that the solutions of (CH) may form singularity in finitely many piecewise $C^2$ curves in the domain $[0, T] \times \mathbb{R}$, for any $T > 0$, the singularity of
(2CH) generically only can occur at finitely many isolated points within the same domain (see Theorem 4.3).

The content of this paper is following. In section 2, we recall the basic definitions of transversality and genericity. Section 3 deals with the one-component Camassa-Holm equation. Subsection 3.1 reviews the fundamental results and presents our theorem (Theorem 3.3) of generic regularity of solutions to the Camassa-Holm equation. This theorem is proved in subsections 3.2 – 3.4. In subsection 3.5, we prove the asymptotic behavior for generic singularities. In section 4, we consider the two-component Camassa-Holm system. After reviewing the fundamental results of 2CH and presenting our results, Theorem 4.3 and Theorem 4.4 in subsection 4.1, We prove these two theorems in subsections 4.2 – 4.4.

2 Preliminaries

2.1 Transversality and genericity

The content of this section can be found in many books or monographs [1, 12, 22]. We collect them here for reader’s convenience.

Definition 2.1. Let $F : X \to Y$ be a smooth map from manifold $X$ to manifold $Y$. $W$ is a submanifold of $Y$. We say $F$ is transverse to $W$ at a point $x \in X$, denote by $F \pitchfork_x W$ if

- either $F(x) \notin W$,
- or $F(x) \in W$ and $T_{F(x)}Y = (dF)_x(T_xX) + T_{F(x)}W$.

Here $T_xX$ means the tangent space of $X$ at point $x$.

If $F \pitchfork_x W$ for every $x \in X$, we say $F$ is transverse to $W$, and denote as $F \pitchfork W$.

Definition 2.2. Let $F : X \to Y$ be a smooth map from manifold $X$ to $Y$. A point $y \in Y$ is a regular value if for every $x \in X$, one has $T_yY = (dF)_x(T_xX)$.

In the special case where $W = \{y\}$ consists of a single point, $f \pitchfork W$ if and only if $y$ is a regular value of $f$.

Theorem 2.1 (Thom’s transversality Lemma[1, 12]). Let $X, \Theta,$ and $Y$ be smooth manifolds, $W$ a submanifold of $Y$. Let $\theta \mapsto \phi^\theta$ be a smooth map which to each $\theta \in \Theta$ associates a function $\phi^\theta \in C^\infty(X,Y)$, and define $\Phi : X \times \Theta \to Y$ by setting $\Phi(x, \theta) = \phi^\theta(x)$. If $\Phi \pitchfork W$ then the set $\{\theta \in \Theta; \phi^\theta \pitchfork W\}$ is dense in $\Theta$.

The generic property is a property which is satisfied by almost all elements of the whole set, i.e. by an open and dense subset. Considering our PDE problem here, generic regularity is the regularity of the solutions solved from an open and dense subset of initial data.
3 Camassa-Holm equation

3.1 Basic definitions and results for CH

Recall Camassa-Holm equation (CH)

\[ u_t + \left(\frac{u^2}{2}\right)_x + P_x = 0, \tag{3.1} \]

where the nonlocal source term \( P \) is defined as a convolution:

\[ P = \frac{1}{2} e^{-|x|} * \left( u^2 + \frac{u_x^2}{2} \right), \tag{3.2} \]

The initial data is specified as

\[ u(0, x) = u_0(x). \tag{3.3} \]

To make sense of the source term \( P \), at each time \( t \) we require that the function \( u(t, \cdot) \) lies in the space \( H^1(\mathbb{R}) \) of absolutely continuous functions \( u \in L^2(\mathbb{R}) \) with derivative \( u_x \in L^2(\mathbb{R}) \), endowed with the norm

\[ \|u\|_{H^1} = \left( \int_{\mathbb{R}} \left[ u^2(x) + u_x^2(x) \right] dx \right)^{1/2}. \]

For \( u \in H^1(\mathbb{R}) \), Young’s inequality ensures that

\[ P = (1 - \partial_x^2)^{-1} \left( u^2 + \frac{u_x^2}{2} \right) \in H^1(\mathbb{R}). \]

For future use we record the following inequalities, valid for any function \( u \in H^1(\mathbb{R}) \):

\[ \|u\|_{L^\infty} \leq \|u\|_{H^1}, \tag{3.4} \]

\[ \|P\|_{L^\infty}, \; \|P_x\|_{L^\infty} \leq \left\| \frac{1}{2} e^{-|x|} \right\|_{L^\infty} \cdot \left\| u^2 + \frac{u_x^2}{2} \right\|_{L^1} \leq \frac{1}{2} \|u\|_{H^1}^2, \tag{3.5} \]

\[ \|P\|_{L^2}, \; \|P_x\|_{L^2} \leq \left\| \frac{1}{2} e^{-|x|} \right\|_{L^2} \cdot \left\| u^2 + \frac{u_x^2}{2} \right\|_{L^1} \leq \frac{1}{\sqrt{2}} \|u\|_{H^1}^2. \tag{3.6} \]

**Definition 3.1.** By a solution of the Cauchy problem (1.1)-(1.3) on \([0, T]\) we mean a Hölder continuous function \( u = u(t, x) \) defined on \([0, T] \times \mathbb{R}\) with the following properties. At each fixed \( t \) we have \( u(t, \cdot) \in H^1(\mathbb{R}) \). Moreover, the map \( t \mapsto u(t, \cdot) \) is Lipschitz continuous from \([0, T]\) into \( L^2(\mathbb{R}) \) and satisfies the initial condition (1.3) together with

\[ \frac{d}{dt} u = - uu_x - P_x \tag{3.7} \]

for a.e. \( t \). Here (3.7) is understood as an equality between functions in \( L^2(\mathbb{R}) \).

As shown in [4, 5], as soon as the gradient of a solution blows up, uniqueness is lost, in general. To single out a unique solution, some additional conditions are needed.
For smooth solutions, differentiating (3.1) w.r.t. \( x \) one obtains
\[
  u_{xt} + (uu_x)_x = \left( u^2 + \frac{u_x^2}{2} \right) - P. \tag{3.8}
\]

Multiplying (3.1) by \( u \) and (3.8) by \( u_x \), we obtain the two conservation laws with source term
\[
  \left( \frac{u^2}{2} \right)_t + \left( \frac{u^3}{3} + uP \right)_x = u_x P, \tag{3.9}
\]
\[
  \left( \frac{u_x^2}{2} \right)_t + \left( \frac{uu_x^2}{2} - u^3 \right)_x = -u_x P. \tag{3.10}
\]
Summing (3.9) and (3.10), and integrating w.r.t. \( x \), we see that for smooth solutions the total energy
\[
  E(t) = \int_{\mathbb{R}} \left( u^2(t, x) + u_x^2(t, x) \right) dx \tag{3.11}
\]
is constant in time.

**Definition 3.2.** A solution \( u = u(t, x) \) is conservative if \( w = u_x^2 \) provides a distributional solution to the balance law (3.12),
\[
  w_t + (uw)_x = 2 (u^2 - P) u_x, \tag{3.12}
\]

namely
\[
  \int_0^\infty \int \left[ u_x^2 \varphi_t + uu_x^2 \varphi_x + 2 (u^2 - P) u_x \varphi \right] dx dt + \int u_0^2(x) \varphi(0, x) dx = 0 \tag{3.13}
\]
for every test function \( \varphi \in \mathcal{C}^1_c (\mathbb{R}^2) \).

The main result proved in [4], on the global existence of conservative solutions can be stated as follows.

**Theorem 3.1.** For any initial data \( \bar{u} \in H^1(\mathbb{R}) \) the Camassa-Holm equation has a global conservative solution \( u = u(t, x) \). More precisely, there exists a family of Radon measures \( \{ \mu(t) \}, \ t \in \mathbb{R} \), depending continuously on time w.r.t. the topology of weak convergence of measures, such that the following properties hold.

(i) The function \( u \) provides a solution to the Cauchy problem (3.1)-(3.3) in the sense of Definition 1.

(ii) There exists a null set \( \mathcal{N} \subset \mathbb{R} \) with \( \text{meas}(\mathcal{N}) = 0 \) such that for every \( t \notin \mathcal{N} \) the measure \( \mu(t) \) is absolutely continuous and has density \( u_x^2(t, \cdot) \) w.r.t. Lebesgue measure.

(iii) The family \( \{ \mu(t) ; t \in \mathbb{R} \} \) provides a measure-valued solution \( w \) to the linear transport equation with source
\[
  w_t + (uw)_x = 2 (u^2 - P) u_x. \tag{3.14}
\]
At a time \( t \in \mathbb{N} \) the measure \( \mu(t) \) has a nontrivial singular part. For a conservative solution \( u \) which is not smooth, in general we only know that the energy \( E \) in (4.13) coincides a.e. with a constant. Namely,

\[
E(t) = E(0) \quad \text{for} \quad t \notin \mathbb{N}, \quad E(t) < E(0) \quad \text{for} \quad t \in \mathbb{N}.
\]

Following the idea in [4], we also denote

\[
v = 2 \arctan u_x, \quad q = (1 + u_x^2) \cdot \frac{\partial y}{\partial \xi} \tag{3.15}
\]

Then the conservative solution is constructed by following semilinear system

\[
\begin{align*}
\frac{\partial}{\partial t} u(t, \xi) &= -P_x, \\
\frac{\partial}{\partial t} v(t, \xi) &= (u^2 - P)(1 + \cos v) - \sin^2 \frac{v}{2} \\
\frac{\partial}{\partial t} q(t, \xi) &= (u^2 + \frac{1}{2} - P) \sin v \cdot q,
\end{align*}
\tag{3.16}
\]

with

\[
x_\xi(t, \xi) = \frac{q(t, \xi)}{1 + \tan^2(\frac{v(t, \xi)}{2})} = q(t, \xi) \cos^2 \frac{v(t, \xi)}{2} \tag{3.17}
\]

And for every \( \beta \in \mathbb{R} \) we have the initial condition

\[
\begin{align*}
u(0, \beta) &= u_0(x(0, \beta)), \\
v(0, \beta) &= 2 \arctan u_{0,x}(x(0, \beta)), \\
q(0, \beta) &= 1.
\end{align*} \tag{3.18}
\]

**Lemma 3.1.** Let \((u, v, q, x)\) be the solution (3.16)(3.17), with \( q > 0 \). Then the set of points

\[
\{(t, x(t, \xi), u(t, \xi)); (t, \xi) \in \mathbb{R} \times \mathbb{R}\}
\tag{3.19}
\]

is the graph of a conservative solution to the Cammasa-Holm equation (3.1).

The conservative weak solution constructed above is unique [3]:

**Theorem 3.2.** For any initial data \( u_0 \in H^1(\mathbb{R}) \), the Cauchy problem (3.1)-(3.3) has a unique conservative solution \( u = u(t, x) \) satisfying (i)-(iii) in Theorem 3.1.

To recover the singularities of the solution \( u \) of (3.1) in the original \( t - x \) plane, it now suffices to study the level sets

\[
\{v(t, \xi) = \pi\}.
\]

Since \( u, v \) and \( q \) are smooth, the generic structure of these level sets can be analyzed by techniques of singularity theory, relying on Thom’s transversality theorem as in section 2. Our first theorem in this part is
Theorem 3.3 (Generic regularity). For any $T > 0$ fixed, there exists an open dense set of initial data \[ D \subset \left( C^3(\mathbb{R}) \cap H^1(\mathbb{R}) \right), \] such that for $u_0 \in D$, the conserved solution $u = u(t,x)$ of Camassa-Holm equation (3.1)-(3.3) is twice continuously differentiable in the complement of finitely many characteristic curves $\gamma_i$, within the domain $[0,T] \times \mathbb{R}$.

