Self-contracted curves in Riemannian manifolds

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Abstract It is established that every self-contracted curve in a Riemannian manifold has finite length, provided its image is contained in a compact set.

Key words Self-contracted curve, self-expanded curve, rectifiable curve, length, secant, Riemannian manifold

AMS Subject Classification *Primary* 28A75, 52A41 ; *Secondary* 37N40, 53A04, 53B20.

Contents

1. Introduction

Self-contracted curves were introduced in [3, Definition 1.2.] to capture the behaviour of gradient orbits of a quasiconvex potential $\dot{x} = -\nabla f(x)$, of polygonal curves generated by a proximal algorithm applied to a convex function, or finally of generalized orbits (continuous curves) of a convex foliation. The definition of self-contractedness —recalled below in a more general setting (Riemannian manifolds)— is purely metric, without requiring prior smoothness/continuity assumption on the curve. Although self-contracted curves can be discontinuous (without admitting a continuous self-contracted extension) the left and right limits at each point always exist [3, Proposition 2.2]. In a Euclidean setting it has been established in [4, Section 3] (and independently in [7] for continuous curves) that self-contracted curves are rectifiable. In both cases the proof was based on an old result of Manselli-Pucci [9] which allows to deduce that all self-contracted curves lying in a given ball have lengths which are uniformly bounded. Applications of this fact have been discussed in [4, Section 4], [2].

The results of [9], [3], [4], [7] are all heavily based on the Euclidean structure. In [5] the authors consider (under a different terminology) absolutely continuous self-contracted curves in a bounded convex subset of a two-dimensional complete surface of constant Gaussian curvature, and provide an upper bound for the length, but in case of a surface of positive curvature (sphere), they made the additional assumption that the diameter of this subset was strictly less than $\pi/2$.

In this work we establish that any (possibly discontinuous) self-contracted curve in a compact set of a smooth Riemannian manifold has finite length. This result generalizes the results mentioned above. In particular, contrary to [5] it does not require any assumption on the curvature or on the dimension of the manifold. Moreover, our result holds in the case of discontinuous self-contracted curves. The proof relies on an appropriate localization argument which allows to define a finite family of Lyapunov functions.

2. Main result

Let $(\mathcal{M}, g)$ be a smooth complete Riemannian manifold whose geodesic distance is denoted by d_g. This work is devoted to the study of self-contrated curves.

Definition 2.1 (self-contracted curve)**.** Given an interval $I = [0, T_\infty)$ with $T_\infty \in [0, \infty) \cup \{\infty\}$, a curve $\gamma : I \to \mathcal{M}$ is called *self-contracted*, if for every $t_1 \le t_2 \le t_3$ in I we have

$$d_g(\gamma(t_1), \gamma(t_3)) \ge d_g(\gamma(t_2), \gamma(t_3)). \tag{2.1}$$

In other words, for every $t_3 \in [0, T_\infty)$ the function $t \mapsto d_g(\gamma(t), \gamma(t_3))$ is nonincreasing on $[0, t_3]$.

Given an interval $I = [0, T_\infty)$ with $T_\infty \in [0, \infty) \cup \{\infty\}$, the *length* of a curve $\gamma : I \to \mathcal{M}$ is defined as

$$\ell(\gamma) := \sup \left\{ \sum_{i=0}^{m-1} d_g(\gamma(t_i), \gamma(t_{i+1})) \right\}, \tag{2.2}$$

where the supremum is taken over all finite increasing sequences $t_0 < t_1 < \cdots < t_m$ that lie in the interval I. We say that a (possibly discontinuous) curve $\gamma : I \to \mathcal{M}$ has *finite length* if $\ell(\gamma)$ is finite. Any continuous curve $\gamma : I \to \mathcal{M}$ with finite length can be reparametrized into a Lipschitz curve on $[0, \ell(\gamma)]$ with speed of constant norm *a.e.* equal to 1. The following extends previous results by [5], [4], [7].

Theorem 2.2 (Main result)**.** *Let $(\mathcal{M}, g)$ be a smooth Riemannian manifold, $\mathcal{K}$ be a compact subset of $\mathcal{M}$ and $\gamma : I \to \mathcal{K}$ be a self-contracted curve. Then γ has finite length.*

The rest of the paper is devoted to the proof of Theorem 2.2.

3. Proof

3.1. **Notation.** The symbol $\mathcal{M}$ will always stand for a smooth manifold of dimension $d \ge 2$ whose tangent bundle is denoted by $T\mathcal{M}$. An element of $T\mathcal{M}$ is denoted by $\xi := (x, v_x)$ with $v_x \in T_x\mathcal{M}$ (for simplicity we often drop the index x from v_x when no confusion arises). Given a smooth Riemannian metric g, we denote the metric at $x \in \mathcal{M}$ alternatively as $g_x(\cdot, \cdot)$ or $\langle \cdot, \cdot \rangle_x$ and its norm by $|\cdot|_x$. We sometimes omit x if no ambiguity arises. The geodesic distance is denoted by d_g and the open geodesic ball centered at x of radius $r \ge 0$ is denoted by $B_g(x, r)$. For every $x \in \mathcal{M}$, we denote by $\exp_x : T_x\mathcal{M} \to \mathcal{M}$ the exponential mapping at x. We denote by $\hat{B}_x$ the balls in $T_x\mathcal{M}$ (with respect to the Euclidean metric g_x in $T_x\mathcal{M}$). We denote the unit tangent bundle associated with g by $U\mathcal{M}$, that is,

$$U\mathcal{M} := \{\xi = (x, u) \in T\mathcal{M} : |u|_x = 1\}.$$

We consider a canonical Riemannian metric on the unit bundle, whose associated distance is denoted by D_g. We may assume that for every $\xi = (x, u)$, $\bar{\xi} = (\bar{x}, \bar{u})$ in $U\mathcal{M}$ it holds

$$D_g(\xi, \bar{\xi}) \ge d_g(x, \bar{x}). \tag{3.1}$$

We refer to [1], [6] for prerequisites on Riemannian manifolds.

3.2. **Exponential map - Cosine law.** We first notice that for every $x \in \mathcal{M}$, there exists $r > 0$, such that the exponential function $\exp_x$ (that we sometimes denote by ϕ_x, especially when we want to abbreviate notation for its inverse ϕ_x^{-1} or its differential $d\phi_x$) is a smooth diffeomorphism between the open ball $\hat{B}_x(0,r)$ of $T_x\mathcal{M}$ onto the open geodesic ball $B_g(x,r)$ in M. The following lemma is an easy consequence of the compactness of $\mathcal{K}$ and the smoothness of the geodesic flow.

Lemma 3.1. *There exists $\rho > 0$ such that for every $x \in \mathcal{K}$, $\phi_x := \exp_x$ is a smooth diffeomorphism from the ball $\hat{B}_x(0,2\rho)$ to its image $B_g(x,2\rho)$.*

Given two unit vectors $v, w \in T_x\mathcal{M}$ (we denote this by $v, w \in U_x\mathcal{M}$), we define the function $\Phi_x^{v,w} : (-\rho,\rho) \times (-\rho,\rho) \to \mathbb{R}$ by

$$\Phi_x^{v,w}(t_1,t_2) = d_g\left(\exp_x(t_1 v), \exp_x(t_2 w)\right)^2, \qquad \forall t_1,t_2 \in (-\rho,\rho). \tag{3.2}$$

The following result asserts that small geodesic triangles almost satisfy the classical law of cosines, see Fig. 1 for an illustration.

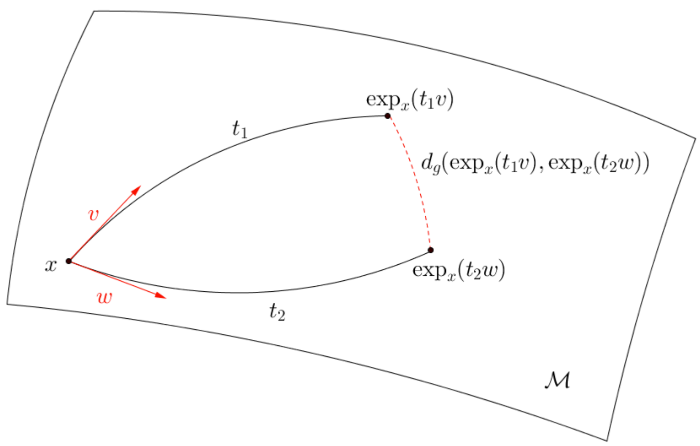

Figure 1. Cosine law in Riemannian manifolds.

Lemma 3.2 (Cosine law in manifolds)**.** *There is $K > 0$ such that for every $x \in \mathcal{K}$ and every $v, w \in U_x\mathcal{M}$,*

$$\left|\Phi_x^{v,w}(t_1,t_2) - t_1^2 - t_2^2 + 2\,t_1\,t_2\,\langle v,w\rangle_x\right| \le K\,t_1^2\,t_2^2, \qquad \forall\, t_1,t_2 \in (-\rho,\rho). \tag{3.3}$$

Proof. Let $x \in \mathcal{K}$ and $v, w \in U_x\mathcal{M}$ be fixed. We check easily that for every $t_1, t_2 \in (-\rho,\rho)$,

$$\Phi_x^{v,w}(t_1,0) = t_1^2, \quad \Phi_x^{v,w}(0,t_2) = t_2^2,$$

and

$$\frac{\partial \Phi_x^{v,w}}{\partial t_1}(0,t_2) = -2\,t_2\,\langle v,w\rangle_x, \quad \frac{\partial \Phi_x^{v,w}}{\partial t_2}(t_1,0) = -2\,t_1\,\langle v,w\rangle_x.$$

Then we infer that

$$\frac{\partial^2 \Phi_x^{v,w}}{\partial t_1 \partial t_2}(0,0) = -2\,\langle v,w\rangle_x$$

and for every integer $k \in \{2,3\}$,

$$\frac{\partial^{k+1} \Phi_x^{v,w}}{\partial t_1^k \partial t_2}(0,0) = \frac{\partial^{k+1} \Phi_x^{v,w}}{\partial t_1 \partial t_2^k}(0,0) = 0.$$

We conclude by considering the Taylor expansion formula of order 4 for the function $\Phi_x^{v,w}$ together with compactness of $\mathcal{K}$. $\square$

Remark 3.3 (Adapting the constant ρ)**.** Let $K > 0$ be given by Lemma 3.2. We may always shrink $\rho > 0$ of Lemma 3.1 to ensure

$$\rho \le \sqrt{\frac{1}{K}} \qquad \Longleftrightarrow \qquad K\,\rho^2 \le 1\,. \tag{3.4}$$

The above will be used in Section 3.5.2 where we derive some technical estimations. Therefore we shall eventually assume that (3.4) holds true.

