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Abstract

We investigate the effect of ambiguity and ambiguity attitude on the shape and properties of the optimal contract in an adverse selection model with a continuum of types, using the parametric model of ambiguity and ambiguity aversion called the NEO-additive model (Chateauneuf, Eichberger, and Grant, 2007). We show that it necessarily features efficiency and a jump at the top and pooling at the bottom of the distribution. Conditional on the degree of ambiguity, the pooling section may or may not be supplemented by a separating section. As a result, ambiguity adversely affects the principal's ability to solve the adverse selection problem and therefore the least efficient types benefit from ambiguity with respect to risk. Conversely, ambiguity is

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detrimental to the most efficient types. This is confirmed in the comparative statics section.

Keywords Adverse selection, ambiguity, ambiguity aversion, NEO-additive model, non-expected utility models, behavioral economics.

JEL Classification Numbers D81, D82.

1 Introduction

1.1 Motivation

Most principal-agent models with adverse selection share the assumption that the principal knows at least the probability distribution of the agent's type. For instance, in a regulatory setting, the regulator knows the distribution from which the marginal cost of a natural monopoly is drawn. How does the regulator come up with this knowledge, though?

As observed by Sappington and Weisman (1996, p. 115)

> Considerable expertise in a variety of areas is required merely to understand central issues in telecommunications regulation [...] The regulatory process tends to be an adversarial process in which interested parties provide evidence that supports their positions. Regulators [...] must assess the evidence brought before them, and make decisions based on their assessment of the evidence.

In standard contract theory, it is assumed that the outcome of this information gathering process is a precisely defined probability distribution. However, this may not necessarily be the case. The evidence might be conflicting, even when the information providers are not biased; they might just not have access to the same data. The conflict of evidence may therefore not always be solvable without a serious loss of information. In that case, the regulator faces a trade-off:

- Either summarize all the evidence by a single probability distribution, at the cost of omitting all conflicting evidence, however valuable it may be,

- Or make do with a less precise representation of uncertainty so as to keep as much of the information provided as possible.

It might therefore be more prudent to work with an imprecise probability distribution in order to keep more of the available information, even though it is potentially conflicting.

In this paper, our aim is to study the impact of an imprecisely known probability distribution on the second best optimal contract in an adverse selection model with a continuum of types.

Poor knowledge of the probability distribution of efficiencies is an instance of the general concept of ambiguity (Ellsberg, 1961). This powerful concept has spawned numerous decision making models.[1] Among them, the NEO–additive model (Chateauneuf et al., 2007) has particularly appealing properties. Indeed, it ties the concepts of ambiguity and ambiguity attitude to two additional parameters, without abandoning the probabilistic approach altogether. Comparison with the expected utility model is therefore made easier in this model.

1.2 Main findings

To be consistent with the stylized facts described above, we consider a Baron-Myerson's model where the agent privately knows his type (i.e. his marginal cost).
With no ambiguity, equivalently in the expected utility model, the contract exhibits the well-known rent-extraction efficiency trade-off: downward distortions from efficiency are imposed on production (except for the most efficient type) in order to reduce the adverse selection agency cost (i.e. the information rent). Moreover, under usual monotonicity conditions, the contract is smooth and separating.
By contrast, the optimal contract under ambiguity exhibits the following properties: there is still no distortion at the top, but a jump at the top and pooling at the bottom. These departures from the expected utility model arise because of the following main changes. The presence of ambiguity, combined with ambiguity seeking, or optimism, (resp. ambiguity aversion, or pessimism), leads to the introduction of a mass point for the most (resp.

[1] See Etner, Jeleva, and Tallon (2012) for a survey of these models.

the least) efficient agent. This has two consequences.

First, this places a positive weight on the need of the principal to secure efficient production for those types. She is thus tempted to eliminate output distortions at the top and at the bottom. Second, this increases the agency cost of adverse selection with respect to the expected utility case. The principal is then induced to increase distortions for all types in order to limit the rent of the most efficient type.

Clearly, the two consequences work in opposing directions. The simplest way for the principal to satisfy these opposing effects would be to give up the continuity of the production schedule. More precisely, she would like to implement a downward jump at the top, and an upward jump at the bottom. But, to get a truthful report from the agent, the contract must ensure a production scheme which is non increasing with efficiency (i.e. the implementability condition). It follows that the former jump does not violate this condition, whereas the latter does. So the jump at the top is optimal, the jump at the bottom is not. The principal keeps the first jump in the contract and replaces the second by pooling for a whole range of inefficient types. We can conclude that ambiguity and optimism sharpen the rent-extraction efficiency trade-off, whereas ambiguity and pessimism blunt it.

1.3 Related literature

The literature on the impact of ambiguity on contract theory is relatively scarce, and focuses more on moral hazard: a pioneering but yet unpublished paper is Ghirardato (1994). Other references include Rigotti (1998); Karni (2009); Weinschenk (2010). For the case of adverse selection, the paper closest to ours is Mondello (2012), which deals with the two types case in the NEO-additive framework. The author claims to have found the counterintuitive result that the optimal contract under ambiguity leads to a higher production requirement than in the risk case for the low-skill type. However, this is based on the wrong assessment of the sign of a certain quantity, even though the computations are otherwise right, and in fact the results of that paper are consistent with ours. Although not dealing with ambiguity, Lewis and Sappington (1993) obtain results similar to ours.

We shall comment on how their findings are related to ours in due course.

1.4 Organization of the paper

The model is laid out in section 2. In section 3, the characteristics of the optimal contract under ambiguity are presented. Comparative statics on these characteristics are discussed in the final section. Proofs and additional material are presented in the appendices.

2 The model

2.1 Background

A risk-neutral principal derives utility $S(q)$ from the production of a quantity q of a good by a risk-neutral agent. We assume that S is twice continuously differentiable with $S' > 0$, $S'' < 0$ and $\lim_{q \to 0} S'(q) = +\infty$. The principal pays a transfer t to the agent, and thus the principal's net payoff is

$$V = S(q) - t. \tag{1}$$

The agent's privately known marginal cost (or efficiency type) is $c \in \mathcal{C} = [\underline{c}, \overline{c}]$. Agent c's net payoff is thus

$$U = t - cq. \tag{2}$$

Appealing to the revelation principle, we focus on the direct revelation mechanism. Thus a contract is a schedule that assigns a transfer and a required quantity $\langle t(\hat{c}), q(\hat{c}) \rangle$ to each agent's report $\hat{c} \in [\underline{c}, \overline{c}]$.

2.2 First best problem and solution

The overall benchmark situation occurs when the principal can observe the agent's efficiency type.

Participation constraint. In that case, since the agent cannot be forced to accept the contract at a lower utility than its reservation level, normalized to 0, the principal faces the agent's participation constraint, $\forall c \in \mathcal{C}$

$$U(c) = t(c) - cq(c) \geq 0. \tag{PC}$$

The problem and its solution. Combining (1) and (2), we get

$$V(c) = S(q(c)) - cq(c) - U(c). \tag{3}$$

The principal must maximize (3) subject to (PC). The first best contract is, $\forall c \in \mathcal{C}$

$$\begin{cases} U^{FB}(c) = 0, \\ q^{FB}(c) = S'^{-1}(c). \end{cases}$$

It is characterized by no rent left to the agent because the participation constraint is binding, and by efficient production since it maximizes the social surplus $S(q(c)) - cq(c)$.

Moreover, it is straightforward to see that the principal's net payoff is strictly decreasing in efficiency. Thus she still prefers to contract with a more efficient agent.

2.3 Second best problem

In this setting, the principal cannot observe the agent's efficiency.

Principal's beliefs. Let $\Delta(\mathcal{C})$ be the set of c.d.f. on $\mathcal{C}$. For $G \in \Delta(\mathcal{C})$, abusing notation we denote

$$G(B) = \int_{\underline{c}}^{\overline{c}} \mathbb{1}_B \mathrm{d}G$$

for every Borel set $B \subseteq \mathcal{C}$. That is, we denote both a distribution function and the measure it induces with the same letter.

We assume that the principal believes that there exists a distribution F on $\mathcal{C}$, with density f, $f > 0$ on $\mathcal{C}$, and $\alpha \in [0,1]$, such that the true distribution of types lies in the set[2]

$$\begin{aligned} \Delta_{F,\alpha} &= \{G \in \Delta(\mathcal{C}) \mid G(B) \geq (1-\alpha)F(B), \text{ for every Borel set } B\} \\ &= \{\alpha H + (1-\alpha)F \mid H \in \Delta(\mathcal{C})\} \\ &= \alpha\Delta(\mathcal{C}) + (1-\alpha)F. \end{aligned}$$

[2]The equality of the two expressions of the set requires a (simple) proof, available upon request.

Clearly, the larger α, the larger that set is. Moreover, whenever $\alpha < 1$, if $G \in \Delta_{F,\alpha}$, then F is absolutely continuous with respect to G. Thus, F has a density with respect to G, denoted $\frac{\mathrm{d}F}{\mathrm{d}G}$, and we may compute the relative entropy or Kullback-Leibler divergence of G from F, $D_{KL}(F \mid\mid G)$, defined by

$$D_{KL}(F \mid\mid G) = \int_{\underline{c}}^{\overline{c}} \ln \frac{\mathrm{d}F}{\mathrm{d}G} \mathrm{d}F.$$

Now it can be shown[3] that, whenever $G \in \Delta_{F,\alpha}$

$$D_{KL}(F \mid\mid G) \leq \ln \frac{1}{1-\alpha}.$$

Since the relative entropy is a measure of the divergence of the distribution G from the distribution F, this set is included in a "ball"[4] centered on F with ray $\ln \frac{1}{1-\alpha}$. Hence, the larger α is, the larger the ray is. Therefore, α can be interpreted as measuring the *degree of ambiguity*, and equivalently $1-\alpha$ as the *degree of confidence* the principal has when she estimates the distribution of types using F.

Objective function. We assume that the principal's objective function is a convex combination of the optimistic criterion (the best possible expected utility given the set of priors) and the pessimistic criterion (the worst possible expected utility). Let β be the *optimism* parameter used for this combination (by contrast $1-\beta$ represents the level of *pessimism*). The principal's objective function is therefore:

$$W = \beta \max_{G \in \Delta_{F,\alpha}} \int_{\underline{c}}^{\overline{c}} V(c) \mathrm{d}G(c) + (1-\beta) \min_{G \in \Delta_{F,\alpha}} \int_{\underline{c}}^{\overline{c}} V(c) \mathrm{d}G(c).$$

As it turns out, given the shape of the set of priors, the objective function can be rewritten in the following way[5]

[3]See Appendix section A.1.1.

[4]This is not strictly speaking a ball as the relative entropy is not a distance in the mathematical sense.

[5]Indeed, we have $\max_{G \in \Delta_{F,\alpha}} \int_{\underline{c}}^{\overline{c}} V(c) \mathrm{d}G(c) = \max_{H \in \Delta} \left(\alpha \int_{\underline{c}}^{\overline{c}} V(c) \mathrm{d}H(c) + (1-\alpha) \int_{\underline{c}}^{\overline{c}} V(c) \mathrm{d}F(c) \right)$. This is also equal to $\alpha \left(\max_{H \in \Delta} \int_{\underline{c}}^{\overline{c}} V(c) \mathrm{d}H(c) \right) + (1-\alpha) \int_{\underline{c}}^{\overline{c}} V(c) \mathrm{d}F(c) = \alpha \max_{c \in \mathcal{C}} V(c) + (1-\alpha) \int_{\underline{c}}^{\overline{c}} V(c) \mathrm{d}F(c)$. Similarly, we get $\min_{G \in \Delta_{F,\alpha}} \int_{\underline{c}}^{\overline{c}} V(c) \mathrm{d}G(c) = \alpha \min_{c \in \mathcal{C}} V(c) + (1-\alpha) \int_{\underline{c}}^{\overline{c}} V(c) \mathrm{d}F(c)$.

$$
\begin{aligned}
W &= \alpha[\beta \max_c V(c) + (1-\beta)\min_c V(c)] + (1-\alpha)\int_{\underline{c}}^{\overline{c}} V(c)f(c)\mathrm{d}c. \\
&= \alpha[\beta \max_c (S(q(c)) - t(c)) + (1-\beta)\min_c (S(q(c)) - t(c))] \qquad (4) \\
&\qquad\qquad + (1-\alpha)\int_{\underline{c}}^{\overline{c}} (S(q(c)) - t(c))\, f(c)\mathrm{d}c.
\end{aligned}
$$

This is actually the formula for NEO-additive Expected Utility (Chateauneuf et al., 2007) with respect to F.

Incentive constraints. Due to the non observability of efficiency, the principal must offer a contract such that the agent gets a higher net payoff when he reports truthfully. It follows that the incentive constraints are, for all $c, \hat{c} \in \mathcal{C}$

$$U(c) \geq t(\hat{c}) - cq(\hat{c}). \qquad \text{(IC)}$$

Principal's problem. The principal's problem is to $\max_{\langle t(.), q(.)\rangle}(4)$, subject to (PC) and (IC). This needs to be reformulated before it can be solved. First, following standard computations in incentives theory, the incentive and participation constraints are respectively, for piecewise continuously differentiable functions,[6] for all $c \in \mathcal{C}$

$$U'(c) = -q(c), \qquad \text{(IC1)}$$

$$q(.) \text{ non increasing}, \qquad \text{(IC2)}$$

and

$$U(\overline{c}) = 0. \qquad \text{(PC1)}$$

Second, we will focus on the case where

$$
\begin{cases}
\underline{c} \in \arg\max_c V(c), \\
\overline{c} \in \arg\min_c V(c).
\end{cases} \qquad (5)
$$

[6]We will use the following definition for piecewise continuous differentiability: a function g is piecewise continuously differentiable on an interval $[a, b]$ if there exists a finite subdivision $a = x_1 < \ldots < x_i < \cdots < x_n = b$ such that g is continuously differentiable (and thus continuous) on each interval (x_i, x_{i+1}), and both g and g' have left and right limits at each x_i.

