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Abstract. Let X be a real-valued stationary Gaussian random field defined on $\mathbb{R}^d$ ($d \geq 1$), with almost every realization of class $\mathcal{C}^2$. This paper is concerned with the random variable giving the number of points in T (a compact set of $\mathbb{R}^d$) where the gradient X' takes a fixed value $v \in \mathbb{R}^d$, $N^{X'}(T,v) = \{t \in T : X'(t) = v\}$. More precisely, it deals with the finiteness of the variance of $N^{X'}(T,v)$, under some non-degeneracy hypothesis on X. For $d = 1$, the so-called "Geman condition" has been proved to be a sufficient condition for $N^{X'}(T,v)$ to admit a finite second moment. This condition on the fourth derivative $r^{(4)}$ of the covariance function of X does not depend on v and requires $t \mapsto \frac{r^{(4)}(0)-r^{(4)}(t)}{t}$ to be integrable in a neighbourhood of zero. We prove that for $d \geq 1$, a generalization of the Geman condition remains a sufficient condition for $N^{X'}(T,v)$ to admit a second moment. No assumption of isotropy is required.

Keywords: Gaussian fields; stationary points; Geman condition; crossings
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Introduction

Let d be a positive integer and let $X : \Omega \times \mathbb{R}^d \to \mathbb{R}$ be a stationary Gaussian random field. We assume that almost every realization is of class $\mathcal{C}^2$ on $\mathbb{R}^d$. Let T be a compact set in $\mathbb{R}^d$ such that the boundary of T has a finite $(d-1)$-dimensional Lebesgue measure. For any $v \in \mathbb{R}^d$, we consider the number $N^{X'}(T,v)$ of points in T where the gradient of X, denoted by X', reaches the value v:

$$N^{X'}(T,v) = \#\{t \in T : X'(t) = v\}.$$

For $v = 0$, it is nothing but the number of stationary points of X in T. In this paper, we establish a sufficient condition on the covariance function r of the random field X in order that $N^{X'}(T,v)$ admits a finite variance.

The existence of the second moment of $N^{X'}(T,v)$ has been studied since the late 60s, first in dimension one and for a level equal to the mean, *i.e.* $v = 0$. Cramér and Leadbetter were the first to propose in [6] a sufficient condition on the covariance function r in order that $N^{X'}(T,0)$ belongs to $L^2(\Omega)$. If X satisfies some non-degeneracy assumptions, this simple condition requires that the fourth derivative $r^{(4)}$ satisfies

$$\exists \delta > 0, \quad \int_0^\delta \frac{r^{(4)}(0) - r^{(4)}(t)}{t}\, dt \ < \ +\infty.$$

It is known as the Geman condition for Geman proved some years after in [10] that it was not only sufficient but also necessary. The issue of the finiteness of the higher moments of $N^{X'}(T,0)$ has also been discussed in many papers (see [5, 7, 12] for instance and references therein). Kratz and León generalized Geman's result in

[11] to the number of crossings of any level $v \in \mathbb{R}$ and also to the number of a curve crossings.

Concerning the problem in higher dimension, it has been an open question for a long time. Elizarov gave in [8] a sufficient condition for $N^{X'}(T,0)$ to be in $L^2(\Omega)$. Even though his condition is weaker than ours, his proof is short and elliptical and it only concerns the number of stationary points. Under the additional hypothesis that X is isotropic and of class $\mathcal{C}^3$, Estrade and León proved in [9] that for any $v \in \mathbb{R}^d$, $N^{X'}(T,v)$ admits a finite second moment.

Beside the specific works already mentioned, we will intensively refer in the present paper to [2] and [4] as recent and complete books dedicated to the geometry of random fields.

The paper is organized as follows. In Section 1, we introduce our notations and assumptions. Our proof begins with the use of Rice formulas in Section 2.1 to give an expression of $N^{X'}(T,v)$ in an integral form. It allows us to restrict the problem to the one of the integrability in a neighbourhood of zero in $\mathbb{R}^d$ of the function

$$t \mapsto \mathbb{E}[\,(\det X''(0))^2 \,/\, X'(0) = X'(t) = v\,]\, \|t\|^{-d}\,.$$

We are able to bound this function and, thanks to a regression method implemented in Section 2.2, to study the asymptotic properties of the bound around zero. Section 3 is devoted to the main result of this paper, namely Theorem 3.1. It gives an extension of Geman condition in dimension $d > 1$ that is sufficient to establish that $N^{X'}(T,v)$ is square integrable for any v.

1. Notations and derivatives

We deal with a stationary Gaussian field $X = \{X(t),\, t \in \mathbb{R}^d\}$ and we denote by r its covariance function $t \mapsto \mathrm{Cov}(X(0), X(t))$. We assume that almost every realization of X is of class $\mathcal{C}^2$ on $\mathbb{R}^d$. That implies that r is of class $\mathcal{C}^4$ on $\mathbb{R}^d$.