3.3. **Dealing with discontinuities.** Let $I = [0, T_\infty)$ ($T_\infty \in [0,\infty) \cup \{\infty\}$) and let $\gamma : I \to \mathcal{M}$ be a self-contracted curve. For every $\tau \in I$, we define the set $\Gamma(\tau)$ (tail of γ at $x = \gamma(\tau)$) by

$$\Gamma(\tau) := \Big\{\gamma(t)\,|\,t \ge \tau\Big\}.$$

We also denote by $\gamma(\tau^-)$ the left limit of γ at τ, that is,

$$\gamma(\tau^-) := \lim_{s \nearrow \tau} \gamma(s), \tag{3.5}$$

where the notation $s \nearrow \tau$ means that $s \le \tau$ and $s \to \tau$. Notice that this limit always exists as consequence of Definition 2.1 (self-contractedness) —the same proof as in the Euclidean case applies (*c.f.* [3, Proposition 2.2]).

We further denote by

$$D^- := \{\tau \in I : \gamma(\tau) \ne \gamma(\tau^-)\}\,,$$

that is, the set of points where γ is not left continuous. Notice that

$$D^- = \bigcup_{n\in\mathbb{N}} D_n^-\,, \quad \text{where } \; D_n^- := \left\{\tau \in I : d_g(\gamma(\tau), \gamma(\tau^-)) > \frac{4}{n}\right\}.$$

For the needs of the following lemma, let us denote by $|S|$ the cardinality of a set S.

Lemma 3.4 (Local count of left discontinuities)**.** *For any ball $B_g(x, \frac{1}{n})$ of $\mathcal{M}$ we have:*

$$\Big|\gamma(D_n^-) \cap B_g\,(x, 1/n)\Big| \le 2. \tag{3.6}$$

In particular, D^- is at most countable.

Proof. Let $\tau_1, \tau_2, \tau_3 \in D_n^-$ with $\tau_1 < \tau_2 < \tau_3$, be such that $\{\gamma(\tau_i)\}_{i=1}^3 \subset B_g(x, \frac{1}{n})$. Set $x_i = \gamma(\tau_i)$ and $x_i' = \gamma(\tau_i^-)$, $i \in \{1,2,3\}$. It follows that $\{x_i\}_{i=1}^3 \subset B_g(x, \frac{1}{n})$ and $\{x_i'\}_{i=1}^3 \subset \mathcal{M} \setminus B_g(x, \frac{1}{n})$. Since γ is self-contracted, we obtain a contradiction:

$$\frac{2}{n} < d_g(x_2', x_2) - d_g(x_2, x_3) \le d_g(x_2', x_3) \le d_g(x_1, x_3) \le \frac{2}{n}\,.$$

The assertion of the lemma follows. $\square$

Remark 3.5 (Cardinality of D_n^-)**.** As a consequence of self-contractedness, the sets D_n^- (subset of $(0, T_\infty)$) and $\gamma(D_n^-)$ (subset of $\mathcal{K}$) have the same cardinality, for every $n \in \mathbb{N}$. Compactness of $\mathcal{K}$ together with Lemma 3.4 yield that this cardinality is bounded by $2\,N$, where N is the number of balls of radius $1/n$ that can cover $\mathcal{K}$.

Analogous results hold for right discontinuities, that is, points $\tau \in [0, T_\infty)$ such that

$$\gamma(\tau^+) := \lim_{t \searrow \tau} \gamma(t) \neq \gamma(\tau),$$

where the notation $t \searrow \tau$ means that $t > \tau$ and $t \to \tau$. In order to establish forthcoming intermediate results concerning behaviour/properties of the curve γ around a point $\gamma(\tau)$, we shall be led to consider separately the case where γ is left continuous at τ and the case where it is not, but the jump is within a certain threshold. The question of whether or not γ is right continuous at τ will not appear until the very last section (see Corollary 3.22) where we shall evoke both a left and a right-discontinuity threshold. Let us record this notation for later reference.

Remark 3.6 (Thresholding left discontinuities)**.** Let $\eta > 0$ (it will be fixed in Section 3.7). Then the following sets of left/right discontinuities with jump beyond the η-threshold are finite:

(i) (left-η-threshold)

$$\mathcal{D}^-(\eta) := \left\{\tau \in (0, T_\infty) : \quad d_g(\gamma(\tau), \gamma(\tau^-)) \geq \eta\right\}. \tag{3.7}$$

(ii) (right-η-threshold)

$$\mathcal{D}^+(\eta) := \left\{\tau \in (0, T_\infty) : \quad d_g(\gamma(\tau), \gamma(\tau^+)) \geq \eta\right\}. \tag{3.8}$$

In both cases, the cardinality of the set is bounded by $2N$, where N is the minimal number of balls of radius $\eta/4$ that need to be used to cover the compact set $\mathcal{K}$.

3.4. **Describing backward secants.** Before we proceed, we introduce some extra notation. For any $x \in \mathcal{K}$, and $z \in B_g(x, \rho)$ we set

$$v_x(z) := \phi_x^{-1}(z) \in T_x\mathcal{M} \quad \text{and} \quad u_x(z) := \frac{v_x(z)}{|v_x(z)|_x} \in U_x\mathcal{M} \quad (\text{provided } z \neq x). \tag{3.9}$$

By construction, $v_x(z)$ is the initial velocity of the geodesic $\theta : [0,1] \to \mathcal{M}$ joining x to z, so we have $|v_x(z)|_x = d_g(x, z)$. Let us now fix $\tau \in (0, T_\infty)$ and let us define the set of all possible limits of *backward secants* at $x = \gamma(\tau)$ as follows:

$$\sec^-(\tau) := \left\{ q \in U_x\mathcal{M} : \quad q = \lim_{s_k \nearrow \tau} u_x\big(\gamma(s_k)\big) \right\},$$

where $\{s_k\}_k \nearrow \tau$ indicates that $\{s_k\}_k \to \tau$ and $s_k < \tau$ for all k. Notice that $\sec^-(\tau) \neq \emptyset$ for every $\tau > 0$ (*c.f.* compactness of the unit sphere).

Let us now introduce for ease reference the notion of truncated (localized) tail of the curve: given $\tau \in (0, T_\infty)$ and an open neighborhood $\mathcal{U}$ of $x = \gamma(\tau)$, the *$\mathcal{U}$-truncated tail* of γ at x is defined by

$$\Gamma_{\mathcal{U}}(\tau) := \Gamma(\tau) \cap \mathcal{U}. \tag{3.10}$$

The next result is important for our purposes: it asserts that every backward secant at a point $x = \gamma(\tau)$ where the curve is left continuous, is normal to all tangent vectors in $T_x\mathcal{M}$ generated by the truncated tail $\Gamma_{\mathcal{U}}(\tau)$ via the inverse exponential mapping.

Lemma 3.7 (Backward secants)**.** *Let $\mathcal{U}$ be an open neighborhood of $x = \gamma(\tau)$ with* $\operatorname{diam}\mathcal{U} \leq \rho$. *Set $x' = \gamma(\tau^-)$ and recall the notation of* (3.10).

(I) *If $x = x'$ (that is, γ is left continuous at τ), then*

$$\sec^-(\tau) \subset N_{\exp_x^{-1}(\Gamma_{\mathcal{U}}(\tau))}(x) \tag{3.11}$$

that is,

$$\langle q, u_x(z)\rangle_x \le 0\,, \quad \textit{for all } q \in \sec^-(\tau) \textit{ and } z \in \Gamma_{\mathcal{U}}(\tau)\setminus\{x\}.$$

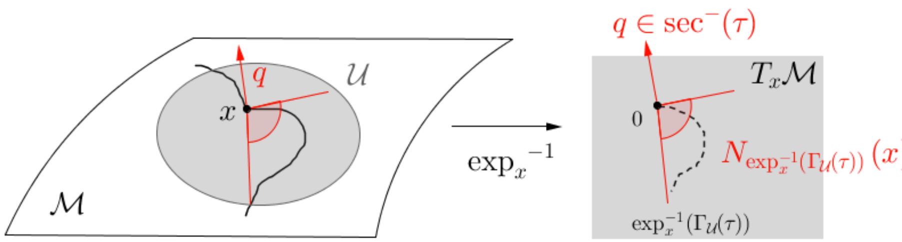

FIGURE 2. $\sec^-(\tau) := \{u_x(x')\} \not\subseteq N_{\exp_x^{-1}(\Gamma_{\mathcal{U}}(\tau))}(x)$

(II) *If* $x \neq x'$ *and* $x' \in B_g(x, 2\rho)$ *then*

$$\sec^-(\tau) = \{u_x(x')\}.$$

Proof. (I) Let $q \in \sec^-(\tau)$. Then for some $s_k \nearrow \tau$ we have

$$v_k := \exp_x^{-1}(\gamma(s_k)) \quad \text{and} \quad q = \lim_{k\to\infty} \frac{v_k}{|v_k|_x} \quad (\text{in } T_x\mathcal{M}).$$

Clearly $\mathcal{U} \subset B_g(x, 2\rho)$. We may also assume that $\Gamma_{\mathcal{U}}(\tau)\setminus\{x\} \neq \emptyset$ (else the conclusion follows trivially) and $\{\gamma(s_k)\}_k \subset \mathcal{U}$. Pick any $z \in \Gamma_\rho(\tau)\setminus\{x\}$ and notice that since γ is self-contracted, we have for all $k \in \mathbb{N}$

$$d_g\left(\exp_x(v_k), z\right) \ge d_g(x, z).$$

Applying (3.3) for $u := u_x(\gamma(s_k))$, $w := u_x(z)$, $t_1 := d_g(x, \gamma(s_k)) = |v_k|_x$ and $t_2 := d_g(x, z) = |v_x(z)|_x$ we infer that

$$-|v_k|_x^2 + 2\, d_g(x, z)\, \langle v_k, u_x(z)\rangle_x \le K\, |v_k|_x^2\, d_g(x, z)^2.$$

Dividing by $|v_k|$ and passing to the limit as $k \to \infty$ we conclude easily.