Based on this generic regularity, we can obtain a detailed asymptotic description of the solution in a neighborhood of each singular point, where $|u_x| \to \infty$.

Theorem 3.4 (Singular behavior). Consider generic initial data $u_0 \in D$ as in Theorem 3.3, with $u_0 \in C^\infty(\mathbb{R})$. Call $(u,v,q,x,t)$ the corresponding solution of the semilinear system (3.16) and let $u = u(x,t)$ be the solution to the original equation (3.1). Consider a singular point $P = (t_0, \xi_0)$ where $v = \pi$, and set $(x_0,t_0) = (x(t_0,\xi_0), t(t_0,\xi_0))$. Generically, at the singular point, $u$ has following parametric expression.

\[(i)\] If $P$ is a point of Type I: $v = \pi, v_\xi \neq 0$, 
\[
u(t,x) = u(t_0,x_0) + A|x - x_0 - u(t_0,x_0)(t-t_0)|^{2/3} + B_2(t-t_0)|x - x_0 - u(t_0,x_0)(t-t_0)|^{1/3} + B_1(t-t_0) + O(1)(|t-t_0|^2, |x - x_0 - u(t_0,x_0)(t-t_0)|^{4/3}), \] for some constants $A, B_1, B_2$.

\[(ii)\] If $P$ is a point of Type II: $v = \pi, v_\xi = 0, v_\xi \neq 0, v_t \neq 0$, 
\[
u(t,x) = u(t_0,x_0) + A|x - x_0 - u(t_0,x_0)(t-t_0)|^{3/5} + B_2(t-t_0)|x - x_0 - u(t_0,x_0)(t-t_0)|^{1/5} + B_1(t-t_0) + O(1)(|t-t_0|^2, |x - x_0 - u(t_0,x_0)(t-t_0)|^{4/5}), \] for some constants $A, B_1, B_2$.

3.2 Families of perturbed solutions

We can construct several families of perturbations of a given solution of (3.16). The main goal of this subsection is to prove

Lemma 3.2. let $(u,v,q)$ be a smooth solution of the semilinear system (3.16), and let a point $(t_0, \xi_0) \in \mathbb{R}_+ \times \mathbb{R}$ be given.

\[(1)\] If $(v, v_\xi, v_{\xi\xi})(t_0, \xi_0) = (\pi, 0, 0)$, then there exists a 3-parameter family of smooth solutions $(u^\theta, v^\theta, q^\theta)$, depending smoothly on $\theta \in \mathbb{R}^3$, such that the following holds.

\[(i)\] When $\theta = 0 \in \mathbb{R}^3$, one recovers the original solution, namely $(u^0, v^0) = (u,v)$.

\[(ii)\] At the point $(t_0, \xi_0)$, when $\theta = 0$ one has 
\[
\text{rank } D_\theta(v^\theta, v^\theta_\xi, v^\theta_{\xi\xi}) = 3. \tag{3.22}
\]
(2) If \((v, v_\xi, v_t) = (\pi, 0, 0)\), then there exists a 3-parameter family of smooth solutions \((u^0, v^0, q^0)\), depending smoothly on \(\theta \in \mathbb{R}^3\), satisfying (i)-(ii) above, with (4.61) replaced by
\[
\text{rank } D\theta (v^0, v^0_t, v^0) = 3. \tag{3.23}
\]

**Proof.** Let \((u, v, q)\) be a smooth solution of the semilinear system. Given the point \((t_0, \xi_0)\).

By taking derivatives to the equation of \(v\) in the semilinear system (3.16), we can obtain
\[
\frac{\partial}{\partial t} v(t, \xi) = (2uu \xi - P) (1 + \cos v) - (u^2 - P + \frac{1}{2}) \sin v v \xi 
\tag{3.24}
\]
\[
\frac{\partial}{\partial t} v_t(t, \xi) = (-2uP_x - P_1) (1 + \cos v) - (u^2 - P + \frac{1}{2}) \sin v (u^2 - P) (1 + \cos v) - \sin^2 \frac{v}{2} 
\tag{3.25}
\]
\[
\frac{\partial}{\partial t} v_{\xi}(t, \xi) = (2uu \xi + 2u_\xi - P_\xi) (1 + \cos v) - 2(2uu \xi - P_\xi) \sin v v \xi 
\tag{3.26}
\]
\[
\frac{\partial}{\partial t} q(t, \xi) = (u^2 + \frac{1}{2} - P) \sin v \cdot q_x + (u^2 + \frac{1}{2} - P) \cos v v q + (2uu \xi - P x \xi) \sin v \cdot q 
\tag{3.27}
\]

with
\[
u_\xi = \frac{1}{2} \sin v q, \quad u_\xi = \frac{1}{2} \sin v q + \frac{1}{2} \cos v q v \xi. \tag{3.28}
\]

We now construct families \((\bar{u}^\theta, \bar{v}^\theta, \bar{q}^\theta)\) of perturbations of the initial data as
\[
\bar{u}^\theta(\xi) = \bar{u}(\xi) + \sum_{i=1,2,3} \theta_i U_i(\xi) 
\tag{3.29}
\]
\[
\bar{v}^\theta(\xi) = \bar{v}(\xi) + \sum_{i=1,2,3} \theta_i V_i(\xi) 
\tag{3.30}
\]
\[
\bar{q}^\theta(\xi) = \bar{q}(\xi) + \sum_{i=1,2,3} \theta_i Q_i(\xi) 
\tag{3.31}
\]

System (3.16) together with (3.24), (3.27) and (3.26) forms a complete system.

Next, the following technical lemma will be used to get the rank which we desired.

**Lemma 3.3.** Consider an ODE system
\[
\frac{d}{dt} u^\varepsilon = f(u^\varepsilon), \quad u^\varepsilon(0) = u_0 + \varepsilon_1 v_1 + \ldots + \varepsilon_m v_m, 
\tag{3.32}
\]
where \(u^\varepsilon(t) : \mathbb{R} \to \mathbb{R}^n\). \(f\) is a Lipschitz function. The system is well-posed in \([0, T]\). Assume the matrix
\[
D_\varepsilon u_0^\varepsilon = (v_1, v_2, \ldots, v_m) \in \mathbb{R}^{n \times m}, 
\tag{3.33}
\]
and the rank of this matrix is
\[
\text{rank}(D_\varepsilon u_0^\varepsilon) = k. \tag{3.34}
\]
Then for any \(t \in [0, T]\), \(\text{rank}(D_\varepsilon u^\varepsilon(t)) = k\).
\textbf{Proof.} Take derivative to $\varepsilon$, 
\[ \frac{d}{dt} D_\varepsilon u^\varepsilon = D_\varepsilon u^\varepsilon \cdot \nabla f(u^\varepsilon). \] 
(3.35)

Then we have 
\[ D_\varepsilon u^\varepsilon(t) = D_\varepsilon u^\varepsilon(0) \cdot \exp \left( \int_0^t \nabla f(u^\varepsilon(s))ds \right). \] 
(3.36)

Since $f$ is Lipschitz, $\nabla f$ is finite. So $\exp \left( \int_0^t \nabla f(u^\varepsilon(s))ds \right)$ is of full rank, i.e. rank $n$. Because $D_\varepsilon u^\varepsilon(0)$ has rank $k$, $D_\varepsilon u^\varepsilon(t)$ has rank $k$. □

Now we prove Lemma 3.2. Combining (3.16)(3.24)(3.26), we obtain an ODE system of dimension 5. That is

\[ \frac{\partial}{\partial t} \begin{pmatrix} u \\ v \\ q \\ v_\xi \\ v_{\xi\xi} \end{pmatrix} = \begin{pmatrix} -P_x \\ (u^2 - P)(1 + \cos v) - \sin^2 v/2 \\ (u^2 + \frac{1}{2} - P) \sin v \cdot q \\ (2u_\xi v - P_r)(1 + \cos v) - (u^2 - P + \frac{1}{2}) \sin vv_\xi \\ (2u_\xi v + 2u^2 - P_r)(1 + \cos v) - 2(2u_\xi v - P_r) \sin vv_\xi - \\ (u^2 - P + \frac{1}{2})(\sin vv_{\xi\xi} + \cos vv_\xi^2) \end{pmatrix}, \] 
(3.37)

We construct a family solution $(u^\theta, v^\theta, q^\theta)$ to (3.37) of perturbations of the initial data as in (3.29)-(3.31). Take derivative to $\theta$,

\[ \frac{\partial}{\partial t} \begin{pmatrix} D_\theta u^\theta_1 \\ D_\theta v^\theta_1 \\ D_\theta q^\theta_1 \\ D_\theta v_\xi^\theta_1 \\ D_\theta v_{\xi\xi}^\theta_1 \end{pmatrix} = \begin{pmatrix} D_\theta f_1^\theta \\ D_\theta f_2^\theta \\ D_\theta f_3^\theta \\ D_\theta f_4^\theta \\ D_\theta f_5^\theta \end{pmatrix}, \] 
(3.38)

where $f_1^\theta, ..., f_5^\theta$ are the perturbation of the right-hand-side of (3.38).

Then we can get

\[ \frac{\partial}{\partial t} \begin{pmatrix} D_\theta u^\theta_1 \\ D_\theta v^\theta_1 \\ D_\theta q^\theta_1 \\ D_\theta v_\xi^\theta_1 \\ D_\theta v_{\xi\xi}^\theta_1 \end{pmatrix} = \begin{pmatrix} D_{u_1} f_1^\theta & D_{u_2} f_1^\theta & D_{u_3} f_1^\theta & D_{u_4} f_1^\theta & D_{u_5} f_1^\theta \\ D_{v_1} f_2^\theta & D_{v_2} f_2^\theta & D_{v_3} f_2^\theta & D_{v_4} f_2^\theta & D_{v_5} f_2^\theta \\ D_{q_1} f_3^\theta & D_{q_2} f_3^\theta & D_{q_3} f_3^\theta & D_{q_4} f_3^\theta & D_{q_5} f_3^\theta \\ D_{v_\xi_1} f_4^\theta & D_{v_\xi_2} f_4^\theta & D_{v_\xi_3} f_4^\theta & D_{v_\xi_4} f_4^\theta & D_{v_\xi_5} f_4^\theta \\ D_{v_{\xi\xi}_1} f_5^\theta & D_{v_{\xi\xi}_2} f_5^\theta & D_{v_{\xi\xi}_3} f_5^\theta & D_{v_{\xi\xi}_4} f_5^\theta & D_{v_{\xi\xi}_5} f_5^\theta \end{pmatrix} \cdot \begin{pmatrix} D_\theta u^\theta_1 \\ D_\theta v^\theta_1 \\ D_\theta q^\theta_1 \\ D_\theta v_\xi^\theta_1 \\ D_\theta v_{\xi\xi}^\theta_1 \end{pmatrix}. \] 
(3.39)

