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Abstract

In this paper, we will give some remarks on links between the spectral gap of the Ornstein-Uhlenbeck operator on the Riemannian path space with lower and upper bounds of the Ricci curvature on the base manifold; this work was motivated by a recent work of A. Naber on the characterization of the bound of the Ricci curvature by analysis of path spaces.

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1 Introduction

Let M be a complete smooth Riemannian manifold of dimension d, and Z a C^1-vector field on M. We will be concerned with the diffusion operator

$$L = \frac{1}{2}(\Delta_M - Z),$$

where Δ_M is the Beltrami-Laplace operator on M. Let ∇ be the Levi-Civita connection and Ric the Ricci curvature tensor on M. We will denote

$$\mathrm{Ric}_Z = \mathrm{Ric} + \nabla Z.$$

It is well-known that the lower bound K_2 of the symmetrized Ric_Z^s, that is,

$$\mathrm{Ric}_Z^s(x) = \frac{1}{2}\Big(\mathrm{Ric}_Z(x) + \mathrm{Ric}_Z^*(x)\Big) \geq K_2\,\mathrm{Id}, \tag{1.1}$$

where Ric_Z^* denotes the transposed matrix of Ric_Z, gives the lower bound of constants in the logarithmic Sobolev inequality with respect to the heat measure $\rho_t(x, dy)$, associated to L; more precisely,

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$$\int_M u^2(y) \log\Big(\frac{u^2(y)}{||u||^2_{\rho_t}}\Big)\, \rho_t(x,dy) \leq 2\,\frac{1-e^{-K_2 t}}{K_2} \int_M |\nabla u(y)|^2\, \rho_t(x,dy), \quad t>0, \tag{1.2}$$

where $||u||^2_{\rho_t} = \int_M u^2(y)\, \rho_t(x,dy)$.

Given now a finite number of times $0 < t_1 < \ldots < t_N$, consider the probability measure $\nu_{t_1,\ldots,t_N}$ on M^N defined by

$$\int_{M^N} f\, d\nu_{t_1,\ldots,t_N} = \int_{M^N} f(y_1,\ldots,y_N)\, p_{t_1}(x,dy_1)p_{t_2-t_1}(y_1,dy_2)\cdots p_{t_N-t_{N-1}}(y_{N-1},dy_N) \tag{1.3}$$

where f is a bounded measurable function on M^N. Then with respect to the correlated metric $|\cdot|_C$ on TM^N (see definition (1.10) below), the logarithmic Sobolev inequality still holds for $\nu_{t_1,\ldots,t_N}$, that is, there is a constant $C_N > 0$ such that

$$\int_{M^N} f^2 \log\Big(\frac{f^2}{||f||^2_{\nu_{t_1,\ldots,t_N}}}\Big)\, d\nu_{t_1,\ldots,t_N} \leq C_N \int_{M^N} |\nabla f|^2_C\, d\nu_{t_1,\ldots,t_N}, \quad f\in C^1(M^N). \tag{1.4}$$

It was proved in [20, 6] that under the hypothesis

$$\sup_{x\in M} |||\mathrm{Ric}_Z(x)||| < +\infty, \tag{1.5}$$

where $|||\cdot|||$ denotes the norm of matrices, the constant C_N in (1.4) can be bounded, that is

$$\sup_{N\geq 1} C_N < +\infty. \tag{1.6}$$

A natural question is whether (1.6) still holds only under Condition (1.1)? In a recent work [21], A. Naber proved that if the uniform bound (1.6) holds, then the Ricci curvature of the base manifold has an upper bound. It is well-known that Inequality (1.2) implies the lower bound (1.1), therefore Condition (1.6) implies (1.5). The main purpose in [21] is to get informations on Ric_Z from the analysis of the Riemannian path space. Let's explain briefly the context.

Let $O(M)$ be the bundle of orthonormal frames and $\pi : O(M) \to M$ the canonical projection. Let $H_1,\ldots,H_d$ be the canonical horizontal vector fields on $O(M)$, consider the Stratanovich stochastic differential equation (SDE) on $O(M)$:

$$du_t(w) = \sum_{i=1}^d H_i(u_t(w))\circ dw_t^i - \frac{1}{2} H_Z(u_t(w))dt, \quad u_0(w) = u_0 \in \pi^{-1}(x), \tag{1.7}$$

where H_Z denotes the horizontal lift of Z to $O(M)$, that is, $\pi'(u)\cdot H_Z(u) = Z(\pi(u))$. It is well-known that under Condition (1.1), the life-time τ_x of the SDE (1.7) is infinite. Let

$$\gamma_t(w) = \pi(u_t(w)). \tag{1.8}$$

Then $\{\gamma_t(w); t\geq 0\}$ is a diffusion process on M, having L as generator. The probability measure $\nu_{t_1,\ldots,t_N}$ considered in (1.3) is the law of $w \to (\gamma_{t_1}(w),\ldots,\gamma_{t_N}(w))$ on M^N. Now consider the following path space

$$W_x^T(M) = \big\{\gamma : [0,T] \to M \text{ continuous},\ \gamma(0) = x\big\}.$$

The law $\mu_{x,T}$ on $W_x^T(M)$ of $w \to \gamma_\cdot(w)$ is called the Wiener measure on $W_x^T(M)$. The integration by parts formula for $\mu_{x,T}$ was first estalished in the Seminal book [5], then developed in [16, 10]; the Cameron-Martin type quasi-invariance of $\mu_{x,T}$ was first proved by B. Driver [9], completed and simplified in [18, 19, 13]. By means of Cameron-Martin, we consider the space

$$\mathbb{H} = \Big\{ h : [0,T] \to \mathbb{R}^d \text{ absolutely continuous; } h(0) = 0, |h|_{\mathbb{H}}^2 = \int_0^T |\dot{h}(s)|_{\mathbb{R}^d}^2\, ds < +\infty \Big\}$$

where the dot denotes the derivative with respect to the time t. Let $F : W_x^T(M) \to \mathbb{R}$ be a cylindrical function in the form: $F(\gamma) = f(\gamma(t_1), \cdots, \gamma(t_N))$ for some $N \geq 1, 0 \leq t_1 < t_2 < \cdots < t_N \leq 1$, and $f \in C_b^1(M^N)$. The usual gradient of F in Malliavin calculus is defined by