This is akin to implicitly assuming that $V(.)$ is decreasing in c. This allows us to simplify the analysis, but it is important to notice that we will show that (5) holds for the optimal contract.

Third, combining (IC1) and (PC1), the agent gets an information rent given by $U(c) = \int_c^{\overline{c}} q(\varepsilon)d\varepsilon$. Plugging this into (3), then into (4), and using (5), the objective function becomes, after integration by parts

$$W = \alpha\beta\left(S(q(\underline{c})) - \underline{c}q(\underline{c}) - \int_{\underline{c}}^{\overline{c}} q(c)dc\right) + \alpha(1-\beta)\left(S(q(\overline{c})) - \overline{c}q(\overline{c})\right) + (1-\alpha)\int_{\underline{c}}^{\overline{c}}\left(S(q(c)) - cq(c) - \frac{F(c)}{f(c)}q(c)\right)f(c)dc. \tag{6}$$

Collecting terms, we obtain

$$W = \alpha\beta\left(S(q(\underline{c})) - \underline{c}q(\underline{c})\right) + \alpha(1-\beta)\left(S(q(\overline{c})) - \overline{c}q(\overline{c})\right) + \int_{\underline{c}}^{\overline{c}}(1-\alpha)\left(S(q(c)) - cq(c) - \frac{F(c)+\frac{\alpha\beta}{1-\alpha}}{f(c)}q(c)\right)f(c)dc. \tag{7}$$

At this stage, it is important to notice that the term $\frac{F(c)+\frac{\alpha\beta}{1-\alpha}}{f(c)}q(c)$ in (7) is the agency cost of the adverse selection, i.e. the cost of information rent. It can be decomposed into two parts

- the standard no ambiguity cost $\frac{F(c)}{f(c)}q(c)$;
- an additional cost due to ambiguity: $\frac{\alpha\beta}{(1-\alpha)f(c)}q(c)$.

The former is the cost incurred by the principal when using the distribution F, even if this distribution is the right one; the latter can be interpreted as the additional cost incurred when it is the wrong distribution. We assume that the cost of information rent satisifies the following property.

Assumption 1 (Monotone adjusted hazard rate)**.** *The hazard rate* $\frac{F(c)+\frac{\alpha\beta}{1-\alpha}}{f(c)}$ *is increasing in* c *over* $\mathcal{C}$.

This is a version of the Monotone Hazard Rate (MHR) used in contract theory. It avoids pooling contracts due to non monotonicity.[7]

Finally, the second best problem is to maximize (7) with respect to $q(.)$ subject to (IC2).

[7]With differentiable f this assumption is equivalent to $\left(\frac{F(c)}{f(c)}\right)' > \frac{\alpha\beta}{1-\alpha}\frac{f'(c)}{(f(c))^2}$. If $f' \geq 0$, this assumption

3 Properties of the Optimal Contract

3.1 No ambiguity: $\alpha = 0$

When there is no ambiguity, the objective function (7) is the usual one in expected utility models.

Proposition 1. *The optimal production, $q^*(c)$, is such that:*

$$q^*(c) = S'^{-1}\left(c + \frac{F(c)}{f(c)}\right) := q^{EU}(c). \tag{EU}$$

Proof. See Laffont and Martimort (2002, p. 87). □

With no ambiguity, the optimal production reflects a conflict between incentives and efficiency. To ensure a truthful report, the principal must leave an information rent to the agent. This leads to the agency cost $\frac{F(c)}{f(c)}q(c)$ in (7), with $\alpha = 0$. Efficiency is related to the presence of the social surplus $S(q(c)) - cq(c)$ in (7) still with $\alpha = 0$.

Consequently, the production is distorted downward from its efficient level in order to reduce the agency cost. The quantity $q^*(c)$ reflects the standard rent extraction efficiency trade-off in expected utility models.

Four other properties must be noted. First, the most efficient agent, $\underline{c}$, has an efficient production, $q^*(\underline{c}) = q^{FB}(\underline{c})$, because the agency cost is null since $F(\underline{c}) = 0$. Second, the production schedule is continuous in the agent's efficiency. Third, under assumption 1, the contract is separating. Finally, the principal's optimal net payoff increases with the agent's efficiency (i.e. decreases with the agent's type, his cost). The robustness of these properties to the introduction of ambiguity must be investigated.

3.2 General properties of the optimal contract under ambiguity

We present here properties of the optimal contract that hold under any positive value of α. They are stated in the next proposition.

is stronger than the standard MHR assumption. If $f' \leq 0$, it is implied by it. In general, however, they are independent. The uniform distribution satisfies both.

Proposition 2. *Let q^* be a solution to the SBP. If $0 < \alpha \leq 1$, then*

(i) ***No distorsion at the top:*** $q^*(\underline{c}) = q^{FB}(\underline{c})$

(ii) ***Jump at the top, except for*** $\beta = 0$***:*** $\lim_{c \downarrow \underline{c}} q^*(c) < q^*(\underline{c})$ *if and only if* $\beta > 0$.

(iii) ***Pooling at the bottom, except for*** $\beta = 1$***:*** *There exists* $c^* \in \mathcal{C}$ *such that* $q^*(c) = q^*(\overline{c}) := \overline{q}$ *for all* $c \in (c^*, \overline{c}]$ *and* $c^* < \overline{c}$ *if and only if* $\beta < 1$.

Two observations can be made compared to the no ambiguity case: a discontinuity at $\underline{c}$ and pooling over $(c^*, \overline{c}]$ emerge. How can this be explained?

Jump at the top. While the absence of distorsion at the top is standard, the jump is not, and can be understood in the following way. The presence of ambiguity ($\alpha > 0$) and of a certain degree of ambiguity seeking (or *optimism)* ($\beta > 0$) sharpens the standard conflict between efficiency and rent extraction for efficient types. Indeed, it introduces (in the NEO-additive functional) a mass point at $\underline{c}$.[8] Both the social surplus $S(q(\underline{c})) - \underline{c}q(\underline{c})$ and the rent, $U(\underline{c})$ are allocated a strictly positive weight. For the principal, the first effect reinforces (with respect to EU) the will to get efficient production at $\underline{c}$. The second effect, on the other hand, increases (with respect to EU) the agency cost that is the sum of the standard no ambiguity cost and the additional cost due to ambiguity. This implies the need for a further distortion (with respect to EU) for types $c > \underline{c}$ in order to limit the information rent of the type $\underline{c}$. This exacerbated conflict between efficiency and rent extraction results in a jump.

Note that when ambiguity is present but the principal is fully ambiguity averse (*pessimistic* ($\beta = 0$), the jump disappears (because the mass point does). Thus it requires some amount of ambiguity seeking or *optimism*.

Pooling at the bottom. The presence of ambiguity and a certain degree of ambiguity aversion *pessimism* blunt the conflict between efficiency and rent extraction for inefficient types by introducing (in the NEO-additive functional) a mass point at $\overline{c}$. This raises the

[8]Note that this is possible here only because the beliefs of the principal are actually nonadditive (or even better said this mass point creates the nonadditivity). Thus the presence of a mass point here is a genuine consequence of ambiguity.

importance for the principal of the social surplus at $\bar{c}$. By contrast, it does not affect the information rent that is null at that point because of the rationality constraint (PC1). This effect implies that the principal wishes to eliminate production distorsions at this point. But, because of the agency cost, downward distortions exist for types slightly more efficient than $\bar{c}$. A separating contract would violate incentive compatibility (IC2): slightly more efficient types would have an incentive to claim to be the most inefficient type.
The principal must therefore give up the separation of inefficient types and offer a pooling contract. The principal gives up important distorsions with respect to EU, so that the conflict between efficiency and rent extraction is blunt for inefficient types.

Again, note that if ambiguity is present but there is no ambiguity aversion at all ($\beta = 1$), pooling disappears, because the principal no longer cares about efficient production of the least efficient type $\bar{c}$.

This shape of the optimal contract is akin to the shape of the optimal contract in Lewis and Sappington (1993). In this paper, both the principal and the agent ignore the agent's type when the contract is signed, but the agent receives an imperfectly informative signal about his production cost. The signal is perfectly informative with probability $1 - p$. This leads to the appearance of a mass point (of mass p) at a value that is endogenously defined by the first order conditions. By contrast, we introduce two mass points that are also endogenously located in the distribution. The consequences of the presence of these mass points are similar, namely the presence of a discontinuity and a pooling zone. The main difference is that the discontinuity is at the top in our case, while it is at an interior point in Lewis and Sappington (1993). As we will see in the next section, a (possibly degenerate) separating zone follows the discontinuity in both cases.

3.3 Full shape of the optimal contract under ambiguity

To study the full shape of the optimal contract, we need a lemma and a notation.

Lemma 1. *For all $\beta \in (0,1)$, the equation in α*

$$\frac{(1-\alpha\beta)\alpha\beta}{(1-\alpha)} = f(\underline{c})(\bar{c} - \underline{c}) \tag{8}$$

has a unique solution in $(0,1)$, that we denote $\alpha^*(\beta)$.

This will allow us to distinguish between *low ambiguity* whenever $0 < \alpha < \alpha^*(\beta)$ and *high ambiguity* whenever $\alpha^*(\beta) \leq \alpha \leq 1$.

3.3.1 Low ambiguity: $0 < \alpha < \alpha^*(\beta)$

Proposition 3. *Under low ambiguity, the optimal production, $q^*(c)$, is given by*

$$q^*(c) = \begin{cases} S'^{-1}(\underline{c}) & \text{if } c = \underline{c}, \\ S'^{-1}\left(c + \frac{F(c)+\frac{\alpha\beta}{1-\alpha}}{f(c)}\right) & \text{if } c \in (\underline{c}, c^*], \\ S'^{-1}\left(c^* + \frac{F(c^*)+\frac{\alpha\beta}{1-\alpha}}{f(c^*)}\right) & \text{if } c \in (c^*, \overline{c}]. \end{cases} \tag{9}$$

with $c^* \in (\underline{c}, \overline{c}]$ implicitly defined by

$$\alpha(1-\beta)\left(c^* + \frac{F(c^*) + \frac{\alpha\beta}{1-\alpha}}{f(c^*)} - \overline{c}\right) = \int_{c^*}^{\overline{c}} (1-\alpha)\left(c + \frac{F(c) + \frac{\alpha\beta}{1-\alpha}}{f(c)} - c^* - \frac{F(c^*) + \frac{\alpha\beta}{1-\alpha}}{f(c^*)}\right) f(c)dc \tag{10}$$

whenever $\beta < 1$ and $c^* = \overline{c}$ whenever $\beta = 1$.

Under low ambiguity, the effect of ambiguity on the conflict between efficiency and rent extraction, and particularly its blunting effect for inefficient types, is not strong enough to completely preclude the possibility of screening some of the efficient types.

When $\alpha < \alpha^*(\beta)$, the ambiguity is sufficiently low for the conflicts created by the mass points at $\underline{c}$ and $\overline{c}$ not to interact which each other.

In the interval of types between the jump and the pooling zone, ambiguity implies a usual rent extraction efficiency trade-off, except that the agency cost is higher than without ambiguity. The contract is separating. The rent extraction-efficiency trade-off implies that production is such that marginal social surplus $S'(q(c)) - c$ equals the sum of marginal information cost $\frac{F(c)}{f(c)}$ and marginal ambiguity cost $\frac{\alpha\beta}{(1-\alpha)f(c)}$.

The limit between the separating part of the contract and the pooling part is determined by (10). To interpret this equation, let

$$q^s(c) := (S')^{-1}\left(c + \frac{F(c) + \frac{\alpha\beta}{1-\alpha}}{f(c)}\right).$$

which is a decreasing function of c because of assumption 1 and $(S')^{-1}$ is decreasing.

Thus, (10) can be rewritten

$$\alpha(1-\beta)\left(S'(q^s(c^*)) - S'(q^{FB}(\overline{c})\right) = \int_{c^*}^{\overline{c}} (1-\alpha)\left(S'(q^s(c)) - S'(q^s(c^*))\right) f(c)dc. \tag{11}$$

Let c^{**} be such that $q^s(c^{**}) = q^{FB}(\overline{c})$ whenever this equation has a solution on $\mathcal{C}$,[9] and let $c^{**} = \underline{c}$ otherwise. Equation 10 tells us that the contract is separating on $[c^{**}, c^*]$ and pooling on $[c^*, \overline{c}]$. How is c^* defined though? Let us examine the LHS of (11), then the RHS.

Since $c^* \in [c^{**}, \overline{c}]$ and q^s is decreasing, we have $q^s(c^*) \leq q^s(c^{**}) \leq q^{FB}(\overline{c})$.[10] Thus, since S' is decreasing, the LHS is nonnegative. More precisely, the LHS is null if $c^* = c^{**}$, positive if $c^* > c^{**}$, and increases as c^* moves away from c^{**}. So the more separating the contract is, the higher the LHS is. The LHS is thus the cost of separating on $[c^{**}, c^*]$, i.e. the cost of not having efficient production at $\overline{c}$.