According to above lemma, we only need to prove the Lipschitz continuity of $f$. Because of the smoothness of $(u, v, P)$, we just need to consider the Lipschitz continuity of the nonlocal term $P$ and $P_x$. Recall

\[ P(t, \xi) = \frac{1}{2} \int_{-\infty}^{\infty} \exp \left\{ - \left| \int_\xi^{\xi'} \cos^2 \frac{v(t, \xi')}{2} \cdot q(t, \xi) d\xi' \right| \} \cdot \left[ u^2(t, \xi') \cos^2 \frac{v(t, \xi')}{2} + \frac{1}{2} \sin^2 \frac{v(t, \xi')}{2} \right] q(t, \xi') d\xi'. \] 
(3.40)

\[ P_x(t, \xi) = \frac{1}{2} \left( \int_\xi^{\infty} - \int_{-\infty}^{\xi} \right) \exp \left\{ - \left| \int_\xi^{\xi'} \cos^2 \frac{v(t, \xi')}{2} \cdot q(t, \xi) d\xi' \right| \} \] 
(3.41)

\[ \cdot \left[ u^2(t, \xi') \cos^2 \frac{v(t, \xi')}{2} + \frac{1}{2} \sin^2 \frac{v(t, \xi')}{2} \right] q(t, \xi') d\xi'. \]
Indeed, we have

\[
\left| \frac{\partial P}{\partial u}(t, \xi) \right| = \frac{1}{2} \left| \int_{-\infty}^{\infty} \exp \left\{ - \int_{\xi}^{\xi'} \cos^2 \frac{v(t, \xi)}{2} \cdot q(t, \xi') \, d\xi' \right\} \cdot \left[ 2u(t, \xi') \cos^2 \frac{v(t, \xi')}{2} \right] q(t, \xi') \, d\xi' \right|
\]

\[
\leq \int_{-\infty}^{\infty} \exp \{ -|x - x'| \} \cdot u(t, x) \, dx
\]

\[
\leq \left( \int_{-\infty}^{\infty} \exp \{ -2|x - x'| \} \, dx \right)^{1/2} \left( \int_{-\infty}^{\infty} u^2(t, x) \, dx \right)^{1/2}
\]

\[
\leq E(0)^{1/2}
\]

\[
\left| \frac{\partial P}{\partial v}(t, \xi) \right| \leq \frac{1}{2} \int_{-\infty}^{\infty} \exp \left\{ - \int_{\xi}^{\xi'} \cos^2 \frac{v(t, \xi)}{2} \cdot q(t, \xi') \, d\xi' \right\} \cdot \left| \int_{\xi}^{\xi'} \left| \sin \frac{v(t, \xi)}{2} \right| q(t, \xi') \, d\xi' \right|
\]

\[
\cdot \left[ u^2 \cos^2 \frac{v}{2} + \frac{1}{2} \sin^2 \frac{v}{2} \right] (t, \xi') q(t, \xi') \, d\xi'
\]

\[
+ \frac{1}{2} \int_{-\infty}^{\infty} \exp \left\{ - \int_{\xi}^{\xi'} \cos^2 \frac{v(t, \xi)}{2} \cdot q(t, \xi') \, d\xi' \right\} \cdot \left( \left| u^2(t, \xi') \frac{\sin v(t, \xi)}{2} \right| + \left| \frac{\sin v(t, \xi')}{4} \right| \right) q(t, \xi') \, d\xi'
\]

\[
\leq \frac{1}{2} \int_{-\infty}^{\infty} e^{-|x - x'|} \int_{x'}^{x} |u_x(t, y)| \, dy \left| u^2(t, x') \right| + \frac{1}{2} u_x^2(t, x') \, dx' + \frac{1}{2} \int_{-\infty}^{\infty} e^{-|x - x'|} \left| u_x^2(t, x') \right| + \frac{1}{2} u^2(t, x') \, dx'
\]

\[
\leq \frac{1}{2} \int_{-\infty}^{\infty} e^{-|x - x'|} |E(0)|^{1/2} |x - x'|^{1/2} |u^2(t, x')| + \frac{1}{2} u_x^2(t, x') \, dx'
\]

\[
+ \frac{1}{2} \int_{-\infty}^{\infty} e^{-|x - x'|} |E(0)| + \frac{1}{2} |u_x(t, x')| \, dx'
\]

\[
\leq \frac{\sqrt{2}}{4} e^{-1/2} |E(0)|^{3/2} + \frac{1}{2} |E(0)| + \frac{1}{2} |E(0)|^{1/2}
\]  \hspace{1cm} (3.43)

Similarly, considering the fact that \( q \) is bounded above and below, we obtain the boundedness of \( \left| \frac{\partial P}{\partial q} \right|, \left| \frac{\partial P}{\partial \epsilon} \right|, \left| \frac{\partial P}{\partial \xi} \right|, \left| \frac{\partial P}{\partial \xi'} \right| \).

By choosing suitable perturbation \( V_i(i = 1, 2, 3) \) we can make

\[
\text{rank}D_{\theta} \left( \begin{array}{c}
v \\
v_{\xi} \\
v_{\xi\xi}
\end{array} \right) = 3,
\]  \hspace{1cm} (3.44)

when \( \theta = 0 \).

Since the system (4.25) together with (3.24) and (3.25) forms a complete system, by choosing suitable perturbation \( V_i(i = 1, 2, 3) \) we can make

\[
\text{rank}D_{\theta} \left( \begin{array}{c}
v \\
v_{\xi} \\
v_{t}
\end{array} \right) = 3,
\]  \hspace{1cm} (3.45)

when \( \theta = 0 \).
3.3 Generic solutions of the semilinear system

In this subsection we study smooth solutions to the semilinear system (3.16), determining the generic structure of the level sets \( \{ v(t, \xi) = \pi \} \).

Lemma 3.4. Consider the compact domain of form

\[ \Gamma := \{(t, \xi); 0 \leq t \leq T, |\xi| \leq M \}. \tag{3.46} \]

Call \( S \) the family of all \( C^2 \) solutions to the semilinear system, with \( q > 0 \) for all \((t, \xi) \in \mathbb{R}_+ \times \mathbb{R} \). Moreover call \( S' \subset S \) the subfamily of all solutions \((u, v, q)\) such that for \((t, \xi) \in \Gamma\), none of the following values is attained:

\[ (v, v_\xi, v_{\xi\xi}) = (\pi, 0, 0), \quad (v, v_\xi, v_i) = (\pi, 0, 0). \tag{3.47} \]

Then \( S' \) is a relatively open and dense subset of \( S \), in the topology induced by \( C^2(\Gamma) \).

Proof. 1. Denote \( S_1 \) to be the set of solutions for which \((v, v_\xi, v_{\xi\xi}) = (\pi, 0, 0)\) is not attained on \( \Gamma \), and \( S_2 \) the set of solutions for which \((v, v_\xi, v_i) = (\pi, 0, 0)\) is not attained on \( \Gamma \). Their intersection is \( S' = S_1 \cap S_2 \). Since \( \Gamma \) is a compact domain, each \( S_i \) is a relatively open subset of \( S \), in the topology of \( C^2(\Gamma) \). We then will prove that \( S_1 \) and \( S_2 \) are dense in \( S \).

2. Let \((u, v, q)\) be any \( C^2 \) solution of the semilinear system, with \( q > 0 \).

Given any point \((t_0, \xi_0) \in \Gamma\), two cases can occur.

CASE 1: \((v, v_\xi, v_{\xi\xi})(t_0, \xi_0) \neq (\pi, 0, 0)\). In this case, by continuity, there is an open neighborhood \( \mathcal{N} \) of \((t_0, \xi_0)\) in \( t, \xi \)-plane where \((v, v_\xi, v_{\xi\xi}) \neq (\pi, 0, 0)\).

CASE 2: \((v, v_\xi, v_{\xi\xi})(t_0, \xi_0) = (\pi, 0, 0)\). By Lemma 3.2 we can find a 3-parameter family of solutions \((u^\theta, v^\theta, q^\theta)\) such that the 3 \( \times \) 3 Jacobian matrix of the map

\[ (\theta_1, \theta_2, \theta_3) \mapsto (v^\theta(t, \xi), v_\xi^\theta(t, \xi), v_{\xi\xi}^\theta(t, \xi)) \tag{3.48} \]

has rank 3 at the point \((t_0, \xi_0)\), when \( \theta = 0 \). By continuity, this matrix still has rank 3 on a neighborhood \( \mathcal{N}' \) of \((t_0, \xi_0)\), for \( \theta \) small enough.

Now we choose finitely many points \((t_i, \xi_i), i = 1, \ldots, n\), such that the corresponding open neighborhoods \( \mathcal{N}_{(t_i, \xi_i)} \) cover the compact set \( \Gamma \). Call \( n_T \) the cardinality of the set of indices

\[ \mathcal{I} := \{ i; (v, v_\xi, v_{\xi\xi})(t_i, \xi_i) = (\pi, 0, 0) \} \tag{3.49} \]

so that CASE 2 applies, and set \( N = 3n_T \).

3. Let \( \Omega \supset \Gamma \) be an open set contained in the union of the neighborhoods \( \mathcal{N}_{(t_i, \xi_i)} \), and call \( B_{\varepsilon} := \{ \theta \in \mathbb{R}^N; |\theta| < \varepsilon \} \) the open ball of radius \( \varepsilon \) in \( \mathbb{R}^N \). We shall construct a family \((u^\theta, v^\theta, q^\theta)\) of smooth solutions to the semilinear system, such that the map

\[ (t, \xi, \theta) \mapsto (v^\theta(t, \xi), v_\xi^\theta(t, \xi), v_{\xi\xi}^\theta(t, \xi)) \tag{3.50} \]

from \( \Omega \times B_{\varepsilon} \) into \( \mathbb{R}^3 \) has \((\pi, 0, 0)\) as a regular value. Toward this goal, we need to combine perturbations based at possibly different points \((t_i, \xi_i)\) into a single \( N \)-parameter family of perturbed solutions. Let \((u, v, q)(t, \xi)\) be a solution to the semilinear system. For each \( k = 1, \ldots, N \), let a point \((t_k, \xi_k)\) be given, together with a number \( U_k \in \mathbb{R} \) and functions \( V_k, Q_k \in C^\infty(\mathbb{R}) \). By the previous analysis, a 1-parameter family of perturbed solutions to the semilinear system is then determined as follows. For \( |\varepsilon| < \varepsilon_k \) sufficiently small, let

\[ (u^\varepsilon, v^\varepsilon, q^\varepsilon) := \Psi_k^\varepsilon(u, v, q) \tag{3.51} \]
be the unique solution of the perturbed semilinear system. Given \((\theta_1, \ldots, \theta_N)\), a perturbation of the original solution \((u, v, q)\) is defined as the composition of \(N\) perturbations:

\[
(u^\theta, v^\theta, q^\theta) := \Psi_N^{\theta_N} \circ \cdots \circ \Psi_1^{\theta_1}(u, v, q).
\]

4. At every point \((t, \xi)\) where \(i \in I\), using Lemma 3.2, we can obtain a \(3\)-parameter families of perturbed solutions so that Jacobian matrix of (3.48) has rank \(3\) on neighborhood \(N\) of point \((t_0, \xi_0)\), for \(\theta\) sufficiently small.

Now we can choose finitely many points \((t_i, \xi_i)\), \(i = 1, \ldots, n\), such that the corresponding open neighborhood \(N_{(t_i, \xi_i)}\) cover the compact set \(\Gamma\). So we obtain an \(N\)-parameter family of solutions s.t. the value \((\pi, 0, 0)\) is a regular value for the map (3.48) from \(\Omega \times B_\varepsilon\) into \(\mathbb{R}^3\). By transversality theorem, for a.e. \(\theta\), the map \((t, \xi) \mapsto (v^\theta(t, \xi), v^\theta_3(t, \xi), v^\theta_\zeta(t, \xi))\) from \(\Omega\) into \(\mathbb{R}^3\) is transverse to \(\{(\pi, 0, 0)\}\). For convenience, denote this map as \(F\). By definition of transversality, either

\[
F(t, \xi) \neq (\pi, 0, 0)
\]

or

\[
F(t, \xi) = (\pi, 0, 0) \quad \text{and} \quad T_{(\pi, 0, 0)}\mathbb{R}^3 = (dF)(t, \xi)(T_{(t, \xi)}\Omega).
\]

Since \(\Omega\) is only two-dimensional, \(T_{(\pi, 0, 0)}\mathbb{R}^3 = (dF)(t, \xi)(T_{(t, \xi)}\Omega)\) cannot happen. So the only choice is \(F(t, \xi) \neq (\pi, 0, 0)\).