We fix an othonormal basis of $\mathbb{R}^d$, according to the canonical scalar product that we denote by $\langle \cdot\,, \cdot \rangle$. We consider the partial derivatives of X and r computed in this basis. We write $(X'_i)_{1\le i\le d}$ and $(X''_{i,j})_{1\le i,j\le d}$ the partial derivatives of X of first and second order, respectively, and r'_i, $r''_{i,j}$, $r^{(3)}_{i,j,m}$ and $r^{(4)}_{i,j,m,n}$ the partial derivatives of r, from order one to four, respectively. We refer to the gradient of X at t as $X'(t)$ and to the Hessian matrix of X at t as $X''(t)$. Similarly, we write $r''(t)$ the Hessian of r at t. We will sometimes denote by $r^{(3)}_{i,j}(t)$ the vector $(r^{(3)}_{i,j,m}(t))_{1\le m\le d}$ and by $r^{(4)}_{i,j}(t)$ the matrix $(r^{(4)}_{i,j,m,n}(t))_{1\le m,n\le d}$. We also use the same notation for $t \in \mathbb{R}^d$ and the column vector containing its coordinates.

In every space $\mathbb{R}^m$ (m is any positive integer), we denote by $\|\cdot\|$ the norm associated to the canonical scalar product. We use the standard notations $o(\cdot)$ and $O(\cdot)$ to describe the behaviour of some functions in a neighbourhood of zero.

In this paper, we will make extensive use of the relationships between the partial derivatives of r and the covariances between the partial derivatives of X. We recall them here. For $s,t \in \mathbb{R}^d$ and for $1 \le i,j,m,n \le d$, the following relations hold:

$$\begin{array}{ll}
\mathrm{Cov}(X(s), X(t)) = r(s-t) & \mathrm{Cov}(X'_i(s), X(t)) = r'_i(s-t) \\
\mathrm{Cov}(X'_i(s), X'_j(t)) = -r''_{i,j}(s-t) & \mathrm{Cov}(X''_{i,j}(s), X(t)) = r''_{i,j}(s-t) \\
\mathrm{Cov}(X''_{i,j}(s), X'_m(t)) = -r^{(3)}_{i,j,m}(s-t) & \mathrm{Cov}(X''_{i,j}(s), X''_{m,n}(t)) = r^{(4)}_{i,j,m,n}(s-t).
\end{array}$$

We will need the assumption that for any $t \in \mathbb{R}^d \backslash \{0\}$, the vector

$$(X'(0), X'(t), (X''_{i,j}(0))_{1\le i\le j\le d}, (X''_{i,j}(t))_{1\le i\le j\le d})$$

is not degenerate. As a consequence, $\mathrm{Var}[X(t)] = r(0) \neq 0$ and so we may assume that $r(0) = 1$. As another consequence, the covariance matrix of $X'(0)$ is not degenerate, which allows us to assume that $-r''(0) = I_d$ or, equivalently, that the first-order derivatives of X are uncorrelated and of unit variance. This assumption is taken from the proof of Lemma 11.7.1 in [2]. We explain it here in a few words. The covariance matrix of $X'(0)$ is $-r''(0)$. A square root Q of $(-r''(0))^{-1}$ will satisfy $-Q\, r''(0)\, Q = I_d$. We now define a new random field X^Q on $\mathbb{R}^d$ by $X^Q(t) = X(Q\, t)$. It is not hard to see that X^Q is still stationary, with unit variance, and that the covariance matrix of $\left(X^Q\right)'(0)$ is I_d. Note that this does not imply that X^Q is isotropic. From now on, we will abandon the notation X^Q, although we will still assume that $-r''(0) = I_d$.

We gather all the assumptions made on X in one assumption referred to as **(H)**:

$$\textbf{(H)} \begin{cases} \text{almost every realization of } X \text{ is of class } \mathcal{C}^2, \\ \forall t \neq 0,\ \mathrm{Cov}\left((X'(0), X'(t), (X''_{i,j}(0))_{1\le i\le j\le d}, (X''_{i,j}(t))_{1\le i\le j\le d})\right) \text{ is of full rank,} \\ r(0) = 1 \quad \text{and} \quad -r''(0) = I_d. \end{cases}$$

Note that the major assumptions in condition **(H)** are the first two ones. The last assumption has been added to make the intermediate proofs and computations easier, but the main result of our paper remains true if we remove it.

With these assumptions in mind, we are able to write the next Taylor formulas around 0 for the covariance function r and its derivatives:

$$\begin{cases} r(t) = 1 - \dfrac{1}{2} \displaystyle\sum_{1\le i\le d} t_i^2 + \dfrac{1}{4!} \displaystyle\sum_{i,j,m,n} r^{(4)}_{i,j,m,n}(0)\, t_i\, t_j\, t_m\, t_n + o(\|t\|^4) \\ r''(t) = -I_d + \dfrac{1}{2}\Theta(t) + o(\|t\|^2) \\ r^{(3)}_{i,j}(t) = r^{(4)}_{i,j}(0)t + o(\|t\|), \quad \text{for all } \ 1 \le i,j \le d \\ r^{(4)}_{i,j}(t) = r^{(4)}_{i,j}(0) + o(1), \ \text{for all } \ 1 \le i,j \le d\,, \end{cases}$$