The RHS is nonnegative because q^s and S' are decreasing. More precisely, the RHS is null if $c^* = \overline{c}$, strictly positive if $c^* < \overline{c}$, and increases as c^* moves away from $\overline{c}$. So the more pooling the contract is, the higher the RHS is. The RHS is thus the cost of pooling on $[c^*, \overline{c}]$, i.e. the cost of giving up producing $q^s(c)$ over this interval. Specifically, at c, the decrease in marginal utility suffered by the principal if she pools is $S'(q^s(c)) - S'(q^s(c^*))$.

Since the cost of separating on $[c^{**}, c^*]$ is increasing in c^* and the cost of pooling on $[c^*, \overline{c}]$ is decreasing in c^*, there is a tradeoff between these two costs. Equation (10) tells us that c^* is the result of resolving this conflict by equating these costs.

3.3.2 High ambiguity: $\alpha^*(\beta) \leq \alpha \leq 1$

Proposition 4. *Under high ambiguity, the optimal production, $q^*(c)$, is given by*

$$q^*(c) = \begin{cases} S'^{-1}(\underline{c}) & \text{if } c = \underline{c}, \\ S'^{-1}\left(\overline{c} + \frac{\alpha\beta}{1-\alpha\beta}(\overline{c} - \underline{c})\right) & \text{if } c \in (\underline{c}, \overline{c}], \end{cases} \tag{12}$$

[9] That is, whenever $q^s(\underline{c}) \geq q^{FB}(\overline{c})$, i.e. $\overline{c} - \underline{c} \geq \frac{\alpha\beta}{(1-\alpha)f(\underline{c})}$.

[10] This is obvious if $q^s(c^{**}) = q^{FB}(\overline{c})$; if $c^{**} = \underline{c}$, then $q^s(c^{**}) = q^s(\underline{c}) < q^{FB}(\overline{c})$ by definition of c^{**}.

whenever $\alpha < 1$ *or* $\alpha = 1$ *and* $\beta < 1$, *and*

$$q^*(c) = \begin{cases} q^{FB}(c) & \text{if } c = \underline{c} \\ 0 & \text{if } c \in (\underline{c}, \overline{c}]. \end{cases} \tag{13}$$

whenever $\alpha = 1$ *and* $\beta = 1$.[11]

Comparing with low ambiguity, the optimal production is still characterized by a jump and pooling, but there is no longer separation. In this case, the ambiguity is high enough for the conflicts generated by the mass points at $\underline{c}$ and $\overline{c}$ to interact with each other. In other words, ambiguity creates an overall conflict between optimism, pessimism, and incentives. Because of the presence of optimism (resp. pessimism), there is a jump (resp. pooling). But, a high level of ambiguity would imply a downward jump at $\underline{c}$ at least as large as the efficient production gap between the most and the least efficient agents.

Note that when $\alpha < 1$ or $\alpha = 1$ and $\beta < 1$, the principal behaves under high ambiguity as she would if she considered only two types and maximized expected utility: the production corresponds to the standard expected utility model with two types, $\underline{c}$ and $\overline{c}$, with probabilities $\alpha\beta$ and $1 - \alpha\beta$.

3.4 Shutdown

Given the properties of the optimal contract and Assumption 1, it is easy to check that the integrand in (7) is decreasing. Therefore, it can be optimal for the principal to allow for some shutdown of inefficient types. Let c_* be the threshold type such that all types $c > c_*$ are excluded from the contract by the principal. The objective function becomes

$$\begin{aligned} W = {} & \alpha\beta \left(S(q(\underline{c})) - \underline{c}q(\underline{c})\right) + \alpha(1-\beta)\left(S(q(c_*)) - c_*q(c_*)\right) \\ & + \int_{\underline{c}}^{c_*} (1-\alpha)\left(S(q(c)) - cq(c) - \frac{F(c) + \frac{\alpha\beta}{1-\alpha}}{f(c)}q(c)\right) f(c)dc. \end{aligned} \tag{14}$$

We state the following proposition.

[11] Actually, for the case $\alpha = 1$ and $\beta = 1$ we may set $q^*(c) = q^{FB}(c)$ if $c = \underline{c}$ and $q^*(c) = \bar{q}$ if $c \in (\underline{c}, \overline{c}]$ for any $0 \le \bar{q} \le q^{FB}(\overline{c})$, since in this case only the value at $\underline{c}$ is taken into account. Setting $\bar{q} = 0$, however, is the only way of making $q^*(c, \alpha, \beta)$ be continuous w.r.t. α and β.

Proposition 5. *The threshold c_* is implicitly defined by*

$$(1-\alpha)\left(S(q^*(c_*)) - c_*q^*(c_*) - \frac{F(c_*)+\frac{\alpha\beta}{1-\alpha}}{f(c_*)}q^*(c_*)\right)f(c_*) = \alpha(1-\beta)q^*(c_*). \tag{15}$$

The marginal benefit from c_* corresponds to the payoff it provides to the principal. This is the LHS of (15). But the presence of ambiguity implies that the principal would like this type to produce an efficient quantity. This entails that she must incur the marginal cost with respect to efficiency of this production. This corresponds to the RHS of (15).
It is interesting to note that (15) can be rewritten as

$$\left(S(q(c_*)) - c_*q(c_*) - \frac{F(c_*)}{f(c_*)}q(c_*)\right)f(c_*) = \frac{\alpha}{1-\alpha}q(c_*) > 0.$$

Thus, the principal excludes more types than with expected utility (where the RHS is 0).

3.5 An example

To illustrate the results and summarize them in a graph, consider the following specification. We let $\underline{c} = 1$, $\overline{c} = 2$, $S(q) = \sqrt{q}$ and $F(c) = c - 1$, the uniform distribution on $\mathcal{C} = [1, 2]$. Then, the first best contract is given by

$$q^{FB}(c) = \frac{1}{4c^2},$$

while the optimal second best contract under expected utility is given by

$$q^{EU}(c) = \frac{1}{4(2c-1)^2}.$$

The optimal second best production under ambiguity is given by, in the case of low ambiguity

$$q^*(c) = \begin{cases} \frac{1}{4} & \text{if } c = \underline{c} \\ \frac{(1-\alpha)^2}{4[(2c-1)(1-\alpha)+\alpha\beta]^2} & \text{if } c \in (\underline{c}, c^*] \\ \frac{(1-\alpha)^2}{4[3-\alpha(1+\beta)-2\sqrt{\alpha(1-\beta)}]^2} & \text{if } c \in (c^*, \overline{c}], \end{cases}$$

where

$$c^* = \frac{2-\alpha(1+\beta)-\sqrt{\alpha(1-\beta)}}{1-\alpha}.$$

Moreover, low ambiguity arises when

$$\alpha < \alpha^*(\beta) = \frac{1+\beta-\sqrt{1+2\beta-3\beta^2}}{2\beta^2}.$$

In the case of high ambiguity, optimal production is such that

$$q^*(c) = \begin{cases} \frac{1}{4} & \text{if } c = \underline{c}, \\ \frac{(1-\alpha\beta)^2}{4[2-\alpha\beta]^2} & \text{if } c \in (\underline{c}, \overline{c}]. \end{cases}$$

Figure 1 shows the shape of the optimal contract under different configurations of ambiguity and ambiguity attitude.

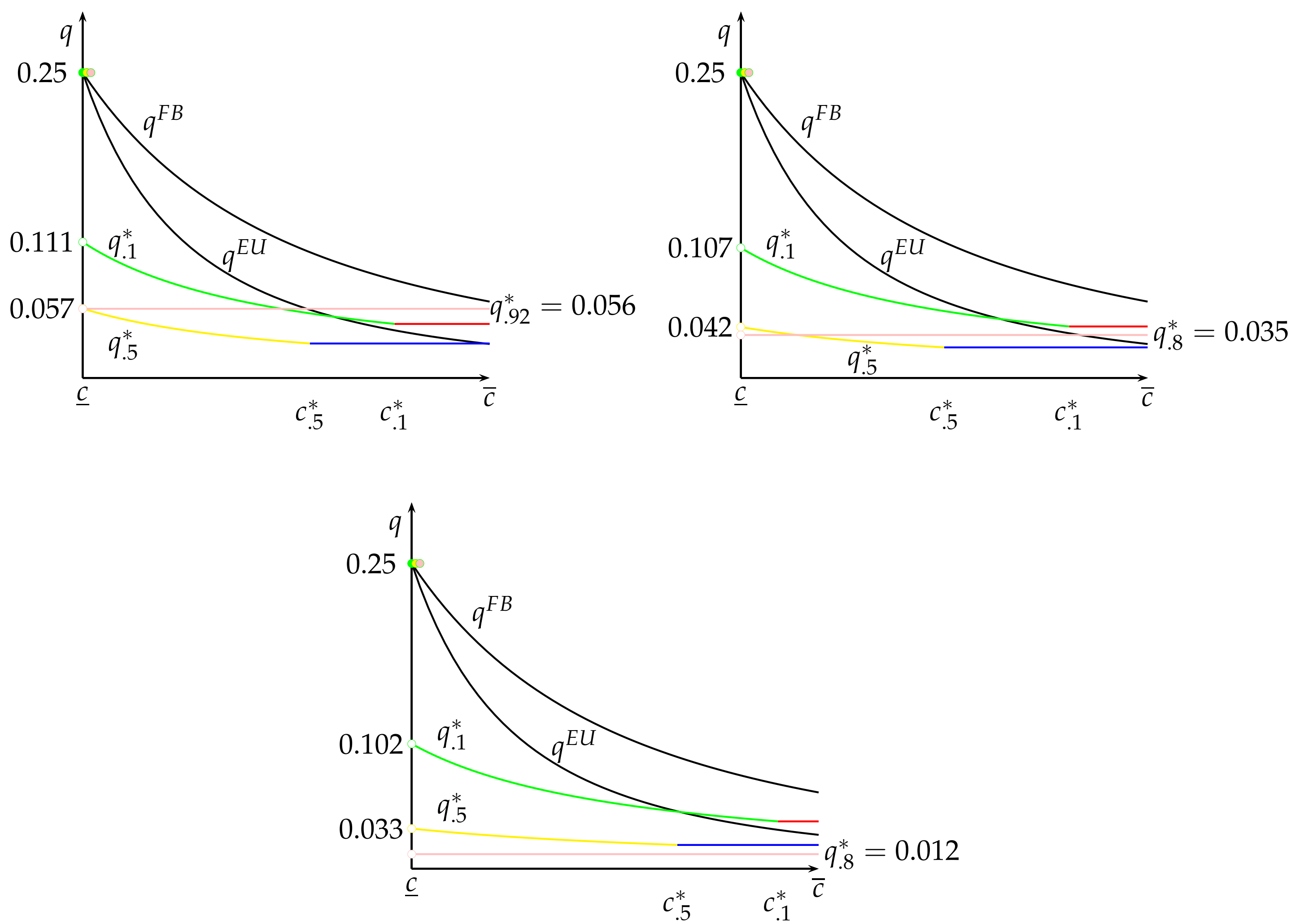

Figure 1: The optimal contract for $\beta = .1$, $\beta = .5$ and $\beta = .9$ and various values of α (subscripted).

4 Comparative Statics

Ambiguity and ambiguity attitude have two consequences for the objective function. They both modify the agency cost and the weights assigned to the three components of the objective function. Hence they imply several effects

- qualitative effects, through the trade-off between separating and pooling,
- quantitative effects, through the production distortions,
- allocative effects, through the payoffs secured by the principal and the agent.

These are studied in turn.

4.1 Qualitative effects

Notice that the trade-off between separating and pooling is relevant only for low ambiguity, since in this case, $c^* \in]\underline{c}, \overline{c}[$.

Proposition 6. *Under assumption* 1*, when* $0 < \alpha < \alpha^*(\beta)$

- c^* *is a decreasing function of* α*,*
- *there exists a unique* $\hat{\beta} \in [0,1]$ *such that* c^* *is a decreasing function of* β *if* $\beta < \hat{\beta}$ *and an increasing function of* β *if* $\beta > \hat{\beta}$*.*

Recall that c^* is such that the cost of separating below c^* is equal to the cost of pooling above c^* (c.f. equation (11)). The effect of α and β on these two costs are the following.

On the cost of separating side, a rise in α or β has two different effects. On the one hand, this leads to a decrease in the level of production $q^s(c)$ for all c, in particular at c^*, because the agency cost is increased. This raises the cost of separating because the production is further away from the first best at $\overline{c}$. On the other hand, α raises the weight attached to getting efficient production from the least efficient agent, whereas β decreases it. So α reinforces the former effect on the cost of separating; the opposite is true for β. Taken together, these two effects lead to an increase in the cost of separating in the case of α, and a U-shaped effect in the case of β.

On the cost of pooling side, a rise in α and β reduces $q^s(c^*)$, but also $q^s(c)$ for all $c \geq c^*$ because the agency cost is raised. Thus, the effect on the difference between these two levels, which contributes to the cost of pooling, is not clear-cut. Moreover, when α increases, the cost of pooling falls because the principal's confidence in the distribution F is reduced. By contrast, β is neutral. Ultimately, α tends to reduce the cost of pooling while β has an unclear effect.

Therefore, the overall effect on c^* is the following. An increase in ambiguity lowers c^* because it reduces the cost of separating and increases the cost of pooling. The pooling zone is thus increased.[12] By contrast, a decrease in ambiguity aversion (an increase in optimism) does not have an unequivocal overall effect on c^*. As a result, the c^* function is a U-shaped function of β.

4.2 Quantitative effects

Two aspects are examined (1) the size of the jump (2) the level of the constant production on the pooling zone. This is done in the next three propositions. The first two are devoted to the case of low ambiguity, the last one to the case of high ambiguity.