This leads to for a.e. \(\theta\), the corresponding solution \((u^\theta, v^\theta, q^\theta)\) has property that \((v^\theta, v^\theta_3, v^\theta_\zeta) \neq (\pi, 0, 0)\), for all \((X, Y) \in \Gamma\). This proves that the set \(S_1\) of solutions is dense in \(S\).

5. Repeating the above construction, we obtain that \(S_2\) is a relatively open, dense subset of \(S\). So the intersection \(S'\) is a relatively open, dense subset of \(S\).

### 3.4 Proof of Theorem 3.3.

Consider the space \(S = C^3(\mathbb{R}) \cap H^1(\mathbb{R})\), with norm

\[
\|u_0\|_S := \|u_0\|_{C^3} + \|u_0\|_{H^1}.
\]

Given initial data \(\hat{u}_0 \in S\), consider the open ball

\[
B_\delta := \{u_0 \in S; \|u_0 - \hat{u}_0\| < \delta\}.
\]

Theorem 3.3 will be proved by showing that, for any \(\hat{u}_0 \in S\), there exists a radius \(\delta > 0\) and an open dense subset \(D \subset B_\delta\) such that for every initial data \(u_0 \in D\), the conservative solution \(u = u(t, x)\) is of class \(C^2\) in the complement of finitely many characteristic curves \(\gamma_i\), within the domain \([0, T] \times \mathbb{R}\).

1. (Construction of \(D\)) If \(u_{0,x} = O(x)\), then the blow-up time of \(u_x\) along the characteristics \(y = y(t, x)\) is of order \(\frac{1}{2}\). Since \(u_{0,x} \in S\), we know \(u_{0,x}(x) \to 0\) as \(x \to \infty\). So we can take \(r > 0\) big enough such that when \(|x| > r\), \(|u_x| < \frac{1}{r}\). This means the singularity of \(u(t, x)\) in set \([0, T] \times \mathbb{R}\) only appears in the compact set \(\mathfrak{A} := [0, T] \times [-r - \|u\|_{\infty} T, r + \|u\|_{\infty} T]\), where \(\|u\|_{\infty} := \max\{u(t, x), (t, x) \in [0, T] \times \mathbb{R}\}\). In \((t, \xi)\) plane, we can take a domain \(\Gamma\) such that \(\mathfrak{A} \subset \Lambda(\Gamma)\), where \(\Lambda\) is the map from \((t, \xi)\) to \((t, x(t, \xi))\).

The subset \(D \subset B_\delta\) is defined as follows. \(u_0 \in D\) if \(u_0 \in B_\delta\) and for the corresponding solution \((u, v, q)\) of (3.16) with initial data, the values (4.58) are never attained for \((t, x) \in \mathfrak{A}\).

2. (\(D\) is open.) Take a sequence of initial data \(u_0^n \notin D\), and converges to \(u_0\). By definition of \(D\), there is point \((t^n, \xi^n)\) satisfying

\[
(v^n, v^n_3, v^n_\zeta(t^n, \xi^n)) = (\pi, 0, 0), \quad (t^n, x^n(t^n, \xi^n)) \in \mathfrak{A}
\]
for all $\nu \geq 1$. Since the domain $\mathcal{R}$ is compact, by taking a subsequence, we can assume $(t^\nu, \xi^\nu)$ converges to some point $(t, \xi)$ and

$$(v, v_\xi, v_\xi)(t, \xi) = (\pi, 0, 0), \quad (t, x(t, \xi)) \in \mathcal{R},$$

which implies $u_0 \notin D$. The other case $(v, v_\xi, v_t) = (\pi, 0, 0)$ can be proved in the same way. So $D$ is open.

3. $(D$ is dense.) Given $u_0 \in B_d$, by a small perturbation, we can assume $u_0 \in C^\infty$.

By Lemma 3.4, we can construct a sequence of solutions $(u^\nu, v^\nu, q^\nu)$ of (3.16) such that:

(i) For every bounded set $\Omega \subset \mathbb{R}^2$ and $k \geq 1$,

$$\lim_{\nu \to \infty} \left\| (u^\nu - u, v^\nu - v, q^\nu - q, x^\nu - x) \right\|_{C^k(\Omega)} = 0.$$  (3.56)

(ii) For every $\nu \geq 1$, the values in (3.47) are never attained for any $(t, \xi) \in \Gamma$.

Consider the sequence of solution $u^\nu(t, x)$ with graph

$$\left\{ (u^\nu(t, \xi), v^\nu(t, \xi), x^\nu(t, \xi)); (t, \xi) \in \mathbb{R}^2 \right\} \subset \mathbb{R}^3.$$  (3.57)

The corresponding sequence of initial data satisfies

$$\left\| u^\nu(0, \cdot) - u_0 \right\|_{C^k(I)} \to 0, \quad \text{as } \nu \to \infty,$$  (3.58)

for every bounded set $I$.

However, this is not enough. We still need the convergence for the far field. To this end, we modify the sequence a little bit. Define a cutoff function $\varphi \in C^\infty_c$,

$$\varphi(x) = 1, \quad \text{if } |x| \leq \varrho,$$

$$\varphi(x) = 0, \quad \text{if } |x| \geq \varrho + 1,$$  (3.59)

with $\varrho \gg r + \|u\|_\infty T$ sufficiently large. For each $\nu \geq 1$, consider the initial data

$$\vec{u}_0^\nu := \varphi u_0^\nu + (1 - \varphi)u_0.$$  (3.60)

We have

$$\lim_{\nu \to \infty} \left\| \vec{u}_0^\nu - u_0 \right\|_S = 0.$$  (3.61)

Moreover, if $\varrho > 0$ large enough, we have

$$\vec{u}^\nu(t, x) = u^\nu(t, x) \quad \text{for all } (t, x) \in \mathcal{R},$$  (3.62)

while $\vec{u}^\nu$ is $C^2$ on the outer domain. This implies $\vec{u}_0^\nu \in D$ for all $\nu \geq 1$ sufficiently large, which leads to $D$ is dense on $B_d$.

4. $(u$ is piecewise $C^2$.) By previous argument, we know that $u$ is $C^2$ on the outer domain $\{(t, x)|0 \leq t \leq T, |x| > \|u\|_\infty T\}$. So it suffices to study the singularity of $u$ on the inner domain $\mathcal{R}$. Consider a point $(t_0, \xi_0) \in \Gamma$. Following two cases can happen.

(I) $v(t_0, \xi_0) \neq \pi$. By (3.17),

$$0 < x_\xi < \infty.$$
So the map \((t, \xi) \mapsto (t, x)\) is locally invertible in a neighborhood of \((t_0, \xi_0)\). We then conclude that the function \(u\) is \(C^2\) in a neighborhood of the point \((t_0, x_0(t_0, \xi_0))\).

(II) \(v(t_0, \xi_0) = \pi\). Then by (3.47), \(v_t(t_0, \xi_0) \neq 0\) or \(v_\xi(t_0, \xi_0) \neq 0\). By the implicit function theorem, we can conclude that the set \(S^v := \{(t, \xi) \in \Gamma; v(t, \xi) = \pi\}\) is the union of finitely many \(C^2\) curves.

![Figure 1: The singularity curves of \(\{v = \pi\}\) in \(t-\xi\) plane and \(t-x\) plane.](image)

### 3.5 Generic singularity behavior

For smooth data \(u_0 \in C^\infty(\mathbb{R})\), the solution \((t, \xi) \mapsto (x, t, u, v, q)(t, \xi)\) of the semilinear system (3.16), with initial data as in (3.18), remains smooth on the entire \(t-\xi\) plane. Yet, the solution \(u = u(x, t)\) of (1.4) can have singularities because the coordinate change \(: (\xi, t) \mapsto (x, t)\) is not smoothly invertible. By definitions, its Jacobian matrix is computed by

\[
\begin{pmatrix}
x_\xi & x_t \\
x_t & t_\xi & t_t
\end{pmatrix} = \begin{pmatrix}
q \cos^2 \frac{v}{2} & u \\
0 & 1
\end{pmatrix}.
\]

We recall that \(q\) remain uniformly positive and uniformly bounded on compact subsets of the \(t-\xi\) plane. By Lemma 1, at a point \((t_0, \xi_0)\) where \(v \neq \pi\), this matrix is invertible, having a strictly positive determinant. The function \(u = u(x, t)\) considered at (3.16) is thus smooth on a neighborhood of the point

\[
(x_0, t_0) = (x(t_0, \xi_0), t(t_0, \xi_0)).
\]

To study the set of points \(x-t\) plane where \(u\) is singular, we thus need to look at points where \(v = \pi\).

Our main result provides a detailed description of the solution \(u = u(x, t)\) in a neighborhood of each one of these singular points. For simplicity, we shall assume that the initial data \(u_0\) are smooth, so we shall not need to count how many derivatives are actually used to derive the Taylor approximations.

**Theorem 3.5** (Singular behavior). Consider generic initial data \(u_0 \in D\) as in Theorem 3.3, with \(u_0 \in C^\infty(\mathbb{R})\). Call \((u, v, q, x, t)\) the corresponding solution of the semilinear system (3.16) and let \(u = u(x, t)\) be the solution to the original equation (3.1). Consider a singular point \(P = (t_0, \xi_0)\) where \(v = \pi\), and set \((x_0, t_0) = (x(t_0, \xi_0), t(t_0, \xi_0))\). Generically, at the singular point, \(u\) has following parametric expression.
(i) If $P$ is a point of Type I: $v = \pi, v_\xi \neq 0$,
\[
u(t, x) = u(t_0, x_0) + A[x - x_0 - u(t_0, x_0)(t - t_0)] + B_1(t - t_0)(x - x_0 - u(t_0, x_0)(t - t_0))^{1/3} + B_1(t - t_0) + O(1)(|t - t_0|^2, |x - x_0 - u(t_0, x_0)(t - t_0)|^{1/3}),
\]
for some constants $A, B_1, B_2$.

(ii) If $P$ is a point of Type II: $v = \pi, v_\xi = 0, v_\eta \neq 0, v_t \neq 0$,
\[
u(t, x) = u(t_0, x_0) + A[x - x_0 - u(t_0, x_0)(t - t_0)]^{3/5} + B_2(t - t_0)(x - x_0 - u(t_0, x_0)(t - t_0))^{1/5} + B_1(t - t_0) + O(1)(|t - t_0|^2, |x - x_0 - u(t_0, x_0)(t - t_0)|^{4/5}),
\]
for some constants $A, B_1, B_2$.