where the $d\times d$ matrix $\Theta(t)$ is defined by $\Theta(t)_{m,n} = \langle r^{(4)}_{m,n}(0)t\,, t\rangle = \sum_{1\le i,j\le d} r^{(4)}_{i,j,m,n}(0) t_i t_j$. We note that, for any $t \neq 0$, $\Theta(t)$ is inversible. Indeed, since $\Theta(t)$ is the covariance matrix of vector $X''(0)\, t$, if it was not inversible, $X''(0)\, t$ would be a degenerate Gaussian vector and so there would exist a linear dependence between the coordinates of $X''(0)$. That would be inconsistent with assumption **(H)**. Hence, in what follows, we denote by $\Delta(t)$ the inverse matrix of $\Theta(t)$ for $t \neq 0$. Besides, we also remark that $t \mapsto \Theta(t)$ and $t \mapsto \Delta(t)$ are homogeneous functions of respective degrees 2 and -2.

We fix a compact set T in $\mathbb{R}^d$, such that the boundary of T has a finite $(d-1)$-dimensional Lebesgue measure. For instance, T can be a bounded rectangle in $\mathbb{R}^d$.

2. Preliminary results

2.1. **Rice formula.** For any $v \in \mathbb{R}^d$, $N^{X'}(T,v)$ is the number of roots in T of the vectorial random field $X'-v$. The well-known Rice formula ([4] Theorem 6.2 or [2] Corollary 11.2.2) not only gives a closed formula for the expectation of $N^{X'}(T,v)$ but also states that it is finite in our context. So the variance of $N^{X'}(T,v)$ is finite if and only if its second-order factorial moment is finite. Another Rice formula gives the second factorial moment of $N^{X'}(T,v)$ under hypothesis **(H)** ([4] Theorem 6.3 or [2] Corollary 11.5.2):

$$\begin{aligned}&\mathbb{E}[N^{X'}(T,v)(N^{X'}(T,v)-1)]\\&=\int_{T\times T}\mathbb{E}[|\det X''(s)\ \det X''(t)|\,/\,X'(s)=X'(t)=v]\,p_{s,t}(v,v)\,ds\,dt\,,\end{aligned}$$

where $p_{s,t}$ denotes the probability density function of the Gaussian vector $(X'(s),X'(t))$. This formula holds whether both sides are finite or not. We introduce

$$F(v,t)=\mathbb{E}[|\det X''(0)\ \det X''(t)|\,/\,X'(0)=X'(t)=v]\ ;\ v,t\in\mathbb{R}^d\,,$$

and we use the stationarity of X to transform the double integral in the Rice formula into a simple integral:

$$\mathbb{E}[N^{X'}(T,v)(N^{X'}(T,v)-1)]=\int_{T_0}|T\cap(T-t)|\,F(v,t)\,p_{0,t}(v,v)\,dt,$$

where $|T\cap(T-t)|$ is the Lebesgue measure of $T\cap(T-t)$ and $T_0=\left\{t-t',\ (t,t')\in T^2\right\}$. This formula allows us to give a simple criteria for $N^{X'}(T,v)$ to be square integrable.

Notation. Let $u:\mathbb{R}^d\to\mathbb{R}^m$. We write $u\in L^1(\mathcal{V}_0,\|t\|^{-d}\,dt)$ if there exists a positive constant δ such that $\int_{B(0,\delta)}\frac{\|u(t)\|}{\|t\|^d}\,dt<+\infty$.

Lemma 2.1. *Assume that X fulfills condition **(H)**. For any $v\in\mathbb{R}^d$, we introduce*

$$G(v,\cdot):t\in\mathbb{R}^d\longmapsto G(v,t)=\mathbb{E}[(\det X''(0))^2\,/\,X'(0)=X'(t)=v]\,.$$

Then

$$G(v,\cdot)\in L^1(\mathcal{V}_0,\|t\|^{-d}dt)\Rightarrow N^{X'}(T,v)\in L^2(\Omega).$$

Proof. Note that the function $t\mapsto|T\cap(T-t)|\,F(v,t)\,p_{0,t}(v,v)$ is continuous on $\mathbb{R}^d\backslash\{0\}$, because the random field X is Gaussian. So it is integrable in every bounded domain that does not include a neighbourhood of zero.
We are now concerned with its behaviour in a neighbourhood of zero. We first remark that, as t tends to 0, the term $|T\cap(T-t)|$ is equivalent to $|T|$. Next, we use Cauchy-Schwarz inequality and stationarity to write

$$F(v,t)\le(G(v,t)\,G(v,-t))^{1/2}\,.$$

Let us now study $t\mapsto p_{0,t}(v,v)$ as t tends to 0. We know that

$$p_{0,t}(v,v)\le p_{0,t}(0,0)=(2\pi)^{-d/2}\,(\det\Gamma(t))^{-1/2}\,,$$

where $\Gamma(t)$ is the covariance matrix of the $2d$-dimensional Gaussian vector $(X'(0), X'(t))$. It is given blockwise by $\Gamma(t) = \begin{pmatrix} I_d & -r''(t) \\ -r''(t) & I_d \end{pmatrix}$ and so