(II) It is straightforward since $x \neq x'$ and x' is the limit of $\gamma(s)$ as $s \nearrow \tau$. $\square$

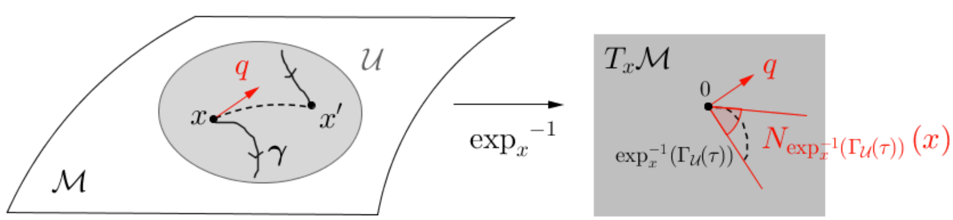

FIGURE 3. $\sec^-(\tau) := \{u_x(x')\} \not\subseteq N_{\exp_x^{-1}(\Gamma_{\mathcal{U}}(\tau))}(x)$

Remark 3.8. Notice that for $\tau \in D^-$, the backward secant is unique (*c.f.* Lemma 3.7 (II)), but (3.11) may fail. An illustration is given in FIG. 3.

3.5. **Aperture of the truncated tail.** Given any subset C of the unit sphere of $\mathbb{R}^d$, we define its *aperture* $A(C)$ as follows:

$$A(C) := \inf\left\{ \langle u_1, u_2 \rangle \; : \; u_1, u_2 \in C \right\} . \tag{3.12}$$

More generally, for every $y \in \mathcal{M}$ and $\Gamma \subset B_g(x, 2\rho)$, setting

$$C := \{u_y(z) : z \in \Gamma, z \neq x\},$$

we define (the *aperture* of $\Gamma \subset \mathcal{M}$ at $y \in \mathcal{M}$):

$$A_y(\Gamma) := \Big(A(C) = \Big) \; \inf\Big\{ \langle u_y(z_1), u_y(z_2) \rangle_x \; : \; z_1, z_2 \in \Gamma \setminus \{y\} \Big\}. \tag{3.13}$$

Roughly speaking, the aperture of a subset Γ of a manifold $\mathcal{M}$ (with respect to a point $y \in \mathcal{M}$) measures the size of the cone generated by the unit tangents $u \in T_y\mathcal{M}$ at y corresponding to all points $z \in \Gamma \setminus \{y\}$ via the mapping $\phi_x^{-1} := \exp_x^{-1}$ (that is, $u = u_y(z)$, according to the notation (3.9)).

The aperture will play a major role in the sequel. The set Γ will be taken to be the (truncated) tail $\Gamma_{\mathcal{U}}(\tau)$ of the self-contracted curve γ (determined by $\tau \in (0, T_\infty)$ and an open neighborhood $\mathcal{U}$ of $x = \gamma(\tau)$), and the point $y \in \mathcal{M}$ at which the aperture is taken will be either:

(i) the point $x = \gamma(\tau) \in \Gamma_{\mathcal{M}}(\tau)$ if the curve γ is continuous at τ ; or

(ii) a point $\bar{x}$ lying in the minimal geodesic joining $x = \gamma(\tau)$ to $x' = \gamma(\tau^-)$ (see (3.5)), if γ is left discontinuous at τ.

3.5.1. *Left-continuous case.* Let us assume $\tau \in (0, T_\infty) \setminus D^-$, and let us set $x = \gamma(\tau)$ and consider any open neighborhood $\mathcal{U}$ of x with $\operatorname{diam}\mathcal{U} \leq \rho$. We set

$$C_{x,\mathcal{U}} := \Big\{ u_x(z) \; : \; z \in \Gamma_{\mathcal{U}}(\tau) \setminus \{x\} \Big\} \subset U_x\mathcal{M} \, . \tag{3.14}$$

Lemma 3.9 (Aperture of $\Gamma_{\mathcal{U}}(\tau)$ at x)**.** *Let $r \in (0, \rho)$ and $\mathcal{U}$ be any nonempty open subset of $\mathcal{M}$ with $\operatorname{diam}\mathcal{U} \leq r$. Then for every $\tau \in (0, T_\infty)$ with $x = \gamma(\tau) \in \mathcal{U}$ the following property holds:*

$$A_x(\Gamma_{\mathcal{U}}(\tau)) \left(= A(C_{x,\mathcal{U}}) \right) \; \geq \; -\frac{Kr^2}{2} \, . \tag{3.15}$$

Proof. Set $x := \gamma(\tau)$ and for $i \in \{1, 2\}$ let $z_i = \gamma(t_i) \in \Gamma_{\mathcal{U}}(\tau) \setminus \{x\}$ with $\tau < t_1 \leq t_2$. Self-contractedness of γ yields that $d_g(x, z_2) \geq d_g(z_1, z_2)$, or equivalently,

$$\begin{aligned} |v_x(z_2)|_x^2 = d_g\left(x, \exp_x(v_x(z_2))\right)^2 \;\; &\geq \;\; d_g\left(\exp_x(v_x(z_1)), \exp_x(v_x(z_2))\right)^2 \\ &= \;\; \Phi_x^{u_x(z_1), u_x(z_2)}\left(d_g(x, z_1), d_g(x, z_2)\right) . \end{aligned}$$

where we use the notations introduced in (3.2) and (3.9). Therefore, applying (3.3) with

$$t_i := d_g(x, z_i) = |v_x(z_i)| \, , \quad i \in \{1, 2\}$$

we get

$$-\left|v_x(z_1)\right|_x^2 + 2\left\langle v_x(z_1), v_x(z_2)\right\rangle_x \geq -K\left|v_x(z_1)\right|_x^2 \left|v_x(z_2)\right|_x^2 ,$$

which yields

$$\langle u_x(z_1), u_x(z_2) \rangle_x \geq -\frac{Kr^2}{2} \, .$$

This proves the assertion. $\square$

Remark 3.10. Roughly speaking, the above result asserts that the cone generated by the $\mathcal{U}$-truncated tail $\Gamma_{\mathcal{U}}(\tau)$ at $T_x\mathcal{M}$ has angle almost equal (a bit more than) $\pi/2$, for any open neighborhood $\mathcal{U}$ of x of sufficiently small diameter. This is the Riemannian analogue of [9, Section 3, Formula (2)] (see also [4, Fig. 1]).

3.5.2. *Left-discontinuous case.* Let $\tau \in D^-$ (that is, γ is left-discontinuous at $x = \gamma(\tau)$). In this case, for reasons that will become transparent in Section 3.6 (see also Remark 3.8), we need to consider the aperture of the truncated tail $\Gamma_{\mathcal{U}}(\tau)$ with respect to a different point $\bar{x}$ (other than $x = \gamma(\tau)$). This point will be taken on the minimal geodesic joining x to x' and relatively close to $x' := \gamma(\tau^-)$. To define this geodesic, notice that $\hat{q} := u_x(x')$ is the (unique) left secant of γ at τ (*c.f.* Lemma 3.7 (II)), that is, the initial velocity of the unit speed geodesic $\theta : [0, d_g(x,x')] \to \mathcal{M}$ joining x to x'. For any $\beta \in (0, 1/2)$ we set

$$\bar{x} = \theta\left((1-\beta)\, d_g(x,x')\right) \quad \text{and} \quad \bar{q} = \dot{\theta}\left((1-\beta)\, d_g(x,x')\right) = u_{\bar{x}}(x'). \tag{3.16}$$

(The exact location of the point $\bar{x}$ will be determined in a uniform manner in Lemma 3.17 where we fix a common value $\bar{\beta}$ for all points of left discontinuity $\tau \in D^-$.)

Assuming for the moment that this has been done (therefore, given x (and x') the point $\bar{x}$ is determined unambiguously), we set

$$C_{\bar{x},\mathcal{U}} := \left\{ u_{\bar{x}}(z) \,:\, z \in \Gamma_{\mathcal{U}}(\tau) \right\} \subset U_{\bar{x}}\mathcal{M}\,. \tag{3.17}$$

We seek for good lower bound estimations for the aperture $A_{\bar{x}}(\Gamma_{\mathcal{U}}(\tau)) := A(C_{\bar{x},\mathcal{U}})$.

This would not be an easy task though: Indeed, since $\bar{x}$ is not a point of γ, the previous argument (*c.f.* proof of Lemma 3.9), based on self-contractedness, is no longer valid. Our new task will require several technical estimations (see forthcoming Lemma 3.12 and Lemma 3.13), the adaptation of the constant ρ (given in Remark 3.3) as well as estimating the aperture of $\Gamma_{\mathcal{U}}(\tau)$ at the point x' (which might not be a point of the curve, but belongs to its closure).

Lemma 3.11 (Aperture of $\Gamma_{\mathcal{U}}(\tau)$ at x')**.** *Let $\mathcal{U}$ be an open subset of $\mathcal{M}$ with diameter $r := \operatorname{diam} \mathcal{U} \in (0,\rho)$ and let $\tau \in D^-$ be such that both $x = \gamma(\tau)$ and $x' := \gamma(\tau^-)$ are in $\mathcal{U}$. It holds:*

$$A_{x'}(\Gamma_{\mathcal{U}}(\tau)) \,\geq\, -\frac{Kr^2}{2}\,. \tag{3.18}$$

Proof. The proof is essentially the same as in Lemma 3.9. Since $x' := \lim_{s \nearrow \tau} \gamma(s)$ is a limit of points of γ, the estimation (3.15) holds true for the aperture of $\Gamma_{\mathcal{U}}(\tau)$ at $\gamma(s)$ for all $s \in (0,\tau)$ sufficiently close to τ so that $\mathcal{U} \subset B_g(\gamma(s),\rho)$. We conclude easily by a standard continuity argument. $\square$.

