Mechanical response of fabric sheets to three-dimensional bending, twisting, and stretching

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Abstract A model for the mechanics of woven fabrics is developed in the framework of two-dimensional elastic surface theory. Thickness effects are modeled indirectly in terms of appropriate constitutive equations. The model accounts for the strain of the fabric and additional effects associated with the normal bending, geodesic bending, and twisting of the constituent fibers.

Keywords Woven fabrics · Elastic surface theory · Strain gradients

1 Introduction

We present a continuum model for the mechanics of fabric sheets, regarded as elastic surfaces endowed with kinematical, dynamical, and constitutive structures that reflect the main features of the mechanical response of fabric. The main microstructural feature of woven fabric is the local oscillatory out-of-plane curvature of the yarns, known as *crimp*. As the fabric is stretched, the yarns straighten, or *decrimp*, and the lengths of their projections onto the tangent plane of the surface increase [1].

Dedicated to the memory of Chien Wei-Zang.

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We refer to these projected curves as *fibers*. We do not model the small-scale crimping/decrimping mechanism directly, but instead model the stretches of the projected fibers. The mechanical behavior of fabric sheets affords an interesting setting for the development and application of extended models of elastic surfaces. In particular, the fibers of fabrics offer the kind of elastic resistance to stretching and shearing normally associated with membrane effects, while simultaneously furnishing elastic resistance to bending and twisting. Bending deformations are described by the curvature vector of a fiber. The component of this vector in the direction of the surface normal is associated with the second fundamental form of the surface; this arises from the embedding of the surface in 3-space, and confers the kind of bending elasticity encountered in standard plate and shell theories. In contrast, the component of the curvature vector in the tangent plane of the surface—the geodesic curvature—is associated entirely with the surface metric. This gives rise to a non-standard strain-gradient effect in the elastic response. Moreover, the twist of a yarn is in principle independent of both the embedding geometry and the metric; in general it is determined by a rotation tensor that is partly independent of the surface deformation. Thus, at this level of generality, the mechanics of fabrics, regarded as elastic surfaces, embodies elements of both the strain-gradient and Cosserat theories of elasticity. We show, however, that the local interactions of the interwoven yarns effectively constrain the rotation in such a way that twist is ultimately determined by the metric and embedding geometry of the fabric surface.

In Sect. 2 we develop the kinematical foundations of the model in terms of the differential geometry of the fiber network. Section 3 is concerned with the constitutive structure. This entails the specification of an appropriate strain-energy function and associated response functions. These in turn are used in Sect. 4 to obtain the equilibrium equations via a variational argument. Finally, in Sect. 5, we apply the model to the solution of a simple illustrative problem.

Related work by Wang and Pipkin [2,3] accounts for the effects of normal and geodesic bending in networks consisting of two families of inextensible fibers, thereby generalizing earlier efforts on the modeling of fabrics [4–6]. The present work extends this further to accommodate fiber stretch and twist. In particular, the twisting and bending energies of fibers, regarded here as spatial Kirchhoff rods, are comparable in magnitude in a general deformation; this implies that twisting effects should be taken into account whenever bending effects are non-negligible. Moreover, fiber stretch is included to accommodate elastic resistance to the crimping/decrimping mechanism associated with woven fabrics. Preliminary work on the related problem of twisting resistance in inextensible fabrics is discussed in Refs. [7,8]; see also in Ref. [9].

We work in the setting of the direct theory of elastic surfaces. Thus we do not model the three-dimensional aspects of fabric material explicitly. Our preference for this framework is due in no small part to the difficulty in identifying a three-dimensional structure that reflects the salient features of the two-dimensional weave pattern of a typical fabric sheet. Thus, in general, there is no three-dimensional parent model that can be used to effect a dimension reduction procedure leading to a two-dimensional model of the kind desired.

2 Differential geometry of fabric deformation

2.1 Surface geometry

The theory of fabric deformation affords a particularly strong demonstration of the inextricable link between Mechanics and Differential Geometry. In this section we build on a number of associated results derived in Ref. [10] that bear directly on the present work.

We use convected coordinates θ^α to label material points of the fabric, regarded as a two-dimensional manifold. The function $\boldsymbol{x}(\theta^\alpha)$ is an embedding of this manifold into 3-space, and serves to define position of a material point on a fixed reference surface Ω. Position of the same material point on a typical deformed surface ω is denoted by $\boldsymbol{r}(\theta^\alpha)$. The latter parametrization induces the associated basis elements $\boldsymbol{a}_\alpha = \boldsymbol{r}_{,\alpha} \in T_\omega$, the tangent plane to ω at the point with coordinates θ^α; the metric $a_{\alpha\beta} = \boldsymbol{a}_\alpha \cdot \boldsymbol{a}_\beta$; the dual metric $(a^{\alpha\beta}) = (a_{\alpha\beta})^{-1}$ and the dual tangent basis $\boldsymbol{a}^\alpha = a^{\alpha\beta}\boldsymbol{a}_\beta$. These in turn yield the local orientation of ω in terms of its unit normal $\boldsymbol{n}$, defined by $\epsilon_{\alpha\beta}\boldsymbol{n} = \boldsymbol{a}_\alpha \times \boldsymbol{a}_\beta$, where $\epsilon_{\alpha\beta} = \sqrt{a}e_{\alpha\beta}$ is the covariant permutation tensor, with $a = \det(a_{\alpha\beta})$ and $e_{\alpha\beta}$ the unit alternator ($e_{12} = +1$, etc.). The contravariant permutation tensor is $\epsilon^{\alpha\beta} = e^{\alpha\beta}/\sqrt{a}$, where $e^{\alpha\beta} = e_{\alpha\beta}$.

The Gauss and Weingarten equations play a central role in the development of the theory. These are

$$\boldsymbol{r}_{,\alpha\beta} = \Gamma^\lambda_{\alpha\beta}\boldsymbol{a}_\alpha + b_{\alpha\beta}\boldsymbol{n} \quad \text{and} \quad \boldsymbol{n}_{,\alpha} = -b_{\alpha\beta}\boldsymbol{a}^\beta, \tag{1}$$

where $\Gamma^\lambda_{\alpha\beta}$ are the Levi-Civita connection coefficients induced by the coordinates on ω and $b_{\alpha\beta}$ is the covariant curvature tensor (the coefficients of the second fundamental form).

The deformation gradient $\boldsymbol{F} = \nabla\boldsymbol{r}$ is given by

$$\boldsymbol{F} = \boldsymbol{a}_\alpha \otimes \boldsymbol{e}^\alpha, \tag{2}$$

where $\boldsymbol{e}^\alpha$ are the duals on T_Ω of the basis elements $\boldsymbol{e}_\alpha$ induced by the coordinates on Ω via $\boldsymbol{e}_\alpha = \boldsymbol{x}_{,\alpha}$, and the Cauchy-Green deformation tensor is

$$\boldsymbol{C} = \boldsymbol{F}^t\boldsymbol{F} = a_{\alpha\beta}\boldsymbol{e}^\alpha \otimes \boldsymbol{e}^\beta. \tag{3}$$

The fabric is assumed to consist of two families of fibers that are continuously distributed over the surface, thus every material point lies at the intersection of a pair of fibers, each modeled as a mathematical curve endowed with kinematical and constitutive structures intended to capture the main features of the local macroscopic behavior of fabrics. The unit tangents to the fibers on ω are denoted by $\boldsymbol{l}$ and $\boldsymbol{m}$, and their counterparts on Ω by $\boldsymbol{L}$ and $\boldsymbol{M}$. These are necessarily tangential to ω and Ω, respectively, and so admit the representations

$$\begin{aligned}&\boldsymbol{l} = l^\alpha\boldsymbol{a}_\alpha, \quad \boldsymbol{m} = m^\alpha\boldsymbol{a}_\alpha \quad \text{and}\\ &\boldsymbol{L} = L^\alpha\boldsymbol{e}_\alpha, \quad \boldsymbol{M} = M^\alpha\boldsymbol{e}_\alpha,\end{aligned} \tag{4}$$

in terms of contravariant components, for example. The fibers are presumed to be convected as material curves with no relative slipping; this is realistic in the presence of sufficient friction between overlapping yarns of the actual weave or, in the case of a coarse-mesh network [10], if the fibers are tied together at their points of intersection. Thus, [10]

$$\lambda\boldsymbol{l} = \boldsymbol{F}\boldsymbol{L} \quad \text{and} \quad \mu\boldsymbol{m} = \boldsymbol{F}\boldsymbol{M}, \tag{5}$$