Low ambiguity. When ambiguity is low, recall that the size of the jump is given by $q^{FB}(\underline{c}) - q^s(\underline{c})$, and the constant production $\bar{q}$ is such that, using (12), $\bar{q} = q^*(c^*) = (S')^{-1}\left(c^* + \frac{F(c^*)+\frac{\alpha\beta}{1-\alpha}}{f(c^*)}\right)$.

Proposition 7. *When $0 < \alpha < \alpha^*(\beta)$, the size of the jump at the top is increasing in α and β.*

When α and β rise, $q^s(c)$ decreases for all c because the agency cost increases, in particular at $\underline{c}$, whereas $q^{FB}(\underline{c})$ remains unchanged. So the jump increases. As mentioned above, ambiguity and ambiguity seeking (optimism) sharpen the rent extraction efficiency trade-off at the top. Thus, when the corresponding parameters increase, the conflict is heightened and the jump increases.

[12]This is in line with other similar results in the literature, for instance the fact that ambiguity aversion leads to absence of trade under certain circumstances (Dow and da Costa Werlang, 1992).

Proposition 8. *Let $\bar{q}$ be the production of inefficient types when $0 < \alpha < \alpha^*(\beta)$,*

- $\frac{\partial \bar{q}}{\partial \alpha} > 0$ *if and only if $F(c^*) > \beta$. Equivalently, for all β there exists $\tilde{\alpha} \in (0, \alpha^*(\beta)]$ such that $\frac{\partial \bar{q}}{\partial \alpha} \geq 0$ on $(0, \tilde{\alpha})$ and $\frac{\partial \bar{q}}{\partial \alpha} \leq 0$ on$(\tilde{\alpha}, \alpha^*(\beta)]$.*
- $\frac{\partial \bar{q}}{\partial \beta} < 0$*.*

As α or β increase, the agency cost rises all other things being equal. Therefore, the constant production tends to be reduced. However, as previously noted in Proposition 6, a rise in α or β affect the trade-off between separating and pooling at the bottom of the distribution, and therefore the size of the pooling zone through the value of c^*. In the case of a rise in α, c^* decreases which tends to increase the constant production. The overall effect of ambiguity on the constant production thus results in an approximately inverted U-shaped curve: the agency cost effect trumps the effect on c^* for small values of α and is trumped by it for large values. In the case of a rise in β, c^* can rise or fall. However, some effects offset each other, resulting in an overall decrease in constant production.

High ambiguity. Let us analyze the case of high ambiguity when $\alpha\beta < 1$. Using Proposition 4, the jump is equal to $q^{FB}(\underline{c}) - \bar{q}$, with $\bar{q} = S'^{-1}\left(\bar{c} + \frac{\alpha\beta}{1-\alpha\beta}(\bar{c} - \underline{c})\right)$ for all $c \in (\underline{c}, \bar{c}]$. In this case, the size of the jump and the constant production covary in opposite directions. We have the following proposition.

Proposition 9. *Let $\bar{q}$ be the production of all types except $\underline{c}$ when $\alpha^*(\beta) \leq \alpha \leq 1$. Then $\frac{\partial \bar{q}}{\partial \alpha} < 0$ and $\frac{\partial \bar{q}}{\partial \beta} < 0$.*

As in the preceding proposition, a rise in α and β increases the agency cost. So this reduces the constant production. This corresponds also to the overall effect now because, unlike in the previous proposition, c^* no longer contributes to defining $\bar{q}$. As mentioned, the jump varies in the opposite direction and thus increases when ambiguity and ambiguity seeking increase.

4.3 Allocative effects

The way the size of the jump and the quantity produced by inefficient types in the optimal contract vary with ambiguity and ambiguity attitude suggests, given the connection between quantity produced and rent, that it is worth studying the effect of ambiguity and ambiguity aversion on the rent of the agent and the surplus of the principal.

4.3.1 The agent's rent

Let

$$U^*(c) = \int_c^{\overline{c}} q^*(x)dx$$

be the rent of an agent of type $c \in \mathcal{C}$. We then have the following proposition.

Proposition 10.

- *For all* $\alpha \in (0,1)$, $\frac{\partial U^*(c)}{\partial \beta} < 0$ *for all* $c \in (\underline{c}, \overline{c}]$.
- *If* $0 < \alpha < \alpha^*(\beta)$, *then there exists* $K > 0$ *such that* $\frac{\partial U^*(c)}{\partial \alpha} > 0$ *for all* $c \in (\underline{c}, \overline{c}]$ *if and only if* $\frac{F(c^*)}{\beta} \geq 1 + K$.
- *If* $\alpha^*(\beta) \leq \alpha < 1$, $\frac{\partial U^*(c)}{\partial \alpha} < 0$ *for all* $c \in (\underline{c}, \overline{c}]$.

What this proposition shows is first, that all agents benefit from the fact that the principal is ambiguity averse, independently of the level of ambiguity she perceives. Indeed, ambiguity aversion leads to the principal being more attentive to the objective of getting the least efficient type to produce the first best quantity and raises the agency cost, and both effects work in the direction of raising the production of the least efficient types, which in terms of rent also benefits the most efficient types. From a more psychological point of view, a more ambiguity averse principal will prefer less ambiguous, i.e. "flatter" contracts, to more variable ones, which also benefits all agents. Second, this proposition shows that, when the principal perceives a low ambiguity, more ambiguity favors all the agents as long as the principal is sufficiently ambiguity averse. Indeed, more ambiguity will lead the principal to prefer even more unambiguous contracts if she is sufficiently ambiguity averse. Conversely, what this proposition says is that if she is not sufficiently ambiguity averse, then some agents, at the top of the distribution, i.e. efficient ones, might suffer from an increase in ambiguity, while the agents at the bottom of the distribution, inefficient ones, might still benefit from it.

Finally, when the principal perceives a lot of ambiguity, we know from proposition 4 that she behaves as if there were only two types and the probability of the efficient type $\underline{c}$ were $\alpha\beta$, and the probability of the inefficient type(s) $1 - \alpha\beta$. Thus rises in α (more ambiguity perceived) or β (less ambiguity aversion) lead to a decrease in the latter probability, and as a consequence, the principal will allocate a lower rent to the inefficient type, since the objective of productive efficiency of the less efficient type is less pressing.

4.3.2 Social surplus and principal's payoff

Let

$$V^*(c) = S(q^*(c)) - cq^*(c)) - U^*(c) = S(q^*(c)) - cq^*(c)) - \int_c^{\overline{c}} q^*(x)dx$$

be the principal's ex post net payoff when she actually faces an agent of type $c \in \mathcal{C}$. The effects of ambiguity and of her ambiguity attitude on this payoff are not as clear cut as the effects they have on the agent. What we can show, however, is how they affect the total surplus $V^*(c) + U^*(c)$, and the consequences we can draw from this and proposition 10.

Proposition 11.

- *For all $\alpha \in (0,1)$, $\frac{\partial V^*(c)+U^*(c)}{\partial \beta} < 0$ for all $c \in (\underline{c}, \overline{c}]$.*
- *If $0 < \alpha < \alpha^*(\beta)$, $\frac{\partial V^*(c)+U^*(c)}{\partial \alpha} < 0$ for all $c \in (\underline{c}, \overline{c}]$ if and only if $F(c^*) < \beta$.*
- *If $\alpha^*(\beta) \leq \alpha < 1$, $\frac{\partial V^*(c)+U^*(c)}{\partial \alpha} < 0$ for all $c \in (\underline{c}, \overline{c}]$.*

This proposition shows that ambiguity aversion increases the size of the total surplus. We know from proposition 10 that ambiguity aversion raises the agent's rent, and this proposition shows that, it may or may not lower the the principal's payoff, but in any case the size of the rent increase is always sufficient to compensate for it.

As far as ambiguity is concerned, let us consider first the case of high ambiguity. In this case, more ambiguity entails a decrease in total surplus. Therefore, since we know from proposition 10 that the rent of the inefficient agents in this case decreases with ambiguity, we see that, again, even though the principal might gain from ambiguity, her gain can never exceed the absolute value of the loss incurred by the agent.

Consider now the case of low ambiguity. In this case, the total surplus increases with ambiguity, if and only if $\beta < F(c^*)$, i.e. if and only if the principal is sufficiently ambiguity averse. In that case, let K be the constant identified in proposition 10 such that $\frac{\partial U^*(c)}{\partial \alpha} > 0$ if and only if $\frac{F(c^*)}{\beta} > 1 + K$. Then, we have the following corollary:

Corollary 1. *If* $0 < \alpha < \alpha^*(\beta)$ *and* $1 < \frac{F(c^*)}{\beta} < 1 + K$*. Then* $\frac{\partial V^*(c)}{\partial \alpha} > 0$ *for all* $c \in (\underline{c}, \overline{c}]$*.*

This corollary identifies a sufficient condition for which the principal's payoff might increase with ambiguity. She should be ambiguity averse but not too much so.

5 Conclusion

The optimal contract in an adverse selection model with a continuum of types and a parametric model of ambiguity and ambiguity aversion, namely the NEO-additive model, necessarily involves efficiency and a jump at the top of the distribution and pooling at the bottom of the distribution. As a result, ambiguity adversely affects the principal's ability to solve the adverse selection problem and therefore, if the principal is not very ambiguity averse, the least efficient types benefit from ambiguity with respect to risk, while ambiguity is detrimental to the most efficient types.

A Appendices

A.1 Miscellaneous results.

A.1.1 Result on the Kulback-Leibler Divergence

$$
\begin{aligned}
G(B) \geq (1-\alpha)F(B) &\iff \int_{\underline{c}}^{\overline{c}} \mathbb{1}_B \mathrm{d}G \geq (1-\alpha)\int_{\underline{c}}^{\overline{c}} \mathbb{1}_B \mathrm{d}F \quad \forall B \\
&\iff \int_{\underline{c}}^{\overline{c}} \mathbb{1}_B \mathrm{d}G \geq (1-\alpha)\int_{\underline{c}}^{\overline{c}} \mathbb{1}_B \frac{\mathrm{d}F}{\mathrm{d}G}\mathrm{d}G \quad \forall B \\
&\iff \int_{\underline{c}}^{\overline{c}} \mathbb{1}_B \left(1-(1-\alpha)\frac{\mathrm{d}F}{\mathrm{d}G}\right)\mathrm{d}G \geq 0 \quad \forall B. \\
&\Rightarrow 1-(1-\alpha)\frac{\mathrm{d}F}{\mathrm{d}G} \geq 0 \quad G\text{-a.s.} \\
&\Rightarrow \frac{1}{1-\alpha} \geq \frac{\mathrm{d}F}{\mathrm{d}G} \quad G\text{-a.s.} \\
&\Rightarrow \ln\left(\frac{1}{1-\alpha}\right) \geq \ln\left(\frac{\mathrm{d}F}{\mathrm{d}G}\right) \quad G\text{-a.s.} \\
&\Rightarrow \ln\left(\frac{1}{1-\alpha}\right) \geq \ln\left(\frac{\mathrm{d}F}{\mathrm{d}G}\right) \quad F\text{-a.s.} \\
&\Rightarrow \ln\left(\frac{1}{1-\alpha}\right) \geq \int_{\underline{c}}^{\overline{c}} \ln\left(\frac{\mathrm{d}F}{\mathrm{d}G}\right)\mathrm{d}F.
\end{aligned}
$$

A.1.2 Proof of Lemma 1.

Let

$$\varphi_\beta(\alpha) = \beta^2\alpha^2 - (\beta + f(\underline{c})(\overline{c}-\underline{c}))\alpha + f(\underline{c})(\overline{c}-\underline{c}).$$

Then $\alpha \in (0,1)$ is a solution to the above equation if and only if it is a solution to the equation $\varphi_\beta(\alpha) = 0$.

The discriminant of this equation of degree two in α is

$$\Delta = \beta^2 + (f(\underline{c})(\overline{c}-\underline{c}))^2 + 2\beta f(\underline{c})(\overline{c}-\underline{c})(1-2\beta).$$

Since $1-2\beta \in (-1,1)$ for $\beta \in (0,1)$, $\Delta > [\beta + f(\underline{c})(\overline{c}-\underline{c})]^2 > 0$. So this equation has two roots

$$\alpha^*(\beta) := \frac{\beta + f(\underline{c})(\overline{c}-\underline{c}) - \sqrt{\beta^2 + (f(\underline{c})(\overline{c}-\underline{c}))^2 + 2\beta f(\underline{c})(\overline{c}-\underline{c})(1-2\beta)}}{2\beta^2}$$

and

$$\hat{\alpha}_{+}(\beta) := \frac{\beta + f(\underline{c})(\overline{c}-\underline{c}) + \sqrt{\beta^2 + (f(\underline{c})(\overline{c}-\underline{c}))^2 + 2\beta f(\underline{c})(\overline{c}-\underline{c})(1-2\beta)}}{2\beta^2}.$$

Now, $\varphi_\beta(0) = f(\underline{c})(\overline{c}-\underline{c}) > 0$ and $\varphi_\beta(1) = \beta(\beta-1) < 0$, so there is at least one solution in $(0,1)$ and since φ_β is a convex polynomial of the second degree it has at most two roots, the one that lies in $(0,1)$ with $\varphi_\beta(0) > 0$ and $\varphi_\beta(1) < 0$ is on the decreasing branch, so the second one must be on the increasing branch, so the root in $(0,1)$ is unique. Since $\alpha^*(\beta) < \hat{\alpha}_{+}(\beta)$, this root must be $\alpha^*(\beta)$.