Lemma 3.12 (Technical estimations - I)**.** *Let $\mathcal{U}$ be an open neighborhood of $x = \gamma(\tau)$, where $\tau \in D^-$. Fix any $\beta \in (0, 1/2)$, set $x' = \gamma(\tau^-)$ and $\bar{x} = \theta\left((1-\beta)\, d_g(x,x')\right)$. Then for every $z \in \Gamma_{\mathcal{U}}(\tau)$ one has:*

$$\left(\frac{1-2\beta}{2}\right) d_g(x,x') \leq d_g(\bar{x},z) \tag{3.19}$$

and

$$\left(\frac{1-4\beta}{1-2\beta}\right) \leq \frac{d_g(x',z)}{d_g(\bar{x},z)} \leq \left(\frac{1}{1-2\beta}\right). \tag{3.20}$$

Proof. Since γ is self-contracted, it follows easily that $d_g(x,z) \leq d_g(x',z)$. Therefore

$$d_g(x,x') \leq d_g(x,z) + d_g(x',z) \leq 2d_g(x',z).$$

It follows by (3.16) that $d_g(\bar{x},x')=\beta d_g(x,x')$. Thus, we deduce

$$\frac{1}{2}d_g(x,x')\le d_g(x',z)\le d_g(\bar{x},z)+d_g(\bar{x},x')=d_g(\bar{x},z)+\beta d_g(x,x').$$

which yields (3.19). On the other hand

$$d_g(\bar{x},z)\le d_g(\bar{x},x')+d_g(x',z)=\beta d_g(x,x')+d_g(x',z),$$

which combined with (3.19) yields

$$\left(\frac{1-4\beta}{2}\right)d_g(x,x')\le d_g(x',z). \tag{3.21}$$

Therefore we get

$$d_g(\bar{x},z)\le d_g(\bar{x},x')+d_g(x',z)\le \beta d_g(x,x')+d_g(x',z)\le\left(\frac{1-2\beta}{1-4\beta}\right)d_g(x',z).$$

In an analogous way, using (3.19) again, we deduce

$$d_g(x',z)\le d_g(x',\bar{x})+d_g(\bar{x},z)\le \beta d_g(x,x')+d_g(\bar{x},z)\le\left(\frac{1}{1-2\beta}\right)d_g(\bar{x},z).$$

We conclude easily. □

Lemma 3.13 (Technical estimations - II)**.** *Let $r\in(0,\rho)$, $\beta\in(0,1/8)$ and $\tau\in D^-$ and set $x=\gamma(\tau)$, $x'=\gamma(\tau^-)$ and $\bar{x}=\theta\left((1-\beta)\,d_g(x,x')\right)$ (according to the notation of (3.16)). Then for every open subset $\mathcal{U}$ of $\mathcal{M}$ with $\operatorname{diam}\mathcal{U}\le r$ and $\{x,\bar{x},x'\}\subset\mathcal{U}$ and every $z\in\Gamma_{\mathcal{U}}(\tau)$ we have*

$$d_g(\bar{x},z)^2-d_g(x',x)^2\ge-(1+Kr^2)d_g(\bar{x},x')^2-2d_g(\bar{x},x')d_g(\bar{x},z), \tag{3.22}$$

and

$$d_g(\bar{x},z)^2-d_g(x',x)^2\le 2d_g(\bar{x},x')^2+d_g(\bar{x},x')d_g(x',z)Kr^2. \tag{3.23}$$

Proof. Let $z\in\Gamma_{\mathcal{U}}(\tau)$. Since $x\in\Gamma_{\mathcal{U}}(\tau)$ and $x\neq x'$ we deduce by Lemma 3.11 that

$$c'(x,z):=\langle u_{x'}(x),u_{x'}(z)\rangle_{x'}\ge-\frac{Kr^2}{2}\,. \tag{3.24}$$

Let us set $\sigma:=d_g(x,x')$ (so that $d_g(\bar{x},x')=\beta\sigma$),

$$\bar{d}:=d_g(\bar{x},z)\qquad\text{and}\qquad d':=d_g(x',z).$$

We also set

$$\bar{c}(x',z):=\langle u_{\bar{x}}(x'),u_{\bar{x}}(z)\rangle_{\bar{x}}\ \big(\ge-1\big).$$

Let us first apply Lemma 3.2 for the function $\Phi_{\bar{x}}^{u_{\bar{x}}(x'),u_{\bar{x}}(z)}$, with $t_1=d_g(\bar{x},x')=\beta\sigma$ and $t_2=\bar{d}:=d_g(\bar{x},z)$. Observing that

$$d'=d_g(x',z)=\left[\Phi_{\bar{x}}^{u_{\bar{x}}(x'),u_{\bar{x}}(z)}(\beta\sigma,\bar{d})\right]^{1/2},$$

we deduce readily

$$\left|d'^2-(\beta\sigma)^2-\bar{d}^2+2\beta\sigma\,\bar{d}\,\bar{c}(x',z)\right|\le K\,(\beta\sigma)^2\,\bar{d}^2.$$

Since $\bar{c}(x',z)\ge-1$ and $\bar{d}\le r$ (recall that $\bar{x},\,z\in\mathcal{U}$) we deduce

$$\bar{d}^2-d'^2\ge-K\,(\beta\sigma)^2\,\bar{d}^2-(\beta\sigma)^2+2\beta\sigma\,\bar{d}\,\bar{c}(x',z)\ge-(1+Kr^2)\,(\beta\sigma)^2-2\,(\beta\sigma)\,\bar{d},$$

thus (3.22) holds.

To establish (3.23), we apply Lemma 3.2 for $\Phi_{x'}^{u_{x'}(\bar{x}),u_{x'}(z)}$, with $t_1 = d_g(\bar{x},x') = \beta\sigma$ and $t_2 = d' := d_g(x',z)$. In this case we have

$$\bar{d} := d_g(\bar{x},z) = \left[\Phi_{x'}^{u_{x'}(\bar{x}),u_{x'}(z)}(\beta\sigma,d')\right]^{1/2}.$$

Setting

$$c'(\bar{x},z) := \langle u_{x'}(\bar{x}),u_{x'}(z)\rangle_{x'} = \langle u_{x'}(x),u_{x'}(z)\rangle_{x'} \;\left(\geq -\frac{Kr^2}{2}\right) \tag{3.25}$$

we obtain readily

$$\left|\bar{d} - (\beta\sigma)^2 - d'^2 + 2\,(\beta\sigma)\;d'\,c'(\bar{x},z)\right| \leq K\,(\beta\sigma)^2\,d'^2.$$

In view of (3.25) the above yields

$$\bar{d}^2 - d'^2 \leq (1+Kd'^2)\,(\beta\sigma)^2 + (\beta\sigma)\,d'Kr^2.$$

Since $d' \leq r$ and $Kr^2 \leq 1$ (*c.f.* (3.4)) we conclude easily. $\square$

We are now ready to state the following quantitative result for the aperture of $\Gamma_{\mathcal{U}}(\tau)$ with respect to $\bar{x}$. Let now $\tau \in D^-$ and let $\mathcal{U}$ be open subset of $\mathcal{M}$ with $\mathrm{diam}\,\mathcal{U} \leq \bar{r}$ containing $\theta([0,d_g(x,x'])$ (*c.f.* notation of (3.16)).

Proposition 3.14 (Aperture of $\Gamma_{\mathcal{U}}(\tau)$ at $\bar{x}$)**.** *Let $r \in (0,\rho)$, $\beta \in (0,1/8)$ and $\tau \in D^-$ and set $x = \gamma(\tau)$, $x' = \gamma(\tau^-)$ and $\bar{x} = \theta\left((1-\beta)\,d_g(x,x')\right)$ (according to the notation of (3.16)). Then for every open subset $\mathcal{U}$ of $\mathcal{M}$ with $\mathrm{diam}\,\mathcal{U} \leq r$ and $\{x,\bar{x},x'\} \subset \mathcal{U}$ we have*

$$A_{\bar{x}}(\Gamma_{\mathcal{U}}(\tau)) \geq -4K\,r^2 - 8\,\beta\,.$$

Proof. Since $x' \notin \Gamma_{\mathcal{U}}(\tau)$ we deduce by Lemma 3.11 that for every $z_1,z_2 \in \Gamma_{\mathcal{U}}(\tau)$ ut holds

$$c'(z_1,z_2) := \langle u_{x'}(z_1),u_{x'}(z_2)\rangle_{x'} \geq -\frac{Kr^2}{2}\,. \tag{3.26}$$

In order to simplify notation, let us set, as before, $\sigma := d_g(x,x')$ and

$$\begin{cases} d_i := d_g(x,z_i) \\ \bar{d}_i := d_g(\bar{x},z_i) \\ d'_i := d_g(x',z_i) \end{cases} \qquad \text{for } i \in \{1,2\}.$$

Applying Lemma 3.2 for $\Phi_{x'}^{u_{x'}(z_1),u_{x'}(z_2)}$ and setting

$$e := d_g(z_1,z_2) = \left[\Phi_{x'}^{u_{x'}(z_1),u_{x'}(z_2)}(d'_1,d'_2)\right]^{1/2}$$

we obtain

$$\left|e^2 - d'^2_1 - d'^2_2 + 2\,d'_1\,d'_2\,c'(z_1,z_2)\right| \leq K d'^2_1\,d'^2_2. \tag{3.27}$$

In an analogous manner, applying Lemma 3.2 for $\Phi_{\bar{x}}^{u_{\bar{x}}(z_1),u_{\bar{x}}(z_2)}$ and setting again

$$e := d_g(z_1,z_2) = \left[\Phi_{\bar{x}}^{u_{\bar{x}}(z_1),u_{\bar{x}}(z_2)}(\bar{d}_1,\bar{d}_2)\right]^{1/2}$$

and

$$\bar{c}(z_1,z_2) := \langle u_{\bar{x}}(z_1),u_{\bar{x}}(z_2)\rangle_{\bar{x}}$$

we obtain

$$\left|e^2 - \bar{d}_1^2 - \bar{d}_2^2 + 2\bar{d}_1\bar{d}_2\bar{c}(z_1,z_2)\right| \leq K\bar{d}_1^2\bar{d}_2^2. \tag{3.28}$$

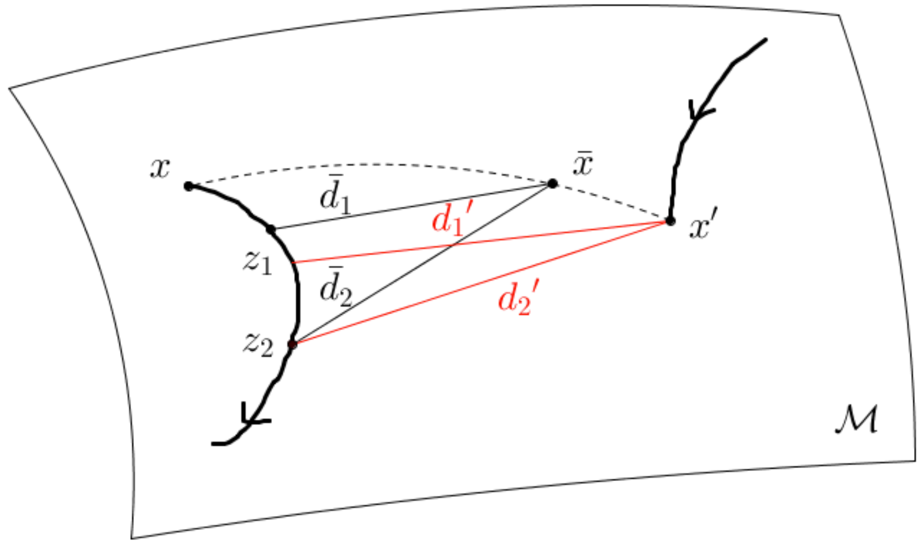

FIGURE 4. $\sec^-(\tau) := \{u_x(x')\} \not\subseteq N_{\exp_{\bar{x}}^{-1}(\Gamma_{\mathcal{U}(\tau)})}(x)$

