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A criterion for $p$-henselianity in characteristic $p$

Zoé Chatzidakis$^*$ and Milan Perera

Abstract

Let $p$ be a prime. In this paper we give a proof of the following result: A valued field $(K, v)$ of characteristic $p > 0$ is $p$-henselian if and only if every element of strictly positive valuation of the form $x^p - x$ for some $x \in K$.

Preliminaries

Throughout this paper, all fields have characteristic $p > 0$. First we recall some definitions and notations. Let $\mathcal{O}_v := \{x \in K \mid v(x) \geq 0\}$ be the valuation ring associated with $v$. It is a local ring, and $\mathcal{M}_v := \{x \in K \mid v(x) > 0\}$ is its maximal ideal. Let $\overline{K}_v := \mathcal{O}_v / \mathcal{M}_v = \{\overline{a} = a + \mathcal{M}_v \mid a \in \mathcal{O}_v\}$ be the residue field (or $\overline{K}$ when there is no danger of confusion). We let $K(p)$ denote the compositum of all finite Galois extensions of $K$ of degree a power of $p$.

A valued field $(K, v)$ is $p$-henselian if $v$ extends uniquely to $K(p)$. Equivalently (see [1], Thm 4.3.2), if it satisfies a restricted version of Hensel’s lemma (which we call $p$-Hensel lemma): $K$ is $p$-henselian if and only if every polynomial $P \in \mathcal{O}_v[X]$ which splits in $K(p)$ and with residual image in $\overline{K}_v[X]$ having a simple root $\alpha$ in $\overline{K}_v$, has a root $a$ in $\mathcal{O}_v$ with $\overline{a} = \alpha$. Furthermore, another result (see [1], Thm 4.2.2) shows that $(K, v)$ is $p$-henselian if and only if $v$ extends uniquely to every Galois extension of degree $p$.

The aim of this note is to give a complete proof of the following result:

**Theorem.** Let $(K, v)$ be a valued field. $(K, v)$ is $p$-henselian if and only if $\mathcal{M}_v \subseteq \{x^p - x \mid x \in K\}$.

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This result was announced in [3], Proposition 1.4, however the proof was not complete. The notion of \( p \)-henselianity is important in the study of fields with definable valuations, and in particular it is important to show that the property of \( p \)-henselianity is an elementary property of valued fields.

The proof we give is elementary, and uses extensively pseudo-convergent sequences and their properties. Recall that a sequence \( \{a_\rho\}_{\rho<\kappa} \in K^\kappa \) indexed by an ordinal \( \kappa \) is said to be \textit{pseudo-convergent} if for all \( \alpha < \beta < \gamma < \kappa \):

\[
v(a_\beta - a_\alpha) < v(a_\gamma - a_\beta). \tag{1}\]

A pseudo-convergent sequence \( \{a_\rho\}_{\rho<\kappa} \) is called \textit{algebraic} if there is a polynomial \( P \) in \( K[X] \) such that \( v(P(a_\alpha)) < v(P(a_\beta)) \) ultimately for all \( \alpha < \beta \), i.e:

\[
\exists \lambda < \kappa \forall \alpha, \beta < \kappa \; (\lambda < \alpha < \beta) \implies v(P(a_\alpha)) < v(P(a_\beta)). \tag{2}\]

Otherwise, it is called \textit{transcendental}.

We assume familiarity with the properties of pseudo-convergent sequences, see [2] for more details, and in particular Theorem 3, Lemmas 4 and 8.

\section*{Proof of the theorem}

First, we prove a lemma in order to restrict our study to immediate extensions:

\textbf{Observation.} Let \((K,v)\) be a valued field and \((L,w)\) be a Galois extension of degree a prime \( \ell \). Then, if \((L,w)/(K,v)\) is residual or ramified, \( w \) is the unique extension of \( v \) to \( L \).

\textit{Proof.} The fundamental equality of valuation theory (see [1], Thm 3.3.3) tells us that if \( L \) is a Galois extension of \( K \), then

\[
[L : K] = e(L/K)f(L/K)gd \tag{3}\]

where \( e(L/K) \) is the ramification index, \( f(L/K) \) the residue index, \( g \) the number of extensions of \( v \) to \( L \) and \( d \), the defect, is a power of \( p \).

Thus, as \( \ell \) is a prime, if \( e(L/K)f(L/K) > 1 \), then necessarily \( g = d = 1 \), and in particular, \( v \) has a unique extension to \( L \).

\[\square\]
Now, let us prove the result announced in the preliminaries:

**Theorem.** Let \((K, \mathcal{O}_v)\) be a valued field of characteristic \(p\). Then, \((K, \mathcal{O}_v)\) is \(p\)-henselian if and only if \(\mathcal{M}_v \subseteq K^{(p)} := \{x^p - x \mid x \in K\}\).

**Proof.** The forward direction is an immediate application of the \(p\)-Hensel Lemma.

Conversely, assume that \(\mathcal{M}_v \subseteq K^{(p)} := \{x^p - x \mid x \in K\}\). Every Galois extension of \(K\) of degree \(p\) is an Artin-Schreier extension, \(i.e\) is generated over \(K\) by a root \(a\) of a polynomial \(X^p - X - b = 0\), with \(b \in K \setminus K^{(p)}\). The previous observation gives us the result when \(K(a)/K\) is not immediate. Let \(L\) be an immediate Galois extension of degree \(p\) and \(\tilde{v}\) an extension of \(v\) to \(L\) (hence with the same value group \(\Gamma\) and residue field \(\mathcal{L} = \mathcal{K}\) as \(K\)). We can write \(L = K(a)\) where \(a^p - a = b \in K \setminus K^{(p)}\).