A.2 Optimal Second Best Contracts

A.2.1 Maximum principle and identification of the argmax and argmin of V

Let $c_0 \in \arg\max V(c)$ and $c_1 \in \arg\min V(c)$.

Assume first that $c_0 \leq c_1$. After similar computations as in (6) and (7), we obtain

$$\begin{aligned}
W(q) = {} & \alpha\beta(S(q(c_0)) - c_0 q(c_0)) + \alpha(1-\beta)(S(q(c_1)) - c_1 q(c_1)) \\
& + (1-\alpha)\int_{\underline{c}}^{c_0} \left(S(q(c)) - cq(c) - \frac{F(c)}{f(c)} q(c)\right) f(c)dc \\
& + (1-\alpha)\int_{c_0}^{c_1} \left(S(q(c)) - cq(c) - \frac{F(c) + \frac{\alpha\beta}{1-\alpha}}{f(c)} q(c)\right) f(c)dc \\
& + (1-\alpha)\int_{c_1}^{\overline{c}} \left(S(q(c)) - cq(c) - \frac{F(c) + \frac{\alpha}{(1-\alpha)}}{f(c)} q(c)\right) f(c)dc.
\end{aligned} \tag{16}$$

Let i be such that

$$i = \begin{cases} 1 \text{ if } c \in [\underline{c}, c_0] := \mathcal{C}_1 \\ 2 \text{ if } c \in [c_0, c_1] := \mathcal{C}_2 \\ 3 \text{ if } c \in [c_1, \overline{c}] := \mathcal{C}_3 \end{cases} \tag{17}$$

The principal's problem is to maximize (16) subject to (IC2). This is a multi-stage control problem. Let μ_i be the costate variable associated with the state q and y the control such that $q' = y$. The Hamiltonians are, $i = 1, 2, 3$,

$$H_i = (1-\alpha)\left(S(q(c)) - cq(c) - T_i(c, \alpha, \beta) q(c)\right) f(c) + \mu_i(c) y(c), \tag{18}$$

with

$$T_i(c,\alpha,\beta) = \begin{cases} \dfrac{F(c)}{f(c)} \text{ if } i=1 \\ \dfrac{F(c)+\frac{\alpha\beta}{1-\alpha}}{f(c)} \text{ if } i=2 \\ \dfrac{F(c)+\frac{\alpha}{(1-\alpha)}}{f(c)} \text{ if } i=3 \end{cases} \tag{19}$$

From the maximum principle (see Amit, 1986; Tomiyama, 1985) , we know that

$$\partial H_i/\partial y = \mu_i(c) \geq 0, \quad y(c)\partial H_i/\partial y = \mu_i(c)y(c) = 0; \tag{20}$$

$$\begin{aligned} \mu_i'(c) &= -\partial H_i/\partial q = -\left((1-\alpha)\left(S'(q(c)) - c - T_i(c,\alpha,\beta)\right)\right) f(c) \\ &= (1-\alpha)f(c)\left(c + T_i(c,\alpha,\beta) - S'(q(c))\right) \\ &\text{except at points of discontinuities of } y(c). \end{aligned} \tag{21}$$

Moreover, transversality conditions are

$$\mu_1(\underline{c}) = 0 \tag{22}$$

$$\mu_3(\overline{c}) = 0 \tag{23}$$

Finally, transition conditions are

$$\mu_1(c_0) - \alpha\beta\left(S'(q(c_0)) - c_0\right) = \mu_2(c_0) \tag{24}$$

$$\mu_2(c_1) - \alpha(1-\beta)\left(S'(q(c_1)) - c_1\right) = \mu_3(c_1) \tag{25}$$

Because $S'' < 0$, necessary and transversality conditions are also sufficient.

We will now proceed to show that at the optimum $c_0 = \underline{c}$ and $c_1 = \overline{c}$.

In order to do that, we need a series of lemmata.

We are looking for a piecewise C^1 solution q^*. Let $N \subset \mathcal{C}$ be the finite set of points where q' is not continuous. Note that in principle N can be empty. Let $\mathcal{C}_i^* := \mathcal{C}_i \setminus N$ and $\mathcal{C}^* = \mathcal{C}_1^* \cup \mathcal{C}_2^* \cup \mathcal{C}_3^*$. Condition (21) holds at every point of $\mathcal{C}^*$, hence, since q^* is continuous on every interval of $\mathcal{C}^*$, it implies that μ_i' exists and is continuous on every interval of $\mathcal{C}_i^*$, but may not exist at some point in N.

Introduce a piece of notation. For all $c \in \mathcal{C}$, let

$$q_i^s(c) := S'^{-1}\left(c + T_i(c, \alpha, \beta)\right) \tag{26}$$

and

$$q^s(c) := q_i^s(c) \text{ whenever } c \in \mathcal{C}_i. \tag{27}$$

Let also $\mu(c) := \mu_i(c)$ whenever $c \in \mathcal{C}_i$. Finally, let

$$c^i := \begin{cases} c_0 \text{ if } i = 1 \\ c_1 \text{ if } i = 2 \\ \bar{c} \text{ if } i = 3 \end{cases}$$

Lemma 2. *Under assumption* 1, q_i^s *and* q^{FB} *are differentiable on* $\mathcal{C}$, $(q_i^s)'(c) < 0$ *and* $(q^{FB})'(c) < 0$ *for all* $c \in \mathcal{C}$.

Proof. Straightforward. □

Lemma 3. *If* μ_i^* *is a solution and* μ_i^* *is constant on some interval* $I \subset \mathcal{C}_i$, *then* $\mu_i^*(c) = 0$ *for all* $c \in I$.

Proof. Assume μ_i^* is constant on I. Then, for all $c \in I \cap \mathcal{C}^*$, $(\mu_i^*)'(c) = 0$, and thus by (21) $q^*(c) = q_i^s(c)$. Therefore, by lemma 2 $(q_i^*)'(c) < 0$ for all $c \in I \cap \mathcal{C}^*$, and thus by (20) $\mu_i^*(c) = 0$ for all $c \in I \cap \mathcal{C}^*$. Since μ_i^* is continuous on I and N is finite, this implies $\mu_i^*(c) = 0$ for all $c \in I$. □

Lemma 4. *If* (q^*, μ^*) *is a solution such that* $(\mu^*)'(c_0^i) > 0$ *for some* $c_0^i \in \mathcal{C}_i^*$, *then* $(\mu_i^*)'(c) > 0$ *for all* $c \in [c_0^i, c^i] \cap \mathcal{C}_i^*$, *and* $\mu_i^*(c) > 0$ *for all* $c \in [c_0^i, c^i]$.

Proof. Assume $(\mu^*)'(c_0^i) > 0$ for some $c_0^i \in \mathcal{C}_i^*$. Then since $(\mu_i^*)'$ is continuous at c_0^i and $\mu_i^* \geq 0$, we may w.l.o.g. assume that $\mu_i^*(c_0^i) > 0$.

Now, assume by contradiction that there exists $c_1^i \in (c_0^i, c^i]$ such that $(\mu_i^*)'(c_1^i) \leq 0$. Note that it is impossible that $(\mu_i^*)'(c) = 0$ for all $c \in (c_0^i, c^i)$, because this would imply that μ_i^* is constant on this interval, and thus by lemma 3 that $\mu_i^*(c) = 0$ for all $c \in (c_0^i, c^i)$, which would be a contradiction since $\mu_i^*(c_0^i) > 0$ and μ_i^* is continuous. Thus w.l.o.g. we can assume that $(\mu_i^*)'(c_1^i) < 0$. Again by continuity, we may therefore assume w.l.o.g. that $\mu_i^*(c_1^i) > 0$. Now since μ_i^* is continuous over the compact interval $[c_0^i, c_1^i]$, it has a global maximum there, reached at c_2^i. Since $\mu_i^*(c_0^i) > 0$ and $\mu_i^*(c_1^i) > 0$, we have $\mu_i^*(c_2^i) > 0$, therefore there exists an open interval $(c_2^i - \varepsilon, c_2^i + \varepsilon)$ such that $\mu_i^* > 0$ over this interval. By (20), this implies that q^* is constant over this interval. On the other hand, by the same argument as before $(\mu_i^*)'$ is not always 0 over this interval, and, since μ^* is piecewise C^1, there exist $c_3^i, c_4^i \in (c_2^i - \varepsilon, c_2^i) \cup (c_2^i, c_2^i + \varepsilon) \cap \mathcal{C}_i^*$ such that $c_3^i < c_4^i$, $(\mu_i^*)'(c_3^i) > 0$ and $(\mu_i^*)'(c_4^i) < 0$. Thus, $q^*(c_3^i) > q_i^s(c_3^i) > q_i^s(c_4^i) > q^*(c_4^i)$ and q^* is not constant on $(c_2^i - \varepsilon, c_2^i + \varepsilon)$, which is a contradiction. □

Now, assume $\mu_1^*(c) = 0$ for all $c \in \mathcal{C}_1$. Then, $(\mu_1^*)'(c) = 0$ for all $c \in \mathcal{C}_1^*$. Then $q^*(c) = q_1^s(c)$ on $\mathcal{C}_1^*$ by (21), and, in particular,

$$\lim_{c \uparrow c_0} q^*(c) \leq q_1^s(c_0) < q^{FB}(c_0), \tag{28}$$

since $\underline{c} < c_0$). On the other hand, condition (24) implies that

$$-\alpha\beta \left(S'(q^*(c_0)) - c_0 \right) = \mu_2^*(c_0) \geq 0,$$

thus $q^*(c_0) \geq q^{FB}(c_0)$. But this is incompatible with equation (28) given (IC2).

Let $M_i := \{c \in \mathcal{C}_i \mid \mu_i^*(c) > 0\}$. We have just shown that $M_1 \neq \varnothing$. Since it is bounded below by $\underline{c}$, it has a lower bound. Denote it c_0^*. Since μ_1^* is continuous, M_1 is open, and thus $\mu_1^*(c_0^*) = 0$. Let $c \in M_1$ be such that μ_1 is differentiable on (c, c_0^*). Then, $\mu_1^*(c) > \mu_1^*(c_0^*)$ and there exists $c' \in (c_0^*, c)$ such that $(\mu_1^*)'(c') = \frac{\mu_1^*(c) - \mu_1^*(c_0^*)}{c - c_0^*} > 0$. By lemma 4, this implies that $\mu^*(c'') > 0$ for all $c_0 > c'' > c'$. Since this is true for every $c \in M_1 \cap \mathcal{C}_1^*$, $(\mu_1^*)(c) > 0$ for all $c > c_0^*$, i.e. $M_1 = (c^*, \bar{c}]$. In particular, q^* is constant over M_1 by (20). Denote the values it takes on this interval by $\bar{q}_1$.

Summarizing, on $\mathcal{C}_1$, we have

$$q^*(c) = \begin{cases} q_1^s(c) & \text{if } c \in [\underline{c}, c_0^*] \\ \bar{q}_1 & \text{if } c \in [c_0^*, c_0], \end{cases}$$

and therefore, since $(q_1^s)'(c) < 0$ and $q_1^s(c) \leq q^{FB}(c)$ on $[\underline{c}, c_0^*]$, we have

$$V'(c) = \begin{cases} (S'(q_1^s(c)) - c)(q_1^s)'(c) < 0 & \text{if } c \in [\underline{c}, c_0^*) \\ 0 & \text{if } c \in [c_0^*, c_0]. \end{cases}$$

Thus V is nonincreasing on $\mathcal{C}_1$ and therefore we cannot have $c_0 > \underline{c}$.

On $\mathcal{C}_3$, on the other hand, there is no contradiction with condition (25) in assuming that $\mu_3 = 0$, since condition (25) implies $q^*(c_1) \leq q^{FB}(c_1)$, and thus $q^*(c) = q_3^s(c)$. But then $V'(c) = (S'(q_1^s(c)) - c)(q_1^s)'(c) < 0$ on $\mathcal{C}_3$, contradicting the fact that $c_1 \in \arg\min(V(c))$, unless $c_1 = \overline{c}$.

If $c_0 > c_1$, then

$$\begin{aligned} W(q) =& \alpha\beta(S(q(\theta_0) - \theta_0 q(\theta_0)) + \alpha(1-\beta)(S(q(\theta_1)) - \theta_1 q(\theta_1)) \\ & \quad + (1-\alpha)\int_{\theta_1}^{\theta_0} \left(S(q(\theta)) - \theta q(\theta) - \frac{F(\theta) + \frac{\alpha(1-\beta)}{1-\alpha}}{f(\theta)} q(\theta) \right) f(\theta) d\theta \\ & + (1-\alpha)\int_{\underline{\theta}}^{\theta_1} \left(S(q(\theta)) - \theta q(\theta) - \frac{F(\theta)}{f(\theta)} q(\theta) \right) f(\theta) d\theta \\ & \quad + (1-\alpha)\int_{\theta_0}^{\overline{\theta}} \left(S(q(\theta)) - \theta q(\theta) - \frac{(1-\alpha)F(\theta) + \alpha}{(1-\alpha)f(\theta)} q(\theta) \right) f(\theta) d\theta. \end{aligned} \tag{29}$$

In that case, similar arguments as above show that the optimality conditions imply that V is decreasing and continuous on $[c_1, c_0]$. But this is incompatible with $V(c_1) < V(c_0)$ and $c_1 < c_0$. Thus the latter case is impossible at the optimum.

To conclude, $c_0 = \underline{c}$ and $c_1 = \overline{c}$.