$$D_\tau F(\gamma(w)) = \sum_{j=1}^N u_{t_j}(w)^{-1} (\partial_j f)(\gamma_{t_1}(w), \cdots, \gamma_{t_N}(w))\, \mathbf{1}_{(\tau \leq t_j)}, \tag{1.9}$$

where ∂_j is the gradient with respect to the j-th component. The correlated norm of ∇f is

$$|\nabla f|_C^2 = \sum_{j,k=1}^N \langle u_{t_j}(w)^{-1}(\partial_j f), u_{t_k}(w)^{-1}(\partial_k f)\rangle\, t_j \wedge t_k, \tag{1.10}$$

where $t_j \wedge t_k$ denotes the minimum between t_j and t_k. Notice that the norm $|\nabla f|_C$ is random. The generator $\mathcal{L}_T^x$ associated to the Dirichlet form

$$\mathscr{E}(F,F) = \int_{W_x^T(M)} \Big(\int_0^T |D_\tau F|^2(\gamma)\, d\tau \Big)\, d\mu_{x,T}(\gamma)$$

is called the Ornstein-Uhlenbeck operator. The powerful tool of Γ_2 of Bakry and Emery [3] is not applicable to $\mathcal{L}_T^x$, the reason for this is the geometry of $W_x^T(M)$ inherted from $\mathbb{H}$ is quite complicated, the associated "Ricci tensor" being a divergent object (see [7, 8, 12]). When the base manifold M is compact, the existence of the spectral gap for $\mathcal{L}_T^x$ has been proved in [14]. The logarithmic Sobolev inequality for $D_\tau F$ defined in (1.9) has been established in [2], as well as in [20] or [6] where the constant was estimated using the bound of Ricci curvature tensor of the base manifold M. The method used in [14] is the martingale representation, which takes advantage the Itô filtration; this method has been developed in [12] to deal with the problem of vanishing of harmonic forms on $W_x^T(M)$. The purpose in [21] is to proceed in the opposite direction, to get the bound for Ricci curvature tensor of the base manifold M from the analysis of the path space $W_x^T(M)$.

The organization of the paper is as follows. In section 2, we will recall briefly basic objets in Analysis of $W_x^T(M)$. On the path space $W_x^T(M)$, there exist two type of gradients: the usual one is more related to the geometry of the base manifold, while the damped one is easy to be handled. In section 3, we will make estimation of the spectral gap of $\mathcal{L}_T^x$ as explicitly as possible in function of lower bound K_2 and upper bound K_1 of Ric. In section 4, we will study the behaviour of the spectral gap $Spect(\mathcal{L}_T^x)$ as $T \to 0$. Roughly speaking, we will get the following result:

$$1 - \frac{K_1 T}{2} + o(T) \leq Spect(\mathcal{L}_T^x) \leq 1 + \frac{K_2 T}{2} + o(T), \quad \text{as } T \to 0$$

under the following condition (3.1).

2 Framework of the Riemannian path space

We shall keep the notations of Section 1, and throughout this section, $u_t(w)$ denotes always the solution of (1.7) and $\gamma_t(w)$ the path defined in (1.8). For any $h \in \mathbb{H}$, we introduce first the usual gradient on the path space $W_x^T(M)$, which gives Formula (1.9) when the functional F is a cylindrical function. To this end, let

$$q(t,h) = \int_0^t \Omega_{u_s(w)}\Big(h(s), \circ dw(s) - \frac{1}{2} u_s(w)^{-1} Z_{\gamma_s(w)} ds\Big) \tag{2.1}$$

where Ω_u is the equivariant representation of the curvature tensor on M. Let ric_Z be the equivariant representation of Ric_Z, that is,

$$\mathrm{ric}_Z(u) = u^{-1} \circ \mathrm{Ric}_Z(\pi(u)) \circ u, \quad u \in O(M).$$

Consider $\hat{h}(w) \in \mathbb{H}$ defined by

$$\dot{\hat{h}}_t(w) = \dot{h}(t) + \frac{1}{2} \mathrm{ric}_Z(u_t(w))\, h(t). \tag{2.2}$$

Let $F : W_x^T(M) \to \mathbb{R}$ be a functional, we denote $\tilde{F}(w) = F(\gamma_\cdot(w))$. Then according to [16], we define

$$(D_h F)(\gamma_\cdot(w)) = \Big\{ \frac{d}{d\varepsilon} \tilde{F}\Big(\int_0^\cdot e^{\varepsilon q(s,h)}\, dw(s) + \varepsilon\, \hat{h} \Big) \Big\}_{\varepsilon=0}. \tag{2.3}$$

By [5, 16], if F is a cylindrical function on $W_x^T(M)$, then

$$(D_h F)(\gamma_\cdot(w)) = \int_0^T \langle D_\tau F(\gamma_\cdot(w)), \dot{h}(\tau) \rangle\, d\tau$$

where $D_\tau F$ was given in (1.9). Consider the following resolvent equation

$$\frac{dQ_{t,s}}{dt} = -\frac{1}{2} \mathrm{ric}_Z(u_t(w))\, Q_{t,s}, \quad t \geq s,\ Q_{s,s} = \mathrm{Id}. \tag{2.4}$$

For a cylindrical function F on $W_x^T(M)$ given by $F(\gamma) = f(\gamma(t_1), \cdots, \gamma(t_N))$ with $f \in C_b^1(M^N)$, following [16], we define the damped gradient $\tilde{D}_\tau F$ of F by

$$\tilde{D}_\tau F(\gamma_\cdot(w)) = \sum_{j=1}^N Q^*_{t_j,\tau} \big(u_{t_j}(w)^{-1} \partial_j f \big)\, \mathbf{1}_{(\tau \leq t_j)}, \tag{2.5}$$

where $Q^*_{\tau,s}$ is the transpose matrix of $Q_{\tau,s}$. The damped gradient $\tilde{D}_\tau F$ on the path space $W_x^T(M)$ plays a basic role in Analysis of $W_x^T(M)$. Let $(v_t)_{t\geq 0}$ be a $\mathbb{R}^d$-valued process, adapted to the Itô filtration $\mathscr{F}_t$ generated by $\{w(s);\ s \leq t\}$ such that $\mathbb{E}(\int_0^T |v_t|^2\, dt) < +\infty$. Consider two maps $v \to \tilde{v}$ and $v \to \hat{v}$ defined respectively by

$$\tilde{v}_t = v_t - \frac{1}{2} \mathrm{ric}_{u_t(w)} \int_0^t Q_{t,s} v_s\, ds, \tag{2.6}$$

and

$$\hat{v}_t = v_t + \frac{1}{2} \mathrm{ric}_{u_t(w)} \int_0^t v_s\, ds. \tag{2.7}$$