where $\lambda(= |\boldsymbol{F}\boldsymbol{L}|)$ and $\mu(= |\boldsymbol{F}\boldsymbol{M}|)$ are the fiber stretches; i.e., the stretches of the projections of the woven yarns onto the tangent plane of Ω. These yield the useful connections

$$\lambda l^\alpha = L^\alpha \quad \text{and} \quad \mu m^\alpha = M^\alpha \tag{6}$$

relating contravariant components only. The covariant components are related by [10, Eqs. 5.20]

$$l_\alpha = \lambda L_\alpha + \mu\sin\gamma M_\alpha \quad \text{and} \quad m_\alpha = \mu M_\alpha + \lambda\sin\gamma L_\alpha, \tag{7}$$

where γ is the fiber shear angle on ω, defined by

$$\sin\gamma = \boldsymbol{l}\cdot\boldsymbol{m}. \tag{8}$$

We suppose, with minor loss of generality, that the fibers are everywhere orthogonal on Ω, so that [10]

$$\boldsymbol{e}^{\alpha} = L^{\alpha}\boldsymbol{L} + M^{\alpha}\boldsymbol{M} \quad \text{and} \quad \boldsymbol{e}_{\alpha} = L_{\alpha}\boldsymbol{L} + M_{\alpha}\boldsymbol{M}, \tag{9}$$

yielding the representation

$$\delta^{\alpha}_{\beta} = L^{\alpha}L_{\beta} + M^{\alpha}M_{\beta} \tag{10}$$

of the Kronecker delta. The contravariant permutation tensor on Ω is defined by $\mu^{\alpha\beta} = e^{\alpha\beta}/\sqrt{e}$, where e is the determinant of the metric induced by the coordinates. For orthogonal fibers this may be expressed in the form [10]

$$\mu^{\alpha\beta} = L^{\alpha}M^{\beta} - M^{\alpha}L^{\beta}. \tag{11}$$

Equations (2)–(5) combine to furnish

$$\boldsymbol{F} = \lambda\boldsymbol{l}\otimes\boldsymbol{L} + \mu\boldsymbol{m}\otimes\boldsymbol{M}; \tag{12}$$

equivalently,

$$\boldsymbol{a}_{\alpha} = \lambda\boldsymbol{l}L_{\alpha} + \mu\boldsymbol{m}M_{\alpha}. \tag{13}$$

The Cauchy-Green tensor follows from Eq. (3), with

$$a_{\alpha\beta} = \lambda^2 L_{\alpha}L_{\beta} + \mu^2 M_{\alpha}M_{\beta} + \lambda\mu\sin\gamma(L_{\alpha}M_{\beta} + M_{\alpha}L_{\beta}). \tag{14}$$

Using this and the dual metric induced by Eq. $(9)_1$, it may be shown that the area stretch, $J = \sqrt{a/e}$, is [10]

$$J = \lambda\mu\,|\cos\gamma|\,, \tag{15}$$

and that

$$\boldsymbol{l}\times\boldsymbol{m} = |\cos\gamma|\,\boldsymbol{n}. \tag{16}$$

Further, these results yield the curvature tensor in the form [10]

$$b_{\alpha\beta} = \lambda^2\kappa_l L_{\alpha}L_{\beta} + \mu^2\kappa_m M_{\alpha}M_{\beta} + \lambda\mu\tau(L_{\alpha}M_{\beta} + M_{\alpha}L_{\beta}), \tag{17}$$

where

$$\kappa_l = b_{\alpha\beta}l^{\alpha}l^{\beta} \quad \text{and} \quad \kappa_m = b_{\alpha\beta}m^{\alpha}m^{\beta} \tag{18}$$

are the normal curvatures of the deformed fibers, and

$$\tau = b_{\alpha\beta}l^{\alpha}m^{\beta} \tag{19}$$

is the torsion.

Of central importance in this work are the geodesic curvatures η_l and η_m of the fibers, defined by [10]

$$l^{\alpha}\boldsymbol{l}_{,\alpha} = \eta_l\boldsymbol{p} + \kappa_l\boldsymbol{n} \quad \text{and} \quad m^{\alpha}\boldsymbol{m}_{,\alpha} = \eta_m\boldsymbol{q} + \kappa_m\boldsymbol{n}, \tag{20}$$

with

$$\boldsymbol{p} = \boldsymbol{n}\times\boldsymbol{l} \quad \text{and} \quad \boldsymbol{q} = \boldsymbol{n}\times\boldsymbol{m}. \tag{21}$$

We shall also require [10]

$$m^{\alpha}\boldsymbol{l}_{,\alpha} = \phi_l\boldsymbol{p} + \tau\boldsymbol{n} \quad \text{and} \quad l^{\alpha}\boldsymbol{m}_{,\alpha} = \phi_m\boldsymbol{q} + \tau\boldsymbol{n}, \tag{22}$$

where ϕ_l and ϕ_m are the Tchebychev curvatures of the fibers.

It is well known that the geodesic curvatures are determined by the surface metric. In the present context the explicit expressions are [10]

$$\begin{aligned} J\eta_l &= (\mu\sin\gamma L^{\alpha} - \lambda M^{\alpha})_{|\alpha} \quad \text{and} \\ J\eta_m &= (\mu L^{\alpha} - \lambda\sin\gamma M^{\alpha})_{|\alpha}, \end{aligned} \tag{23}$$

where $(\cdot)_{|\alpha}$ is the covariant derivative on Ω. Explicit expressions for the Tchebychev curvatures are derived in Sect. 2.4.

2.2 Fiber kinematics

Consider the orthonormal basis $\{\boldsymbol{l}_i\} = \{\boldsymbol{l}, \boldsymbol{p}, \boldsymbol{n}\}$ with $\boldsymbol{p}$ given by Eq. $(21)_1$. This consists of the unit tangent $\boldsymbol{l}$ to the first fiber trajectory, and two vectors—$\boldsymbol{p}$ and the surface normal $\boldsymbol{n}$—spanning the cross-sectional plane of the fiber. Let $\{\boldsymbol{L}_i\} = \{\boldsymbol{L}, \boldsymbol{M}, \boldsymbol{N}\}$, where $\boldsymbol{N}$ is the unit normal to Ω and $\boldsymbol{M} = \boldsymbol{N}\times\boldsymbol{L}$. Then there is a rotation tensor, $\boldsymbol{R}_{(l)}$, such that $\boldsymbol{l}_i = \boldsymbol{R}_{(l)}\boldsymbol{L}_i$. The rate of change of the basis $\{\boldsymbol{l}_i\}$ with respect to arclength along the $\boldsymbol{L}$-trajectory, denoted by $(\cdot)'$, is

$$\boldsymbol{l}_i' = L^{\alpha}\boldsymbol{l}_{i,\alpha} = \omega_{(l)}\times\boldsymbol{l}_i, \tag{24}$$

where $\omega_{(l)}$ is the axial vector of the skew tensor $\Omega_{(l)} = \boldsymbol{R}'_{(l)}\boldsymbol{R}^t_{(l)}$. This has the representation $\omega_{(l)} = \omega_{(l)i}\boldsymbol{l}_i$, where $\omega_{(l)i} = \frac{1}{2}e_{ijk}\Omega_{(l)kj}$ and $\Omega_{(l)kj} = \boldsymbol{l}_k\cdot\Omega_{(l)}\boldsymbol{l}_j$, in which e_{ijk} is the usual permutation symbol ($e_{123} = +1$, etc.). Here, $\omega_{(l)1}$ is the fiber twist, whereas $\omega_{(l)2}$ and $\omega_{(l)3}$ are the curvatures of the fiber due to flexure. In view of Eq. (12), we also have the connection $\boldsymbol{F}\boldsymbol{L} = \lambda\boldsymbol{R}_{(l)}\boldsymbol{L}$ between the rotation and the surface deformation.

We suppose that the fiber behaves mechanically like a spatial Kirchhoff rod. This model is known to apply to sufficiently thin filaments in the presence of small axial extensions [11] on the order of those typically encountered in the deformations of woven fabrics. In this model the rod responds constitutively to the Galilean-invariant curvature-twist vector $\boldsymbol{R}^t_{(l)}\omega_{(l)} = \omega_{(l)i}\boldsymbol{L}_i$. Our objective in this subsection is to show that this may be specified entirely in terms of surface geometry and hence in terms of the deformation of the surface.