So let us denote

$$H = H_2, \mu(c) = \mu_2(c) \text{ and } T(c, \alpha, \beta) = T_2(c, \alpha, \beta), \tag{30}$$

with μ absolutely continuous. Conditions (20)-(25) reduce to

$$\mu(c) \geq 0, y(c)\mu(c) = 0; \tag{31}$$

$$\mu'(c) = (1-\alpha)f(c)\left(c + T(c,\alpha,\beta) - S'(q(c))\right)$$

$$\text{except at points of discontinuities of } y(c) \tag{32}$$

$$\mu(\underline{c}) = -\alpha\beta\left(S'(q(\underline{c}))) - \underline{c}\right) \tag{33}$$

$$\mu(\overline{c}) = \alpha(1-\beta)\left(S'(q(\overline{c})) - \overline{c}\right) \tag{34}$$

From (31), we know that $\mu(c) \geq 0$. Therefore, since $S'' < 0$, we get, using (FB) with (33), then with (34)

$$\mu(\underline{c}) \geq 0 \Leftrightarrow \alpha\beta = 0 \text{ or } (\alpha\beta > 0 \text{ and } q(\underline{c}) \geq q^{FB}(\underline{c})) \tag{35}$$

$$\mu(\overline{c}) \geq 0 \Leftrightarrow \alpha(1-\beta) = 0 \text{ or } (\alpha(1-\beta) > 0 \text{ and } q(\overline{c}) \leq q^{FB}(\overline{c})) \tag{36}$$

Parallel arguments show that, on $\mathcal{C}_2$,

$$q^*(c) = \begin{cases} q_2^s(c) & \text{if } c \in (\underline{c}, c^*] \\ \bar{q}_2 & \text{if } c \in [c^*, \overline{c}], \end{cases}$$

for some values c^* and $\bar{q}_2$.

We may now proceed to prove the propositions in the text.

A.2.2 Proof of Proposition 2

(i) If q^* is a solution, $q^*(\underline{c}) = q^{FB}(\underline{c})$.

Proof. By (33), $q^*(\underline{c}) \geq q^{FB}(\underline{c})$, so we must show the reverse inequality. If $\underline{c} \in \mathcal{C}^*$, then this follows from the following lemma.

Lemma 5. *If q^* is a solution, for all $c \in \mathcal{C}^*$, $q^*(c) \leq q^{FB}(c)$.*

Proof. By contradiction, consider $c_0 \in \mathcal{C}^*$ such that $q^*(c_0) > q^{FB}(c_0)$. Then, $q^*(c_0) > q^s(c_0)$, thus $S'(q^*(c_0)) < c_0 + T(c_0)$, so that, by (32), $(\mu^*)'(c_0) > 0$. By lemma 4, therefore, $(\mu^*)'(c) > 0$ for all $c \in [c_0, \overline{c}]$, and $\mu^*(c) > 0$ for all $c \in (c_0, \overline{c}]$. So q^* is constant on $(c_0, \overline{c}]$. Let $\overline{q}$ be its value. Then, for all $c \in (c_0, \overline{c}]$, $q^*(c) = \overline{q} \geq q^*(c_0) > q^{FB}(c_0) > q^{FB}(c)$. In particular, $q^*(\overline{c}) > q^{FB}(\overline{c})$. However, since $\mu^*(\overline{c}) > 0$, by (34) $q^*(\overline{c}) < q^{FB}(\overline{c})$, which is a contradiction. □

So assume that $\underline{c} \notin \mathcal{C}^*$ and $q^*(\underline{c}) > q^{FB}(\underline{c})$. Then, by (33) again $\mu(\underline{c}) > 0$. But then, since μ is continuous, $\mu > 0$ on some interval $[\underline{c}, c_1)$. Thus, by (31), $y = 0$ on $[\underline{c}, c_1)$, hence $\lim_{c \to \underline{c}} y(c) = 0 = y(\underline{c})$ thus q' is continuous at $\underline{c}$: $\underline{c} \in \mathcal{C}^*$, contradicting the initial assumption. □

(ii) If q^* is a solution and $\beta > 0$, q^* is not continuous at $\underline{c}$.

Proof. To prove the claim, we shall need the following lemma:

Lemma 6. *If q^* is a solution, for all $c \in \mathcal{C} \setminus \{\underline{c}\}$, $q^*(c) \leq q^{FB}(c)$.*

Proof. Lemma 5 shows that this is true for $c \in \mathcal{C}^*$. So consider $c_0 \in N \setminus \{\underline{c}\}$. There is $c_1 \in \mathcal{C}^*$, $c_1 < c_0$ such that $(c_1, c_0) \subset \mathcal{C}^*$. Thus, for all $c \in (c_1, c_0)$, since q^* is non increasing, $q^*(c_0) \leq q^*(c) \leq q^{FB}(c)$ by lemma 2. By continuity of q^{FB}, this implies $q^*(c_0) \leq q^{FB}(c_0)$. □

Assume on the contrary that q^* is continuous at $\underline{c}$. Since q^* is piecewise continuously differentiable, both q^* and $(q^*)'$ have a right limit at $\underline{c}$ and the assumption implies that $(q^*)'(\underline{c})$ exists and is equal to $\lim_{c \downarrow \underline{c}} (q^*)'(c)$. Thus our assumption implies that $\underline{c} \in \mathcal{C}^*$. In particular, this implies that $(\mu^*)'(\underline{c})$ exists and is given by (32). We will therefore consider various possibilities for the sign of $(\mu^*)'(\underline{c})$ and show that all of them lead to a contradiction, and thus to the rejection of the assumption that q^* is continuous at $\underline{c}$. Note that, for all $0 < h \leq \overline{c} - \underline{c}$, $\mu^*(\underline{c} + h) \geq 0$. Thus, since $\mu^*(\underline{c}) = 0$ by (33) and point (i)

$$\frac{\mu^*(\underline{c} + h) - \mu^*(\underline{c})}{h} = \frac{\mu^*(\underline{c} + h)}{h} \geq 0,$$

so that $(\mu^*)'(\underline{c}) \geq 0$. Therefore we have only two cases to consider

Case 1: $(\mu^*)'(\underline{c}) = 0$. Since $\underline{c} \in \mathcal{C}^*$ by assumption, (32) applies and implies in this case that $q^*(\underline{c}) = q^s(\underline{c})$. Yet, $q^s(\underline{c}) < q^{FB}(\underline{c}) = q^*(\underline{c})$; a contradiction.

Case 2: $(\mu^*)'(\underline{c}) > 0$. Then, by lemma 4, $(\mu^*)'(c) > 0$ for all $c \in \mathcal{C}$, therefore $\mu > 0$ on $(\underline{c}, \overline{c}]$ and q^* is constant on $(\underline{c}, \overline{c}]$ by (31). Since moreover we have assumed that it was continuous at $\underline{c}$, it is constant on $\mathcal{C}$. Denote $\overline{q}$ its value. Then, for all $c \in (\underline{c}, \overline{c}]$, $q^*(c) = \overline{q} \geq q^*(\underline{c})$. But $q^*(\underline{c}) = q^{FB}(\underline{c})$ by point (i) and $q^{FB}(\underline{c}) > q^{FB}(c)$ by lemma 2, so $q^*(c) > q^{FB}(c)$; a contradiction by lemma 6.

To sum up, assuming that q^* is continuous at $\underline{c}$ leads to a contradiction because $(\mu^*)'(\underline{c})$ must be either 0 or positive, yet both cases are incompatible with this assumption. Therefore q^* is not continuous at $\underline{c}$. □

(iii)

Claim 1. *If μ^* is a solution, then there exists $c^* \in \mathcal{C}$ such that $\mu^*(c) = 0$ for all $c \in [\underline{c}, c^*]$ and $\mu^*(c) > 0$ for all $c \in (c^*, \overline{c}]$. Moreover, $(c^*, \overline{c}] \subset \mathcal{C}^*$ and $c^* < \overline{c}$ if and only if $\beta < 1$.*

Proof. The arguments are similar to the one used above but we repeat them here for completeness.

Let $M := \{c \in \mathcal{C} \mid \mu^*(c) > 0\}$. We must show that there exists $c^* \in \mathcal{C}$ such that $M = (c^*, \overline{c}]$ and $(c^*, \overline{c}] \subset \mathcal{C}^*$. We will first prove that $M \neq \varnothing$. Assume by contradiction that $\mu^*(c) = 0$ for all $c \in \mathcal{C}$. Then, $(\mu^*)'(c) = 0$ for all $c \in \mathcal{C}$, and therefore, by (32), $q^*(c) = q^s(c)$ for all $c \in \mathcal{C}^*$. Assume $\beta < 1$. If $\overline{c} \in \mathcal{C}^*$, we also have by (34) $q^*(\overline{c}) = q^{FB}(\overline{c}) > q^s(\overline{c})$; a contradiction. If $\overline{c} \notin \mathcal{C}^*$, there exists an interval $(c_0, \overline{c}) \subset \mathcal{C}^*$ such that $q^s(c) = q^*(c) \geq q^*(\overline{c}) = q^{FB}(\overline{c})$ for all $c \in (c_0, \overline{c})$, thus by continuity of q^s, $q^s(\overline{c}) \geq q^{FB}(\overline{c})$; a contradiction since $q^s(\overline{c}) < q^{FB}(\overline{c})$. We therefore proved that M is not empty. Since it is bounded below by $\underline{c}$, it has a lower bound. Denote it c^*. Since μ^* is continuous, M is open, and thus $\mu^*(c^*) = 0$. Therefore, for all $c \in M$, $\mu^*(c) > \mu^*(c^*)$ and there exists $c' \in (c^*, c)$ such that $(\mu^*)'(c') = \frac{\mu^*(c) - \mu^*(c^*)}{c - c^*} > 0$. By lemma 4, this implies that $(\mu^*)'(c) > 0$ for all $c > c'$. Since this is true for every $c \in M$, $(\mu^*)'(c)$ exists for all $c > c^*$ and $(\mu^*)'(c) > 0$. This implies that $(\mu^*)(c) > 0$ for all $c > c^*$, i.e. $M = (c^*, \overline{c}]$. In particular, q^* is constant over M by (31), so that it is C^1 on M, and thus $M \subset \mathcal{C}^*$. Moreover, since M is open, this proves in particular that $c^* < \overline{c}$ whenever $\beta < 1$. If $\beta = 1$, then $\mu^*(\overline{c}) = 0$ and since, moreover, $\mu^*(\underline{c}) = 0$, if $M \neq \varnothing$, then, there exists $c_0 \in \mathcal{C}^*$ such that $\mu'(c_0) > 0$, and thus by lemma 4, this implies that $(\mu^*)'(c) > 0$ for all $c > c_0$, thus $\mu^*(\overline{c}) > 0$; a contradiction. Thus $M = \varnothing$, i.e. $c^* = \overline{c}$. $\square$

This claim proves point (iii), as, if $\mu^*(c) > 0$ for all $c > c^*$, then by (31) q^* is constant on $(c^*, \overline{c})$.

A.2.3 Proof of Propositions 3 and 4

For $\beta = 1$, we know from claim 1 that $c^* = \bar{c}$. So from now on, we assume that $\beta < 1$, and in that case $c^* < \bar{c}$.

Let us show

Claim 2. *If $c^* > \underline{c}$, for all $c \in (\underline{c}, c^*]$, we have $q^*(c) = q^s(c)$.*

Proof. Assume $c^* > \underline{c}$. Then, for all $c \in (\underline{c}, c^*)$, $\mu^*(c) = 0$, hence $(\mu^*)'(c) = 0$. If $c \in \mathcal{C}^*$, this implies by (32) that $q^*(c) = q^s(c)$. If $c \notin \mathcal{C}^*$, there exists c', c'', with $c' < c < c''$, such that $(c', c) \cup (c, c'') \subset \mathcal{C}^*$. Then for all $\tau \in (c', c)$, $\tau' \in (c, c'')$,

$$q^s(\tau) = q^*(\tau) \geq q^*(c) \geq q^*(\tau') = q^s(\tau'),$$

hence taking limits as $\tau \to c$ and $\tau' \to c$ and since q^s is continuous, $q^*(c) = q^s(c)$. $\square$

Claim 3. *If $c^* > \underline{c}$, for all $c \in (c^*, \bar{c}]$, we have $q^*(c) = q^s(c^*)$.*

Proof. By (31), since $\mu^* > 0$ on $(c^*, \bar{c}]$, q^* is constant on $(c^*, \bar{c}]$. Denote its value $\bar{q}$. Moreover, since $(c^*, \bar{c}] \subset \mathcal{C}^*$, (32) applies, therefore for all $c \in (c^*, \bar{c}]$, $q^s(c) < \bar{q}$. As $c \to c^*$, this implies $q^s(c^*) \leq \bar{q}$. On the other hand, q^* is non-increasing, thus, if $c^* > \underline{c}$, for all $c \in (\underline{c}, c^*)$, we have $q^s(c) = q^*(c) \geq \bar{q}$, thus again as $c \to c^*$, we have $q^s(c^*) \geq \bar{q}$. Thus $q^s(c^*) \geq \bar{q}$. Finally, $\bar{q} = q^s(c^*)$. $\square$

Proposition 3. Proposition 3 follows from claims 2 and 3, once we have characterized c^* when $c^* > \underline{c}$. Let us now do this. We know from claim 1 that

$$c^* = \sup\{c \in \mathcal{C} \mid \mu^*(c) = 0\}.$$

Moreover, since (q^*, μ^*) is a solution, (34) must hold. Given the previous analysis, this condition writes

$$\alpha(1-\beta)\left(S'(q^s(c^*)) - \bar{c}\right) = \int_{c^*}^{\bar{c}} (1-\alpha)\left(c + T(c,\alpha,\beta) - S'(q^s(c^*))\right) f(c)dc \tag{37}$$