Then $\hat{\tilde{v}} = \tilde{\hat{v}} = v$. The two gradients D_tF and $\tilde{D}_tF$ are linked by the following formula

$$\int_0^T \langle \tilde{D}_tF, v_t\rangle\, dt = \int_0^T \langle D_tF, \tilde{v}_t\rangle\, dt. \tag{2.8}$$

The good feather of the damped gradient is that it admits a nice martingale representation

$$F = \mathbb{E}(F) + \int_0^T \langle \mathbb{E}^{\mathscr{F}_t}(\tilde{D}_tF), dw_t\rangle$$

where $\mathbb{E}^{\mathscr{F}_t}$ denotes the conditional expectation with respect to $\mathscr{F}_t$. The following logarithmic Sobolev inequality holds ([11, 17]):

$$\mathbb{E}\Big(F^2 \log \frac{F^2}{\|F\|_{L^2}^2}\Big) \le 2\mathbb{E}\Big(\int_0^T |\tilde{D}_tF|^2 dt\Big). \tag{2.9}$$

3 Precise lower bound on the spectral gap

The inconvenient of Inequality (2.9) is that the geometric information of the base manifold M is completely hidden. Now we use the usual gradient D_tF to make involving the geometry of M. By (2.9), the matter is now to estimate $\int_0^T |\tilde{D}_tF|^2 dt$ by $|D_tF|$. We assume that

$$K_2\,\mathrm{Id} \le \mathrm{ric}_Z^s, \quad |||\mathrm{ric}_Z||| \le K_1 \tag{3.1}$$

for two constants K_1, K_2 with $K_1 \ge 0$ and $K_1 + K_2 \ge 0$.

Theorem 3.1. *Let $0 < t \le T$. Set*

$$\begin{aligned}\Lambda(t,T) = 1 &+ \frac{K_1}{K_2}\Big(1 - \mathrm{e}^{-\frac{K_2(T-t)}{2}}\Big) + \frac{K_1}{K_2}\Big(1 - \mathrm{e}^{-\frac{K_2 t}{2}}\Big) \\ &+ \Big(\frac{K_1}{K_2}\Big)^2\Big[\Big(1 - \mathrm{e}^{-\frac{K_2 t}{2}}\Big) + \frac{1}{2}\Big(\mathrm{e}^{-\frac{K_2(T+t)}{2}} - \mathrm{e}^{-\frac{K_2(T-t)}{2}}\Big)\Big].\end{aligned} \tag{3.2}$$

Then we have the relation:

$$\int_0^T |\tilde{D}_tF|^2 dt \le \int_0^T \Lambda(t,T)|D_tF|^2 dt. \tag{3.3}$$

Proof. From (2.5) and (2.8), we have

$$\tilde{D}_tF = D_tF - \frac{1}{2}\int_t^T Q_{s,t}^*\, ric_{u_s}^* D_sF ds. \tag{3.4}$$

Thus,

$$\begin{aligned}|\tilde{D}_tF|^2 &= |D_tF|^2 - \Big\langle D_tF, \int_t^T Q_{s,t}^* ric_{u_s}^* D_sF ds\Big\rangle + \frac{1}{4}\Big|\int_t^T Q_{s,t}^* ric_{u_s}^* D_sF ds\Big|^2 \\ &:= I_1 + I_2 + I_3 \text{ respectively.}\end{aligned}$$

In the following we will estimate the term of I_2 and I_3. Under the lower bound in (3.1),

$$|||Q_{s,t}^*||| \le e^{-\frac{K_2(s-t)}{2}}, \quad s \ge t.$$

Let

$$\Lambda_1(t,T) := \int_t^T \left(\mathrm{e}^{-\frac{K_2(s-t)}{4}}\right)^2 ds.$$

Then

$$\begin{aligned}
|I_2| &\le |D_tF| \int_t^T \mathrm{e}^{-\frac{K_2(s-t)}{2}} K_1 |D_sF| ds \\
&\le |D_tF| \sqrt{K_1 \int_t^T \left(\mathrm{e}^{-\frac{K_2(s-t)}{4}}\right)^2 ds} \sqrt{K_1 \int_t^T \mathrm{e}^{-\frac{K_2(s-t)}{2}} |D_sF|^2 ds} \\
&= |D_tF| \sqrt{K_1 \Lambda_1(t,T)} \sqrt{K_1 \int_t^T \mathrm{e}^{-\frac{K_2(s-t)}{2}} |D_sF|^2 ds} \\
&\le \frac{1}{2}\Big\{ |D_tF|^2 K_1 \Lambda_1(t,T) + K_1 \int_t^T \mathrm{e}^{-\frac{K_2(s-t)}{2}} |D_sF|^2 ds \Big\}.
\end{aligned}$$

and

$$\begin{aligned}
|I_3| &\le \frac{1}{4} \Big| \int_t^T \mathrm{e}^{-\frac{K_2(s-t)}{2}} K_1 |D_sF| ds \Big|^2 \\
&\le \frac{1}{4} K_1^2 \Lambda_1(t,T) \int_t^T \mathrm{e}^{-\frac{K_2(s-t)}{2}} |D_sF|^2 ds.
\end{aligned}$$