For example, from Eq. (24) we have

$$\boldsymbol{l}' = \omega_{(l)} \times \boldsymbol{l} = \omega_{(l)2}\boldsymbol{p} \times \boldsymbol{l} + \omega_{(l)3}\boldsymbol{n} \times \boldsymbol{l}. \tag{25}$$

Combining this with Eq. $(6)_1$, and comparing the result to Eq. $(20)_1$, we conclude that

$$\omega_{(l)2} = -\lambda\kappa_l \quad \text{and} \quad \omega_{(l)3} = \lambda\eta_l. \tag{26}$$

The reduction of the twist $\omega_{(l)1}$ is substantially more involved. First, we form $\boldsymbol{p}' = \boldsymbol{\omega}_{(l)} \times \boldsymbol{p}$ and use the fact that $\boldsymbol{p}' = L^\alpha \boldsymbol{p}_{,\alpha}$, together with Eq. $(6)_1$, to obtain

$$\lambda l^\alpha \boldsymbol{p}_{,\alpha} = \omega_{(l)1}\boldsymbol{n} - \omega_{(l)3}\boldsymbol{l}. \tag{27}$$

Differentiating Eq. $(21)_1$, using the Weingarten equation and invoking Eq. (17), we derive

$$l^\alpha \boldsymbol{p}_{,\alpha} = \eta_l \boldsymbol{n} \times \boldsymbol{p} + b_{\alpha\beta} l^\alpha p^\beta \boldsymbol{n}, \tag{28}$$

where $p^\beta = \epsilon^{\alpha\beta} l_\alpha$ and $\epsilon^{\alpha\beta}$ is the contravariant permutation tensor. This combines with Eqs. (27) and $(26)_2$ to deliver $\omega_{(l)1} = \lambda b_{\alpha\beta} l^\alpha p^\beta$, and hence the conclusion that the fiber twist is proportional to the torsion of the surface on the orthonormal $\{\boldsymbol{l}, \boldsymbol{p}\}$−axes. However, for the purposes of the constitutive theory to be discussed in Sect. 3, it is more convenient to have an expression for the twist in terms of the components of the representation Eq. (17). To this end, we use Eq. $(6)_1$ to write $\omega_{(l)1} = b_{\alpha\beta} L^\alpha p^\beta$, and then use Eq. (17) to reduce this to

$$\omega_{(l)1} = \lambda^2 \kappa_l (L_\beta p^\beta) + \lambda\mu\tau(M_\beta p^\beta). \tag{29}$$

We recall that the permutation tensor is $\epsilon^{\alpha\beta} = e^{\alpha\beta}/\sqrt{a} = J^{-1}e^{\alpha\beta}/\sqrt{e}$, and so $\epsilon^{\alpha\beta} = J^{-1}\mu^{\alpha\beta}$. Using the representation Eq. (11), together with Eq. $(7)_1$, we thus derive

$$J p^\beta = \lambda M^\beta - \mu \sin\gamma L^\beta, \tag{30}$$

which combines with Eq. (29) to give

$$\omega_{(l)1} = \lambda^2 \mu J^{-1}(\tau - \kappa_l \sin\gamma). \tag{31}$$

Proceeding in the same way, we may show, with some effort, that the components $\omega_{(m)i}$ of the curvature-twist vector $\boldsymbol{\omega}_{(m)}$ of the second fiber family are given by

$$\begin{aligned} &\omega_{(m)1} = \lambda\mu^2 J^{-1}(\kappa_m \sin\gamma - \tau), \\ &\omega_{(m)2} = -\mu\kappa_m \quad \text{and} \quad \omega_{(m)3} = \mu\eta_m. \end{aligned} \tag{32}$$

Because we regard the fabric as a surface consisting of crossed rods of the Kirchhoff type, it is natural to assume that it responds constitutively to the curvature-twist vectors of the two fiber families. In view of the results obtained here, this is equivalent to a constitutive sensitivity to the surface strain, the surface curvature, and the geodesic curvatures. We show in the next subsection that these variables are in turn determined by the first and second gradients of the deformation.

Remark It is appropriate to regard Kirchhoff rod theory as a special case of Cosserat elasticity theory, insofar as the kinematics of the former involve position and rotation fields that are at least partly independent. This viewpoint is advanced in Refs. [12,13] to construct a three-dimensional model for nonlinearly elastic solids reinforced by a continuous distribution of fibers with bending and twisting resistance. In that context the fiber twist cannot be determined from the deformation of the underlying continuum; instead, it is computed from the Cosserat rotation field.

Here, however, the rotations of the two fiber families are effectively constrained to have a concurrent axis. That is, the rotation tensor of each fiber family acts on the initial surface normal $\boldsymbol{N}$ to yield the normal $\boldsymbol{n}$ to the deformed surface: $\boldsymbol{R}_{(l)}\boldsymbol{N} = \boldsymbol{R}_{(m)}\boldsymbol{N} = \boldsymbol{n}$. This reflects the nature of the interactions of the yarns comprising the fabric weave; thus at the points of the actual fabric where the two fiber families cross, the fibers may pivot about a common axis (the surface normal) while maintaining congruency with the deformed surface. This implies that the normal to the surface is effectively embedded in the plane of a fiber cross section. This is the reason why fiber twist is ultimately determined by surface deformation, yielding a dramatic simplification in that the relevant kinematical fields are computable from surface geometry alone. In effect, then, the present model is subsumed under the second-gradient theory of elasticity instead of the more complicated Cosserat theory.

2.3 Second gradient of the deformation

The second gradient $\nabla\boldsymbol{F} = \nabla\nabla\boldsymbol{r}$ of the deformation is the third-order tensor $\nabla\boldsymbol{F} = \boldsymbol{F}_{,\alpha} \otimes \boldsymbol{e}^\alpha$. Using Eq. (2), this is found to be

$$\nabla\nabla\boldsymbol{r} = S^\lambda_{\alpha\beta}\boldsymbol{a}_\lambda \otimes \boldsymbol{e}^\alpha \otimes \boldsymbol{e}^\beta - \boldsymbol{n} \otimes \kappa, \tag{33}$$

where

$$S^\lambda_{\alpha\beta} = \Gamma^\lambda_{\alpha\beta} - \bar{\Gamma}^\lambda_{\alpha\beta}, \tag{34}$$

in which $\Gamma^\lambda_{\alpha\beta}$ and $\bar{\Gamma}^\lambda_{\alpha\beta}$ are the Levi-Civita connection coefficients induced by the coordinates on ω and Ω, respectively, and

$$\kappa = -b_{\alpha\beta}\boldsymbol{e}^\alpha \otimes \boldsymbol{e}^\beta \tag{35}$$

in which the $b_{\alpha\beta}$ is the covariant curvature on ω. Here, we have used the Gauss formula $(1)_1$ on ω together with its counterpart

$$\boldsymbol{e}^{\alpha}_{,\beta} = -\bar{\Gamma}^{\alpha}_{\lambda\beta}\boldsymbol{e}^{\lambda} \tag{36}$$

on Ω, which we assume, with minor loss of generality, to be a plane. The sign in Eq. (35) conforms to a widely adopted convention in the literature on shell theory [14].

We observe that the difference of two sets of connection coefficients induced by a given (convected) coordinate system is a (third-order) tensor, whereas the connection coefficients themselves do not possess tensor character [15]. Further, as is well known, the $\Gamma^{\lambda}_{\alpha\beta}$ and $\bar{\Gamma}^{\lambda}_{\alpha\beta}$ are determined entirely by the metrics induced by the coordinates on ω and Ω, respectively [14]. Accordingly, the coefficients $S^{\lambda}_{\alpha\beta}$ represent strain gradients. These, and the $b_{\alpha\beta}$, are easily seen to be Galilean invariant.

Our further work is facilitated by using the combination

$$\boldsymbol{r}_{|\alpha\beta} = \boldsymbol{r}_{,\alpha\beta} - \bar{\Gamma}^{\lambda}_{\alpha\beta}\boldsymbol{r}_{,\lambda}. \tag{37}$$

This is the second covariant derivative of the deformation with respect to the metric of Ω. The Gauss equation $(1)_1$ then furnishes

$$\boldsymbol{r}_{|\alpha\beta} = S^{\lambda}_{\alpha\beta}\boldsymbol{r}_{,\lambda} + b_{\alpha\beta}\boldsymbol{n} \tag{38}$$

and Eq. (33) reduces to

$$\nabla\nabla\boldsymbol{r} = \boldsymbol{r}_{|\alpha\beta} \otimes \boldsymbol{e}^{\alpha} \otimes \boldsymbol{e}^{\beta}. \tag{39}$$