$$\begin{aligned}
\det \Gamma(t) &= \det(I_d - r''(t)^2) = \det\left(\Theta(t) + o(\|t\|^2)\right) \\
&= \det \Theta(t) \, \det\left(I_d + o(\|t\|^2)\,\Delta(t)\right) \\
&= \|t\|^{2d} \det \Theta(\frac{t}{\|t\|}) \, \det\left(I_d + o(1)\,\Delta(\frac{t}{\|t\|})\right),
\end{aligned}$$

where we have used the homogeneity properties of Θ and Δ. Since $\min_{u\in\mathbb{S}^{d-1}} \det\Theta(u)$ is strictly positive and $t \mapsto \Delta(\frac{t}{\|t\|})$ is bounded, there exists $c > 0$ such that $\det\Gamma(t) \geq c\,\|t\|^{2d}$ for t in a neighbourhood of zero. Hence, for some positive constant C, $p_{0,t}(v,v) \leq C\,\|t\|^{-d}$.

Consequently, if $G(v,\cdot) \in L^1(\mathcal{V}_0, \|t\|^{-d}\,dt)$ then $t \mapsto |T \cap (T-t)|\, F(v,t)\, p_{0,t}(v,v)$ is bounded by a function that is integrable in a neighbourhood of 0, thanks to Cauchy-Schwarz inequality. That concludes the proof of the lemma. □

Our aim is now to study the behavior of $G(v,t)$ as $t \to 0$, for a fixed $v \in \mathbb{R}^d$. Precisely, we will provide a sufficient condition for $G(v,\cdot)$ to belong to $L^1(\mathcal{V}_0, \|t\|^{-d}dt)$.

2.2. **Regression.** In order to get an estimate for $G(v,t)$, we compute the conditional law of $X''(0)$ with respect to the event $\{X'(0) = X'(t) = v\}$. Let $K = d(d+1)/2$. We consider the symmetric matrix $X''(0)$ as a K-dimensional Gaussian column vector by putting the coefficients of its upper triangular part in a vector that we write $\nabla^2 X(0)$. So the indices $1 \leq \mathbf{k} \leq K$ of this vector have to be seen as double indices ($\mathbf{k} = (i,j)$ with $1 \leq i \leq j \leq d$). For $t \neq 0$, we write the following K-dimensional regression system:

$$\nabla^2 X(0) = A(t)\,X'(0) + B(t)\,X'(t) + Z(t)\,, \tag{1}$$

where $A(t)$ and $B(t)$ are matrices of size $K \times d$ and $Z(t)$ is a K-dimensional centered Gaussian vector, independent from $X'(0)$ and $X'(t)$. Hence, conditioned on $\{X'(0) = X'(t) = v\}$, $\nabla^2 X(0)$ is a Gaussian vector with mean $(A(t) + B(t))v$ and covariance matrix $\Gamma^Z(t)$. Next proposition is simply the result of a Gaussian computation, formulated according to our future needs.

Proposition 2.2. *If X fulfills condition **(H)**, then the regression coefficients of system (1) are given by*

$$A(t) = r^{(3)}(t)\,N_2(t) \quad \textit{and} \quad B(t) = r^{(3)}(t)\,N_1(t)\,, \tag{2}$$

where $r^{(3)}(t)$ has to be considered as a $K \times d$ matrix and $N_1(t)$ and $N_2(t)$ are two $d \times d$ matrices defined on $\mathbb{R}^d \backslash \{0\}$ by

$$N_1(t) = (I_d - (r''(t))^2)^{-1} \quad \textit{and} \quad N_2(t) = r''(t)\,(I_d - (r''(t))^2)^{-1}. \tag{3}$$

Besides, the covariance matrix $\Gamma^Z(t)$ of the K-dimensional Gaussian vector $Z(t)$ is such that for any $1 \leq \mathbf{k}, \mathbf{l} \leq K$ and for $t \in \mathbb{R}^d \backslash \{0\}$,

$$\Gamma^Z(t)_{\mathbf{k},\mathbf{l}} = \operatorname{Cov}(Z(t)_{\mathbf{k}}, Z(t)_{\mathbf{l}}) = r^{(4)}_{\mathbf{k},\mathbf{l}}(0) - \langle r^{(3)}_{\mathbf{k}}(t)\,, N_1(t)\, r^{(3)}_{\mathbf{l}}(t)\rangle\,. \tag{4}$$

Proof. We denote by $\mathbf{X_1}$ the vector $\nabla^2 X(0)$ of size K and by $\mathbf{X_2}$ the vector $(X'(0), X'(t))$ of size $2d$. We write C_1 the $K \times K$ covariance matrix of $\mathbf{X_1}$, C_2 the $2d \times 2d$ covariance matrix of $\mathbf{X_2}$ and C_{12} the $K \times 2d$ matrix of the covariances between the coordinates of $\mathbf{X_1}$ and these of $\mathbf{X_2}$. Then, let us recall that the conditional distribution of $\mathbf{X}_1$ with respect to $\mathbf{X_2}$ (that are both centered) is Gaussian, with mean vector $C_{12}C_2^{-1}\mathbf{X_2}$ and covariance matrix $C_1 - C_{12}C_2^{-1}C_{12}^T$.