After some algebra and noting that

$$\begin{aligned}\int_{c^*}^{\bar{c}} (1-\alpha)\left(c + T(c,\alpha,\beta)\right) f(c)dc &= \int_{c^*}^{\bar{c}} c(1-\alpha)f(c) + (1-\alpha)F(c) + \alpha\beta dc \\ &= \int_{c^*}^{\bar{c}} \frac{\mathrm{d}\, c((1-\alpha)F(c) + \alpha\beta)}{\mathrm{d}c} dc,\end{aligned}$$

one can show that this equation implies that whenever $c^* > \underline{c}$, c^* satisfies

$$(1-(1-\alpha)f(c^*)T(c^*,\alpha,\beta))T(c^*,\alpha,\beta)+c^*-\overline{c}=0. \tag{38}$$

Let, for all α, β in $[0,1]$ and $c \in \mathcal{C}$,

$$\Phi(c,\alpha,\beta) := (1-(1-\alpha)f(c)T(c,\alpha,\beta))T(c,\alpha,\beta)+c-\overline{c}. \tag{39}$$

Since

$$\begin{aligned}
\frac{\partial \Phi}{\partial c}(c,\alpha,\beta) &= 1+(1-\alpha\beta)\frac{\partial T}{\partial c}(c,\alpha,\beta)-(1-\alpha)\left(f(c)T(c,\alpha,\beta)+F(c)\frac{\partial T}{\partial c}(c,\alpha,\beta)\right)\\
&= 1+(1-\alpha\beta-(1-\alpha)F(c))\frac{\partial T}{\partial c}(c,\alpha,\beta)-(1-\alpha)f(c)T(c,\alpha,\beta)\\
&= 1+(1-\alpha\beta-(1-\alpha)F(c))\frac{\partial T}{\partial c}(c,\alpha,\beta)-\alpha\beta-(1-\alpha)F(c)\\
&= \left(1+\frac{\partial T}{\partial c}(c,\alpha,\beta)\right)-(\alpha\beta+(1-\alpha)F(c))\frac{\partial T}{\partial c}(c,\alpha,\beta)-(\alpha\beta+(1-\alpha)F(c))\\
&= \left(1+\frac{\partial T}{\partial c}(c,\alpha,\beta)\right)(1-(\alpha\beta+(1-\alpha)F(c)))>0,
\end{aligned} \tag{40}$$

Φ is increasing in c. Moreover,

$$\Phi(\overline{c},\alpha,\beta)=\mu(\overline{c})>0$$

whenever $\beta<1$.

Thus, $c^* > \underline{c}$ can be the (unique) solution in $\mathcal{C}$ to the equation

$$\Phi(c,\alpha,\beta)=0 \tag{41}$$

only if

$$\Phi(\underline{c},\alpha,\beta)=\frac{(1-\alpha\beta)\alpha\beta}{(1-\alpha)f(\underline{c})}+\underline{c}-\overline{c}<0.$$

Therefore,

$$\begin{aligned}
c^* \in (\underline{c},\overline{c}) &\iff \frac{(1-\alpha\beta)\alpha\beta}{(1-\alpha)f(\underline{c})}+\underline{c}-\overline{c}<0\\
&\iff \frac{(1-\alpha\beta)\alpha\beta}{(1-\alpha)}<f(\underline{c})(\overline{c}-\underline{c})\\
&\iff \alpha<\alpha^*(\beta) \quad \text{by lemma 1.}
\end{aligned}$$

Thus claims 2 and 3 and this condition show that the expression given in proposition 3 for q^* is necessary for q^* to be the solution. Conversely, it is a routine matter to check that q^* is indeed a solution given μ^* as above. So this proves proposition 3.

Proposition 4. Now for proposition 4 consider first the case $\alpha < 1$ and

$$\frac{(1-\alpha\beta)\alpha\beta}{(1-\alpha)} \geq f(\underline{c})(\overline{c}-\underline{c}),$$

i.e., by lemma 1, $\alpha \geq \alpha^*(\beta)$. Then equation 41 does not have a solution in $(\underline{c}, \overline{c})$, and thus $c^* = \underline{c}$. Therefore

$$q^*(c) = \begin{cases} q^{FB}(c) & \text{if } c = \underline{c} \\ \overline{q} & \text{if } c \in (\underline{c}, \overline{c}], \end{cases}$$

where by (34) and (32) $\overline{q}$ satisfies

$$(1-\alpha)\int_{\underline{c}}^{\overline{c}} \left(c + T(c^*, \alpha, \beta) - S'(\bar{q})\right) f(c)\,\mathrm{d}c = \alpha(1-\beta)\left(S'(\bar{q}) - \overline{c}\right).$$

Solving for $\bar{q}$ yields

$$\overline{q} = S'^{-1}\left(\underline{c} + \frac{\overline{c}-\underline{c}}{1-\alpha\beta}\right) = S'^{-1}\left(\overline{c} + \frac{\alpha\beta}{1-\alpha\beta}(\overline{c}-\underline{c})\right).$$

Note that $\bar{q} < q^{FB}(c)$ for $c > \underline{c}$ because $\overline{c} - \underline{c} - \frac{\overline{c}-\underline{c}}{1-\alpha\beta} = (\overline{c}-\underline{c})\frac{-\alpha\beta}{1-\alpha\beta} < 0$.

Consider now the case $\alpha = 1$. Using (32), we get $\mu'(c) = \beta > 0$ for all $c \in \mathcal{C}^*$. Since μ is piecewise C^1 this implies that $\mu'(c) = \beta$ for all $c \in \mathcal{C}$. Then since $\mu(\underline{c}) = 0$, this implies $\mu(c) = \beta(c-\underline{c}) > 0$ for all $c \in (\underline{c}, \overline{c}]$, hence by (31) $(q^*)'(c) = 0$ on the same interval, thus q^* is constant over this interval. Thus $q^*(c) = q^{FB}(c)$ at $\underline{c}$ by (33) and $q^*(c) = \bar{q}$ on $(\underline{c}, \overline{c}]$, with $\bar{q}$ given by, using (34)

$$\beta(\overline{c}-\underline{c}) = (1-\beta)(S'(\bar{q}) - \overline{c}).$$

A.2.4 Proof of Proposition 5

Let $R(q,c) := \alpha(1-\beta)\left(S(q) - cq\right)$. The objective function (14) assigns the scrap value $R(q(c_*), c_*)$ to the type c_*. So according to the maximum principle (Seierstad and Sydsaeter, 1986, p. 184), the condition defining c_* is

$$H(c_*) + \frac{\partial R}{\partial c}(q(c_*), c_*) = 0.$$

From (18), (19) and (30), we get

$$(1-\alpha)\left(S(q(c_*)) - c_* q(c_*) - T(c^*,\alpha,\beta)q(c_*)\right) f(c_*) + \mu(c)y(c) - \alpha(1-\beta)q(c_*) = 0.$$

But, $\mu(c)y(c) = 0$ by (31).

A.3 Comparative statics

A.3.1 Proof of Proposition 6

By definition c^* is given by

$$\Phi(c^*,\alpha,\beta) = c^* + (1-\alpha\beta-(1-\alpha)F(c^*))T(c^*,\alpha,\beta) - \bar{c} = 0.$$

Thus, by the implicit function theorem,

$$\frac{\partial c^*}{\partial \alpha}(\alpha,\beta) = -\frac{\frac{\partial \Phi}{\partial \alpha}}{\frac{\partial \Phi}{\partial c}}(c^*,\alpha,\beta) \quad \text{and} \quad \frac{\partial c^*}{\partial \beta}(\alpha,\beta) = -\frac{\frac{\partial \Phi}{\partial \beta}}{\frac{\partial \Phi}{\partial c}}(c^*,\alpha,\beta).$$

Now, by equation (40), $\frac{\partial \Phi}{\partial c}(c,\alpha,\beta) > 0$; on the other hand,

$$\begin{aligned}
\frac{\partial \Phi}{\partial \alpha}(c,\alpha,\beta) &= (1-\alpha\beta-(1-\alpha)F(c))\frac{\partial T}{\partial \alpha}(c,\alpha,\beta) + (F(c)-\beta)T(c,\alpha,\beta) \\
&= (1-\alpha\beta-(1-\alpha)F(c))\frac{\beta}{(1-\alpha)^2 f(c)} + (F(c)-\beta)\frac{(1-\alpha)F(c)+\alpha\beta}{(1-\alpha)f(c)} \\
&= \frac{[1-\alpha\beta-(1-\alpha)F(c)]\beta + (1-\alpha)(F(c)-\beta)[(1-\alpha)F(c)+\alpha\beta)]}{(1-\alpha)^2 f(c)} \\
&= \frac{\beta-\beta^2+(1-\alpha)^2(\beta-F(c))^2}{(1-\alpha)^2 f(c)} > 0
\end{aligned}$$

hence

$$\frac{\partial c^*}{\partial \alpha}(\alpha,\beta) < 0.$$

In turn,

$$\begin{aligned}
\frac{\partial \Phi}{\partial \beta}(c,\alpha,\beta) &= (1-\alpha\beta-(1-\alpha)F(c))\frac{\partial T}{\partial \beta}(c,\alpha,\beta) - \alpha T(c,\alpha,\beta) \\
&= (1-\alpha\beta-(1-\alpha)F(c))\frac{\alpha}{(1-\alpha)f(c)} - \alpha\frac{\alpha\beta+(1-\alpha)F(c))}{(1-\alpha)f(c)} \\
&= \frac{\alpha}{(1-\alpha)f(c)}(1-2(\alpha\beta+(1-\alpha)F(c))).
\end{aligned}$$

Thus

$$\frac{\partial \Phi}{\partial \beta}(c,\alpha,\beta) > 0 \iff 1 - 2(\alpha\beta + (1-\alpha)F(c)) > 0,$$

and, since $\frac{\partial \Phi}{\partial c}(c,\alpha,\beta) > 0$,

$$\frac{\partial c^*}{\partial \beta}(\alpha,\beta) \lesseqgtr 0 \iff \alpha\beta + (1-\alpha)F(c^*) \lesseqgtr \frac{1}{2} \iff \beta \lesseqgtr \frac{1}{2\alpha} - \frac{1-\alpha}{\alpha}F(c^*).$$

From this we can derive the existence and uniqueness of a $\hat{\beta} \in [0,1]$ such that c^* is a decreasing function of β if $\beta < \hat{\beta}$, an increasing function of β if $\beta > \hat{\beta}$ and $\frac{\partial c^*}{\partial \beta}(\hat{\beta}) = 0$. Indeed, let

$$g(\beta) := \alpha\beta + (1-\alpha)F(c^*) - \frac{1}{2}.$$

Clearly, g is continuous and differentiable and

$$\begin{aligned}
g'(\beta) &= \alpha + (1-\alpha)\frac{\partial c^*}{\partial \beta}(\alpha,\beta)f(c^*) \\
&= \alpha - \frac{\alpha(1-2(\alpha\beta + (1-\alpha)F(c^*)))}{(1-\alpha)f(c^*)\left(1+\frac{\partial T}{\partial c}(c^*,\alpha,\beta)\right)(1-(\alpha\beta+(1-\alpha)F(c^*)))}(1-\alpha)f(c^*) \\
&= \alpha\left(1 - \frac{(1-2(\alpha\beta+(1-\alpha)F(c^*)))}{\left(1+\frac{\partial T}{\partial c}(c^*,\alpha,\beta)\right)(1-(\alpha\beta+(1-\alpha)F(c^*)))}\right) \\
&= \frac{\alpha\left(\left(1+\frac{\partial T}{\partial c}(c^*,\alpha,\beta)\right)(1-(\alpha\beta+(1-\alpha)F(c^*))) - 1 + 2(\alpha\beta+(1-\alpha)F(c^*))\right)}{\left(1+\frac{\partial T}{\partial c}(c^*,\alpha,\beta)\right)(1-(\alpha\beta+(1-\alpha)F(c^*)))} \\
&= \frac{\alpha\left(\frac{\partial T}{\partial c}(c^*,\alpha,\beta)(1-(\alpha\beta+(1-\alpha)F(c^*))) + \alpha\beta + (1-\alpha)F(c^*)\right)}{\left(1+\frac{\partial T}{\partial c}(c^*,\alpha,\beta)\right)(1-(\alpha\beta+(1-\alpha)F(c^*)))} > 0.
\end{aligned}$$

Three cases may appear.

Case 1. $g(0) < 0$ and $g(1) > 0$. In this case, there is a unique $\hat{\beta} \in [0,1]$ such that $g(\beta) = 0$. Moreover, since g is increasing this implies that $g(\beta) < 0$ for all $\beta \in [0,\hat{\beta})$ and $g(\beta) > 0$ for all $\beta \in (\hat{\beta},1]$, thus, $\frac{\partial c^*}{\partial \beta}(\alpha,\beta) < 0$ for all $\beta \in [0,\hat{\beta})$ and $\frac{\partial c^*}{\partial \beta}(\alpha,\beta) > 0$ for all $\beta \in (\hat{\beta},1]$.

Case 2. $g(0) \geq 0$. Then, we can set $\hat{\beta} = 0$.

Case 3. $g(1) \leq 0$. Then we can set $\hat{\beta} = 1$.