2.4 Fiber decompositions

The identity

$$S^{\lambda}_{\alpha\beta} = S^{\lambda}_{\mu\gamma}\delta^{\mu}_{\alpha}\delta^{\gamma}_{\beta} \tag{40}$$

may be combined with Eq. (10) and the symmetry condition $S^{\lambda}_{\alpha\beta} = S^{\lambda}_{\beta\alpha}$ to establish the useful representation

$$S^{\lambda}_{\alpha\beta}\boldsymbol{a}_{\lambda} = \boldsymbol{g}_l L_\alpha L_\beta + \boldsymbol{g}_m M_\alpha M_\beta + \boldsymbol{\Gamma}(L_\alpha M_\beta + M_\alpha L_\beta), \tag{41}$$

where

$$\begin{aligned} &\boldsymbol{g}_l = S^{\lambda}_{\alpha\beta}L^{\alpha}L^{\beta}\boldsymbol{a}_{\lambda}, \quad \boldsymbol{g}_m = S^{\lambda}_{\alpha\beta}M^{\alpha}M^{\beta}\boldsymbol{a}_{\lambda} \quad \text{and} \\ &\boldsymbol{\Gamma} = S^{\lambda}_{\alpha\beta}L^{\alpha}M^{\beta}\boldsymbol{a}_{\lambda}. \end{aligned} \tag{42}$$

In view of the symmetry already noted, the last of these is equivalent to

$$\boldsymbol{\Gamma} = S^{\lambda}_{\alpha\beta}M^{\alpha}L^{\beta}\boldsymbol{a}_{\lambda}. \tag{43}$$

To obtain explicit expressions for the coefficient vectors in Eq. (41) we differentiate Eq. (13) directly and equate the result to the right-hand side of Eq. (41), thus

$$\begin{aligned} S^{\lambda}_{\alpha\beta}\boldsymbol{r}_{,\lambda} + b_{\alpha\beta}\boldsymbol{n} = (\lambda\boldsymbol{l})_{,\beta}L_\alpha + (\mu\boldsymbol{m})_{,\beta}M_\alpha + \lambda\boldsymbol{l}L_{\alpha|\beta} \\ + \mu\boldsymbol{m}M_{\alpha|\beta}. \end{aligned} \tag{44}$$

From Eq. $(9)_1$ we see that the gradient of a function $f(\theta^\alpha)$ may be written in the form

$$\nabla f = f_{,\alpha}\boldsymbol{e}^{\alpha} = L^{\alpha}f_{,\alpha}\boldsymbol{L} + M^{\alpha}f_{,\alpha}\boldsymbol{M}, \tag{45}$$

yielding

$$f_{,\beta} = L^{\alpha}f_{,\alpha}L_\beta + M^{\alpha}f_{,\alpha}M_\beta. \tag{46}$$

In particular,

$$(\lambda\boldsymbol{l})_{,\beta} = L^{\alpha}(\lambda\boldsymbol{l})_{,\alpha}L_\beta + M^{\alpha}(\lambda\boldsymbol{l})_{,\alpha}M_\beta, \tag{47}$$

where, from Eqs. $(6)_1$ and $(20)_1$,

$$L^{\alpha}(\lambda\boldsymbol{l})_{,\alpha} = (\boldsymbol{L}\cdot\nabla\lambda)\boldsymbol{l} + \lambda^2(\eta_l\boldsymbol{p} + \kappa_l\boldsymbol{n}). \tag{48}$$

Using this and other similarly derived formulas, we find that

$$(\lambda\boldsymbol{l})_{,\beta} = \lambda_{,\beta}\boldsymbol{l} + \lambda^2(\eta_l\boldsymbol{p} + \kappa_l\boldsymbol{n})L_\beta + \lambda\mu(\phi_l\boldsymbol{p} + \tau\boldsymbol{n})M_\beta \tag{49}$$

and

$$(\mu\boldsymbol{m})_{,\beta} = \mu_{,\beta}\boldsymbol{l} + \mu^2(\eta_m\boldsymbol{q} + \kappa_m\boldsymbol{n})M_\beta + \lambda\mu(\phi_m\boldsymbol{q} + \tau\boldsymbol{n})L_\beta. \tag{50}$$

Substituting into Eq. (44) and projecting the resulting equation onto the surface normal leads directly to Eq. (17); the remaining part of this equation yields

$$\begin{aligned} S^{\lambda}_{\alpha\beta}\boldsymbol{a}_{\lambda} = {} & (\lambda L_\alpha)_{|\beta}\boldsymbol{l} + (\lambda^2\eta_l L_\alpha L_\beta + \lambda\mu\phi_l L_\alpha M_\beta)\boldsymbol{p} \\ & + (\mu M_\alpha)_{|\beta}\boldsymbol{m} \\ & + (\mu^2\eta_m M_\alpha M_\beta + \lambda\mu\phi_m M_\alpha L_\beta)\boldsymbol{q}, \end{aligned} \tag{51}$$

and hence Eq. (41), with

$$\begin{aligned} &\boldsymbol{g}_l = \lambda^2\eta_l\boldsymbol{p} + L^{\alpha}L^{\beta}[(\lambda L_\alpha)_{|\beta}\boldsymbol{l} + (\mu M_\alpha)_{|\beta}\boldsymbol{m}] \quad \text{and} \\ &\boldsymbol{g}_m = \mu^2\eta_m\boldsymbol{q} + M^{\alpha}M^{\beta}[(\lambda L_\alpha)_{|\beta}\boldsymbol{l} + (\mu M_\alpha)_{|\beta}\boldsymbol{m}]. \end{aligned} \tag{52}$$

The two equivalent expressions for $\boldsymbol{\Gamma}$ are

$$\begin{aligned} \boldsymbol{\Gamma} &= \lambda\mu\phi_l\boldsymbol{p} + L^{\alpha}M^{\beta}[(\lambda L_\alpha)_{|\beta}\boldsymbol{l} + (\mu M_\alpha)_{|\beta}\boldsymbol{m}] \\ &= \lambda\mu\phi_m\boldsymbol{q} + M^{\alpha}L^{\beta}[(\lambda L_\alpha)_{|\beta}\boldsymbol{l} + (\mu M_\alpha)_{|\beta}\boldsymbol{m}]. \end{aligned} \tag{53}$$

The Tchebychev curvatures may be eliminated by making the compatibility conditions $S^{\lambda}_{\alpha\beta} = S^{\lambda}_{\beta\alpha}$ explicit. These in turn are equivalent to the vector equation

$$\mu^{\alpha\beta} S^{\lambda}_{\alpha\beta} \boldsymbol{a}_{\lambda} = \boldsymbol{0}, \tag{54}$$

where $\mu^{\alpha\beta}$ is the referential permutation symbol; combining this in the form Eq. (11), with Eq. (51), leads, with some effort, to

$$\lambda\eta_L \boldsymbol{l} + \mu\eta_M \boldsymbol{m} = (\boldsymbol{M}\cdot\nabla\lambda)\boldsymbol{l} - (\boldsymbol{L}\cdot\nabla\mu)\boldsymbol{m} + \lambda\mu(\phi_l \boldsymbol{p} - \phi_m \boldsymbol{q}), \tag{55}$$

where

$$\eta_L = -M^{\beta}_{|\beta} \quad \text{and} \quad \eta_M = L^{\beta}_{|\beta} \tag{56}$$

are the geodesic curvatures of the fibers on Ω (cf. 23). Proceeding as in Ref. [10], we then obtain

$$\begin{aligned} J\phi_l &= J\eta_m + \lambda \boldsymbol{M}\cdot\nabla(\sin\gamma) \quad \text{and} \\ J\phi_m &= J\eta_l - \mu \boldsymbol{L}\cdot\nabla(\sin\gamma), \end{aligned} \tag{57}$$

which combines with Eq. (23) to yield expressions in terms of the fiber stretches and shear angle. The same results follow from the final equality in Eq. (53).

3 Constitutive framework

In view of the complexity of the equations obtained thus far we confine our further development to the important special case in which both fiber families are initially straight and so take $\boldsymbol{L}$ and $\boldsymbol{M}$ to be fixed. This entails a minor loss of generality, which is, however, offset by increased clarity and tractability. Thus, we have the simplifications

$$\boldsymbol{g}_l = \lambda^2\eta_l \boldsymbol{p} + (\boldsymbol{L}\cdot\nabla\lambda)\boldsymbol{l},\ \boldsymbol{g}_m = \mu^2\eta_m \boldsymbol{q} + (\boldsymbol{M}\cdot\nabla\mu)\boldsymbol{m} \tag{58}$$

and

$$\boldsymbol{\Gamma} = (\boldsymbol{L}\cdot\nabla\mu)\boldsymbol{m} + \lambda\mu\phi_m \boldsymbol{q} = (\boldsymbol{M}\cdot\nabla\lambda)\boldsymbol{l} + \lambda\mu\phi_l \boldsymbol{p}. \tag{59}$$

The final equality is, of course, simply the compatibility condition of Eq. (55) with $\eta_L = \eta_M = 0$.