Thanks to the relations recalled in Section 1, we have

$$C_1 = \left(r^{(4)}(0)\right), \; C_2 = \begin{pmatrix} I_d & -r''(t) \\ -r''(t) & I_d \end{pmatrix}, \; C_{12} = \begin{pmatrix} O_{K,d} & r^{(3)}(t) \end{pmatrix},$$

where $r^{(4)}(0)$ stands for the $K \times K$ matrix $(r^{(4)}_{\mathbf{k},\mathbf{l}}(0))_{1\le \mathbf{k},\mathbf{l}\le K}$, $O_{K,d}$ for the $K \times d$ zero matrix and, for any $t \in \mathbb{R}^d$, $r^{(3)}(t)$ stands for the $K \times d$ matrix $(r^{(3)}_{\mathbf{k},i}(t))_{1\le \mathbf{k}\le K,\, 1\le i\le d}$.

Let us note that C_2, which is the covariance matrix of $(X'(0), X'(t))$, is not degenerate for $t \neq 0$ because of hypothesis **(H)**. We note $N(t)$ its inverse. It is not hard to find that $N(t) = \begin{pmatrix} N_1(t) & N_2(t) \\ N_2(t) & N_1(t) \end{pmatrix}$ where $N_1(t)$ and $N_2(t)$ are two square matrices of dimensions $d \times d$. To show (3), we just have to solve the system

$$\begin{cases} N_1(t) - r''(t)\, N_2(t) = I_d \\ -r''(t)\, N_1(t) + N_2(t) = 0. \end{cases}$$

Computing the conditional mean of $\mathbf{X_1}$ with respect to $\mathbf{X_2}$, we get

$$C_{12}C_2^{-1}\mathbf{X_2} = r^{(3)}(t)\, N_2(t)\, X'(0) + r^{(3)}(t)\, N_1(t)\, X'(t),$$

and thus we deduce the regression coefficients as announced in (2). Moreover, the covariance matrix of the conditional distribution of $\mathbf{X}_1$ with respect to $\mathbf{X_2}$ is given by

$$C_1 - C_{12}C_2^{-1}C_{12}^T = r^{(4)}(0) - r^{(3)}(t)\, N_1(t)\, r^{(3)}(t)^T .$$

Its coefficients are exactly those written in formula (4). That concludes the proof. □

3. Sufficient Geman condition

We now state our main result. Assumption **(H)** is still in force and we introduce a new condition:

$$\textbf{(G)} \begin{cases} \text{there exists } \delta > 0 \text{ such that } \displaystyle\int_{B(0,\delta)} \frac{\|r^{(4)}(0) - r^{(4)}(t)\|}{\|t\|^d}\, dt < +\infty . \end{cases}$$

Condition **(G)** is weaker than X almost surely of class $\mathcal{C}^3$, since in that case, $r^{(4)}(0) - r^{(4)}(t) = o(\|t\|)$ as t tends to zero. It is a generalization of Geman condition known in dimension $d = 1$. In this particular case, it has been proved to be a sufficient and necessary condition to have $N^{X'}(T,v) \in L^2(\Omega)$ for any $v \in \mathbb{R}$ (see [11]).

It turns out that our condition **(G)** remains a sufficient condition in dimension $d > 1$ for $N^{X'}(T,v)$ to be in $L^2(\Omega)$.

Theorem 3.1. *If X fulfills conditions **(H)** and **(G)**, then*

$$\text{for any } v \in \mathbb{R}^d, \; N^{X'}(T,v) \in L^2(\Omega) .$$

Proof of Theorem 3.1. We will proceed in several steps.

First step: study of function G. Recall that G has been introduced in Lemma 2.1.

Lemma 3.2. *Suppose that X fulfills condition **(H)** and let $\mathcal{V} \subset \mathbb{R}^d$ be a compact set. Then*

(i) for any $v \in \mathcal{V}$, $G(v,t) = G(0,t) + o(\|t\|)$,

(ii) there exists a homogeneous polynomial $Q_{(d)}$ of degree d, which does not depend on X, such that $G(0,t) = Q_{(d)}(\Gamma^Z(t))$, where $Q_{(d)}(\Gamma^Z(t))$ is the evaluation of the polynom $Q_{(d)}$ at the coefficients of matrix $\Gamma^Z(t)$.