**Step 1:** (Claim) The set \(C = \{v(x^p - x - b) \mid x \in K\} = v(K^{(p)} - b)\) is contained in \(\Gamma_{<0}\) and has no last element.

First observe that \(C \subseteq \Gamma_{\leq 0}\) if \(v(c^p - c - b) > 0\), then the equation \(X^p - X + (c^p - c - b)\) has a root in \(K\), so that \((a - c) \in K\): contradiction. Let \(\gamma \in \Gamma\), \(d \in K\) such that \(v(d^p - d - b) = \gamma\). As \(L/K\) is immediate there is \(c \in K\) such that \(\tilde{v}(a - (d + c)) > \tilde{v}(a - d)\). If \(\tilde{v}(a - d) = 0\) then \(\tilde{v}(a - (d + c)) > 0\) and \((d + c)^p - (d + c) - b = (d + c - a)^p - (d + c - a)\) in \(\mathcal{M}_v\), which as above give a contradiction. Hence \(\tilde{v}(a - d) < 0\), and from \(d^p - d - b = (d - a)^p - (d - a)\), we deduce that \(\gamma = v(d^p - d - b) > 0\), and \(v((d + c)^p - (d + c) - b) = p\tilde{v}(a - (d + c)) > \gamma\). This shows the claim.

**Step 2:** We extract a strictly well-ordered increasing and cofinal sequence from \(C\). If we write \(P(X) := X^p - X - b\), we get a sequence \(\{a_\rho\}_{\rho < \kappa}\) in \(K\) such that the sequence \(\{v(P(a_\rho))\}_{\rho < \kappa}\) is strictly increasing and cofinal in \(C\). Thus, the sequence \(\{P(a_\rho)\}_{\rho < \kappa}\) is pseudo-convergent (with 0 one of its limits). As \(v(P(a_\alpha)) < 0\), we have \(v(a_\beta - a_\alpha) = \max\{v(P(a_\alpha))\} = v_\alpha\) for \(\alpha < \beta < \kappa\). Thus, the sequence \(\{a_\rho\}_{\rho < \kappa}\) is also pseudo-convergent. Furthermore, \(\{a_\rho\}_{\rho < \kappa}\) has no limit in \(K\): if \(l \in K\) is a limit of \(\{a_\rho\}_{\rho < \kappa}\) then \(P(l)\) is a limit of \(\{P(a_\rho)\}_{\rho < \kappa}\). As \(v(P(a_\rho))\) is cofinal in \(C\), \(v(P(l))\) would be a maximal element of \(C\): contradiction.

**Step 3:** (Claim) Let \(P_0(X) \in K[X]\), and assume that \(v(P_0(a_\alpha))\) is strictly increasing ultimately. Then \(\deg(P_0(X)) \geq p\).

We take such a \(P_0\) of minimal degree, assume this degree is \(n < p\), and will
derive a contradiction. One consequence of Lemma 8 in [2] is that:

\[ v(P_0(a_{\rho})) = \delta' + \gamma_{\rho} \text{ ultimately for } \rho < \kappa \]  

(4)

where \( \delta' \) is the ultimate valuation of \( P_0'(a_{\rho}) \) and \( \gamma_{\rho} \) is the valuation of \( (a_{\sigma} - a_{\rho}) \) for \( \rho < \sigma < \kappa \) (which does not depend on \( \sigma \) as \( \{a_{\rho}\}_{\rho<\kappa} \) is pseudo-convergent).

We write \( P(X) = \sum_{i=0}^{m} h_i(X)P_0(X)^i \) with \( \deg(h_i) < n, \forall i \in \{1, \ldots, m \} \). Then, \( \{h_i(a_{\rho})\}_{\rho<\kappa} \) is ultimately of constant valuation, and we let \( \lambda_i \) be this valuation. As \( \{a_{\rho}\}_{\rho<\kappa} \) has no limit in \( K \), it is easy to see that \( n > 1 \), so that \( m < p \). By Lemma 4 in [2], there is an integer \( i_0 \in \{1, \ldots, m \} \) such that we have ultimately:

\[ \forall i \neq i_0 \quad (\lambda_i + i\delta') + i\gamma_{\rho} > (\lambda_{i_0} + i_0\delta') + i_0\gamma_{\rho}. \]  

(5)

Then, ultimately:

\[ p\gamma_{\rho} = v(P(a_{\rho})) = v\left(\sum_{i=0}^{m} h_i(a_{\rho})P_0(a_{\rho})^i\right) = \lambda_{i_0} + i_0(\delta' + \gamma_{\rho}). \]  

(6)

Thus, we have ultimately \( (p-i_0)\gamma_{\rho} = \lambda_{i_0} + i_0\delta' \). As \( p > m \geq i_0 \), the left hand side of the equation increases strictly monotonically with \( \rho \). But the right hand side is constant: it has no dependence in \( \rho \)! We have a contradiction, thus \( n = p \).

Step 4: Clearly, \( \{a_{\rho}\}_{\rho<\kappa} \) is of algebraic type. By Theorem 3 in [2], if \( a_{\infty} \) is a root of \( P \), we get an immediate extension \( (L', v') = (K(a_{\infty}), v') \). Let \( a_{\infty} = a \), we have \( (K(a), v') \) isomorphic to \( (K(a), \tilde{v}) \). Thus:

\[ \forall Q \in K_p[X] \quad \tilde{v}(Q(a)) = v'(Q(a)) = v(Q(a_{\rho})) \text{ ultimately} \]  

(7)

This shows the uniqueness of \( \tilde{v} \) and concludes the proof of the theorem. \( \square \)

References


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