We take for granted the existence of a strain-energy function that depends on the stretches of the fibers as well as their curvatures and twists. Further, as (tangential) stretch gradients appear in the constitutive equations for one-dimensional models of thin fibers that account for finite-thickness effects [16,17], we also allow a constitutive sensitivity to the tangential derivatives of the fiber stretches. Beyond this, we refer to arguments given in Ref. [18] to the effect that in tightly woven fabrics there is likely to be a constitutive sensitivity to the cross derivatives of the fiber stretches and to the gradient of the shear angle between them, the former being simply the derivatives in directions transverse to the fibers. All such effects are contained in the vectors $\boldsymbol{g}_l$, $\boldsymbol{g}_m$ and $\boldsymbol{\Gamma}$, and so it is natural to include these among the arguments of the strain-energy function.

To non-dimensionalize the variables appearing in this function, it is necessary to introduce a local length scale. Candidates for this are the sheet thickness, the characteristic wavelength of the fabric weave, or the widths of the constituent yarns. The latter two are roughly equal in tightly woven fabrics and somewhat larger than the thickness. If any of these are used as the length scale, then in typical applications the non-dimensionalized vectors $\boldsymbol{g}_l$, $\boldsymbol{g}_m$ and $\boldsymbol{\Gamma}$ are so small that the dependence of the strain energy on them is quadratic at leading order, assuming the associated couple stresses and bending/twisting moments to vanish when the fibers are straight and untwisted. A simple strain-energy function of this type is

$$\begin{aligned} W = w(\lambda,\mu,J) &+ \frac{1}{2}A_g\left(|\boldsymbol{g}_l|^2 + |\boldsymbol{g}_m|^2\right) + \frac{1}{2}A_{\boldsymbol{\Gamma}}|\boldsymbol{\Gamma}|^2 \\ &+ \frac{1}{2}k(K_L^2 + K_M^2) + \frac{1}{2}\bar{k}T^2, \end{aligned} \tag{60}$$

where

$$\begin{aligned} &K_L = b_{\alpha\beta}L^{\alpha}L^{\beta} = \lambda^2\kappa_l,\ K_M = b_{\alpha\beta}M^{\alpha}M^{\beta} = \mu^2\kappa_m, \\ &T = b_{\alpha\beta}L^{\alpha}M^{\beta} = \lambda\mu\tau; \end{aligned} \tag{61}$$

and the coefficients A_g, $A_{\boldsymbol{\Gamma}}$, k and $\bar{k}$ may be functions of λ, μ, and J; here, we take them to be constants for the sake of definiteness and tractability. This is a simple generalization of the strain-energy function proposed in Ref. [18] for purely plane deformations, to accommodate three-dimensional flexure and twist. Other forms are, of course, possible. In particular, we might separate out the effects of geodesic curvature and tangential stretch gradient in $\boldsymbol{g}_l$ or $\boldsymbol{g}_m$, and assign different elastic moduli to each, as explained in Ref. [18]. Here, following Ref. [19], we forego such refinements for the sake of simplicity.

The energy W is easily shown to exhibit orthotropic symmetry. In particular, J is determined by λ, μ and $|\sin\gamma|$, which are orthotropic invariants [10]. Similarly, every term in the function W remains invariant if $\boldsymbol{r}_{|\alpha\beta}$ is replaced by $\bar{\boldsymbol{r}}_{|\alpha\beta} = \boldsymbol{r}_{|\lambda\mu}H^{\lambda}_{\alpha}H^{\mu}_{\beta}$, where $\boldsymbol{H} \in \{\pm\boldsymbol{L}\otimes\boldsymbol{L} \pm \boldsymbol{M}\otimes\boldsymbol{M}\}$, with any combination of signs. The response of the fabric, therefore, conforms to orthotropic symmetry relative to the fiber axes on the reference plane. This is an example of *homogenous symmetry* in the general theory of material symmetry for second-grade material surfaces [20] (see also in Ref. [21]).

We take the constants A_g, $A_{\boldsymbol{\Gamma}}$, k and $\bar{k}$ to be strictly positive and observe that the part of the energy depending on the second gradient $\boldsymbol{r}_{|\alpha\beta}$ is then non-negative, vanishing if and only if $\boldsymbol{g}_l$, $\boldsymbol{g}_m$, $\boldsymbol{\Gamma}$, K_L, K_M and T all vanish. It is, thus, a positive definite function of the $\boldsymbol{r}_{|\alpha\beta}$. This conclusion follows easily from

$$\boldsymbol{r}_{|\alpha\beta} = L_\alpha L_\beta(\boldsymbol{g}_l + K_L\boldsymbol{n}) + M_\alpha M_\beta(\boldsymbol{g}_m + K_M\boldsymbol{n}) \\ + (L_\alpha M_\beta + M_\alpha L_\beta)(\boldsymbol{\Gamma} + T\boldsymbol{n}), \tag{62}$$

which is obtained by combining Eqs. (17) and (41). As such, the energy is a strictly convex function of $\boldsymbol{r}_{|\alpha\beta}$, which is enough to secure the existence of solutions to conservative boundary value problems characterized by a potential energy functional [22]; we discuss an example in Sect. 4 below. Trivially, it then satisfies the Legendre–Hadamard necessary condition for energy minimizers; this requires that the strain energy deliver a positive value when $\boldsymbol{r}_{,\alpha}$ and $\boldsymbol{r}_{|\alpha\beta}$ in its arguments are replaced by $\boldsymbol{0}$ and $\boldsymbol{c}b_\alpha b_\beta$, respectively, in which b_α is an arbitrary non-zero 2-vector and $\boldsymbol{c}$ is any non-zero 3-vector [23].

Let $\boldsymbol{r}(\theta^\alpha;\epsilon)$ be a one-parameter family of deformations, and let $\boldsymbol{u} = \dot{\boldsymbol{r}}$ be the derivative with respect to the parameter at a certain fixed value; $\epsilon = 0$, say. The relevant response functions of the theory are simply the coefficient vectors $\boldsymbol{N}^\alpha$ and $\boldsymbol{M}^{\alpha\beta}$ in the associated expression

$$\dot{W} = \boldsymbol{N}^\alpha \cdot \boldsymbol{u}_{,\alpha} + \boldsymbol{M}^{\alpha\beta} \cdot \boldsymbol{u}_{|\alpha\beta}, \tag{63}$$

for the derivative of the energy, in which we have used the fact that W depends on the deformation through $\boldsymbol{r}_{,\alpha}$ and $\boldsymbol{r}_{|\alpha\beta}$; that is, $\boldsymbol{N}^\alpha = \partial W/\partial \boldsymbol{r}_{,\alpha}$ and $\boldsymbol{M}^{\alpha\beta} = \partial W/\partial \boldsymbol{r}_{|\alpha\beta}$. In particular, we may impose $\boldsymbol{M}^{\alpha\beta} = \boldsymbol{M}^{\beta\alpha}$ without loss of generality. To derive the required expressions we use Eq. (60) and obtain

$$\dot{W} = \dot{w} + A_g(\boldsymbol{g}_l \cdot \dot{\boldsymbol{g}}_l + \boldsymbol{g}_m \cdot \dot{\boldsymbol{g}}_m) + A_{\boldsymbol{\Gamma}}\boldsymbol{\Gamma} \cdot \dot{\boldsymbol{\Gamma}} \\ + k(K_L\dot{K}_L + K_M\dot{K}_M) + \bar{k}T\dot{T}, \tag{64}$$

where

$$\dot{w} = w_\lambda\dot{\lambda} + w_\mu\dot{\mu} + w_J\dot{J}. \tag{65}$$

The objective is thus to express this as a bilinear form in $\boldsymbol{u}_{,\alpha}$ and $\boldsymbol{u}_{|\alpha\beta}$.