Combining (3.27) and (3.28) we deduce

$$2\bar{d}_1\bar{d}_2\bar{c}(z_1,z_2) - 2d_1'd_2'c'(z_1,z_2) \geq -K\left(\bar{d}_1^2\bar{d}_2^2 + d_1'^2d_2'^2\right) + \bar{d}_1^2 - d_1'^2 + \bar{d}_2^2 - d_2'^2,$$

thus in particular

$$\bar{c}(z_1,z_2) \geq \left(\frac{d_1'd_2'}{\bar{d}_1\bar{d}_2}\right)c'(z_1,z_2) - \frac{K}{2}\bar{d}_1\bar{d}_2\left(1 + \left(\frac{d_1'd_2'}{\bar{d}_1\bar{d}_2}\right)^2\right) + \frac{\bar{d}_1^2 - d_1'^2}{2\,\bar{d}_1\,\bar{d}_2} + \frac{\bar{d}_2^2 - d_2'^2}{2\,\bar{d}_1\,\bar{d}_2}. \qquad (3.29)$$

To proceed, we need to bound the last two terms of (3.29). Applying Lemma 3.13 we obtain

$$\bar{d}_i^2 - d_i'^2 \geq -(1+Kr^2)\,(\beta\sigma)^2 - 2\,(\beta\sigma)\,\bar{d}_i\,, \quad \text{for } i \in \{1,2\}\,,$$

thus, dividing by $2\,\bar{d}_1\bar{d}_2$ we deduce in view of (3.19) and (3.4) that

$$\frac{\bar{d}_i^2 - d_i'^2}{2\,\bar{d}_1\bar{d}_2} \geq -\frac{4\beta^2}{(1-2\beta)^2} - \frac{2\beta}{1-2\beta} = -\frac{2\beta}{(1-2\beta)^2}.$$

Using the above estimation, together with (3.26), (3.19) and (3.20), we deduce from (3.29) that

$$\bar{c}(z_1,z_2) \geq -\left(\frac{1-4\beta}{1-2\beta}\right)^2\frac{K}{2}r^2 - \frac{K}{2}r^2\left(1 + \frac{1}{(1-2\beta)^4}\right) - \frac{4\beta}{(1-2\beta)^2}.$$

We conclude easily. $\square$

The following result is the analogue of Lemma 3.7 (I) for the left-discontinuous case. Roughly speaking, the result (almost) remedies the failure illustrated in Remark 3.8 by moving the point $x = \gamma(\tau)$ (where γ is left-discontinuous) to $\bar{x} := \theta\left((1-\beta)\,d_g(x,x')\right)$ (see (3.16)) and making a parallel transportation of the secant $q := u_x(x')$ at x to $\bar{q} \in T_{\bar{x}}\mathcal{M}$ along the geodesic θ joining x to $\bar{x}$.

Proposition 3.15 (Transported secant)**.** *Under the above notation and same assumptions as in* Proposition 3.14, *it holds:*

$$\langle \bar q, u_{\bar x}(z)\rangle_{\bar x} := \langle u_{\bar x}(x'), u_{\bar x}(z)\rangle_{\bar x} \le 4\,\beta + 2\,Kr^2, \quad \textit{for all } z \in \Gamma_{\mathcal{U}}(\tau).$$

Proof. Let $z \in \Gamma_{\mathcal{U}}(\tau)$, set

$$\sigma := d_g(x, x'), \quad \bar d = d_g(\bar x, z) \quad \text{and} \quad \bar c(x', z) := \langle u_{\bar x}(x'), u_{\bar x}(z)\rangle_{\bar x}.$$

Similarly to the proof of Lemma 3.13, we apply Lemma 3.2 for the function $\Phi_{\bar x}^{u_{\bar x}(x'), u_{\bar x}(z)}$ with $t_1 = d_g(\bar x, x') = \beta\sigma$, $t_2 = \bar d := d_g(\bar x, z)$ and

$$d' = d_g(x', z) = \left[\Phi_{\bar x}^{u_{\bar x}(x'), u_{\bar x}(z)}(\beta\sigma, \bar d)\right]^{1/2}.$$

We deduce readily

$$d'^2 - (\beta\sigma)^2 - \bar d^2 + 2\beta\sigma\,\bar d\,\bar c(x', z) \le K\,(\beta\sigma)^2\,\bar d^2. \tag{3.30}$$

Notice that (3.23) yields

$$\frac{\bar d^2 - d'^2}{2\beta\sigma\,\bar d} \le \beta\left(\frac{\sigma}{\bar d}\right) + \left(\frac{d'}{\bar d}\right)\frac{K}{2}r^2. \tag{3.31}$$

Combining (3.30) with (3.31) and using (3.19) and (3.20) we deduce (recall that $\beta\sigma \le r$ and $\bar d \le r$) we get

$$\bar c(x', z) \le \frac{K}{2}r^2 + \left(\frac{3\beta\sigma}{2\bar d}\right) + \left(\frac{d'}{\bar d}\right)\frac{K}{2}r^2 \le \frac{K}{2}r^2 + \left(\frac{3\beta}{1-2\beta}\right) + \left(\frac{1}{1-2\beta}\right)\frac{K}{2}r^2.$$

We conclude easily. $\square$

3.6. **Estimations involving "almost secants".** We shall now modify the (backward) secant $q \in sec^-(\tau)$, if γ is left-continuous at τ (respectively the transported secant $\bar q := u_{\bar x}(x')$, if γ is left-discontinuous at τ) to obtain a nearby direction $\hat p$ of $T_x\mathcal{M}$ (respectively of $T_{\bar x}\mathcal{M}$). This direction will be called an "almost secant" at x (respectively at $\bar x$) and will be used to quantify the (backward) growth of the self-contracted curve γ. This will be done for all points of left-continuity as well as for all points of left discontinuous up to a certain discontinuity jump.

Lemma 3.16 (Strong separation lemma)**.** *Let C be a nonempty subset of the unit sphere of $\mathbb{R}^d$ satisfying*

$$\langle u_1, u_2\rangle \ge -\delta, \quad \textit{for all } u_1, u_2 \in C\,, \tag{3.32}$$

where

$$\delta = \frac{1}{2(d+1)}. \tag{3.33}$$

Then

$$\overline{\text{conv}}\,(C) \bigcap B(0,\delta) = \emptyset\,. \tag{3.34}$$

Proof. Let us assume, towards a contradiction, that for some $u \in \overline{\text{conv}}(C)$ we have $||u|| < \delta$. By Caratheodory's lemma there exist $\lambda_0, \ldots, \lambda_d \in [0,1]$ with $\sum_{i=0}^d \lambda_i = 1$ and unit vectors $u_0, \ldots, u_d \in C$ such that

$$\left\|\sum_{i=0}^d \lambda_i u_i\right\| < \delta.$$

Let $i_0 \in \{0, \ldots, d\}$ be such that $\lambda_{i_0} \geq \lambda_i$ for any $i \in \{0, \ldots, d\}$. Then $\lambda_{i_0} \geq 1/(d+1)$ and by the Cauchy-Schwarz inequality

$$\begin{aligned}\delta > \left\langle u_{i_0}, \sum_{i=0}^{d} \lambda_i u_i \right\rangle = \sum_{i=0}^{d} \lambda_i \langle u_{i_0}, u_i \rangle = \lambda_{i_0} + \sum_{i \neq i_0} \lambda_i \langle u_{i_0}, u_i \rangle \\ > \frac{1}{d+1} - \delta \left(\sum_{i \neq i_0} \lambda_i \right) > \frac{1}{d+1} - \delta = \delta,\end{aligned}$$

a clear contradiction. Thus the assertion holds true. □

We shall now fix the values of some constants which will be used in a crucial way in the forthcoming estimations. We set

$$\bar{\alpha} = \frac{1}{32(d+1)^2} \; := \frac{\delta^2}{8}, \qquad (\text{where } d = \dim \mathcal{M}) \tag{3.35}$$

and we chose $\bar{r} \in (0, \rho)$, $\bar{\beta} \in (0, 1/8)$ sufficiently small to ensure that

$$\begin{cases} 4K\,\bar{r}^2 + 8\,\bar{\beta} < \delta \\ 2K\,\bar{r}^2 + 4\,\bar{\beta} < \bar{\alpha}. \end{cases} \tag{3.36}$$

Thus in view of Lemma 3.9 and respectively of Proposition 3.14 we deduce that $A(C_{x,\mathcal{U}}) \geq -\delta$, that is,

$$\langle u_1, u_2 \rangle_x \geq -\delta, \quad \text{for all } u_1, u_2 \in C_{x,\mathcal{U}} \tag{3.37}$$

and respectively $A(C_{\bar{x},\mathcal{U}}) \geq -\delta$, that is,

$$\langle u_1, u_2 \rangle_{\bar{x}} \geq -\delta, \quad \text{for all } u_1, u_2 \in C_{\bar{x},\mathcal{U}}\,. \tag{3.38}$$

We are now ready to state a quantitative result for a bunch of almost secant directions.

Lemma 3.17 (measuring growth using "almost secants"). *Let $\bar{r} \in (0, \rho)$, $\bar{\beta} \in (0, 1/8)$ and $\bar{\alpha} > 0$ be as in* (3.35)–(3.36).