A.3.2 Proof or Proposition 7

Let

$$\Delta^{eff} := q^{EU}(\underline{c}) - q^{s}(\underline{c}) = S'^{-1}(\underline{c}) - S'^{-1}\left(\underline{c} + \frac{\alpha\beta}{(1-\alpha)f(\underline{c})}\right)$$

be the size of the jump at the top. Then

$$\frac{\partial\Delta^{eff}}{\partial\alpha} = -\frac{\beta}{(1-\alpha)^2 f(\underline{c})}\frac{1}{S''(S'^{-1}\left(\underline{c} + \frac{\alpha\beta}{(1-\alpha)f(\underline{c})}\right)} > 0$$

and

$$\frac{\partial\Delta^{eff}}{\partial\beta} = -\frac{\alpha}{(1-\alpha)f(\underline{c})}\frac{1}{S''(S'^{-1}\left(\underline{c} + \frac{\alpha\beta}{(1-\alpha)f(\underline{c})}\right)} > 0.$$

A.3.3 Proof of Proposition 8

If $\alpha < \alpha^*(\beta)$, then $\bar{q} = q^s(c^*) = S'^{-1}(c^* + T(c^*))$. Thus,

$$\begin{aligned}\frac{\partial\bar{q}}{\partial\alpha} &= \left(\frac{\partial c^*}{\partial\alpha} + \frac{\partial c^*}{\partial\alpha}\frac{\partial T}{\partial c}(c^*,\alpha) + \frac{\partial T}{\partial\alpha}(c^*,\alpha)\right)(S'^{-1})'(c^* + T(c^*,\alpha))\\ &= \left(\left(1 + \frac{\partial T}{\partial c}(c^*,\alpha)\right)\frac{\partial c^*}{\partial\alpha} + \frac{\partial T}{\partial\alpha}(c^*,\alpha)\right)\frac{1}{S''(S'^{-1}(c^* + T(c^*,\alpha)))}.\end{aligned}$$

But, as was established in the proof of Proposition 6,

$$\left(1 + \frac{\partial T}{\partial c}(c^*,\alpha)\right)\frac{\partial c^*}{\partial\alpha} = -\frac{\beta - \beta^2 + (1-\alpha)^2(\beta - F(c^*))^2}{(1-\alpha)^2 f(c^*)(1 - (\alpha\beta + (1-\alpha)F(c^*)))}$$

and

$$\frac{\partial T}{\partial\alpha}(c^*,\alpha) = \frac{\beta}{(1-\alpha)^2 f(c^*)},$$

therefore

$$\begin{aligned}S''(S'^{-1}(c^* + T(c^*,\alpha))).\frac{\partial\bar{q}}{\partial\alpha} &= \frac{\beta(1 - (\alpha\beta + (1-\alpha)F(c^*))) - (\beta - \beta^2 + (1-\alpha)^2(\beta - F(c^*))^2)}{(1-\alpha)^2 f(c^*)(1 - (\alpha\beta + (1-\alpha)F(c^*)))}\\ &= \frac{(1-\alpha)\beta^2 - (1-\alpha)\beta F(c^*) - (1-\alpha)^2(\beta - F(c^*))^2}{(1-\alpha)^2 f(c^*)(1 - (\alpha\beta + (1-\alpha)F(c^*)))}\\ &= \frac{(1-\alpha)(\beta - F(c^*))(\alpha\beta + (1-\alpha)F(c^*))}{(1-\alpha)^2 f(c^*)(1 - (\alpha\beta + (1-\alpha)F(c^*)))}.\end{aligned}$$

Thus,

$$\frac{\partial \bar{q}}{\partial \alpha} < 0 \iff \beta > F(c^*).$$

Now, let $h(\alpha) := F(c^*) - \beta$. Then, $h(0) = 1 - \beta > 0$ and $h(\alpha^*(\beta)) = -\beta < 0$. Moreover, $h'(\alpha) = \frac{\partial c^*}{\partial \alpha} f(c^*) < 0$. Therefore, there exists $\tilde{\alpha}(\beta) \in (0, \alpha^*(\beta))$ such that $h(\tilde{\alpha}(\beta)) = 0$, and $\alpha \gtrless \tilde{\alpha}(\beta) \iff F(c^*) \lessgtr \beta \iff \frac{\partial \bar{q}}{\partial \alpha} \lessgtr 0$.

$$\begin{aligned}
\frac{\partial \bar{q}}{\partial \beta} &= \left(\frac{\partial c^*}{\partial \beta} + \frac{\partial c^*}{\partial \beta}\frac{\partial T}{\partial c}(c^*, \beta) + \frac{\partial T}{\partial \beta}(c^*, \beta)\right)(S'^{-1})'(c^* + T(c^*, \beta)) \\
&= \left(\left(1 + \frac{\partial T}{\partial c}(c^*, \beta)\right)\frac{\partial c^*}{\partial \beta} + \frac{\partial T}{\partial \beta}(c^*, \beta)\right)\frac{1}{S''(S'^{-1}(c^* + T(c^*, \beta)))}.
\end{aligned}$$

But, as was established in the proof of Proposition 6,

$$\left(1 + \frac{\partial T}{\partial c}(c^*, \beta)\right)\frac{\partial c^*}{\partial \beta} = -\frac{\alpha(1 - 2(\alpha\beta + (1-\alpha)F(c)))}{(1-\alpha)f(c^*)(1 - (\alpha\beta + (1-\alpha)F(c^*)))}$$

and

$$\frac{\partial T}{\partial \beta}(c^*, \beta) = \frac{\alpha}{(1-\alpha)f(c^*)},$$

therefore

$$\begin{aligned}
\frac{\partial \bar{q}}{\partial \beta} &= \frac{\alpha}{(1-\alpha)f(c^*)}\left(1 - \frac{1 - 2(\alpha\beta + (1-\alpha)F(c))}{1 - (\alpha\beta + (1-\alpha)F(c^*))}\right)\frac{1}{S''(S'^{-1}(c^* + T(c^*, \alpha)))}. \\
&= \frac{\alpha}{(1-\alpha)f(c^*)}\left(\frac{\alpha\beta + (1-\alpha)F(c)}{1 - (\alpha\beta + (1-\alpha)F(c^*))}\right)\frac{1}{S''(S'^{-1}(c^* + T(c^*, \alpha)))}. < 0.
\end{aligned}$$

A.3.4 Proof of Proposition 9

$$\bar{q} = S'^{-1}\left(\underline{c} + \frac{\bar{c} - \underline{c}}{1 - \alpha\beta}\right)$$

Thus

$$\frac{\partial \bar{q}}{\partial \alpha} = \frac{\beta}{(1-\alpha\beta)^2}\frac{1}{S''(S'^{-1}\left(\underline{c} + \frac{\bar{c}-\underline{c}}{1-\alpha\beta}\right)} < 0.$$

and

$$\frac{\partial \bar{q}}{\partial \beta} = \frac{\alpha}{(1-\alpha\beta)^2}\frac{1}{S''(S'^{-1}\left(\underline{c} + \frac{\bar{c}-\underline{c}}{1-\alpha\beta}\right)} < 0.$$

A.3.5 Proof of Proposition 10

$$
\begin{aligned}
U^*(c) &= \int_c^{\overline{c}} q^*(x)dx \\
&= \int_c^{c^*} q^*(x)dx + \int_{c^*}^{\overline{c}} q^*(x)dx \\
&= \int_c^{c^*} (S')^{-1}(x + T(x,\alpha,\beta))dx + \int_{c^*}^{\overline{c}} (S')^{-1}(c^* + T(c^*,\alpha,\beta))dx \\
&= \int_c^{c^*} (S')^{-1}(x + T(x,\alpha,\beta))dx + (\overline{c} - c^*)\bar{q}.
\end{aligned}
$$

Thus,

$$
\frac{\partial U^*(c)}{\partial \beta} = \int_c^{c^*} \frac{\alpha}{(1-\alpha)f(x)} \frac{1}{S''(S'^{-1}(x+T(x)))} dx + (\overline{c} - c^*)\frac{\partial \bar{q}}{\partial \beta}
$$

and, by propositions 8 and 9, we have $\frac{\partial \bar{q}}{\partial \beta} < 0$ for all $\alpha \in (0,1)$.

On the other hand, assume $0 < \alpha < \alpha^*(\beta)$. Then,

$$
\frac{\partial U^*(c)}{\partial \alpha} = \int_c^{c^*} \frac{\beta}{(1-\alpha)^2 f(x)} \frac{1}{S''(S'^{-1}(x+T(x)))} dx + (\overline{c} - c^*)\frac{\partial \bar{q}}{\partial \alpha}
$$

Note that, when $0 < \alpha < \alpha^*(\beta)$, by proposition 8 we have $\frac{\partial \bar{q}}{\partial \alpha} < 0$ if and only if $F(c^*) < \beta$. Therefore, if $F(c^*) \leq \beta$, any K works. Assume on the contrary that $F(c^*) > \beta$. Then, note that $\frac{\partial U^*(c^*)}{\partial \alpha} > 0$ and $\frac{\partial^2 U^*(c)}{\partial \alpha \partial c} = -\frac{\beta}{(1-\alpha)^2 f(c)} \frac{1}{S''(S'^{-1}(c+T(c)))} > 0$. Then there exists $c \in (\underline{c}, c^*)$ such that $\frac{\partial U^*(c)}{\partial \alpha} = 0$ if and only if $\frac{\partial U^*(\underline{c})}{\partial \alpha} \leq 0$. In other words, $\frac{\partial U^*(c)}{\partial \alpha} > 0$ for all $c \in \mathcal{C}$ if and only if $\frac{\partial U^*(\underline{c})}{\partial \alpha} > 0$. Now, by the Mean Value Theorem, there exists $\check{c} \in (\underline{c}, c^*)$ such that

$$
\int_{\underline{c}}^{c^*} \frac{\beta}{(1-\alpha)^2 f(x)} \frac{1}{S''(S'^{-1}(x+T(x)))} dx = \frac{(c^* - \underline{c})\beta}{(1-\alpha)^2 f(\check{c}) S''(q^s(\check{c}))},
$$

and

$$
\begin{aligned}
&\frac{(1-\alpha)(\beta - F(c^*))(\alpha\beta + (1-\alpha)F(c^*))}{(1-\alpha)^2 f(c^*)(1 - (\alpha\beta + (1-\alpha)F(c^*)))} \frac{1}{S''(S'^{-1}(c^* + T(c^*,\alpha)))} \\
&\qquad = \frac{(\beta - F(c^*))T(c^*)}{(1 - (\alpha\beta + (1-\alpha)F(c^*)))} \frac{1}{S''(\bar{q})},
\end{aligned}
$$

therefore

$$\begin{aligned}
\frac{\partial U^*(c)}{\partial \alpha} > 0 &\Leftrightarrow \frac{(c^* - \underline{c})\beta}{(1-\alpha)^2 f(\check{c}) S''(q^s(\check{c}))} + \frac{(\beta - F(c^*))T(c^*)}{(1-(\alpha\beta + (1-\alpha)F(c^*)))} \frac{1}{S''(\bar{q})} > 0 \\
&\Leftrightarrow \frac{(\beta - F(c^*))T(c^*)}{(1-(\alpha\beta + (1-\alpha)F(c^*)))} \frac{1}{S''(\bar{q})} > -\frac{(c^* - \underline{c})\beta}{(1-\alpha)^2 f(\check{c}) S''(q^s(\check{c}))} \\
&\Leftrightarrow \beta - F(c^*) < -\frac{(c^* - \underline{c})\beta(1-(\alpha\beta + (1-\alpha)F(c^*)))S''(\bar{q})}{T(c^*)(1-\alpha)^2 f(\check{c}) S''(q^s(\check{c}))} \\
&\Leftrightarrow 1 - \frac{F(c^*)}{\beta} < -\frac{(c^* - \underline{c})(1-(\alpha\beta + (1-\alpha)F(c^*)))S''(\bar{q})}{T(c^*)(1-\alpha)^2 f(\check{c}) S''(q^s(\check{c}))} \\
&\Leftrightarrow 1 + \frac{(c^* - \underline{c})(1-(\alpha\beta + (1-\alpha)F(c^*)))S''(\bar{q})}{T(c^*)(1-\alpha)^2 f(\check{c}) S''(q^s(\check{c}))} < \frac{F(c^*)}{\beta}.
\end{aligned}$$

Letting $K = \frac{(c^* - \underline{c})(1-(\alpha\beta + (1-\alpha)F(c^*)))S''(\bar{q})}{T(c^*)(1-\alpha)^2 f(\check{c}) S''(q^s(\check{c}))} > 0$ we are done.

Finally, assume $\alpha^*(\beta) \leq \alpha < 1$. Then,

$$\frac{\partial U^*(c)}{\partial \alpha} = (\bar{c} - c^*)\frac{\partial \bar{q}}{\partial \alpha} < 0$$

by proposition 9.

A.3.6 Proof of Proposition 11 and its corollary.

Let $\gamma = \alpha$ or β. Clearly, $\frac{\partial V^*(c) + U^*(c)}{\partial \gamma} = \frac{\partial \bar{q}}{\partial \gamma}(S'(q^*(c)) - c)$. Now, $(S'(q^*(c)) - c) > 0$, hence the results follow from propositions 8 and 9.

For the corollary, if $1 < \frac{F(c^*)}{\beta}$, then $\frac{\partial V^*(c) + U^*(c)}{\partial \alpha} > 0$, i.e $\frac{\partial V^*(c)}{\partial \alpha} > -\frac{\partial U^*(c)}{\partial \alpha}$. On the other hand, if $\frac{F(c^*)}{\beta} < 1 + K$, $\frac{\partial U^*(c)}{\partial \alpha} < 0$. Combining the two we get the result.

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