**Proof.** 1. Let $P$ is a point of Type I. Recall (3.17) and (3.15), then we have
\[
u_\xi = u_x x_\xi = u_x \frac{q}{1 + u_x^2} = \frac{1}{2} q \sin v.
\]
In a similar way, we obtain
\[
u_{\xi \xi} = \frac{1}{2} q \sin v + \frac{1}{2} q \cos vv_{\xi}, \quad \nu_{\xi t} = \frac{1}{2} q t \sin v + \frac{1}{2} q \cos vv_t.
\]
At a singular point $P = (t_0, \xi_0)$ where $v = \pi$ we get
\[
u_\xi(t_0, \xi_0) = 0, \quad \nu_{\xi \xi}(t_0, \xi_0) \neq 0, \quad \nu_{\xi \xi}(t_0, \xi_0) = \frac{1}{2} q(t_0, \xi_0) \neq 0.
\]
So we have the Taylor approximations of $u$ at the singular point $P = (t_0, \xi_0)$:
\[
u(t, \xi) = u(t_0, \xi_0) + B_1(t - t_0) + B_2(\xi - \xi_0)(t - t_0) + B_3(\xi - \xi_0)^2 + O(1)(|t - t_0|^2, |\xi - \xi_0|^3).
\]
Recall (3.17) and (3.15), then we get
\[
u_\xi = \frac{q}{1 + u_x^2} = q \cos^2 \frac{v}{2}
\]
In a similar way, we have
\[
u_{\xi \xi} = q_\xi \cos^2 \frac{v}{2} - \frac{1}{2} q \sin vv_\xi, \quad \nu_{\xi t} = q_\xi \cos^2 \frac{v}{2} - \frac{1}{2} q \sin vv_t,
\]
\[
u_{\xi \xi \xi} = q_{\xi \xi} \cos^2 \frac{v}{2} - q_\xi \sin vv_\xi - \frac{1}{2} q \cos vv_\xi^2 - \frac{1}{2} q \sin vv_\xi.
\]
At a singular point $P = (t_0, \xi_0)$ where $v = \pi$ we get
\[
u_\xi = \nu_{\xi \xi} = \nu_{\xi t} = 0,
\]
and
\[
u_{\xi \xi \xi}(t_0, \xi_0) \neq 0.
\]
So we have the Taylor approximations of $x$ at the singular point $P = (t_0, \xi_0)$:
\[
u(t, \xi) = x(t_0, \xi_0) + u(t_0, \xi_0)(t - t_0) + A_1(\xi - \xi_0)^3 + O(1)(|t - t_0|^2, |\xi - \xi_0|^4).
\]
We combine (3.71) and (3.77) to yield (3.66).
2. If $P$ is a point of Type II. At point $P$, we have
\[ v = \pi, \quad v_x = 0, \quad v_{xx} \neq 0, \quad v_t \neq 0, \]
which imply
\[ u_x(t_0, \xi_0) = 0, \quad u_{xx}(t_0, \xi_0) = 0, \quad u_{xt}(t_0, \xi_0) = 0. \] (3.79)
\[ u_{xxx}(t_0, \xi_0) = \frac{1}{2} q(t_0, \xi_0) v_x(t_0, \xi_0) v_{xx}(t_0, \xi_0) \neq 0. \] (3.80)

So the Taylor approximation of $u$ is
\[ u(t, \xi) = u(t_0, \xi_0) + B_1(t - t_0) + B_2(\xi - \xi_0)(t - t_0) + B_3(\xi - \xi_0)^3 + O(1)(|t - t_0|^2 + |\xi - \xi_0|^4). \] (3.81)

At point $P$,
\[ x_x = 0, \quad x_{xx} = x_{x} = 0, \quad x_{xxx} = 0, \quad x_{xxxx} = 0, \]
\[ x_{xxxxx} = \frac{3}{2} q(t_0, \xi_0) v_x(t_0, \xi_0) \neq 0. \] (3.82)
This leads to
\[ x(t, \xi) = x(t_0, \xi_0) + u(t_0, \xi_0)(t - t_0) + A_1(\xi - \xi_0)^5 + O(1)(|t - t_0|^2 + |\xi - \xi_0|^6). \] (3.83)

Then we can conclude (3.67) from (3.81) and (3.84).

4 Two-component Camassa-Holm system

4.1 Basic definitions and results for 2CH

Recall the two-component Camassa-Holm system
\[ \begin{cases} u_t + \left(\frac{u^2}{2}\right)_x + P_x = 0, \\ \rho_t + (u\rho)_x = 0, \end{cases} \] (4.1)
where the nonlocal source term $P$ is defined as a convolution:
\[ P = \frac{1}{2} e^{-|x|} * \left( u^2 + \left(\frac{u_x^2 + \rho^2}{2}\right) \right). \] (4.2)

The initial data is specified as
\[ u(0, x) = u_0(x), \quad \rho(0, x) = \rho_0(x). \] (4.3)

To make sense of the source term $P$, at each time $t$ we require that the function $u(t, \cdot)$ lies in the space $H^1(\mathbb{R})$ of absolutely continuous functions $u \in L^2(\mathbb{R})$ with derivative $u_x \in L^2(\mathbb{R})$, endowed with the norm
\[ \|u\|_{H^1} = \left( \int_{\mathbb{R}} [u^2(x) + u_x^2(x)] \, dx \right)^{1/2}. \]

For $u \in H^1(\mathbb{R})$ and $\rho \in L^2(\mathbb{R})$, Young’s inequality ensures that
\[ P = (1 - \partial_x^2)^{-1} \left( u^2 + \frac{u_x^2 + \rho^2}{2} \right) \in H^1(\mathbb{R}). \]

For future use we record the following inequalities, valid for any function $u \in H^1(\mathbb{R})$:
\[ \|u\|_{L^\infty} \leq \|u\|_{H^1}, \] (4.4)
\[ \|P\|_{L^\infty} \leq \frac{1}{2} e^{-|x|} \|u\|^2 + \frac{u_x^2 + \rho^2}{2} \|_{L^1} \leq \frac{1}{2} \left( \|u\|_{H^1}^2 + \|\rho\|_{L^2}^2 \right), \quad (4.5) \]

\[ \|P\|_{L^2} \leq \frac{1}{2} e^{-|x|} \|u\|^2 + \frac{u_x^2 + \rho^2}{2} \|_{L^2} \leq \frac{1}{\sqrt{2}} \left( \|u\|_{H^1}^2 + \|\rho\|_{L^2}^2 \right). \quad (4.6) \]

**Definition 4.1.** For any \( T > 0 \), by a solution of the Cauchy problem (4.1)-(4.3) on \([0, T]\) we mean a Hölder continuous function \( u = u(t, x) \) defined on \([0, T] \times \mathbb{R}\) and \( \rho \in L^\infty(0, T; L^2(\mathbb{R})) \) with the following properties. At each fixed \( t \) we have \( u(t, \cdot) \in H^1(\mathbb{R}) \). Moreover, the map \( t \mapsto u(t, \cdot) \) is Lipschitz continuous from \([0, T]\) into \( L^2(\mathbb{R}) \) and satisfies the initial condition (4.3) together with

\[ \frac{d}{dt} u = -u u_x - P_x \quad (4.7) \]

for a.e. \( t \). Here (4.7) is understood as an equality between functions in \( L^2(\mathbb{R}) \). And

\[ \frac{d}{dt} \rho = -(\rho u)_x. \quad (4.8) \]

Here (4.8) is understood in the sense of distribution.

As shown in [10, 13], as soon as the gradient of a solution blows up, uniqueness is lost, in general. To single out a unique solution, some additional conditions are needed.

For smooth solutions, differentiating (4.1) w.r.t. \( x \) one obtains

\[ u_{xx} + (u u_x)_x = \left( u^2 + \frac{u_x^2 + \rho^2}{2} \right) - P. \quad (4.9) \]

Multiplying (4.1) by \( u \), (4.1) by \( \rho \) and (4.9) by \( u_x \), we obtain the three conservation laws with source term

\[ \left( \frac{u^2}{2} \right)_t + \left( \frac{u^3}{3} + u P \right)_x = u_x P, \quad (4.10) \]

\[ \left( \frac{u_x^2}{2} \right)_t + \left( \frac{u u_x^2}{2} - \frac{u^3}{3} \right)_x = \frac{\rho^2}{2} u_x - u_x P, \quad (4.11) \]

\[ \left( \frac{\rho^2}{2} \right)_t + \left( \frac{\rho^2}{2} u_x \right)_x = -\frac{\rho^2}{2} u_x. \quad (4.12) \]

Summing (4.10) and (4.12), and integrating w.r.t. \( x \), we see that for smooth solutions the total energy

\[ E(t) = \int_{\mathbb{R}} \left( u^2(t, x) + u_x^2(t, x) + \rho^2(t, x) \right) dx \quad (4.13) \]

is constant in time.

**Definition 4.2.** A solution \( u = u(t, x) \) is conservative if \( w = u_x^2 + \rho^2 \) provides a distributional solution to the balance law (4.14),

\[ w_t + (u w)_x = 2(u^2 - P) u_x \quad (4.14) \]

namely

\[ \int_0^\infty \int [w \varphi_t + u w \varphi_x + 2(u^2 - P) u_x \varphi] \, dx \, dt + \int \left( u_{0,x}^2(x) + \rho_0^2(x) \right) \varphi(0, x) \, dx = 0 \quad (4.15) \]

for every test function \( \varphi \in C^1_c(\mathbb{R}^2) \).
The main result proved in [14, 17, 18], on the global existence of conservative solutions can be stated as follows.

**Theorem 4.1.** For any initial data \( u_0 \in H^1(\mathbb{R}), \rho_0 \in L^2(\mathbb{R}) \) the 2-component Camassa-Holm equation has a global conservative solution \( u = u(t, x), \rho = \rho(t, x) \). More precisely, there exists a family of Radon measures \( \{\mu(t) \mid t \in \mathbb{R}\} \), depending continuously on time w.r.t. the topology of weak convergence of measures, such that the following properties hold.

(i) The functions \( u \) and \( \rho \) provide a solution to the Cauchy problem (4.1)-(4.3) in the sense of Definition 1.

(ii) There exists a null set \( N \subset \mathbb{R} \) with \( \text{meas}(N) = 0 \) such that for every \( t \notin N \) the measure \( \mu(t) \) is absolutely continuous and has density \( u_x^2(t, \cdot) + \rho^2(t, \cdot) \) w.r.t. Lebesgue measure.

(iii) The family \( \{\mu(t) \mid t \in \mathbb{R}\} \) provides a measure-valued solution \( w \) to the linear transport equation with source

\[
   w_t + (uw)_x = 2(u^2 - P)u_x. \tag{4.16}
\]

At a time \( t \in N \) the measure \( \mu(t) \) has a nontrivial singular part. For a conservative solution \( u \) which is not smooth, in general we only know that the energy \( E \) in (4.13) coincides a.e. with a constant. Namely,

\[
   E(t) = E(0) \quad \text{for} \quad t \notin N, \quad E(t) < E(0) \quad \text{for} \quad t \in N.
\]

In [20] the author proved the uniqueness of the above solution.

**Theorem 4.2.** For any initial data \( u_0 \in H^1(\mathbb{R}) \) and \( \rho_0 \in L^2(\mathbb{R}) \), the Cauchy problem (1.1)-(1.3) has a unique conservative solution.

Our main result is

**Theorem 4.3** (Generic regularity). For initial data \( \rho(0, x) = \rho_0(x) \in L^2(\mathbb{R}) \cap C^1(\mathbb{R}), u(0, x) = u_0(x) \in H^1(\mathbb{R}) \). Assume \( \rho_0 \) has \( n \) zero points \( \{x_1, \ldots, x_n\} \). \( \rho_0(x) \) is in \( C^2 \) in each interval \( I_i := (x_i, x_{i+1}), u_0(x) \) is in \( C^3 \) in each interval \( (x_i, x_{i+1}) \), \( i = 0, \ldots, n \) \( (x_0 = -\infty, x_{n+1} = +\infty) \). Then the weak solution \( \rho \) is twice continuously differentiable and \( u \) is three times continuously differentiable in the complement of finitely many isolated points within the domain \([0, T] \times \mathbb{R}\).

Comparing with the corresponding result to CH (see Theorem 3.3) where the singularity generically happens at finitely many characteristic curves, we improve the regularity of the solution to 2CH dramatically by considering the generic solutions.

In fact, above choice of initial data is generic in the following sense.

**Corollary 4.1** (Generic regularity). For any \( T > 0 \) fixed, there is an open dense set of initial data

\[
   \mathcal{D} \subset \{(\rho_0, u_0) \mid \rho_0 \in L^2(\mathbb{R}) \cap C(\mathbb{R}), u_0 \in H^1(\mathbb{R}) \cap C^3(\mathbb{R})\}, \tag{4.17}
\]

such that for \((\rho_0, u_0) \in \mathcal{D}\), the conserved solution \((\rho(t, x), u(t, x))\) of two-component Camassa-Holm equation satisfies: \( \rho \) is twice continuously differentiable and \( u \) is three times continuously differentiable in the complement of finitely many isolated points within the domain \([0, T] \times \mathbb{R}\).