(i) *for every $\tau \in (0, T_\infty) \setminus D^-$ (that is, $x = \gamma(\tau) = \gamma(\tau^-)$) and for every secant $q \in \sec^-(\tau)$, there exists $\hat{p} \in U_x\mathcal{M}$ such that for every open subset $\mathcal{U}$ of $\mathcal{M}$ with $x \in \mathcal{U}$ and* $\operatorname{diam}\mathcal{U} \leq \bar{r}$, *every $p \in \hat{B}_x(\hat{p}, \bar{\alpha})$ and $u \in C_{x,\mathcal{U}}$ it holds*

$$\langle p, u \rangle_x \leq -\bar{\alpha} \qquad \textit{and} \qquad \langle p, q \rangle_x \geq \bar{\alpha}\,.$$

(ii) *for every $\tau \in D^-$, for $\bar{x}$ and $\bar{q} = u_{\bar{x}}(x')$ (transported secant at $\bar{x}$), there exists $\hat{p} \in U_{\bar{x}}\mathcal{M}$ such that for every open subset $\mathcal{U}$ of $\mathcal{M}$ with $\{x, \bar{x}, x'\} \subset \mathcal{U}$ and* $\operatorname{diam}\mathcal{U} \leq \bar{r}$, *every $p \in \hat{B}_{\bar{x}}(\hat{p}, \bar{\alpha})$ and $u \in C_{\bar{x},\mathcal{U}}$ it holds:*

$$\langle p, u \rangle_{\bar{x}} \leq -\bar{\alpha} \qquad \textit{and} \qquad \langle p, \bar{q} \rangle_{\bar{x}} \geq \bar{\alpha}\,.$$

Proof. Both assertions follow by the same arguments and estimations. In order to present a common proof let us proceed to the following identification:

– If $x = \gamma(\tau) = \gamma(\tau^-)$, we identify the tangent space $T_x\mathcal{M}$ with the Euclidean space $\mathbb{R}^d$ equipped with the scalar product $\langle \cdot, \cdot \rangle_x$.

– If $x = \gamma(\tau) \neq \gamma(\tau^-)$, we identify the tangent space $T_{\bar{x}}\mathcal{M}$ with the Euclidean space $\mathbb{R}^d$ equipped with the scalar product $\langle \cdot, \cdot \rangle_{\bar{x}}$.

In the sequel, we shall denote (in both cases) this scalar product by $\langle \cdot, \cdot \rangle$. We further set

$$C = C_{x,\mathcal{U}} \qquad \text{(respectively} \quad C = C_{\bar{x},\mathcal{U}}\,).$$

In view of (3.37) and (3.38) we deduce that condition (3.32) of Lemma 3.16 holds true. Therefore, the projection of 0 to $\overline{\mathrm{conv}}(C)$, denoted by $\hat{c} \in T_x\mathcal{M}$ satisfies:

$$||\hat{c}|| \geq \delta \qquad \text{and} \qquad \langle -\hat{c}, u - \hat{c} \rangle \leq 0,\ \forall u \in C.$$

Recalling that C is made up of unit vectors, we deduce that for every $u \in C$ it holds:

$$\langle -\hat{c}, u \rangle \leq -||\hat{c}||^2 \leq -\delta^2 = -8\bar{\alpha}. \tag{3.39}$$

(i) Let $\tau \in (0, T_\infty) \setminus D^-$ and fix any backward secant $q \in \sec^-(\tau) \in T_x\mathcal{M} \equiv \mathbb{R}^d$ and set

$$\hat{p} := \frac{q - \hat{c}}{||q - \hat{c}||}.$$

By Lemma 3.7 (I) we get $\langle q, u \rangle \leq 0$, for all $u \in C$. Pick now any $p \in \hat{B}_x(\hat{p}, \bar{\alpha}) \equiv B(\hat{p}, \bar{\alpha})$, that is, $p = \hat{p} + v$ for some $v \in T_x\mathcal{M} \equiv \mathbb{R}^d$ with $||v|| \leq \bar{\alpha}$. Then for every $u \in C$ (unit vector) in view of (3.39) we deduce

$$\langle p, u \rangle = \langle \hat{p} + v, u \rangle \leq \frac{\langle q, u \rangle + \langle -\hat{c}, u \rangle}{||q - \hat{c}||} + ||v|| \leq \frac{0 - ||\hat{c}||^2}{||q - \hat{c}||} + \bar{\alpha} \leq \frac{-8\bar{\alpha}}{||q - \hat{c}||} + \bar{\alpha} \leq -3\bar{\alpha} \leq -\bar{\alpha},$$

where the fact that $||q - \hat{c}|| \leq 2$ is used. Finally,

$$\langle p, q \rangle = \langle \hat{p} + v, q \rangle \geq \frac{||q||^2 + \langle -\hat{c}, q \rangle}{||q - \hat{c}||} - ||v|| \geq \frac{1 + 0}{2} - \bar{\alpha} \geq \bar{\alpha}.$$

(ii) Let $\tau \in D^-$ and consider the transported secant $\bar{q} = u_{\bar{x}}(x') \in T_x\mathcal{M} \equiv \mathbb{R}^d$ at $\bar{x}$. In an analogous manner to the above, we set

$$\hat{p} := \frac{\bar{q} - \hat{c}}{||\bar{q} - \hat{c}||}.$$

By Proposition 3.15 we get

$$\langle \bar{q}, u \rangle \leq \bar{\alpha}, \quad \text{for all } u \in C\,. \tag{3.40}$$

Since $\hat{c} \in C$ we deduce

$$||\bar{q} - \hat{c}||^2 = ||\bar{q}||^2 + ||\hat{c}||^2 - \langle \bar{q}, \hat{c} \rangle \geq 1 + \delta^2 - \bar{\alpha} = 1 + 7\bar{\alpha} \geq 1.$$

In particular

$$1 \;\leq\; ||\bar{q} - \hat{c}|| \;\leq\; 2\,. \tag{3.41}$$

Let $p \in \hat{B}_{\bar{x}}(\hat{p}, \bar{\alpha}) \equiv B(\hat{p}, \bar{\alpha})$, that is, $p = \hat{p} + v$ for some $v \in T_{\bar{x}}\mathcal{M} \equiv \mathbb{R}^d$ with $||v|| \leq \bar{\alpha}$. Then for every $u \in C$ (unit vector) in view of (3.40) we deduce

$$\langle p, u \rangle = \langle \hat{p} + v, u \rangle \leq \frac{\langle \bar{q}, u \rangle + \langle -\hat{c}, u \rangle}{||\bar{q} - \hat{c}||} + ||v|| \leq \frac{\bar{\alpha} - ||\hat{c}||^2}{||\bar{q} - \hat{c}||} + \bar{\alpha} \leq \frac{-7\bar{\alpha}}{||\bar{q} - \hat{c}||} + \bar{\alpha} \leq -\bar{\alpha}.$$

To conclude, using again (3.40) together with (3.41) we get

$$\langle p, \bar{q} \rangle = \langle \hat{p} + v, \bar{q} \rangle \geq \frac{||\bar{q}||^2 + \langle -\hat{c}, \bar{q} \rangle}{||\bar{q} - \hat{c}||} - ||v|| \geq \frac{1 - \bar{\alpha}}{||\bar{q} - \hat{c}||} - \bar{\alpha} \geq 1 - 2\bar{\alpha} \geq \bar{\alpha}.$$

This concludes the proof of the assertion. $\square$

3.7. **Width estimates via external tangents.** We shall now define external functions which play the role of the projected width in the Euclidean case. To this end, for $\rho > 0$ given by Lemma 3.1, and $\xi = (y, p) \in U\mathcal{K}$ (that is, $y \in \mathcal{K}$ and $p \in T_y\mathcal{M}$ with $|p|_y = 1$) we define the smooth function

$$\begin{cases} b_\xi : B_g(y, 2\rho) \to \mathbb{R} \\ b_\xi(z) := b_{(y,p)}(z) = \langle p, v_y(z)\rangle_y. \end{cases} \tag{3.42}$$

We underline two important properties of the above mapping that will be used in the sequel.

Lemma 3.18 (Properties of b_ξ)**.** *The following properties hold:*
(i) *for every $\xi = (x, p)$ in $U\mathcal{K}$ it holds:*

$$\nabla b_{(x,p)}(x) = p \in T_x\mathcal{M} \tag{3.43}$$

(ii) *there exists $L > 0$ such that for every $\xi = (y, p) \in U\mathcal{K}$ and $x, z \in B_g(y, \rho)$ it holds (recall notation* (3.9)*):*

$$|b_\xi(z) - (b_\xi(x) + \langle \nabla b_\xi(x), v_x(z)\rangle_x)| \le L|v_x(z)|_x^2\,. \tag{3.44}$$

Proof. Let us recall from Lemma 3.1 the notation $\phi_x := \exp_x$ for the exponential mapping and let us notice that $b_\xi(z) := \langle p, \phi_x^{-1}(z)\rangle_x$. Since $D\phi_x^{-1}(x)$ is the identity mapping on $T_x\mathcal{M}$ it follows by the chain rule that $Db_\xi(y) := Db_{(x,p)}(x) = \langle p, \cdot\rangle_x$. This proves (i).

Let us now observe that the mapping

$$(\xi, x) := ((y, p), x) \mapsto b_\xi(x) := \langle p, \phi_y^{-1}(x)\rangle_y \tag{3.45}$$

is smooth (whenever it is well-defined, that is, $d_g(x, y) \le 2\rho$). The second assertion follows by considering the exact Taylor expansion of order 2 for the function $z \mapsto b_\xi(z)$ at the point x, together with the compactness of $\mathcal{K}$ and $U\mathcal{K}$ and a standard argument. $\square$.

The following result is crucial for our purposes. Roughly speaking it will be used to associate to each pair $(x, \hat{p}) \in U\mathcal{M}$ (in the left-continuous case – *c.f.* Lemma 3.17(i)) and respectively $(x, \hat{p}) \in U\mathcal{M}$ (in the left-discontinuous case – *c.f.* Lemma 3.17(ii)) an element $\xi = (y, p) \in U\mathcal{K}$ (among a finite prescribed family). Each such ξ will provide an "external" tangent at x (namely, the tangent vector $\nabla b_\xi(x) \in T_x\mathcal{M}$) and respectively at $\bar{x}$ (namely, $\nabla b_\xi(\bar{x}) \in T_{\bar{x}}\mathcal{M}$) which turns out to satisfy almost the same estimations as in Lemma 3.17. In this way we shall eventually replace (the infinite set of) "almost secants" by (the set of) external tangents. As we shall show in the sequel, this later will be described by finitely many generators, as a consequence of the compactness.