Proof.
(i) We use the natural identification between symmetric $d\times d$ matrices and vectors in $\mathbb{R}^K$, where $K = d(d+1)/2$, to define $\tilde{\det}(y)$ as the determinant of the $d\times d$ symmetric matrix whose upper triangular part contains the coordinates of $y \in \mathbb{R}^K$. It is a degree d homogeneous polynomial function of K variables. With this notation, and using the regression system (1), we get for $v \in \mathbb{R}^d$, $t \in \mathbb{R}^d\backslash\{0\}$,

$$G(v,t) \quad = \quad \mathbb{E}\left[\tilde{\det}(S(t)v + Z(t))^2\right] ,$$

where $S(t)$ stands for $A(t) + B(t)$. Thanks to formula (3),

$$S(t) = r^{(3)}(t)\,(N_2(t) + N_1(t)) = r^{(3)}(t)\,(I_d - r"(t))^{-1} ,$$

and since $I_d - r"(t) \to 2I_d$ and $r^{(3)}(t) = O(\|t\|)$ as $t \to 0$, we get the following asymptotics:

$$S(t) = A(t) + B(t) = O(\|t\|) \text{ as } t \to 0. \tag{5}$$

Let us come back to the computation of G. By developping the square of the determinant and bringing together the terms according to the powers of the coordinates $(S(t)v)_{\mathbf{k}}$ of the K-dimensional vector $S(t)v$, we get

$$\begin{aligned}\tilde{\det}\,(S(t)v + Z(t))^2 \quad &= \quad \tilde{\det}(Z(t))^2 + \sum_{1\le \mathbf{k}\le K} (S(t)v)_{\mathbf{k}}\, Q^{\mathbf{k}}_{(2d-1)}(Z(t)) \\ &+ \quad \sum_{1\le \mathbf{k},\mathbf{l}\le K} (S(t)v)_{\mathbf{k}}\,(S(t)v)_{\mathbf{l}}\, Q^{\mathbf{kl}}_{(2d-2)}(S(t)v + Z(t))\,,\end{aligned}$$

where the $Q^{\mathbf{k}}_{(2d-1)}$'s and the $Q^{\mathbf{kl}}_{(2d-2)}$'s are multivariate polynomial functions of respective degrees $2d-1$ and $2d-2$. Note that $\mathbb{E}\left[Q^{\mathbf{k}}_{(2d-1)}(Z(t))\right] = 0$ since $Z(t)$ is a centered Gaussian vector and $Q^{\mathbf{k}}_{(2d-1)}$ has an odd degree. Then, by taking the expectation, applying (5) and the fact that $\Gamma^Z(t)$ is bounded for t in any compact set, we obtain that, uniformly with respect to $v \in \mathcal{V}$,

$$G(v,t) \quad = \quad \mathbb{E}[\tilde{\det}(Z(t))^2] + o(\|t\|) \text{ as } t \to 0. \tag{6}$$

Recall that $G(0,t) = \mathbb{E}[\tilde{\det}(Z(t))^2]$, hence point (ii) is proved.

(ii) We now compute $\mathbb{E}[\tilde{\det}(Z(t))^2]$ by applying Wick's formula. Actually, let us consider for a while a K-dimensional centered Gaussian vector Y and let us compute $\mathbb{E}[\tilde{\det}(Y)^2]$. This quantity is equal to an alternate sum of terms with the following shape: $\mathbb{E}[Y_{i_1}\cdots Y_{i_{2d}}]$ where $i_1,\ldots,i_{2d}$ belong to $\{1,\ldots,K\}$. Wick's formula says that the expectation of the product of an even number, say $2d$, of K centered Gaussian variables can be written as a homogeneous polynomial function of degree d evaluated at the covariances of the K Gaussian variables. Hence, there exists a degree d homogeneous polynomial function $Q_{(d)}$ such that

$$\mathbb{E}[\tilde{\det}(Y)^2] = Q_{(d)}(\Gamma^Y)\,, \tag{7}$$

where Γ^Y is the covariance matrix of Y. Taking $Y = Z(t)$, we deduce from (6) that $G(v,t) = Q_{(d)}(\Gamma^Z(t)) + o(\|t\|)$. Lemma 3.2 is then proved. □

Second step: an auxiliary function. This step is dedicated to the properties of a function that will turn out to be, to some extent, close to $\Gamma^Z(t)$, as t tends to zero. Let us recall that the expression of $\Gamma^Z(t)$ is given by formula (4).

We introduce $\gamma(t) = (\gamma(t)_{\mathbf{k},\mathbf{l}})_{1\le \mathbf{k},\mathbf{l}\le K}$ defined for $t \neq 0$ by

$$\begin{aligned} \gamma(t)_{\mathbf{k},\mathbf{l}} &= r^{(4)}_{\mathbf{k},\mathbf{l}}(0) - \sum_{1\le i,j,m,n\le d} r^{(4)}_{\mathbf{k},i,m}(0) r^{(4)}_{\mathbf{l},j,n}(0)\Delta(t)_{m,n} t_i t_j \\ &= r^{(4)}_{\mathbf{k},\mathbf{l}}(0) - \langle r^{(4)}_{\mathbf{k}}(0)\, t\,, \Delta(t) r^{(4)}_{\mathbf{l}}(0)\, t\rangle, \end{aligned} \tag{8}$$

$\Delta(t)$ being the inverse matrix of $\Theta(t)$ introduced in Section 1. Function γ only depends on r through its fourth-order derivatives at zero. Clearly, it is homogeneous of degree zero: for any t in $\mathbb{R}^d\backslash\{0\}$, $\gamma(t) = \gamma\left(\frac{t}{\|t\|}\right)$.