Combining all the above inequalities, we get

$$\begin{aligned}
|\tilde{D}_tF|^2 &\le \Big(1 + \frac{K_1}{2}\Lambda_1(t,T)\Big)|D_tF|^2 + \Big(1 + \frac{K_1}{2}\Lambda_1(t,T)\Big)\frac{K_1}{2} \int_t^T \mathrm{e}^{-\frac{K_2(s-t)}{2}} |D_sF|^2 ds \\
&= \Big(1 + \frac{K_1}{2}\Lambda_1(t,T)\Big)\Big(|D_tF|^2 + \frac{K_1}{2} \int_t^T \mathrm{e}^{-\frac{K_2(s-t)}{2}} |D_sF|^2 ds\Big).
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
\int_0^T |\tilde{D}_tF|^2 dt &\le \int_0^T \Big(1 + \frac{K_1}{2}\Lambda_1(t,T)\Big)|D_tF|^2 dt \\
&\quad + \int_0^T \Big(1 + \frac{K_1}{2}\Lambda_1(t,T)\Big)\frac{K_1}{2} \int_t^T \mathrm{e}^{-\frac{K_2(s-t)}{2}} |D_sF|^2 ds dt \\
&= \int_0^T \Big(1 + \frac{K_1}{2}\Lambda_1(s,T)\Big)|D_sF|^2 ds \\
&\quad + \int_0^T |D_sF|^2 ds \int_0^s \frac{K_1}{2}\Big(1 + \frac{K_1}{2}\Lambda_1(t,T)\Big)\mathrm{e}^{-\frac{K_2(s-t)}{2}} dt \\
&:= \int_0^T \Big(1 + \frac{K_1}{2}\Lambda_1(s,T)\Big)|D_sF|^2 ds + \int_0^T (J_1(s) + J_2(s))|D_sF|^2 ds,
\end{aligned}$$

where

$$J_1(s) := \int_0^s \frac{K_1}{2}\mathrm{e}^{-\frac{K_2(s-t)}{2}} dt, \;\; J_2(s) := \int_0^s \Big(\frac{K_1}{2}\Big)^2 \Lambda_1(t,T) \mathrm{e}^{-\frac{K_2(s-t)}{2}} dt.$$

Next, then we compute the term $J_1(s)$ and $J_2(s)$. By direct computation, we have

$$J_1(s) = \frac{K_1}{K_2}\Big(1 - \mathrm{e}^{-\frac{K_2 s}{2}}\Big)$$

and

$$\begin{aligned}
J_2(s) &= \Big(\frac{K_1}{2}\Big)^2 \int_0^s \frac{2}{K_2}\Big(1-\mathrm{e}^{-\frac{K_2(T-t)}{2}}\Big)\mathrm{e}^{-\frac{K_2(s-t)}{2}}dt \\
&= \Big(\frac{K_1}{2}\Big)^2 \frac{2}{K_2}\Big[\int_0^s \mathrm{e}^{-\frac{K_2(s-t)}{2}}dt - \mathrm{e}^{-\frac{K_2(T+s)}{2}}\int_0^s \mathrm{e}^{K_2 t}dt\Big] \\
&= \Big(\frac{K_1}{2}\Big)^2 \frac{2}{K_2}\Big[\frac{2}{K_2}\Big(1-\mathrm{e}^{-\frac{K_2 s}{2}}\Big) - \frac{1}{K_2}\mathrm{e}^{-\frac{K_2(T+s)}{2}}(\mathrm{e}^{K_2 s}-1)\Big] \\
&= \Big(\frac{K_1}{2}\Big)^2 \frac{2}{K_2}\Big[\frac{2}{K_2}\Big(1-\mathrm{e}^{-\frac{K_2 s}{2}}\Big) + \frac{1}{K_2}\mathrm{e}^{-\frac{K_2(T+s)}{2}} - \frac{1}{K_2}\mathrm{e}^{-\frac{K_2(T-s)}{2}}\Big].
\end{aligned}$$

Adding $J_1(s)$ to $J_1(s)$ implying that

$$\begin{aligned}
&J_1(s)+J_2(s) \\
&= \frac{K_1}{K_2}\Big(1-\mathrm{e}^{-\frac{K_2 s}{2}}\Big) + \Big(\frac{K_1}{K_2}\Big)^2\Big[\Big(1-\mathrm{e}^{-\frac{K_2 s}{2}}\Big) + \frac{1}{2}\Big(\mathrm{e}^{-\frac{K_2(T+s)}{2}} - \mathrm{e}^{-\frac{K_2(T-s)}{2}}\Big)\Big] := \Lambda_2(s,T)
\end{aligned}$$

Thus,

$$\int_0^T |\tilde{D}_t F|^2 dt \le \int_0^T \Lambda(t,T)|D_t F|^2 dt,$$

with

$$\begin{aligned}
\Lambda(t,T) &= 1 + \frac{K_1}{2}\Lambda_1(t,T) + \Lambda_2(t,T) \\
&= 1 + \frac{K_1}{K_2}\Big(1-\mathrm{e}^{-\frac{K_2(T-t)}{2}}\Big) + \frac{K_1}{K_2}\Big(1-\mathrm{e}^{-\frac{K_2 t}{2}}\Big) \\
&+ \Big(\frac{K_1}{K_2}\Big)^2\Big[\Big(1-\mathrm{e}^{-\frac{K_2 t}{2}}\Big) + \frac{1}{2}\Big(\mathrm{e}^{-\frac{K_2(T+t)}{2}} - \mathrm{e}^{-\frac{K_2(T-t)}{2}}\Big)\Big].
\end{aligned}$$

The proof is completed. □

Notice that as $K_2 \to 0$, by expression (3.2),

$$\Lambda(t,T) \to 1 + \frac{K_1 T}{2} + K_1^2\Big(\frac{Tt}{4} - \frac{t^2}{8}\Big).$$

Now we study the variation of the function $t \to \Lambda(t,T)$. It is quite interesting to remark that its monotonicity is dependent of the sign of K_2.