For example, using $\lambda = |\boldsymbol{FL}|$ we derive $\lambda\dot{\lambda} = \boldsymbol{FL}\cdot\dot{\boldsymbol{F}}\boldsymbol{L} = \lambda L^\alpha \boldsymbol{l} \cdot \boldsymbol{u}_{,\alpha}$; then,

$$\dot{\lambda} = L^\alpha \boldsymbol{l} \cdot \boldsymbol{u}_{,\alpha}, \tag{66}$$

and in the same way we obtain

$$\dot{\mu} = M^\alpha \boldsymbol{m} \cdot \boldsymbol{u}_{,\alpha}. \tag{67}$$

Further, it is easily shown [24] that

$$\dot{J} = J\boldsymbol{a}^\alpha \cdot \boldsymbol{u}_{,\alpha} \tag{68}$$

and so we have

$$\dot{w} = (w_\lambda L^\alpha \boldsymbol{l} + w_\mu M^\alpha \boldsymbol{m} + Jw_J\boldsymbol{a}^\alpha) \cdot \boldsymbol{u}_{,\alpha}. \tag{69}$$

Proceeding, we also have $\dot{K}_L = \dot{b}_{\alpha\beta}L^\alpha L^\beta$, for example, where [14,24]

$$\dot{b}_{\alpha\beta} = \boldsymbol{n} \cdot \boldsymbol{u}_{;\alpha\beta}, \quad \text{with} \quad \boldsymbol{u}_{;\alpha\beta} = \boldsymbol{u}_{,\alpha\beta} - \boldsymbol{\Gamma}^\lambda_{\alpha\beta}\boldsymbol{u}_{,\lambda}. \tag{70}$$

Using

$$\boldsymbol{u}_{|\alpha\beta} = \boldsymbol{u}_{,\alpha\beta} - \bar{\boldsymbol{\Gamma}}^\lambda_{\alpha\beta}\boldsymbol{u}_{,\lambda} \tag{71}$$

we conclude that

$$\boldsymbol{u}_{;\alpha\beta} = \boldsymbol{u}_{|\alpha\beta} - S^\lambda_{\alpha\beta}\boldsymbol{u}_{,\lambda} \tag{72}$$

and hence that

$$\dot{K}_L = L^\alpha L^\beta \boldsymbol{n} \cdot \boldsymbol{u}_{|\alpha\beta} - (\boldsymbol{a}^\alpha \cdot \boldsymbol{g}_l)\boldsymbol{n} \cdot \boldsymbol{u}_{,\alpha}. \tag{73}$$

In the same way, we derive

$$\dot{K}_M = M^\alpha M^\beta \boldsymbol{n} \cdot \boldsymbol{u}_{|\alpha\beta} - (\boldsymbol{a}^\alpha \cdot \boldsymbol{g}_m)\boldsymbol{n} \cdot \boldsymbol{u}_{,\alpha} \tag{74}$$

and

$$\dot{T} = L^\alpha M^\beta \boldsymbol{n} \cdot \boldsymbol{u}_{|\alpha\beta} - (\boldsymbol{a}^\alpha \cdot \boldsymbol{\Gamma})n \cdot \boldsymbol{u}_{,\alpha}. \tag{75}$$

To obtain the remaining terms in Eq. (64) we require (cf. 38)

$$(S^\lambda_{\alpha\beta}\boldsymbol{a}_\lambda)^{\cdot} = \boldsymbol{u}_{|\alpha\beta} - \dot{b}_{\alpha\beta}\boldsymbol{n} - b_{\alpha\beta}\dot{\boldsymbol{n}}. \tag{76}$$

Thus, for example,

$$\boldsymbol{g}_l \cdot \dot{\boldsymbol{g}}_l = L^\alpha L^\beta (S^\lambda_{\alpha\beta}\boldsymbol{a}_\lambda)^{\cdot} = L^\alpha L^\beta \boldsymbol{g}_l \cdot \boldsymbol{u}_{|\alpha\beta} - K_L\boldsymbol{g}_l \cdot \dot{\boldsymbol{n}}. \tag{77}$$

Using $\dot{\boldsymbol{n}} = -(\boldsymbol{n} \cdot \boldsymbol{u}_{,\alpha})\boldsymbol{a}^\alpha$ —obtained by differentiating $\boldsymbol{n} \cdot \boldsymbol{a}_\alpha = 0$—we derive

$$\boldsymbol{g}_l \cdot \dot{\boldsymbol{g}}_l = L^\alpha L^\beta \boldsymbol{g}_l \cdot \boldsymbol{u}_{|\alpha\beta} + K_L(\boldsymbol{g}_l \cdot \boldsymbol{a}^\alpha)\boldsymbol{n} \cdot \boldsymbol{u}_{,\alpha}. \tag{78}$$

Similarly,

$$\boldsymbol{g}_m \cdot \dot{\boldsymbol{g}}_m = M^\alpha M^\beta \boldsymbol{g}_m \cdot \boldsymbol{u}_{|\alpha\beta} + K_M(\boldsymbol{g}_m \cdot \boldsymbol{a}^\alpha)\boldsymbol{n} \cdot \boldsymbol{u}_{,\alpha} \tag{79}$$

and

$$\boldsymbol{\Gamma} \cdot \dot{\boldsymbol{\Gamma}} = L^\alpha M^\beta \boldsymbol{\Gamma} \cdot \boldsymbol{u}_{|\alpha\beta} + T(\boldsymbol{\Gamma} \cdot \boldsymbol{a}^\alpha)\boldsymbol{n} \cdot \boldsymbol{u}_{,\alpha}. \tag{80}$$

These results furnish

$$\begin{aligned} N^{\alpha} =& w_{\lambda} L^{\alpha} \boldsymbol{l} + w_{\mu} M^{\alpha} \boldsymbol{m} + J w_{J} \boldsymbol{a}^{\alpha} \\ &+ \left\{ [(A_{g} - k)(K_{L} \boldsymbol{g}_{l} + K_{M} \boldsymbol{g}_{m}) \right. \\ &\left. + (A_{\Gamma} - \bar{k}) T \boldsymbol{\Gamma}] \cdot \boldsymbol{a}^{\alpha} \right\} \boldsymbol{n} \end{aligned} \tag{81}$$

and

$$\begin{aligned} \boldsymbol{M}^{\alpha\beta} =& L^{\alpha} L^{\beta} (A_{g} \boldsymbol{g}_{l} + k K_{L} \boldsymbol{n}) \\ &+ M^{\alpha} M^{\beta} (A_{g} \boldsymbol{g}_{m} + k K_{M} \boldsymbol{n}) \\ &+ \frac{1}{2} (L^{\alpha} M^{\beta} + M^{\alpha} L^{\beta}) (A_{\boldsymbol{\Gamma}} \boldsymbol{\Gamma} + \bar{k} T \boldsymbol{n}), \end{aligned} \tag{82}$$

in which we have imposed the requisite symmetry with respect to interchange of the superscripts.

4 Equilibrium

The derivation of the Euler equations and boundary conditions in second-gradient elasticity is well known [25–30]. We present it here in outline, and in so doing, establish the constitutive connection between the applied loads and the deformation. To this end, we adopt the virtual-work statement

$$\dot{E} = P, \tag{83}$$

where the superposed dot refers to the variational derivative,

$$E = \int_{\Omega} W \mathrm{d}a \tag{84}$$

is the strain energy and P is the virtual power of the edge loads, the form of which is made explicit below. Conservative loads are characterized by the existence of a potential L such that $P = \dot{L}$, and in this case the problem of determining equilibrium deformations is reduced to the problem of minimizing the potential energy $E - L$.

We have

$$\dot{E} = \int_{\Omega} \dot{W} \mathrm{d}a, \tag{85}$$

where $\dot{W}$ is given by Eq. (63). Let

$$\varphi^{\alpha} = \boldsymbol{T}^{\alpha} \cdot \boldsymbol{u} + \boldsymbol{M}^{\alpha\beta} \cdot \boldsymbol{u}_{,\beta}, \tag{86}$$

with

$$\boldsymbol{T}^{\alpha} = \boldsymbol{N}^{\alpha} - \boldsymbol{M}^{\alpha\beta}_{|\beta} \tag{87}$$

and

$$\boldsymbol{M}^{\alpha\beta}_{|\beta} = \boldsymbol{M}^{\beta\alpha}_{,\beta} + \boldsymbol{M}^{\beta\alpha} \bar{\Gamma}^{\lambda}_{\lambda\beta} + \boldsymbol{M}^{\beta\lambda} \bar{\Gamma}^{\alpha}_{\lambda\beta}. \tag{88}$$

We may thus write

$$\dot{W} = \varphi^{\alpha}_{|\alpha} - \boldsymbol{u} \cdot \boldsymbol{T}^{\alpha}_{|\alpha} \tag{89}$$

and use Stokes' theorem to reduce Eq. (85) to

$$\dot{E} = \int_{\partial\Omega} \varphi^{\alpha} \nu_{\alpha} \mathrm{d}s - \int_{\Omega} \boldsymbol{u} \cdot \boldsymbol{T}^{\alpha}_{|\alpha} \mathrm{d}a, \tag{90}$$

wherein $\boldsymbol{\nu} = \nu_{\alpha} \boldsymbol{e}^{\alpha}$ is the rightward unit normal to $\partial\Omega$.