Remark 3.3. *For any $t \neq 0$, $\gamma(t)$ is the covariance matrix of $\nabla^2 X(0)$ conditioned on $\{X'(0) = X''(0)t = 0\}$.*

Proof of Remark 3.3. The conditional covariance matrix can be computed thanks to the formula recalled in the proof of Proposition 2.2. The covariance matrix of vector $\nabla^2 X(0)$ is the $K \times K$ matrix $C_1 = (r^{(4)}(0))$. The covariance matrix of vector $(X'(0), X''(0)t)$ is the $2d \times 2d$ matrix $C_2 = \begin{pmatrix} I_d & 0 \\ 0 & \Theta(t) \end{pmatrix}$ and the matrix of the covariances between the coordinates of vector $\nabla^2 X(0)$ and these of $(X'(0), X''(0)t)$ is the $K \times 2d$ matrix $C_{12} = \begin{pmatrix} O_{K,d} & (r^{(4)}_{\mathbf{k},i}(0)t)_{\substack{1\le \mathbf{k}\le K \\ 1\le i\le d}} \end{pmatrix}$, where $r^{(4)}_{\mathbf{k},i}(0)$ stands for the d-dimensional line vector $(r^{(4)}_{\mathbf{k},i,j}(0))_{1\le j\le d}$ (i^{th} line of matrix $r^{(4)}_{\mathbf{k}}(0)$). Hence, the covariance matrix of $\nabla^2 X(0)\,/X'(0) = X''(0)t = 0$ is the $K\times K$ matrix $C_1 - C_{12}C_2^{-1}C_{12}^T$. Its $(\mathbf{k},\mathbf{l})$-coefficient is exactly $\gamma(t)_{\mathbf{k},\mathbf{l}}$. □

We now state a property of the auxiliary function γ that is interesting for its own.

Proposition 3.4. *If X satisfies condition **(H)**, then $\forall t \in \mathbb{R}^d\backslash\{0\},\ Q_{(d)}(\gamma(t)) = 0$.*

Proof of Proposition 3.4. We first check the result in the particular case of dimension one. For $d = 1$, $K = 1$ and $Q_{(1)}$ is a one variable polynomial such that, if Y is a Gaussian centered random variable, $Q_{(1)}(\Gamma^Y) = \mathbb{E}\left[\tilde{\det}(Y)^2\right] = \mathbb{E}[Y^2] = \mathrm{Var}[Y] = \Gamma^Y$. Hence, for any $x \in \mathbb{R}$, $Q_{(1)}(x) = x$. Moreover, according to the definition of γ (see (8)), for $t \neq 0$, $\gamma(t) = r^{(4)}(0) - \frac{(r^{(4)}(0)t)^2}{r^{(4)}(0)t^2} = 0$.

By computing explicitely the polynomial $Q_{(2)}$ and the function $\gamma(t)$, we give in the Appendix an alternative proof of Proposition 3.4 in the case $d = 2$.

We now give a general proof. According to Remark 3.3 and to the definition of $Q_{(d)}$ prescribed in (7), we have

$$Q_{(d)}(\gamma(t)) = \mathbb{E}\left[\det(X''(0)^2) \,/\, X'(0) = X''(0)t = 0\right].$$

Besides, one can check the following result that we read in [3]. *Let M be a $d \times d$ symmetric positive matrix and let $(v_i)_{1\leq i\leq d}$ be an orthonormal basis of $\mathbb{R}^d$. Then, denoting by $\widetilde{M}$ the $(d-1)\times(d-1)$ matrix $(\langle Mv_i\,,v_j\rangle)_{2\leq i,j\leq d}$, the following inequality holds:*

$$\det(M) \leq \langle Mv_1\,, v_1\rangle \det(\widetilde{M}).$$

We apply this result with $M = X''(0)^2$, $v_1 = \frac{t}{\|t\|}$, taking for $(v_i)_{2\leq i\leq d}$ any vectors satisfying the required hypothesis. As a result,

$$\det\left(X''(0)^2\right) \leq \left\langle X''(0)^2\frac{t}{\|t\|}\,, \frac{t}{\|t\|}\right\rangle \det(\widetilde{M}) = \|t\|^{-2}\,\langle X''(0)t\,, X''(0)t\rangle \det(\widetilde{M}).$$

So, applying the conditional expectation with respect to the event $\{X'(0) = X''(0)t = 0\}$, we get $Q_{(d)}(\gamma(t)) \leq 0$. That concludes the proof since $Q_{(d)}(\gamma(t))$ only takes non negative values. □

Third step: a comparison between $\Gamma^Z(t)$ and $\gamma(t)$. We introduce the following functions defined on $\mathbb{R}^d$,

$$\varepsilon\ :\ t \mapsto r^{(4)}(0) - r^{(4)}(t), \quad \bar{\varepsilon}\ :\ t \mapsto \int_0^1 \varepsilon(ut)\,du, \quad \hat{\varepsilon}\ :\ t \mapsto \int_0^1 \varepsilon(ut)(1-u)\,du.$$

They all take values in $\mathbb{R}^{d^4}$ and are symmetric functions with respect to the indices $(i,j,m,n) \in [\![1,d]\!]^4$. Since r is $\mathcal{C}^4$, ε is continuous and $\varepsilon(t) = o(1)$ as t tends to 0. The same holds for $\bar{\varepsilon}(t)$ and $\hat{\varepsilon}(t)$.