Proposition 3.2. *(i) If $K_2 < 0$, then $t \to \Lambda(t,T)$ is strictly increasing over $[0,T]$. (ii) If $K_2 > 0$, then the maximum is attained at a point t_0 in $(0,T)$.*

Proof. Taking the derivative of $t \to \Lambda(t,T)$ gives

$$\begin{aligned}
\Lambda'(t,T) &= -\frac{K_1}{2}\mathrm{e}^{-\frac{K_2(T-t)}{2}} + \frac{K_1}{2}\mathrm{e}^{-\frac{K_2 t}{2}} \\
&+ \frac{K_1^2}{2K_2}\mathrm{e}^{-\frac{K_2 t}{2}} - \frac{K_1^2}{4K_2}\mathrm{e}^{-\frac{K_2(T+t)}{2}} - \frac{K_1^2}{4K_2}\mathrm{e}^{-\frac{K_2(T-t)}{2}}.
\end{aligned}$$

In addition, we have

$$\Lambda(0,T) = 1 + \frac{K_1}{K_2}\Big(1-\mathrm{e}^{-\frac{K_2 T}{2}}\Big)$$

and

$$\begin{aligned}
\Lambda(T,T) &= 1+\frac{K_1}{K_2}\Big(1-\mathrm{e}^{-\frac{K_2T}{2}}\Big)+\Big(\frac{K_1}{K_2}\Big)^2\Big[\Big(1-\mathrm{e}^{-\frac{K_2T}{2}}\Big)+\frac{1}{2}\Big(\mathrm{e}^{-K_2T}-1\Big)\Big]\\
&= 1+\frac{K_1}{K_2}\Big(1-\mathrm{e}^{-\frac{K_2T}{2}}\Big)+\frac{1}{2}\Big(\frac{K_1}{K_2}\Big)^2\Big(1-\mathrm{e}^{-\frac{K_2T}{2}}\Big)^2\\
&= \frac{1}{2}+\frac{1}{2}\Big[1+\frac{K_1}{K_2}\Big(1-\mathrm{e}^{-\frac{K_2T}{2}}\Big)\Big]^2=\frac{1}{2}+\frac{1}{2}\Lambda^2(0,T).
\end{aligned}$$

From the second equality in the above, we observe that $\Lambda(T,T)\geq\Lambda(0,T)$. Moreover,

$$\begin{aligned}
\Lambda'(0,T) &= -\frac{K_1}{2}\mathrm{e}^{-\frac{K_2T}{2}}+\frac{K_1}{2}+\frac{K_1^2}{2K_2}-\frac{K_1^2}{4K_2}\mathrm{e}^{-\frac{K_2T}{2}}-\frac{K_1^2}{4K_2}\mathrm{e}^{-\frac{K_2T}{2}}\\
&= \frac{K_1}{2}\Big(1-\mathrm{e}^{-\frac{K_2T}{2}}\Big)+\frac{K_1^2}{2K_2}\Big(1-\mathrm{e}^{-\frac{K_2T}{2}}\Big)\\
&= \frac{K_1}{2}(K_1+K_2)\frac{1-\mathrm{e}^{-\frac{K_2T}{2}}}{K_2}\geq 0;
\end{aligned}\tag{3.5}$$

and

$$\begin{aligned}
\Lambda'(T,T) &= -\frac{K_1}{2}+\frac{K_1}{2}\mathrm{e}^{-\frac{K_2T}{2}}+\frac{K_1^2}{2K_2}\mathrm{e}^{-\frac{K_2T}{2}}-\frac{K_1^2}{4K_2}\mathrm{e}^{-K_2T}-\frac{K_1^2}{4K_2}\\
&= -\frac{K_1}{2}+\frac{K_1}{2}\mathrm{e}^{-\frac{K_2T}{2}}-\frac{K_1^2}{4K_2}\Big(1-2\mathrm{e}^{-\frac{K_2T}{2}}+\mathrm{e}^{-K_2T}\Big)\\
&= -\frac{K_1}{2}\Big(1-\mathrm{e}^{-\frac{K_2T}{2}}\Big)-\frac{K_1^2}{4K_2}\Big(1-\mathrm{e}^{-\frac{K_2T}{2}}\Big)^2.
\end{aligned}\tag{3.6}$$

We see that

$$\begin{cases}\Lambda'(T,T)>0 & if\ K_2<0,\\ \Lambda'(T,T)<0 & if\ K_2>0.\end{cases}\tag{3.7}$$

Now we look for $t\in[0,T]$ such that $\Lambda'(t,T)=0$. We have

$$\begin{aligned}
&\Lambda'(t,T)=0\\
\Leftrightarrow\quad &\Big(-\frac{K_1}{2}\mathrm{e}^{-\frac{K_2T}{2}}-\frac{K_1^2}{4K_2}\mathrm{e}^{-\frac{K_2T}{2}}\Big)\mathrm{e}^{K_2t}+\Big(\frac{K_1}{2}+\frac{K_1^2}{2K_2}-\frac{K_1^2}{4K_2}\mathrm{e}^{-\frac{K_2T}{2}}\Big)=0\\
\Leftrightarrow\quad &-\frac{K_1}{4}\mathrm{e}^{-\frac{K_2T}{2}}\Big(2+\frac{K_1}{K_2}\Big)\mathrm{e}^{K_2t}+\frac{K_1}{4}\Big(2+\frac{2K_1}{K_2}-\frac{K_1}{K_2}\mathrm{e}^{-\frac{K_2T}{2}}\Big)=0\\
\Leftrightarrow\quad &\mathrm{e}^{-\frac{K_2T}{2}}\Big(2+\frac{K_1}{K_2}\Big)\mathrm{e}^{K_2t}=\Big(2+\frac{2K_1}{K_2}-\frac{K_1}{K_2}\mathrm{e}^{-\frac{K_2T}{2}}\Big).
\end{aligned}\tag{3.8}$$

Therefore there exists at most one t such that $\Lambda'(t,T)=0$. For the case where $K_2<0$, if there exists $t_0\in(0,T)$ such that $\Lambda(t_0,T)<0$. Then by (3.5) and (3.7), the equation $\Lambda'(t,T)=0$ has at least two solutions, it is impossible. Therefore for $K_2<0$, $\Lambda'(t,T)\geq 0$. For $K_2>0$, we suppose t_0 such that $\Lambda'(t_0,T)=0$. Let $\beta=\frac{K_1}{K_2}$, then by (3.8)

$$\mathrm{e}^{K_2t_0}=\Big(1+\frac{\beta}{2+\beta}\Big(1-\mathrm{e}^{-\frac{K_2T}{2}}\Big)\Big)\mathrm{e}^{\frac{K_2T}{2}},$$

or $t_0\in(0,T)$ is such that

$$\mathrm{e}^{\frac{K_2t_0}{2}}=\sqrt{1+\frac{\beta}{2+\beta}\Big(1-\mathrm{e}^{-\frac{K_2T}{2}}\Big)}\mathrm{e}^{\frac{K_2T}{4}}.\tag{3.9}$$