In the absence of distributed loads, it follows immediately from Eq. (83) that the relevant Euler–Lagrange equation, holding in Ω, is

$$\boldsymbol{T}^{\alpha}_{|\alpha} = \boldsymbol{0}. \tag{91}$$

Turning to the boundary terms, a standard integration-by-parts procedure [18] is used to recast the first integral in Eq. (90) as

$$\begin{aligned} \int_{\partial\Omega} \varphi^{\alpha} \nu_{\alpha} \mathrm{d}s =& \int_{\partial\Omega} \left\{ \left[\boldsymbol{T}^{\alpha} \nu_{\alpha} - \left(\boldsymbol{M}^{\alpha\beta} \nu_{\alpha} \tau_{\beta} \right)' \right] \cdot \boldsymbol{u} \right. \\ &\left. + \boldsymbol{M}^{\alpha\beta} \nu_{\alpha} \nu_{\beta} \cdot \boldsymbol{u}_{,\nu} \right\} \mathrm{d}s - \sum (\boldsymbol{M}^{\alpha\beta} \nu_{\alpha} \tau_{\beta})_{i} \cdot \boldsymbol{u}_{i}, \end{aligned} \tag{92}$$

where $\boldsymbol{\tau} = \tau_{\alpha} \boldsymbol{e}^{\alpha} = \boldsymbol{N} \times \boldsymbol{\nu}$ is the unit tangent to $\partial\Omega$ (not to be confused with the scalar surface torsion defined in Eq. (19)), $\boldsymbol{u}_{,\nu} = \nu^{\alpha} \boldsymbol{u}_{,\alpha}$ is the normal derivative of $\boldsymbol{u}$, $(\cdot)' = \mathrm{d}(\cdot)/\mathrm{d}s$ and the square bracket refers to the forward jump as a corner of the boundary is traversed. That is, $[\cdot] = (\cdot)_{+} - (\cdot)_{-}$, where the subscripts "$\pm$" identify the limits as a corner located at arclength station s is approached through larger and smaller values of arclength, respectively, and the sum refers to the collection of all corners. Here, we assume the boundary to be piecewise smooth in the sense that its tangent is piecewise continuous.

It follows from Eq. (83) that admissible powers are of the form

$$P = \int_{\partial\Omega_t} \boldsymbol{t} \cdot \boldsymbol{u} \mathrm{d}s + \int_{\partial\Omega_m} \boldsymbol{\mu} \cdot \boldsymbol{u}_{,\nu} \mathrm{d}s + \sum_{*} \boldsymbol{f}_{i} \cdot \boldsymbol{u}_{i}, \tag{93}$$

where

$$\begin{aligned} &\boldsymbol{t} = \boldsymbol{T}^{\alpha} \nu_{\alpha} - (\boldsymbol{M}^{\alpha\beta} \nu_{\alpha} \tau_{\beta})', \quad \boldsymbol{\mu} = \boldsymbol{M}^{\alpha\beta} \nu_{\alpha} \nu_{\beta} \quad \text{and} \\ &\boldsymbol{f}_{i} = -[\boldsymbol{M}^{\alpha\beta} \nu_{\alpha} \tau_{\beta}]_{i} \end{aligned} \tag{94}$$

are the edge traction, edge double force, and the corner force at the ith corner, respectively. Here, $\partial\Omega_t$ and $\partial\Omega_m$, respectively, are parts of $\partial\Omega$ where $\boldsymbol{r}$ and $\boldsymbol{r}_{,\nu}$ are not assigned, and the starred sum ranges over corners where position is not assigned. We suppose that r and $r_{,\nu}$ are assigned on $\partial\Omega / \partial\Omega_t$

and $\partial\Omega/\partial\Omega_m$, respectively, and that position is assigned at the corners not included in the starred sum.

A simple example of conservative loading is furnished by the potential

$$L = \int_{\partial\Omega_t} \boldsymbol{t}\cdot\boldsymbol{r}\,\mathrm{d}s + \int_{\partial\Omega_m} \boldsymbol{\mu}\cdot\boldsymbol{r}_{,\nu}\,\mathrm{d}s + \sum_* \boldsymbol{f}_i\cdot\boldsymbol{r}_i, \tag{95}$$

in which $\boldsymbol{t}$, $\boldsymbol{\mu}$, and $\boldsymbol{f}_i$ are all independent of the deformation.

To understand the role of the double force in mechanical terms, we consider the special case in which no kinematical data are assigned anywhere on $\partial\Omega$, so that rigid-body deformations are kinematically admissible. The variational derivative of such a deformation is expressible in the form $\boldsymbol{u} = \boldsymbol{\omega}\times\boldsymbol{r} + \boldsymbol{c}$, where $\boldsymbol{c}$ and $\boldsymbol{\omega}$ are arbitrary spatially uniform vectors. Because the strain-energy function is invariant under such deformations, we have $\dot{E} = 0$ and Eq. (83) reduces to $P = 0$; i.e.,

$$\begin{aligned}&\boldsymbol{c}\cdot\left(\int_{\partial\Omega} \boldsymbol{t}\,\mathrm{d}s + \sum \boldsymbol{f}_i\right)\\ &\quad + \boldsymbol{\omega}\cdot\left[\int_{\partial\Omega} (\boldsymbol{r}\times\boldsymbol{t} + \boldsymbol{r}_{,\nu}\times\mu)\,\mathrm{d}s + \sum \boldsymbol{r}_i\times\boldsymbol{f}_i\right] = 0.\end{aligned} \tag{96}$$

We then have

$$\begin{aligned}&\int_{\partial\Omega} \boldsymbol{t}\,\mathrm{d}s + \sum \boldsymbol{f}_i = \boldsymbol{0} \quad \text{and}\\ &\int_{\partial\Omega} (\boldsymbol{r}\times\boldsymbol{t} + \boldsymbol{r}_{,\nu}\times\boldsymbol{\mu})\,\mathrm{d}s + \sum \boldsymbol{r}_i\times\boldsymbol{f}_i = \boldsymbol{0},\end{aligned} \tag{97}$$

and hence the interpretation of $\boldsymbol{r}_{,\nu}\times\boldsymbol{\mu}$ is a distribution of edge couples. These couples are configuration dependent in the example of conservative loading described by Eq. (95). In general, as is well known, a non-trivial fixed boundary couple can not be associated with a conservative boundary-value problem.

5 Example: hyperbolic paraboloid

The foregoing theory is quite involved and in practice recourse must be made to numerical methods to obtain solutions. Fortunately the convexity of the strain-energy function with respect to $\boldsymbol{r}_{|\alpha\beta}$ ensures the convergence of minimizing sequences and hence guarantees that solutions are available via the direct method of the calculus of variations; in other words, the theory is amenable to finite-element analysis. We intend to report on its numerical treatment elsewhere.

Here, we merely illustrate the model in terms of an academic problem; in particular, we adopt a semi-inverse strategy and seek conditions under which the generalized hyperbolic paraboloids, defined by

$$\boldsymbol{r}(\theta^\alpha) = \boldsymbol{a}\theta^1\theta^2 + \boldsymbol{b}\theta^1 + \boldsymbol{c}\theta^2 + \boldsymbol{d}, \tag{98}$$

in which $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}$ are fixed vectors, furnish equilibrium deformations. Here, θ^1 and θ^2 are Cartesian coordinates on Ω aligned with the initial fiber directions; i.e., $L_\alpha = \delta^1_\alpha$ and $M_\alpha = \delta^2_\alpha$. Accordingly,

$$\lambda\boldsymbol{l} = L^\alpha\boldsymbol{r}_{,\alpha} = \boldsymbol{a}\theta^2 + \boldsymbol{b} \quad \text{and} \quad \mu\boldsymbol{m} = M^\alpha\boldsymbol{r}_{,\alpha} = \boldsymbol{a}\theta^1 + \boldsymbol{c}. \tag{99}$$

The fiber stretches are easily determined and the shear angle is given by

$$\lambda\mu\sin\gamma = |\boldsymbol{a}|^2\,\theta^1\theta^2 + \boldsymbol{a}\cdot\boldsymbol{b}\theta^1 + \boldsymbol{a}\cdot\boldsymbol{c}\theta^2 + \boldsymbol{b}\cdot\boldsymbol{c}. \tag{100}$$

Further,

$$\begin{aligned}&L^\alpha L^\beta\boldsymbol{r}_{|\alpha\beta} = \boldsymbol{0}, \quad M^\alpha M^\beta\boldsymbol{r}_{|\alpha\beta} = \boldsymbol{0} \quad \text{and},\\ &L^\alpha M^\beta\boldsymbol{r}_{|\alpha\beta} = \boldsymbol{a},\end{aligned} \tag{101}$$

and thus (cf. 62) $\boldsymbol{g}_l$, $\boldsymbol{g}_m$, K_L and K_M all vanish, leaving

$$\boldsymbol{\Gamma} + T\boldsymbol{n} = \boldsymbol{a}. \tag{102}$$

In the somewhat artificial special case when the moduli $A_{\boldsymbol{\Gamma}}$ and $\bar{k}$ coincide we then have

$$\boldsymbol{M}^{\alpha\beta} = \frac{1}{2}A_{\boldsymbol{\Gamma}}(L^\alpha M^\beta + M^\alpha L^\beta)\boldsymbol{a}. \tag{103}$$

Accordingly, $M^{\alpha\beta}_{|\beta}$ vanishes, yielding

$$\boldsymbol{T}^\alpha = \boldsymbol{N}^\alpha = w_\lambda L^\alpha\boldsymbol{l} + w_\mu M^\alpha\boldsymbol{m} + Jw_J\boldsymbol{a}^\alpha, \tag{104}$$

where $w_J = \partial w/\partial J$, etc.