**Proof.** We only need to prove:
In above theorem, singularity can only appear on the characteristics $\phi$ for some constants $\{\}$ starting from Theorem 4.4 (Singular behavior) neighborhood of each singular point which is different with the CH case. Hence, $x, b$ then for any $\{\}$ we can define a cut-off function $D : \mathbb{R}$ parts, we know $\phi$ is dense. Given any $\{\}$ in $C^1(\mathbb{R}) \times C^3(\mathbb{R})$ topology in the outer part $\{x : |x| \geq \ell\}$ still belongs to $\mathcal{D}$. In the outer domain $\{(t, x) | 0 \leq t \leq T, x < x(t; 0, -\ell) \text{ or } x > x(t; 0, \ell)\}$, there is no singularity.

For the inner part, after a small perturbation, $\rho_0$ still has finitely many zero points. Combining two parts, we know $\mathcal{D}$ is open.

Then we prove $\mathcal{D}$ is dense. Given any $\rho_0 \in C^1(\mathbb{R}) \cap L^2(\mathbb{R})$, $u_0(x) \in C^3(\mathbb{R}) \cap H^1(\mathbb{R})$, we can find a suitable $\ell$ such that $|u_{0,x}(x)| < \varepsilon$ when $|x| > \ell$.

Then we can define a cut-off function $\varphi(x) \in C^\infty(\mathbb{R})$, such that

$$\varphi(x) = 1 \text{ when } |x| \leq \ell; \quad \varphi(x) = 0 \text{ when } |x| > \ell + \delta.$$ 

Define

$$\Phi(x, b) := \phi^b(x) := \rho_0(x) - b\varphi(x). \quad (4.18)$$

Then for any $x, b$, the differential $\frac{\partial}{\partial b} \Phi(x, b) = -\varphi(x) = -1$, when $|x| \leq \ell$, which spans $\mathbb{R}$. By transversality lemma, almost all $\phi^b$ are transversal to $W = \{0\}$. That means

$$\phi^b(x_0) = 0 \Rightarrow \frac{\partial}{\partial x} \phi^b(x_0) \neq 0.$$ 

Hence, $x_0$ is an isolated zero point of $\phi^b$. So the set $\mathcal{D}$ is dense.

Based on this generic regularity, we can obtain a detailed asymptotic description of the solution in a neighborhood of each singular point which is different with the CH case.

**Theorem 4.4** (Singular behavior). In above theorem, singularity can only appear on the characteristics starting from $\{x_1, \ldots, x_n\}$ and is isolated. Generically, at the singular point, denoted as $(t_0, x_0)$, $u$ may have following parametric expression:

$$u(t, x) - u(t_0, x_0) = A_1(t - t_0) + A_2[x - x_0 - u(t_0, x_0)(t - t_0)]^{2/3} + A_3(t - t_0)[x - x_0 - u(t_0, x_0)(t - t_0)]^{1/3} + O(1)(|t - t_0|^2 + |x - x_0 - u(t_0, x_0)(t - t_0)|),$$

$$\text{for some constants } A_1, A_2, A_3,$$

$$(4.19)$$

$$u(t, x) - u(t_0, x_0) = A_1(t - t_0) + A_2[x - x_0 - u(t_0, x_0)(t - t_0)] + A_3(t - t_0)[x - x_0 - u(t_0, x_0)(t - t_0)]^{1/3} + O(1)(|t - t_0|^2 + |x - x_0 - u(t_0, x_0)(t - t_0)|^{4/3}),$$

$$\text{for some constants } A_1, A_2, A_3.$$
4.2 Derivation of semilinear system

Instead of the variables \((t, x)\), it is convenient to work with an adapted set of variables \((t, \xi)\), where \(\xi\) is implicitly defined as
\[
\int_0^{\bar{y}(\xi)} (1 + \bar{u}_x^2 + \bar{\rho}^2) \, dx = \xi. \tag{4.21}
\]
Define \(\Lambda(t, \xi) = (t(t, \xi), x(t, \xi))\). The characteristics is corresponding to the curve on which \(\xi\) equal to a constant,
\[
\partial_t y(t, \xi) = u(t, y(t, \xi)), \quad y(0, \xi) = \bar{y}(\xi). \tag{4.22}
\]
In terms of these variables, the solution \(u = u(t, \xi)\) becomes globally Lipschitz continuous. Denote
\[
l = \frac{1}{1 + u_x^2 + \rho^2}, \quad m = \frac{u_x}{1 + u_x^2 + \rho^2}, \quad n = \frac{\rho}{1 + u_x^2 + \rho^2}, \quad q = (1 + u_x^2 + \rho^2) \cdot \frac{\partial y}{\partial \xi}. \tag{4.23}
\]
Denote
\[
L = \frac{q}{1 + u_x^2 + \rho^2}, \quad M = \frac{u_x \cdot q}{1 + u_x^2 + \rho^2}, \quad N = \frac{\rho \cdot q}{1 + u_x^2 + \rho^2}. \tag{4.24}
\]
The conservative solution satisfies following semilinear system
\[
\begin{aligned}
\frac{\partial}{\partial t} u(t, \xi) &= -P_x, \\
\frac{\partial}{\partial t} q(t, \xi) &= M(1 + 2u^2 - 2P), \\
\frac{\partial}{\partial t} L(t, \xi) &= M, \\
\frac{\partial}{\partial t} M(t, \xi) &= -PL + Lu^2 + \frac{1}{2}(q - L), \\
\frac{\partial}{\partial t} N(t, \xi) &= 0.
\end{aligned} \tag{4.25}
\]
Now we derive this semilinear system.

**Step 1. the equation of \( q \)** For any \( t > \tau > 0 \) and \( \xi_1, \xi_2 \) we consider the integral in the energy coordinates \( I_1 \) denoted by
\[
I_1 := \int_{\xi_1}^{\xi_2} q(t, \eta) d\eta - \int_{\xi_1}^{\xi_2} q(\tau, \eta) d\eta. \tag{4.26}
\]
Then we use the definition of \( q \) in (4.23) to pull back \( I_1 \) in the original time-space coordinates.
\[
I_1 = \int_{x(t, \xi_1)}^{x(t, \xi_2)} (1 + u_x^2 + \rho^2)(t, y) dy - \int_{x(\tau, \xi_1)}^{x(\tau, \xi_2)} (1 + u_x^2 + \rho^2)(t, y) dy. \tag{4.27}
\]
Next we use equations (4.11) and (4.12) to simplify \( I_1 \) as
\[
I_1 = \int_{\tau}^{t} \int_{x(s, \xi_1)}^{x(s, \xi_2)} u_x(1 + 2u^2 - 2P) dy ds. \tag{4.28}
\]
Finally we pull forward \( I_1 \) in the energy coordinates.
\[
I_1 = \int_{\tau}^{t} \int_{\xi_1}^{\xi_2} \frac{u_x(1 + 2u^2 - 2P)q}{1 + u_x^2 + \rho^2} d\xi ds = \int_{\tau}^{t} \int_{\xi_1}^{\xi_2} M(1 + 2u^2 - 2P)d\xi ds. \tag{4.29}
\]
When $|\xi_1 - \xi_2| \to 0$ and $|t - \tau| \to 0$ we have the equation of $q$ in (4.25)$_2$.

**Step 2. the equation of $L$** We apply the similar idea used in step 1 to deal with the definition of characteristics.

\[
\int_{\xi_1}^{\xi_2} L(t, \eta) d\eta - \int_{\xi_1}^{\xi_2} L(\tau, \eta) d\eta = \int_{x(t, \xi_1)}^{x(t, \xi_2)} 1 dyds - \int_{x(\tau, \xi_1)}^{x(\tau, \xi_2)} 1 dyds \quad (4.30)
\]

\[
= \int_{\tau}^{t} \int_{x(s, \xi_1)}^{x(s, \xi_2)} u_x(s, y) dy ds.
\]

On the other hand, we have

\[
\int_{\tau}^{t} \int_{x(s, \xi_1)}^{x(s, \xi_2)} u_x(s, y) dy ds
\]

\[
= \int_{\tau}^{t} \int_{\xi_1}^{\xi_2} \frac{u_x \cdot q}{1 + u_x^2 + \rho^2} (s, \xi) d\xi ds = \int_{\tau}^{t} \int_{\xi_1}^{\xi_2} M(s, \xi) d\xi ds. \quad (4.31)
\]

Combining (4.30) and (4.31) and taking $|\xi_1 - \xi_2| \to 0$ and $|t - \tau| \to 0$ we have the equation of $L$ in (4.25)$_3$.

**Step 3. the equation of $M$** Recall the equation (4.25)$_1$

\[
\frac{\partial}{\partial t} u(t, \xi) = -P_x. \quad (4.32)
\]

Then we get

\[
I_2 := \int_{x(t, \xi_1)}^{x(t, \xi_2)} u_x(t, y) dy - \int_{x(\tau, \xi_1)}^{x(\tau, \xi_2)} u_x(\tau, y) dy
\]

\[
= \int_{\tau}^{t} -P_x(s, x(t, \xi_2)) ds - \int_{\tau}^{t} -P_x(s, x(t, \xi_1)) ds
\]

\[
= -\int_{\tau}^{t} \int_{x(s, \xi_1)}^{x(s, \xi_2)} P_{xx}(s, y) dy ds
\]

\[
= \int_{\tau}^{t} \int_{x(s, \xi_1)}^{x(s, \xi_2)} \left[ \left( u^2 + \frac{u_x^2 + \rho^2}{2} \right) - P \right] (s, y) dy ds
\]

\[
= \int_{\tau}^{t} \int_{\xi_1}^{\xi_2} \left[ \left( u^2 + \frac{u_x^2 + \rho^2}{2} \right) - P \right] \frac{q}{1 + u_x^2 + \rho^2} (s, \xi) d\xi ds.
\]

\[
= \int_{\tau}^{t} \int_{\xi_1}^{\xi_2} \left( PL - Lu^2 - \frac{1}{2} (q - L) \right) (s, \xi) d\xi ds.
\]
On the other hand, we obtain
\[ I_2 = \int_{\xi_1}^{\xi_2} \frac{u_x \cdot q}{1 + u_x^2 + \rho^2} (t, \xi) d\xi - \int_{\xi_1}^{\xi_2} \frac{u_x \cdot q}{1 + u_x^2 + \rho^2} (\tau, \xi) d\xi \]
\[ = \int_{\xi_1}^{\xi_2} M(t, \xi) d\xi - \int_{\xi_1}^{\xi_2} M(\tau, \xi) d\xi. \]  
(4.34)

Combining (4.33) and (4.34) and taking \(|\xi_1 - \xi_2| \to 0\) and \(|t - \tau| \to 0\) we have the equation of \(M\) in (4.25).

**Step 4. the equation of \(N\)** From the equation (1.1)₂, we obtain
\[ 0 = \int_{x(t, \xi_1)}^{x(t, \xi_2)} \rho(s, y) dy - \int_{x(\tau, \xi_1)}^{x(\tau, \xi_2)} \rho(s, y) dy \]
\[ = \int_{\xi_1}^{\xi_2} \frac{\rho \cdot q}{1 + u_x^2 + \rho^2} (t, \eta) d\eta - \int_{\xi_1}^{\xi_2} \frac{\rho \cdot q}{1 + u_x^2 + \rho^2} (\tau, \eta) d\eta \]
\[ = \int_{\xi_1}^{\xi_2} N(t, \eta) d\eta - \int_{\xi_1}^{\xi_2} N(\tau, \eta) d\eta. \]
\[ (4.35) \]

Taking \(|\xi_1 - \xi_2| \to 0\) we have the equation of \(N\) in (4.25)₅.