The proof is completed. □

Proposition 3.3. *Let* $\beta = \dfrac{K_1}{K_2}$*, then (i) if* $K_2 > 0$*,*

$$\sup_{t\in[0,T]} \Lambda(t,T) = (1+\beta)^2 - \Big(\beta + \frac{\beta^2}{2}\Big)\sqrt{1 + \frac{\beta}{2+\beta}\Big(1 - \mathrm{e}^{-\frac{K_2 T}{2}}\Big)}\ \mathrm{e}^{-\frac{K_2 T}{4}}$$
$$- \frac{\Big(\beta + \beta^2 - \frac{\beta^2}{2}\mathrm{e}^{-\frac{K_2 T}{2}}\Big)}{\sqrt{1 + \frac{\beta}{2+\beta}\Big(1 - \mathrm{e}^{-\frac{K_2 T}{2}}\Big)}}\mathrm{e}^{-\frac{K_2 T}{4}}. \tag{3.10}$$

(ii) if $K_2 < 0$*,*

$$\sup_{t\in[0,T]} \Lambda(t,T) = \frac{1}{2} + \frac{1}{2}\Big(1 + \frac{K_1}{K_2}\Big[1 - \mathrm{e}^{-\frac{K_2 T}{2}}\Big]\Big)^2. \tag{3.11}$$

Proof. For $K_2 > 0$, we have

$$\begin{aligned}\Lambda(t_0,T) &= 1 + \beta\Big(1 - \mathrm{e}^{-\frac{K_2 T}{2}} \cdot \mathrm{e}^{\frac{K_2 t_0}{2}}\Big) + \beta\Big(1 - \mathrm{e}^{\frac{K_2 t_0}{2}}\Big)\\ &\quad + \beta^2\Big[\Big(1 - \mathrm{e}^{-\frac{K_2 t_0}{2}}\Big) + \frac{1}{2}\Big(\mathrm{e}^{-\frac{K_2 T}{2}} \cdot \mathrm{e}^{\frac{-K_2 t_0}{2}} - \mathrm{e}^{-\frac{K_2 T}{2}} \cdot \mathrm{e}^{\frac{K_2 t_0}{2}}\Big)\Big]\\ &= 1 + 2\beta + \beta^2 - \Big(\beta + \frac{\beta^2}{2}\Big)\mathrm{e}^{-\frac{K_2 T}{2}} \cdot \mathrm{e}^{\frac{K_2 t_0}{2}} - \Big(\beta + \beta^2 - \frac{\beta^2}{2}\mathrm{e}^{-\frac{K_2 T}{2}}\Big)\mathrm{e}^{-\frac{K_2 t_0}{2}}.\end{aligned}$$

Using (3.9) yields (3.10). For $K_2 < 0$, $\sup_{t\in[0,T]} \Lambda(t,T) = \Lambda(T,T)$, which gives (3.11). □

Combining (2.9) and (3.3), we get

Theorem 3.4. *Let* $C(T,K_1,K_2) = \sup\limits_{t\in[0,T]} \Lambda(t,T)$*; then it holds*

$$\mathbb{E}\Big(F^2 \log \frac{F^2}{\|F\|_{L^2}^2}\Big) \le 2C(T,K_1,K_2)\mathbb{E}\Big(\int_0^T |D_t F|^2 dt\Big) \tag{3.12}$$

for any cylindrical function F *on* $W_x^T(M)$*.*

It is well-konwn that the above logarithmic Sobolev inequality implies that the spectral gap of $\mathcal{L}_T^x$, denoted by $Spect(\mathcal{L}_T^x)$, has the following lower bound

$$Spect(\mathcal{L}_T^x) \ge \frac{1}{C(T,K_1,K_2)}.$$

Theorem 3.5. *Assume (3.1) holds, then (i) if* $K_2 > 0$*, we have*

$$Spect(\mathcal{L}_T^x)^{-1} \le \Big(1 + \frac{K_1}{K_2}\Big)^2 - \frac{K_1}{K_2}\sqrt{\Big(2 + \frac{K_1}{K_2}\Big)\Big(2 + 2\frac{K_1}{K_2} - \frac{K_1}{K_2}\mathrm{e}^{-\frac{K_2 T}{2}}\Big)}\ \mathrm{e}^{-\frac{K_2 T}{4}}; \tag{3.13}$$

(ii) if $K_2 < 0$*, we have*

$$Spect(\mathcal{L}_T^x)^{-1} \le \frac{1}{2} + \frac{1}{2}\Big(1 + \frac{K_1}{K_2}\Big[1 - \mathrm{e}^{-\frac{K_2 T}{2}}\Big]\Big)^2. \tag{3.14}$$

Proof. Using the elementary inequality: $A + B \ge 2\sqrt{AB}$ to the last two terms in (3.10) yields (3.13). Inequality (3.14) is obvious. □

It is quite interesting to remark that

Proposition 3.6. *Let* $\psi(T,K_1,K_2)$ *be the right hand side of* (3.13) *when* $K_2>0$ *and the right hand side of* (3.14) *for* $K_2<0$*, then*

$$\psi(T,K_1,K_2)\to 1+\frac{K_1T}{2}+\frac{K_1^2T^2}{8}\quad as\quad K_2\to 0. \tag{3.15}$$

Proof. It is easy to see that the right hand side of (3.14) tends to $1+\frac{K_1T}{2}+\frac{K_1^2T^2}{8}$ as $K_2\to 0$. For the right hand side of (3.13), we first remark that

$$(a)\qquad \frac{K_1}{K_2}e^{-\frac{K_2T}{4}}=\frac{K_1}{K_2}-\frac{K_1T}{4}+\frac{K_1K_2T^2}{32}+o(K_2).$$