Experiments on fabrics [31–33] indicate that their resistance to shear is quite weak unless the fibers are nearly aligned; that is, unless $|\sin\gamma|$ is close to unity. The loci of points where $|\sin\gamma| = 1$ are the curves determined by combining Eqs. (99) with (100). Elsewhere, the shear resistance may be safely ignored; this is tantamount to neglecting w_J in Eq. (104), provided that the fiber stretches remain close to unity. Further, if the mesh of the fabric network is sufficiently coarse, then the mutual interactions of the fibers are also negligible; that is, coarse-mesh nets typically do not exhibit a Poisson effect. In such circumstances, the strain-energy function may be approximated by the sum of two functions, each depending on only one fiber stretch [9]. In this case w_J is negligible and we have the simplification

$$\boldsymbol{N}^\alpha = w_\lambda L^\alpha\boldsymbol{l} + w_\mu M^\alpha\boldsymbol{m}, \tag{105}$$

in which w_λ depends only on $\lambda(\theta^2)$ and w_μ depends only on $\mu(\theta^1)$. We then obtain

$$\boldsymbol{T}^{\alpha}_{|\alpha} = (w_\lambda \boldsymbol{l})_{,1} + (w_\mu \boldsymbol{m})_{,2}. \tag{106}$$

This vanishes identically, and the equilibrium Eq. (91) is satisfied.

Using Eq. (94), we find, on lines of constant θ^1 and θ^2, that $\boldsymbol{t} = w_\lambda \boldsymbol{l}$ and $\boldsymbol{t} = w_\mu \boldsymbol{m}$, respectively, and that $\boldsymbol{\mu}$ vanishes. At the points of intersection of these lines, the corner force is $f = A\boldsymbol{\Gamma}a$. If the sheet is a rectangle with edges parallel to the fibers prior to deformation, and if no corner force is applied, then a vanishes and Eq. (98) reduces to a homogeneous deformation, which is trivially in equilibrium for all strain energies of the form Eq. (60).

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References

1. Warren, W.E.: The elastic properties of woven polymeric fabric. Polym. Eng. Sci. **30**, 1309–1313 (1990)
2. Wang, W.-B., Pipkin, A.C.: Inextensible networks with bending stiffness. Q. J. Mech. Appl. Math. **39**, 343–359 (1986)
3. Wang, W.-B., Pipkin, A.C.: Plane deformations of nets with bending stiffness. Acta Mech. **65**, 263–279 (1986)
4. Pipkin, A.C.: Some developments in the theory of inextensible networks. Q. Appl. Math. **38**, 343 (1980)
5. Pipkin, A.C.: Plane traction problems for inextensible networks. Q. J. Mech. Appl. Math. **34**, 415 (1981)
6. Pipkin, A.C.: Equilibrium of Tchebychev nets. Arch. Ration. Mech. Anal. **85**, 81 (1984)
7. Luo, C., Steigmann, D.J.: Bending and twisting effects in the three-dimensional finite deformations of an inextensible network. In: Durban, D., et al. (eds.) Developments in Shell Theory Technion volume to commemorate the 70th birthday of A. Libai. Advances in the Mechanics of Plates and Shells, pp. 213–228. Kluwer Academic Publishers, London (2001)
8. Luo, C.: Nonlinear three-dimensional mechanics of fabrics. Dissertation, UC Berkeley (2000)
9. Indelicato, G.: The influence of the twist of individual fibers in 2D fibered networks. Int. J. Solids Struct. **46**, 912–922 (2009)
10. Steigmann, D.J., Pipkin, A.C.: Equilibrium of elastic nets. Phil. Trans. R. Soc. Lond. **A335**, 419–454 (1991)
11. Dill, E.H.: Kirchhoff's theory of rods. Arch. Hist. Exact Sci. **44**, 1–23 (1992)
12. Steigmann, D.J.: Theory of elastic solids reinforced with fibers resistant to extension, flexure and twist. Int. J. Non-linear Mech. **47**, 734–742 (2012)
13. Steigmann, D.J.: Effects of fiber bending and twisting resistance on the mechanics of fiber-reinforced elastomers. In: Dorfmann, L., Ogden, R.W. (eds.) CISM Course on Nonlinear Mechanics of Soft Fibrous Tissues. Springer, Wien and New York (in press)
14. Ciarlet, P.G.: An introduction to differential geometry with applications to elasticity. J. Elast. **78–79**, 3–201 (2005)
15. Lovelock, D., Rund, H.: Tensors, Differ. Forms Var. Princ. Dover, New York (1989)
16. Coleman, B.D.: Necking and drawing in polymeric fibers under tension. Arch. Ration. Mech. Anal. **83**, 115–137 (1983)
17. Coleman, B.D., Newman, D.C.: On the rheology of cold drawing: I. elastic materials. J. Polym. Sci. B: Polym. Phys. **26**, 1801–1822 (1988)
18. dell'Isola, F., Steigmann, D.J.: A two-dimensional gradient-elasticity theory for woven fabrics. J. Elast. **118**, 113–125 (2015)
19. Spencer, A.J.M., Soldatos, K.P.: Finite deformations of fibre-reinforced elastic solids with fibre bending stiffness. Int. J. Non-Linear Mech. **42**, 355–368 (2007)
20. Murdoch, A.I., Cohen, H.: Symmetry considerations for material surfaces. Arch. Ration. Mech. Anal. **72**, 61–98 (1979)
21. Indelicato, G., Albano, A.: Symmetry properties of the elastic energy of a woven fabric with bending and twisting resistance. J. Elast. **94**, 33–54 (2009)
22. Ball, J.M., Currie, J.C., Olver, P.J.: Null Lagrangians, weak continuity, and variational problems of arbitrary order. J. Funct. Anal. **41**, 135–174 (1981)
23. Hilgers, M.G., Pipkin, A.C.: The Graves condition for variational problems of arbitrary order. IMA. J. Appl. Math. **48**, 265–269 (1992)
24. Steigmann, D.J., Ogden, R.W.: Elastic surface–substrate interactions. Proc. R. Soc. Lond. **A455**, 437–474 (1999)
25. Toupin, R.A.: Theories of elasticity with couple stress. Arch. Ration. Mech. Anal. **17**, 85–112 (1964)
26. Mindlin, R.D., Tiersten, H.F.: Effects of couple-stresses in linear elasticity. Arch. Ration. Mech. Anal. **11**, 415–448 (1962)
27. Koiter, W.T.: Couple-stresses in the theory of elasticity. Proc. Knononklijke Nederl. Akad. van Wetensch. **B67**, 17–44 (1964)
28. Germain, P.: The method of virtual power in continuum mechanics, part 2: Microstructure. SIAM J. Appl. Math. **25**, 556–575 (1973)
29. dell'Isola, F., Seppecher, P., Madeo, A.: Beyond Euler-Cauchy Continua: The structure of contact actions in N-th gradient generalized continua: A generalization of the Cauchy tetrahedron argument. Variational Models and Methods in Solid and Fluid Mechanics CISM Courses and Lectures, vol. 535, 17–106 (2012)
30. dell'Isola, F., Seppecher, P., Madeo, A.: How contact interactions may depend on the shape of Cauchy cuts in N-th gradient continua: Approach "à la D'Alembert". Z. Angew. Math. Phys. **63**, 1119–1141 (2012)
31. Ferretti, M., Madeo, A., dell'Isola, F., Boisse, P.: Modeling the onset of shear boundary layers in fibrous composite reinforcements by second-gradient theory. Z. Angew. Math. Phys. **65**, 587–612 (2013)
32. Cusick, G.E.: The response of fabric to shearing forces. J. Text. Inst. **52**, T395–T406 (1961)
33. Cao, J., Akkerman, R., Boisse, P., et al. Characterization of mechanical behavior of woven fabrics: experimental methods and benchmark results. Compos. Part A: Appl. Sci. Manuf. **39**, 1037–1053 (2008)