**Remark 4.1.** From the semilinear system, since \(u, P, P_x\) are bounded, \(q, L, M\) satisfy following exponential estimate
\[ \exp \{-C_1 t\} \leq q \leq \exp \{C_1 t\}, \]
\[ C_2 \exp \{-C_1 t\} \leq L \text{ or } M \leq C_2 \exp \{C_1 t\}, \]
(4.36)
(4.37)
where \(C_1, C_2\) are constants related with initial data.

**Lemma 4.1** (Consistency condition). For fixed \(\xi\), if \(L, M, N, q\) satisfy
\[ L^2 + M^2 + N^2 = Lq \]
(4.38)
at initial time, then it also holds for any time \(t\).

**Proof.** It suffices to prove \(\frac{d}{dt} (L^2 + M^2 + N^2 - Lq) = 0\). Indeed, by using system (4.25),
\[ \frac{d}{dt} (L^2 + M^2 + N^2 - Lq) \]
\[ = 2LL_t + 2MM_t + 2NN_t - L_tq - Lq_t \]
\[ = 2LM + 2M[-PL + Lu^2 + \frac{1}{2}(q - L)] - Mq - LM(1 + 2u^2 - 2P) \]
\[ = 0 \]
(4.39)

**Lemma 4.2.** The singularity only appears on the characteristics where \(N(0) = 0\).

**Proof.** Since the initial data is regular, for any characteristics, \(L(0) \neq 0\). From Lemma 1, on the characteristics where \(N(0) \neq 0\), by the equation of \(N\), \(N(t) = N(0) \neq 0\). Solving \(L\) from the consistency condition, we obtain
\[ L = \frac{q \pm \sqrt{q^2 - 4(M^2 + N^2)}}{2}. \]
(4.40)
The smaller root is
\[ L = \frac{q - \sqrt{q^2 - 4(M^2 + N^2)}}{2} = \frac{2(M^2 + N^2)}{q + \sqrt{q^2 - 4(M^2 + N^2)}} \geq \frac{M^2 + N^2}{q} \geq N^2(0) \exp\{-C_1 t\}. \] (4.41)

Considering the definition of $L$ and Remark 1.,
\[ |u_x(t, \xi)| \leq \frac{1}{N(0)} \exp\{C_1 t/2\}. \]

So the singularity point only appears on the characteristics where $N(0) = 0$.

### 4.3 Proof of theorem 4.3

By equation
\[ u_t + uu_x + P_x = 0, \quad \rho_t + (\rho u)_x = 0. \] (4.42)

On the characteristics where $N(0) \neq 0$, we have upper bound
\[ \rho(t) \leq \frac{2}{C_1 N(0)} \exp\{C_1 t/2\}. \] (4.43)

$\rho_x$ and $u_{xx}$ satisfy
\[
\begin{align*}
(u_{xx})_t + u(u_{xx})_x + 3u_x u_{xx} + P_x - (2u u_x + u_x u_{xx} + \rho \rho_x) &= 0, \\
(\rho_x)_t + u(\rho_x)_x + \rho u_{xx} + 2 \rho_x u_x &= 0,
\end{align*}
\] (4.44)

or we can express it as
\[
\frac{d}{dt} \begin{pmatrix} u_{xx} \\ \rho_x \end{pmatrix} = \begin{pmatrix} -2u_x & \rho \\ -\rho & -2u_x \end{pmatrix} \begin{pmatrix} u_{xx} \\ \rho_x \end{pmatrix} + \begin{pmatrix} 2u u_x - P_x \\ 0 \end{pmatrix} \] (4.45)

Consider Lyapunov function
\[ W(t) = \rho_x^2(t, \xi) + u_{xx}^2(t, \xi). \] (4.46)

For fixed $\xi$,
\[
\frac{d}{dt} W(t) = 2 \rho_x (\rho_x)_t + 2 u_{xx} (u_{xx})_t \\
= 2 \rho_x (-\rho u_{xx} - 2 \rho_x u_x) + 2 u_{xx} (-2u_x u_{xx} + \rho \rho_x + 2uu_x - P_x) \\
= -4u_x (\rho_x^2 + u_{xx}^2) + 4uu_x u_{xx} - 2P_x u_{xx} \\
\leq \frac{C}{N} e^{C_1 t/2} W + \frac{C}{N^2} e^{C_1 t} + C,
\] (4.47)

where $C$ is a constant which related with initial data.

Solving this inequality, we obtain the bound for $W'$,
\[ W(t) \leq (W_0 + \int_0^t C e^{C_1 s} ds) e^{C_1 t}. \] (4.48)
The equations for $u_{xxx}$ and $\rho_{xx}$ are

$$
\begin{align*}
\begin{cases}
(u_{xxx})_t + u(u_{xxx})_x + 4u_x u_{xxx} + 3u^2_{xx} + P - u^2 - \frac{u_x^2 + \rho_x^2}{2} = 0, \\
-(2u_x^2 + 2u u_{xx} + u^2_{xx} + u_x u_{xxx} + \rho_x^2 + \rho \rho_{xx}) = 0, \\
(\rho_{xx})_t + u(\rho_{xx})_x + 3u_x \rho_{xx} + 3\rho_x u_{xx} + \rho u_{xxx} = 0.
\end{cases}
\end{align*}
$$

(4.49)

or

$$
\frac{d}{dt} \begin{pmatrix} u_{xxx} \\ \rho_{xx} \end{pmatrix} = \begin{pmatrix} -3u_x & \rho \\ -\rho & -3u_x \end{pmatrix} \begin{pmatrix} u_{xxx} \\ \rho_{xx} \end{pmatrix} + \begin{pmatrix} -2u_x^2 - \frac{3}{2}u_x^2 + 2u_{xx} + \rho_x^2 + u^2 - P + \frac{1}{2}\rho_x^2 \\ -3\rho_x u_{xx} \end{pmatrix}
$$

(4.50)

Define Lyapunov function

$$
V(t) = \rho_{xx}(t, \xi) + u^2_{xxx}(t, \xi).
$$

(4.51)

For fixed $\xi$,

$$
\frac{d}{dt} V(t) = 2\rho_{xx}(\rho_{xx})_t + 2u_{xxx}(u_{xxx})_t
$$

$$
= -3u_x (\rho_x^2 + u^2_{xx}) + \rho_{xx} (-3\rho_x u_{xx})
$$

$$
+ u_{xxx} (-2u_x^2 + \frac{5}{2}u_x^2 + 2u_{xx} + \rho_x^2 + u^2 - P + \frac{1}{2}\rho_x^2)
$$

$$
\leq \frac{C}{N}e^{Ct/2}V + \frac{C}{N^2}e^{Ct} + C.
$$

(4.52)

We obtain

$$
V(t) \leq (V_0 + \int_0^t C e^{Ct} dt)e^{Ct}.
$$

(4.53)

So we proved the point-wise estimate of higher order derivatives of $\rho, u$ when $x \in I_i$ for every $i = 0, \ldots, n$. This also shows that whenever $u_x$ is bounded, $V(t)$ is bounded.

Note that the equation for $M$, we see that $\partial_t M \neq 0$ at the singular point $(t_0, \xi_0)$. We can take a small neighborhood of $t_0$ such that at the time $t = t_0 + \varepsilon$, $M(t_0 + \varepsilon, \xi_0) \neq 0$, and also $\partial_t  \neq 0$ due to the consistency condition. So there is a time $T_0$, such that $u_x$ is bounded in the interval $[t_0 + \varepsilon, T_0 - \varepsilon]$. Since here $\varepsilon$ can be arbitrarily small, so $u_x$ is bounded in any proper sub-interval of $(t_0, T_0)$.

By above estimate, we can conclude that $V(t)$ is also bounded in the proper subset of the time interval $(t_0, T_0)$ along the characteristics $\xi = \xi_0$. This means the singular points are isolated on the characteristic curves where $\rho = 0$.

Now the last step is to prove there are only finitely many singular point on each characteristics in the interval $[0, T]$. From (4.9), on the characteristics where $\rho = 0$,

$$
\frac{d}{dt} u_x = u^2 - P - \frac{1}{2}u_x^2.
$$

(4.54)

On the right-hand-side, $|u^2 - P|$ is bounded by some constant $C$. Then we estimate the life span of the solution to above equation.

**Lemma 4.3.** Consider a family of ODEs

$$
\frac{d}{dt} x(t) = f(x) := C - \frac{1}{2}x^2, \quad x(0) = x_0,
$$

with $C \in [-M, 0)$. The life span of the solution to this ODE, denoted as $T$, is uniformly positive with respect to $C$. 

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\[ \rho = 0 \quad \rho = 0 \quad \rho = 0 \]

\[ 0 \quad x \quad t \]

Figure 2: Left: The singularity points are all on the characteristic curves where \( \rho = 0 \), and each point is isolated, and there are only finitely many singularities in the domain \([0, T] \times \mathbb{R}\). Right: The function of \( L \) along one characteristics. The singularity corresponds to \( L = 0 \). At first sight, it seems that there may be infinitely many singularities on one characteristics. But this is not true. We will exclude this case later.

**Proof.** When \( C < 0 \),

\[ x(t) = \sqrt{-2C} \tan \left( \frac{-C}{2}t + \arctan \frac{x_0}{\sqrt{-2C}} \right). \]

If \( x_0 = 0 \), then \( T = \frac{\pi}{\sqrt{-2C}} \geq \frac{\pi}{\sqrt{2M}} \).

If \( x_0 < 0 \), then \( T = \sqrt{\frac{2}{-C} \left( \frac{\pi}{2} - \arctan \frac{x_0}{\sqrt{-2C}} \right)} \geq \sqrt{\frac{2}{M} \left( \frac{\pi}{2} - \arctan \frac{x_0}{\sqrt{2M}} \right)} > \frac{\pi}{2} \sqrt{\frac{2}{M}}. \)

If \( x_0 > 0 \), since \( f(x) < 0 \), \( x(t) \) is strictly decreasing. So by comparison principle, in this case the life span must be larger than the case when \( x_0 < 0 \).

\[ f(x) \quad f(x) \]

Figure 3: Left: \( C > 0 \). Right: \( C < 0 \).

**Remark 4.2.** When \( C > 0 \), we know from the graph of \( f \) that if \( x_0 \geq -\sqrt{2C} \), then the solution is global in time, that is \( T = \infty \).

From (4.25), we observe that when \( L = 0, M = 0 \) and \( M_t > 0 \). See Fig. 4. This means when \( t \to t_0^- \), \( u_x(t, \xi_0) \to -\infty \); when \( t \to t_0^+ \), \( u_x(t, \xi_0) \to +\infty \). So the period from one singularity to the next singularity on the same characteristics is corresponding to the procedure of \( u_x \) changing from \(-\infty \to +\infty \to -\infty \). By using Lemma 4.3 and Remark 4.2, we obtain that the length of the time interval of \( u_x \) changing from \( +\infty \) to \( -\infty \) is uniformly positive.
Figure 4: The function of $L$ and $M$ along one characteristics. When $L = 0$, $M = 0$ and $M_t > 0$. When $L$ takes local maximum, $M = 0$.

So we conclude that the time interval of two neighboring singularities on the same characteristics is uniformly positive. Hence there are only finitely many singularities on one characteristics in time interval $[0, T]$.

This leads to generic regularity theorem 4.3.