Corollary 3.19 (Approximating "almost secants" by external tangents)**.** *Let $\bar{\alpha} > 0$ be given by* (3.35)*. Then there exists $\hat{r} \in (0, \rho)$ such that for every $\hat{\xi} = (x, \hat{p}) \in U\mathcal{K}$, $\xi = (y, p) \in B_D(\hat{\xi}, \hat{r})$ (Riemannian ball in the unit bundle $U\mathcal{M}$), and $z \in B_g(x, \hat{r})$ we have:*

$$\left|\nabla b_{(y,p)}(x) - \hat{p}\right|_x < \frac{\bar{\alpha}}{4}\,; \tag{3.46}$$

and

$$|b_\xi(z) - (b_\xi(x) + \langle \nabla b_\xi(x), v_x(z)\rangle_x)| \le \frac{\bar{\alpha}}{4}|v_x(z)|_x\,. \tag{3.47}$$

Proof. Since the mapping

$$(\xi, x) := ((y, p), x) \mapsto Db_\xi(x) := \langle p, D\phi_y^{-1}(x)(\cdot)\rangle_y$$

is continuous, we deduce easily from the compactness of $\mathcal{K}$ and $U\mathcal{K}$ and relation (3.43) that there exists $\hat{r} > 0$ such that for all $\hat{\xi} = (x, \hat{p})$, $\xi = (y, p)$ in $U\mathcal{K}$ satisfying $D_g((x, \hat{p}), (y, p)) < \hat{r}$ we have

$$\left|\nabla b_{(y,p)}(x) - \nabla b_{(x,\hat{p})}(x)\right|_x = \left|\nabla b_{(y,p)}(x) - \hat{p}\right|_x < \frac{\bar{\alpha}}{4}.$$

For the second assertion, let $L > 0$ be given by Lemma 3.18 (ii). We shrink $\hat{r} > 0$ if necessary to ensure that $\hat{r} \le \min\{\rho, \bar{\alpha}/4L\}$ and choose any $z \in B_g(x, \hat{r})$. Since $d_g(x, z) = |v_x(z)|_x < \bar{\alpha}/4L$ we deduce readily (3.47) from (3.44). $\square$.

Let us now apply the previous result to the case where $\hat{\xi} = (x, \hat{p}) \in U\mathcal{K}$ is directly related to our self-contracted curve. We consider two cases:

- Case $\tau \in (0, T_\infty) \setminus D^-$ (point of left-continuity of γ): for every backward secant $q \in \sec^-(\tau)$ at $x = \gamma(\tau)$ we associate its (almost secant) approximation $\hat{p} \in U_x\mathcal{K}$ (*c.f.* Lemma 3.17 (i)) and we set
$$\hat{\xi} := (x, \hat{p})\,. \tag{3.48}$$
(Notice that different secants at x might give rise to different $\hat{p} \in U_x\mathcal{K}$ (therefore to different elements $\hat{\xi} \in U\mathcal{K}$).
- Case $\tau \in D^- \setminus \mathcal{D}^-(\eta)$ (point of left-discontinuity of γ where η is determined in the next lemma). In this case the backward secant $q := u_x(x')$ at $x = \gamma(\tau)$ is unique). We set $\bar{x} := \theta\,((1-\beta)\,d_g(x, x'))$, $\bar{q} := u_{\bar{x}}(x')$, see (3.16), and consider $\hat{p} \in U_{\bar{x}}\mathcal{K}$ (*c.f.* Lemma 3.17 (ii)). We set:
$$\hat{\xi} := (\bar{x}, \hat{p}). \tag{3.49}$$

Under the above notation we have:

Lemma 3.20 (Approximating estimations)**.** *There exists $\eta \in (0, \rho)$ such that the following statements hold:*

(I) *Let $\tau \in (0, T_\infty) \setminus D^-$ and $q \in \sec^-(\tau)$, and let $\hat{\xi} := (x, \hat{p})$ be defined as in* (3.48)*. Then for every $\xi = (y, p) \in B_D(\hat{\xi}, \eta)$, setting $\mathcal{U} := B_g(y, 2\eta)$ we have*
$$b_\xi(z) \le b_\xi(x) - \frac{3\bar{\alpha}}{4} d_g(x, z), \quad \forall z \in \Gamma_{\mathcal{U}}(\tau) \setminus \{x\}, \tag{3.50}$$
and
$$\langle \nabla b_\xi(x), q\rangle_x \ge \bar{\alpha}. \tag{3.51}$$

(II) *Let $\mathcal{D}^-(\eta)$ be as in* (3.7) *(for this value of $\eta > 0$) and $\tau \in D^- \diagdown \mathcal{D}^-(\eta)$ (that is, $0 < d_g(x, x') < \eta$). Let $\hat{\xi} := (\bar{x}, \hat{p})$ be defined by (3.49). Then for every $\xi = (y, p) \in B_D(\hat{\xi}, \eta)$, setting $\mathcal{U} = B_g(y, 2\eta)$ we have*
$$b_\xi(z) \le b_\xi(\bar{x}) - \frac{3\bar{\alpha}}{4} d_g(\bar{x}, z), \quad \forall z \in \Gamma_{\mathcal{U}}(\tau), \tag{3.52}$$
and
$$\langle \nabla b_\xi(\bar{x}), \bar{q}\rangle_{\bar{x}} \ge \bar{\alpha}. \tag{3.53}$$

Proof. Let $\bar{r} > 0$ be given by (3.36) and $\hat{r} > 0$ be given by Corollary 3.19. Shrinking the latter if necessary, we may assume $\hat{r} \le \bar{r}/2$. Fix now any $\eta \in (0, \hat{r}/3)$.

We shall first consider the case $\tau \in (0, T_\infty) \setminus D^-$. We fix $q \in \sec^-(\tau)$ and set $\hat{\xi} := (x, \hat{p})$. Let $\xi = (y, p) \in B_D(\hat{\xi}, \eta)$. It follows from (3.1) that $d_g(x, y) < \eta$. Therefore,

$$\mathcal{U} := B_g(y, 2\eta) \subset B_g(x, \hat{r}) \quad \text{and} \quad \operatorname{diam} \mathcal{U} \le \bar{r}.$$

By Corollary 3.19 we obtain $\nabla b_\xi(x) \in \hat{B}_x(\hat{p}, \bar{\alpha}/4)$, therefore we deduce from Lemma 3.17 (i) that for all $z \in \Gamma_{\mathcal{U}}(\tau) \setminus \{x\}$ it holds

$$\langle \nabla b_\xi(x), u_x(z)\rangle_x \le -\bar{\alpha} \qquad \text{and} \qquad \langle \nabla b_\xi(x), q\rangle_x \ge \bar{\alpha}\,.$$

Recalling notation from (3.9) we have for all $z \in \Gamma_{\mathcal{U}}(\tau)$, $z \neq x$

$$v_x(z) = |v_x(z)|_x u_x(z) = d_g(x,z) u_x(z).$$

Using the above we deduce from (3.47) that

$$b_\xi(z) - b_\xi(x) \le \langle \nabla b_\xi(x), v_x(z)\rangle_x + \frac{\bar{\alpha}}{4}|v_x(z)|_x \le \Big(\langle \nabla b_\xi(x), u_x(z)\rangle_x + \frac{\bar{\alpha}}{4}\Big)\, d_g(x,z) \le -\frac{3\bar{\alpha}}{4} d_g(x,z).$$

The case $\tau \in D^- \diagdown \mathcal{D}^-(\eta)$ follows in a similar way under obvious amendments. $\square$

Remark 3.21 (Determining an η-net)**.** We say that a finite subset $\mathcal{F}$ of $U\mathcal{K}$ is an η-net if $\mathcal{F}$ has an nonempty intersection with any ball $B_D(\hat{\xi}, \eta)$ where $\hat{\xi}$ runs throughout $U\mathcal{K}$. Since $U\mathcal{K}$ is compact, using a standard argument we infer from Lemma 3.20 above that there exists a finite η-net

$$\mathcal{F} = \{\xi_i = (y_i, p_i)\}_{i=1}^k \tag{3.54}$$

in $U\mathcal{M}$ (which goes together with the finite family of open sets $\{\mathcal{U}_i := B_g(y_i, 2\eta)\}_{i=1}^k$) such that for every $\tau \in (0, T_\infty) \setminus D^-$ and $q \in \sec^-(\tau)$ (respectively, for every $\tau \in D^- \diagdown \mathcal{D}^-(\eta)$) and every $z \in \Gamma_{\mathcal{U}}(\tau)$ relations (3.50)–(3.51) (respectively (3.52)–(3.53)) hold.

We are now ready to state our fundamental result, which states that the growth of the length of a self-contracted curve is locally controlled by an active external function b_ξ (determined by some $\xi \in \mathcal{F}$). To this end, let us recall from Remark 3.6 the notation of $\mathcal{D}^-(\eta)$ and $\mathcal{D}^+(\eta)$ (η-threshold for left and right discontinuity)

Corollary 3.22 (Determining an η-net)**.** *Under the notation of Remark 3.21 we have:*
(i) for every $\tau \in (0, T_\infty) \setminus D^-$ there exists $\delta > 0$ such that for every $s \in (\tau - \delta, \tau]$ there exists $\xi \in \mathcal{F}$ satisfying

$$b_\xi(\gamma(s)) - b_\xi(\gamma(\tau)) \ \ge\ \frac{\bar{\alpha}}{4}\ d_g(\gamma(s), \gamma(\tau)). \tag{3.55}$$

(ii) for every $\tau \in D^- \setminus \mathcal{D}^-(\eta)$ there exists $\xi \in \mathcal{F}$ and $\delta > 0$ such that for every $s \in (\tau - \delta, \tau]$ it holds:

$$b_\xi(\gamma(s)) - b_\xi(\gamma(\tau)) \ \ge\ \frac{\bar{\alpha}}{4}\ d_g(\gamma(s), \gamma(\tau)). \tag{3.56}$$

In both cases, if $\tau \notin \mathcal{D}^+(\eta)$, then the above formulas (3.55) *and* (3.56) *hold true when we replace $\gamma(\tau)$ by $z = \gamma(t)$ for any $t \in [\tau, \tau + \delta)$.*

Proof. (i). Let $\tau \in (0, T_\infty) \setminus D^-$ (point of left-continuity) and set $x = \gamma(\tau)$. Since $\sec^-(\tau)$ is the set of accumulation points of the subset $\{u_x(\gamma(s))\}$ of $U_x\mathcal{M}$ as $s \nearrow \tau$, and since $U_x\mathcal{M}$ is compact, it follows that there exists $\delta > 0$ such that for every $s \in (\tau - \delta, \tau]$, there exists $q_s \in \sec^-(\tau)$ such that $|q_s - u_x(\gamma(s))|_x < \bar{\alpha}/8$. Applying Remark 3.21 (for $x = \gamma(\tau)$ and $q_s \in \sec^-(\tau)$) we get $\xi = (y, q) \in \mathcal{F}$ (depending on s) and $\mathcal{U} := B_g(y, 2\eta)$ such that (3.50)–(3.51) hold true. It follows from (3.47) that

$$b_\xi(\gamma(s)) - b_\xi(x) \ge \Big(\langle \nabla b_\xi(x), u_x(\gamma(s))\rangle_x - \frac{\bar{\alpha}}{4}\Big)\, |v_x(\gamma(s))|_x. \tag{3.57}$$

Let us provide a lower bound for the right-hand side (we set $u := u_x(\gamma(s))$ and recall that $|u - q_s| < \bar{\alpha}/8$).

$$\begin{aligned} \langle \nabla b_\xi(x), u\rangle_x &= \langle \nabla b_\xi(x), q_s\rangle_x + \langle q_s, u - q_s\rangle_x + \langle \nabla b_\xi(x) - q_s, u - q_s\rangle_x \\ &\geq \bar{\alpha} - (1 + |\nabla b_\xi(x) - q_s|_x)\, |u - q_s|_x \geq \frac{\bar{\alpha}}{2}. \end{aligned} \tag{3.58}$$

We conclude by combining (3.57) and (3.58) and recalling that $|v_x(\gamma(s))|_x = d_g(x, \gamma(s))$.