Secondly

$$\begin{aligned}
&\Big(2+\frac{K_1}{K_2}\Big)\Big(2+2\frac{K_1}{K_2}-\frac{K_1}{K_2}\mathrm{e}^{-\frac{K_2T}{2}}\Big)\\
&=\Big(2+\frac{K_1}{K_2}\Big)\Big(2+\frac{K_1}{K_2}+\frac{K_1T}{2}-\frac{K_1K_2T^2}{8}+o(K_2)\Big)\\
&=\Big(2+\frac{K_1}{K_2}\Big)^2\Bigg(1+\frac{\frac{K_1T}{2}-\frac{K_1K_2T^2}{8}+o(K_2)}{2+\frac{K_1}{K_2}}\Bigg)
\end{aligned}$$

Therefore

$$\begin{aligned}
&\sqrt{\Big(2+\frac{K_1}{K_2}\Big)\Big(2+2\frac{K_1}{K_2}-\frac{K_1}{K_2}\mathrm{e}^{-\frac{K_2T}{2}}\Big)}\\
&=\Big(2+\frac{K_1}{K_2}\Big)\Bigg(1+\frac{1}{2}\frac{\frac{K_1T}{2}-\frac{K_1K_2T^2}{8}+o(K_2)}{2+\frac{K_1}{K_2}}-\frac{K_1K_2T^2}{32}+o(K_2^2)\Bigg)\\
&=\Big(2+\frac{K_1}{K_2}\Big)+\frac{K_1T}{4}-\frac{3K_1K_2T^2}{32}+o(K_2).
\end{aligned}$$

Combining this with (a), we get

$$\begin{aligned}
&\frac{K_1}{K_2}e^{-\frac{K_2T}{4}}\sqrt{\Big(2+\frac{K_1}{K_2}\Big)\Big(2+2\frac{K_1}{K_2}-\frac{K_1}{K_2}\mathrm{e}^{-\frac{K_2T}{2}}\Big)}\\
&=\Big(2+\frac{K_1}{K_2}\Big)\frac{K_1}{K_2}-\frac{K_1T}{2}-\frac{K_1^2T^2}{8}+o(K_2).
\end{aligned}$$

Then (3.15) follows from the right hand side of (3.13). □

Corollary 3.7. *Assume (3.1) holds.*
(1) If $K_1=K_2=K>0$*, then*

$$\psi(T,K,K)=4-\sqrt{3\Big(4-\mathrm{e}^{-\frac{KT}{2}}\Big)\mathrm{e}^{-\frac{KT}{4}}}\to 1\quad as\quad K\to 0.$$

(2) If $K_2=-K_1=-K$*, then*

$$\psi(T,K,-K)=\frac{1}{2}(1+e^{KT}).$$

Remark. Our results improve estimates obtained in [1].

4 Behaviour of $Spect(\mathcal{L}_T^x)$ as $T \to 0$

In this section, we consider the case where $Z = 0$. Then Condition (3.1) can be readed as

$$K_2 \,\mathrm{Id} \le \mathrm{ric} \le K_1 \,\mathrm{Id}, \quad \text{with } K_1 + K_2 \ge 0 \tag{4.1}$$

and SDE (1.7) is reduced to

$$du_t(w) = \sum_{i=1}^d H_i(u_u(w)) \circ dw_t^i, \quad u_0(w) = u_0 \in \pi^{-1}(x). \tag{4.2}$$

The path $\gamma_t(w) = \pi(u_t(w))$ is called Brownian motion path on M. Let $\rho(x, y)$ be the Riemannian distance. By [22, p. 199], there is $\varepsilon > 0$ such that

$$\sup_{t\in[0,T]} \mathbb{E}\Big(\exp\Big(\varepsilon \frac{\rho(x,\gamma_t)^2}{2t}\Big)\Big) < +\infty. \tag{4.3}$$

Assume that the curvature tensor satisfies the following growth condition

$$|||\Omega_u||| + \sum_{i=1}^d |||(L_{H_i}\Omega)_u||| \le C\left(1 + \rho(x, \pi(u))^2\right) \tag{4.4}$$

where L_{H_i} denotes the Lie derivative with respect to H_i.

Let $v \in \mathbb{H}$, consider the functional $F_T : W_x^T(M) \to \mathbb{R}$ defined by

$$F_T(\gamma(w)) = \int_0^T \langle \dot v(t), dw_t\rangle.$$

Let $h \in \mathbb{H}$; then by (2.3), we have (see also [15])

$$(D_h F_T)(\gamma(w)) = \int_0^T \langle \dot v(t), q(t,h) dw_t\rangle + \int_0^T \langle \dot v(t), \dot{\hat h}_t(w)\rangle\, dt. \tag{4.5}$$

Let $a \in \mathbb{R}^d$ and consider $v(t) = ta$ with $|a| = 1$ in (4.5), we have

$$(D_h F_T)(\gamma(w)) = -\int_0^T \langle q(t,h)a, dw_t\rangle + \int_0^T \langle a, \dot{\hat h}_t(w)\rangle\, dt. \tag{4.6}$$

Let $\{e_1, \cdots, e_d\}$ be an orthonormal basis of $\mathbb{R}^d$; define

$$C_i(w, t, \tau) = -\int_\tau^t \Omega_{u_s(w)}\big(e_i, \circ dw(s)\big)\, \mathbf{1}_{(\tau<t)}.$$

Then by Fubini theorem, the term $q(t,h)$ has the expression

$$q(t,h) = -\sum_{i=1}^d \int_0^T \dot h^i(\tau) C_i(w,t,\tau)\, d\tau.$$

According to (4.6), the gradient $D_\tau F_T$ has the following expression:

$$(D_\tau F_T)(\gamma(w)) = \sum_{i=1}^d \Big(\int_\tau^T \langle C_i(w,s,\tau)a, dw_s\rangle\Big) e_i + a + \frac{1}{2}\int_\tau^T \mathrm{ric}_Z(u_s)\, a\, ds. \tag{4.7}$$

We have

$$\mathrm{Var}(F_T) = \mathbb{E}(F_T^2) - \mathbb{E}(F_T)^2 = |a|^2 T = T. \tag{4.8}$$

Proposition 4.1. *Assume* (4.4). *Let*

$$\chi_T = \frac{\mathbb{E}\Big(\int_0^T |D_\tau F|^2\, d\tau\Big)}{\mathrm{Var}(F_T)}.$$

Then

$$\chi_T = 1 + \frac{T}{2}\langle \mathrm{ric}_Z(u_0)a, a\rangle + o(T) \quad \text{as } T \to 0 \tag{4.9}$$

where u_0 *is the initial point of* (4.2).