4.4 Generic singularity behavior

Equations of first order derivatives:

\[
\begin{align*}
\frac{\partial}{\partial t} L_{\xi} &= M_{\xi}, \\
\frac{\partial}{\partial t} L_t &= -PL + Lu^2 + \frac{1}{2}(q - L), \\
\frac{\partial}{\partial t} M_{\xi} &= -P_L L^2 - P_L u^2 + 2M_{\xi} + \frac{1}{2}(q_{\xi} - L_{\xi}), \\
\frac{\partial}{\partial t} q_{\xi} &= M_{\xi}(1 + 2u^2 - 2P) + M(4uu_{\xi} - 2P_x x_{\xi}).
\end{align*}
\]
Equations of second order derivatives:

\[
\begin{align*}
\frac{\partial}{\partial t} L_{\xi \xi} &= M_{\xi \xi}, \\
\frac{\partial}{\partial t} M_{\xi \xi} &= (u^2 - P - \frac{1}{2}) L_{\xi}^2 + \frac{1}{2} q L_{\xi}^2 - 3 P_{x} L L_{\xi} \\
&- (P - u^2 + \frac{1}{2}) L_{\xi \xi} + 4 L_{\xi} u u_{\xi} + 2 L u_{\xi}^2 + 2 L u u_{\xi \xi} + \frac{1}{2} q_{\xi \xi}, \\
\frac{\partial}{\partial t} u_{\xi \xi} &= (u^2 - P - \frac{1}{2}) L_{\xi} + (2 u u_{\xi} - P_{x} L) L + \frac{1}{2} q_{\xi}, \\
\frac{\partial}{\partial t} q_{\xi \xi} &= M_{\xi \xi} (1 + 2 u^2 - 2 P) + 4 M_{\xi} (2 u u_{\xi} - P_{x} L) \\
&+ M \left[ 4 u_{\xi}^2 + 4 u u_{\xi \xi} + 2 L^2 (u^2 - P - \frac{1}{2}) + L q - 2 P_{x} L_{\xi} \right].
\end{align*}
\]  

(4.56)

Based on those closed semilinear system, we can show the next Lemma in a similar way with the one-component Camassa-Holm equation.

**Lemma 4.4.** Consider the compact domain of form

\[
\Gamma := \{(t, \xi); 0 \leq t \leq T, |\xi| \leq C\}. \tag{4.57}
\]

Call \( S \) the family of all \( C^2 \) solutions to the semilinear system, with \( q > 0 \) for all \((t, \xi) \in \mathbb{R}_+ \times \mathbb{R}\). Moreover call \( S' \subset S \) the subfamily of all solutions \((L, M, N, q)\) such that for \((t, \xi) \in \Gamma\), none of the following values is attained:

\[
(L, L_{\xi}, L_{\xi \xi}) = (0, 0, 0), \quad (L, M_{\xi}, M_{\xi \xi}) = (0, 0, 0), \quad (L, N_{\xi}, N_{\xi \xi}) = (0, 0, 0). \tag{4.58}
\]

Then \( S' \) is a relatively open and dense subset of \( S \), in the topology induced by \( C^2(\Gamma) \).

**Proof.** The proof of this lemma is similar with Lemma 3.4. We sketch the procedures in the following. First, we denote \( S_1, S_2, S_3 \) to be the sets of solutions for which the values of (4.58) are not attained in \( \Gamma \) separately. Denote their intersection as \( S' := S_1 \cap S_2 \cap S_3 \), \( S' \) is relatively open subset of \( S \). We then prove \( S' \) is dense. We need to use following Lemma 4.5. Let’s assume it is true for the moment. For any point \((t_0, \xi_0) \in \Gamma\), either \((t_0, \xi_0)\) is one of the three singularity point above, or it is a regular point. If it is a regular point, then there is a neighborhood such that no singularity occurs in it. If it is one of the three types of singular point, then we can make a three-parameter perturbation to the initial data so that after the perturbation, we can find a neighborhood of \((t_0, \xi_0)\) so that there is no singularity in the neighborhood. By a compactness argument, after finitely many times, say \( n \) times, of modification of initial data, we can obtain a regular solution. Then by Lemma 4.5, for each singularity point, the perturbation maps \( \theta \in \mathbb{R}^3 \mapsto (L^0, L_{\xi}^0, L_{\xi \xi}^0), \quad \theta \in \mathbb{R}^3 \mapsto (L^0, M_{\xi}^0, M_{\xi \xi}^0) \) and \( \theta \in \mathbb{R}^3 \mapsto (L^0, N_{\xi}^0, N_{\xi \xi}^0) \) have full rank. By Thom’s transversality Lemma, we can conclude that the perturbed solutions which are regular form a dense set in \( C^2(\Gamma) \).

\( \square \)

**Lemma 4.5.** Let \((u, L, M, N, q)\) be a smooth solution of the semilinear system (4.25), and let a point \((t_0, \xi_0) \in \mathbb{R}_+ \times \mathbb{R}\) be given.

1. If \((L, L_{\xi}, L_{\xi \xi})(t_0, \xi_0) = (0, 0, 0)\), then there exists a 3-parameter family of smooth solutions \((u^0, L^0, M^0, N^0, q^0)\), depending smoothly on \( \theta \in \mathbb{R}^3 \), such that the following holds.

   (i) When \( \theta = 0 \in \mathbb{R}^3 \), one recovers the original solution, namely \((u^0, L^0, M^0, N^0, q^0) = (u, L, M, N, q)\).
(ii) At the point \((t_0, \xi_0)\), when \(\theta = 0\) one has
\[
\text{rank } D_\theta(L^\theta, L_{\xi}^\theta, L_{\xi\xi}^\theta) = 3.
\] (4.59)

(2) If \((L, M_\xi, M_{\xi\xi})(t_0, \xi_0) = (0, 0, 0)\), then there exists a 3-parameter family of smooth solutions
\((u^\theta, L^\theta, M^\theta, N^\theta, q^\theta)\), depending smoothly on \(\theta \in \mathbb{R}^3\), satisfying (i)(ii) above with (4.59) replaced by
\[
\text{rank } D_\theta(L^\theta, M_{\xi}^\theta, M_{\xi\xi}^\theta) = 3.
\] (4.60)

(3) If \((L, M_\xi, M_{\xi\xi})(t_0, \xi_0) = (0, 0, 0)\), then there exists a 3-parameter family of smooth solutions
\((u^\theta, L^\theta, M^\theta, N^\theta, q^\theta)\), depending smoothly on \(\theta \in \mathbb{R}^3\), satisfying (i)(ii) above with (4.59) replaced by
\[
\text{rank } D_\theta(L^\theta, N_{\xi}^\theta, N_{\xi\xi}^\theta) = 3.
\] (4.61)

This lemma can be proved by using Lemma 3.3 with proving boundedness of coefficient matrix of (4.25), (4.55), (4.56).

Assume \((t_0, \xi_0)\) is the point of singularity, then by the definitions, we have
\[
L(t_0, \xi_0) = M(t_0, \xi_0) = N(t_0, \xi_0) = 0, \quad L_t(t_0, \xi_0) = N_t(t_0, \xi_0) = 0.
\] (4.62)

But there holds
\[
M_\xi(t_0, \xi_0) \neq 0, \quad L_{tt}(t_0, \xi_0) \neq 0.
\] (4.63)

This means the singularity of (2CH) is isolated.

Now we prove generically, the singularity may have following possibilities for each quantities
\[
L : \begin{cases} L_\xi \neq 0, \text{ or } L_\xi = 0, L_{\xi\xi} \neq 0; \\
L_\xi = 0, L_{\xi\xi} = 0; \end{cases} \quad M : \begin{cases} M_\xi \neq 0, \text{ or } M_\xi = 0, M_{\xi\xi} \neq 0; \\
M_\xi = 0, M_{\xi\xi} = 0; \end{cases} \quad N : \begin{cases} N_\xi \neq 0, \text{ or } N_\xi = 0, N_{\xi\xi} \neq 0; \\
N_\xi = 0, N_{\xi\xi} = 0. \end{cases}
\] (4.64)

Considering the consistency condition, take derivative \(\partial_\xi\) to (4.38),
\[
2LL_\xi + 2M_{M_\xi} + 2NN_\xi = L_{\xi\xi}q + L_{q\xi}.
\] (4.65)

At the singularity point, we have \(L_\xi = 0\). Taking another derivative, and considering \(L = M = N = L_\xi = 0\), we have
\[
2M_\xi^2 + 2N_\xi^2 = L_{\xi\xi}q.
\] (4.66)

Since the generic singularity is \(L_\xi = 0, L_{\xi\xi} \neq 0\), we can conclude generically \(N_\xi\) and \(M_\xi\) cannot be zero at the same time. So we only have three types of generic singularities:

- Type I: \(L_\xi = 0, L_{\xi\xi} \neq 0, M_\xi = 0, M_{\xi\xi} \neq 0, N_\xi \neq 0;\)
- Type II: \(L_\xi = 0, L_{\xi\xi} \neq 0, M_\xi \neq 0, N_\xi \neq 0;\)
- Type III: \(L_\xi = 0, L_{\xi\xi} \neq 0, M_\xi \neq 0, N_\xi = 0, N_{\xi\xi} \neq 0;\)

For all kinds of singularities,
\[
L(t, \xi) = C_1(t - t_0)^2 + C_2(\xi - \xi_0)^2 + O(1)(|t - t_0|^3 + |\xi - \xi_0|^3),
\] (4.67)

By (4.23),
\[
x(t, \xi) - x_0 = B_1(t - t_0) + B_2(\xi - \xi_0)^3 + B_3(t - t_0)^2(\xi - \xi_0) + O(1)(|t - t_0|^2 + |\xi - \xi_0|^4),
\] (4.68)
For Type II and Type III singularities,

\[ u(t, \xi) - u(t_0, \xi_0) = C_1(t - t_0) + C_2(\xi - \xi_0)^2 + C_3(t - t_0)(\xi - \xi_0) + O(1)(|t - t_0|^2 + |\xi - \xi_0|^3). \]

Thus

\[ u(t, x) - u(t_0, x_0) = A_1(t - t_0) + A_2[x - x_0 - u(t_0, x_0)(t - t_0)]^{2/3} + A_3(t - t_0)[x - x_0 - u(t_0, x_0)(t - t_0)]^{1/3} \]

\[ + O(1)(|t - t_0|^2 + |x - x_0 - u(t_0, x_0)(t - t_0)|). \]  

(4.69)

For Type I singularities,

\[ u(t, \xi) - u(t_0, \xi_0) = C_1(t - t_0) + C_2(\xi - \xi_0)^3 + C_3(t - t_0)(\xi - \xi_0) + O(1)(|t - t_0|^2 + |\xi - \xi_0|^4). \]

In this case,

\[ u(t, x) - u(t_0, x_0) = A_1(t - t_0) + A_2[x - x_0 - u(t_0, x_0)(t - t_0)] + A_3(t - t_0)[x - x_0 - u(t_0, x_0)(t - t_0)]^{1/3} \]

\[ + O(1)(|t - t_0|^2 + |x - x_0 - u(t_0, x_0)(t - t_0)|^{4/3}). \]  

(4.70)

Then we consider the behavior of \( \rho \). For type I and type II singularities

\[ \rho = \frac{N}{L} = \frac{D_1(\xi - \xi_0) + O(1)(|\xi - \xi_0|^2)}{C_1(t - t_0)^2 + C_2(\xi - \xi_0)^2 + O(1)(|t - t_0|^2 + |\xi - \xi_0|^3)}. \]  

(4.71)

For type III,

\[ \rho = \frac{N}{L} = \frac{D_2(\xi - \xi_0)^2 + O(1)(|\xi - \xi_0|^3)}{C_1(t - t_0)^2 + C_2(\xi - \xi_0)^2 + O(1)(|t - t_0|^2 + |\xi - \xi_0|^3)}. \]  

(4.72)

So along the characteristic curve \( \xi = \xi_0, (t_0, \xi_0) \) is a removable singularity of \( \rho \).

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