(ii). Let us now assume $\tau \in D^- \backslash \mathcal{D}^-(\eta)$, set $x = \gamma(\tau)$, $x' = \gamma(\tau^-)$ and $\bar{x} := \theta\left((1 - \bar{\beta})\, d_g(x, x')\right)$. In this case, Remark 3.21 provides $\xi = (y, q) \in \mathcal{F}$ and $\mathcal{U} := B_g(y, 2\eta)$ such that (3.52)–(3.53) hold true. It follows again from (3.47) that

$$b_\xi(x') - b_\xi(\bar{x}) \geq \left(\langle \nabla b_\xi(\bar{x}), u_{\bar{x}}(x')\rangle_x - \frac{\bar{\alpha}}{4}\right) |v_{\bar{x}}(x')|_{\bar{x}}.$$

Since $\bar{q} := u_{\bar{x}}(x')$ and $d_g(\bar{x}, x') = |v_{\bar{x}}(x')|_{\bar{x}}$ the above yields directly in view of (3.53) that

$$b_\xi(x') - b_\xi(\bar{x}) \geq \frac{3\bar{\alpha}}{4} d_g(\bar{x}, x').$$

Combining the above with (3.52) for $z = x$ we deduce easily that

$$b_\xi(x') - b_\xi(x) \geq \frac{3\bar{\alpha}}{4} d_g(x, x').$$

Since $x' = \lim_{s \nearrow \tau} \gamma(s)$ and b_ξ is continuous we obtain via a standard argument that there exists $\delta > 0$ such that (3.56) holds.

Let us finally assume (in both cases (i) and (ii) above) that $\tau \notin \mathcal{D}^+(\eta)$, that is, $d_g(\gamma(\tau), \gamma(\tau^+) < \eta$. Therefore, shrinking δ if necessary we may assume that for all $t \in [\tau, \tau + \delta)$ it holds $d_g(\gamma(\tau), \gamma(t) < \eta$, yielding that $z = \gamma(t) \in \mathcal{U}$, hence (3.50) (in case (i)) and respectively, (3.52) (in case (ii)) hold true. The result follows easily. $\square$

3.8. **Proof of finite length.** Let $\mathcal{F} \subset U\mathcal{M}$ be the finite η-net defined in (3.54). Then for any $i \in \{1, \ldots, k\}$ and $\xi_i = (y_i, p_i) \in \mathcal{F}$, we consider the function b_ξ given by (3.42) and observe that

$$\mathcal{U}_i := B_g(y_i, 2\eta) \subset B_g(y_i, 2\,\rho) := \mathrm{dom}\; b_\xi.$$

We now define the local width of γ at $x = \gamma(\tau)$ with respect to ξ_i as follows:

$$W_i(\tau) := \mathrm{diam}\; \{b_{\xi_i}(z) : z \in \Gamma_{\mathcal{U}_i}(\tau)\} \qquad (\,\mathcal{U}_i := B_g(y_i, 2\eta))\,, \tag{3.59}$$

using the convention that $\mathrm{diam}\, \emptyset = 0$. Notice that for $\tau_1 \leq \tau_2$ we have $\Gamma_{\mathcal{U}_i}(\tau_2) \subset \Gamma_{\mathcal{U}_i}(\tau_1)$, therefore $W_i(\tau_2) \leq W_i(\tau_1)$. In order words, the function $\tau \longmapsto W_i(\tau)$ is decreasing on $[0, T_\infty)$ for every $i \in \{1, \ldots, k\}$. Let us now consider the (decreasing) aggregate function

$$W_{\mathcal{F}}(\tau) := \sum_{i=1}^{k} W_i(\tau). \tag{3.60}$$

The following result holds.

Proposition 3.23. *Let $[a, b] \subset (0, T_\infty) \diagdown (\mathcal{D}^-(\eta) \cup \mathcal{D}^+(\eta))$. Then for every partition*

$$a = t_0 < t_1 < \ldots < t_n = b$$

of $[a, b]$ it holds:

$$\sum_{j=1}^{n} d_g(\gamma(t_{i-1}), \gamma(t_i)) \leq \frac{\bar{\alpha}}{4}\; (W_{\mathcal{F}}(a) - W_{\mathcal{F}}(b))\,. \tag{3.61}$$

Proof. We deduce easily from Corollary 3.6 and the definition of $W_{\mathcal{F}}$ in (3.60) that for every $\tau \notin \mathcal{D}^-(\eta) \cup \mathcal{D}^+(\eta)$ there exists $\delta_\tau > 0$, such that for all $s, t \in (\tau - \delta_\tau, \tau + \delta_\tau)$ with $s \le \tau \le t$ it holds:

$$W_{\mathcal{F}}(s) - W_{\mathcal{F}}(t) \;\ge\; \frac{\bar{\alpha}}{4}\, d_g(\gamma(s), \gamma(t))\,. \tag{3.62}$$

Let $\{t_i\}_{i=0}^n$ be a partition of $[a, b]$. Then for every $i \in \{1, \ldots, n\}$, using a standard compactness argument on $[t_{i-1}, t_i]$ we deduce that (3.62) is true for $s = t_{i-1}$ and $t = t_i$. Summing up these inequalities, for all i we obtain (3.61). □

We are now ready to conclude the proof of Theorem 2.2.

Proof of Theorem 2.2. Let $\gamma : [0, T_\infty) \to \mathcal{M}$ be a self-contracted curve. Set $\mathcal{N} := \mathcal{D}^-(\eta) \cup \mathcal{D}^+(\eta)$ and denote by $|\mathcal{N}|$ its cardinality. Fix $T < T_\infty$ and denote by γ_T the restriction of γ to the compact interval $[0, T]$. We shall prove that γ_T is rectifiable and its length is bounded by $W_F(0) + |\mathcal{N}|\,\Sigma$, where Σ is a strict upper bound for the maximal left or right jump of γ, that is,

$$\Sigma \;>\; \max\left\{ \max_{\sigma \in \widehat{D}} d_g(\gamma(\sigma), \gamma(\sigma^-))\,,\; \max_{\sigma \in \mathcal{D}^+(\eta)} d_g(\gamma(\sigma), \gamma(\sigma^+)) \right\}.$$

Since $\mathcal{N}$ is finite (and the right and left limits exist at every point), we deduce easily that there exists $\delta' > 0$ such that for any $\sigma \in \mathcal{N}$ and any $s, t \in (\sigma - \delta', \sigma + \delta')$ with $s \le \sigma \le t$ it holds

$$d_g(\gamma(s), \gamma(t) \;<\; \Sigma\,. \tag{3.63}$$

Notice that the compact set $[0, T] \setminus \bigcup_{\sigma \in \mathcal{N}} (\sigma - \delta', \sigma + \delta')$ is a finite union of intervals $[a_i, b_i]$, for each of which Proposition 3.23 applies. We deduce easily that

$$\ell(\gamma_T) \le \frac{\bar{\alpha}}{4}\, W_{\mathcal{F}}(0) + |\mathcal{N}|\,\Sigma\,.$$

Since the above bound is independent of T, passing to the limit as $T \to +\infty$ we obtain that the length of γ is bounded by the same constant. □

Remark 3.24. The above proof shows that the upper bound for the length of any self-contracted curve $\gamma : [0, T_\infty) \to \mathcal{K}$ only depends on the dimension of the manifold and the compact set $\mathcal{K}$ (see also Remark 3.6).

References

[1] Boothby, W., *An introduction to differentiable manifolds and Riemannian geometry* (2nd Edition), Pure and Applied Mathematics **120** (Academic Press, 1986).

[2] Daniilidis, A., Drusvyatskiy, D., Lewis, A. S., Orbits of geometric descent, *Canad. Math. Bull.* **58** (2015), 44–50.

[3] Daniilidis, A., Ley, O., Sabourau, S., Asymptotic behaviour of self-contracted planar curves and gradient orbits of convex functions, *J. Math. Pures Appl.* **94** (2010), 183–199.

[4] David, G., Daniilidis, A., Durand-Cartagena, E., Lemenant, A. Rectifiability of Self-contracted curves in the euclidean space and applications, *J. Geom. Anal.* **25** (2015), 1211–1239.

[5] Giannotti, C. Spiro, A., Steepest descent curves of convex functions on surfaces of constant curvature, *Israel J. Math.* **191** (2012), 279–306.

[6] Lee, J., *Manifolds and differential geometry*, Graduate Studies in Mathematics **107** (American Mathematical Society, Providence, RI, 2009).

[7] Longinetti, M., Manselli, P., Venturi, A., On steepest descent curves for quasiconvex families in $\mathbb{R}^n$, *Math. Nachr.* **288** (2015), 420–442.

[8] Manselli, P. Pucci, C., Uniqueness results for evolutes and self-evolvents, *Boll. Un. Mat. Ital. A* **5** (1991), 373–379.

[9] Manselli, P. Pucci, C., Maximum length of steepest descent curves for quasi-convex functions, *Geom. Dedicata* **38** (1991), 211–227.

[10] Palis, J. & De Melo, W., *Geometric theory of dynamical systems. An introduction,* (Translated from the Portuguese by A. K. Manning), Springer-Verlag, New York-Berlin, 1982.

[11] Rockafellar, R.T. & Wets, R., *Variational Analysis*, Grundlehren der Mathematischen, Wissenschaften, Vol. **317**, (Springer, 1998).

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