Proof. We have, using (4.7),

$$\begin{aligned}
|D_\tau F_T|^2 = &\sum_{i=1}^d \Big(\int_\tau^T \langle C_i(w,s,\tau)a, dw_s\rangle\Big)^2 + |a|^2 + \frac{1}{4}\Big|\int_\tau^T \mathrm{ric}(u_s)a\, ds\Big|^2 \\
&+ \Big\langle a, \int_\tau^T \mathrm{ric}(u_s)a\, ds\Big\rangle + 2\sum_{i=1}^d \int_\tau^T \langle C_i(w,s,\tau)a, dw_s\rangle\, a^i \\
&+ 2\int_0^d \int_\tau^T \langle C_i(w,s,\tau)a, dw_s\rangle \cdot \int_\tau^T \langle \mathrm{ric}(u_s)a, e_i\rangle\, ds.
\end{aligned}$$

Put respectively

$$\mathbb{E}\Big(\int_0^T |D_\tau F_T|^2\, d\tau\Big) = I_1(T) + I_2(T) + I_3(T) + I_4(T) + I_5(T) + I_6(T).$$

It is obvious that $I_2(T) = |a|^2 T = T$ and $I_5(T) = 0$. We have

$$I_1(T) = \sum_{i=1}^d \int_0^T \Big(\int_\tau^T \mathbb{E}(|C_i(w,s,\tau)a|^2)\, ds\Big)\, d\tau.$$

Now by growth condition (4.4) and (4.3), there is a constant $\delta > 0$ such that

$$\mathbb{E}(|C_i(w,s,\tau)a|^2) \le \delta\,(s-\tau). \tag{4.10}$$

So that $I_1(T) \le \delta T^3/6$. By condition (4.1), it is easy to see that $I_3(T) \le \frac{K_1^2 T^3}{12}$. It follows that $I_6(T) \le \frac{\sqrt{\delta}K_1}{6}T^3$. Now for $I_4(T)$, we have

$$\lim_{T\to 0} \frac{I_4(T)}{T^2} = \frac{1}{2}\langle \mathrm{ric}(u_0)a, a\rangle.$$

Combining these estimates together with (4.8), we get (4.9). □

Theorem 4.2. *Assume* (4.1) *and* (4.4). *Let* $K_2(x)$ *be the lower bound of* Ric_x. *Then as* $T \to 0$,

$$1 - \frac{K_1 T}{2} + o(T) \le \mathit{Spect}(\mathcal{L}_T^x) \le 1 + \frac{K_2(x)T}{2} + o(T). \tag{4.11}$$

Proof. For $K_2 > 0$, set $\beta = \frac{K_1}{K_2}$. As $T \to 0$, we have

$$\begin{aligned}
&\sqrt{(2+\beta)\Big(2+2\beta-\beta \mathrm{e}^{-\frac{K_2T}{2}}\Big)} = \sqrt{(2+\beta)^2\Big(1+\frac{\beta}{2+\beta}\Big(1-\mathrm{e}^{-\frac{K_2T}{2}}\Big)\Big)}\\
&= (2+\beta)\sqrt{1+\frac{\beta}{2+\beta}\frac{K_2T}{2}+o(T)}\\
&= (2+\beta)\Big(1+\frac{\beta}{2+\beta}\frac{K_2T}{4}+o(T)\Big).
\end{aligned}$$

So, for $K_2 > 0$, as $T \to 0$,

$$\begin{aligned}
&\beta\sqrt{(2+\beta)\Big(2+2\beta-\beta \mathrm{e}^{-\frac{K_2T}{2}}\Big)}\ \mathrm{e}^{-\frac{K_2T}{4}}\\
&= \beta(2+\beta)\Big(1+\frac{\beta}{2+\beta}\frac{K_2T}{4}+o(T)\Big)\Big(1-\frac{K_2}{4}T+o(T)\Big)\\
&= \beta(2+\beta)\Big[1+\frac{T}{4}\Big(\frac{K_1}{2+\beta}-K_2\Big)+o(T)\Big]\\
&= \beta(2+\beta)\Big[1-\frac{K_2T}{2(2+\beta)}+o(T)\Big].
\end{aligned}$$

By (3.13), we get

$$Spect(\mathcal{L}_T^x)^{-1} \le (1+\beta)^2-\beta(2+\beta)\Big[1-\frac{K_2T}{2(2+\beta)}+o(T)\Big] = 1+\frac{K_1T}{2}+o(T),$$

which implies that

$$Spect(\mathcal{L}_T^x) \ge 1-\frac{K_1T}{2}+o(T).$$

For $K_2 < 0$, by (3.14),

$$\begin{aligned}
Spect(\mathcal{L}_T^x)^{-1} &\le \frac{1}{2}+\frac{1}{2}\Big(1+K_1\frac{1-\mathrm{e}^{-\frac{K_2T}{2}}}{K_2}\Big)^2 = \frac{1}{2}+\frac{1}{2}\Big(1+\frac{K_1}{K_2}\Big(\frac{K_2T}{2}+o(T)\Big)\Big)^2\\
&= 1+\frac{K_1T}{2}+o(T),
\end{aligned}$$

which implies again

$$Spect(\mathcal{L}_T^x) \ge 1-\frac{K_1T}{2}+o(T).$$

Now in (4.9), taking the vector a such that $\mathrm{ric}(u_0)a = K_2(x)a$ yields

$$Spect(\mathcal{L}_T^x) \le 1+\frac{K_2(x)T}{2}+o(T).$$

The proof of (4.11) is completed. $\square$

Corollary 4.3. *Assume* (4.4). *In the case where* $\mathrm{Ric} = -K_1\mathrm{Id}$ *with* $K_1 \ge 0$, *we have*

$$\Big|Spect(\mathcal{L}_T^x)-1+\frac{K_1T}{2}\Big| = o(T) \quad \text{as } T \to 0.$$

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