Lemma 3.5. *If X satisfies condition **(H)**, then there exists a neighbourhood $\mathcal{W}$ of zero in $\mathbb{R}^d$ and a positive constant c such that, for any $t \in \mathcal{W}\backslash\{0\}$,*

$$\left\|\Gamma^Z(t) - \gamma(t)\right\| \leq c\,\left(\|\bar{\varepsilon}(t)\| + \|\hat{\varepsilon}(t)\| + \|t\|^2\right).$$

Proof of Lemma 3.5. Formulas (4) and (8) allow us to write:

$$\begin{aligned}
\Gamma^Z(t)_{\mathbf{k},\mathbf{l}} - \gamma(t)_{\mathbf{k},\mathbf{l}} &= \left\langle r^{(4)}_{\mathbf{k}}(0)\,t\,, \Delta(t) r^{(4)}_{\mathbf{l}}(0)\,t\right\rangle - \left\langle r^{(3)}_{\mathbf{k}}(t)\,, N_1(t) r^{(3)}_{\mathbf{l}}(t)\right\rangle \\
&= \left\langle r^{(4)}_{\mathbf{k}}(0)\,t - r^{(3)}_{\mathbf{k}}(t)\,, \Delta(t) r^{(4)}_{\mathbf{l}}(0)\,t\right\rangle \\
&\quad + \left\langle r^{(3)}_{\mathbf{k}}(t)\,, \Delta(t)\left(r^{(4)}_{\mathbf{l}}(0)\,t - r^{(3)}_{\mathbf{l}}(t)\right)\right\rangle \\
&\quad + \left\langle r^{(3)}_{\mathbf{k}}(t)\,, \left(\Delta(t) - N_1(t)\right) r^{(3)}_{\mathbf{l}}(t)\right\rangle.
\end{aligned} \tag{9}$$

We now use Taylor expansions to get precise upperbounds. For any $t \in \mathbb{R}^d$, for any $1 \leq \mathbf{k} \leq K$ and for any $1 \leq i,j \leq d$, let us consider the functions $u \in \mathbb{R} \mapsto r^{(3)}_{\mathbf{k}}(ut)$ and $u \in \mathbb{R} \mapsto r''_{i,j}(ut)$. We can write the following Taylor expansions with integral remainders between $u = 0$ and $u = 1$, up to order zero and to order one, respectively.

That yields:

$$\begin{aligned} r^{(3)}_{\mathbf{k}}(t) &= \int_0^1 r^{(4)}_{\mathbf{k}}(ut)\, t\, du \\ r''_{i,j}(t) &= -\delta_{i,j} + \int_0^1 \left\langle r^{(4)}_{i,j}(ut)\, t, t \right\rangle (1-u)\, du. \end{aligned}$$

Hence, using functions $\bar{\varepsilon}$ and $\hat{\varepsilon}$, and the fact that $\Theta(t)_{i,j} = \left\langle r^{(4)}_{i,j}(0)\, t\, , t \right\rangle$, we get

$$\begin{aligned} r^{(4)}_{\mathbf{k}}(0)\, t - r^{(3)}_{\mathbf{k}}(t) &= \bar{\varepsilon}(t)_{\mathbf{k}}\, t \\ r''_{i,j}(t) &= -\delta_{i,j} + \frac{1}{2}\Theta(t)_{i,j} - \langle \hat{\varepsilon}_{i,j}(t)\, t\, , t \rangle. \end{aligned} \tag{10}$$

We denote by $\widehat{E}(t)$ the $d \times d$ matrix such that $\widehat{E}(t)_{i,j} = 2\langle \hat{\varepsilon}_{i,j}(t)\, t\, , t \rangle$, which allows us to rewrite the last equality $r''(t) = -I_d + \frac{1}{2}\Theta(t) - \frac{1}{2}\widehat{E}(t)$. That yields

$$r''(t)^2 = I_d - \Theta(t) + \widehat{E}(t) + O(\|t\|^4), \tag{11}$$

because $\Theta(t) = O(\|t\|^2)$ and $\widehat{E}(t) = o(\|t\|^2)$. Thanks to (10), we rewrite (9) in the following way:

$$\begin{aligned} \Gamma^Z(t)_{\mathbf{k},\mathbf{l}} - \gamma(t)_{\mathbf{k},\mathbf{l}} = & \left\langle \bar{\varepsilon}(t)_{\mathbf{k}}\, t\ , \Delta(t) r^{(4)}_{\mathbf{l}}(0)\, t \right\rangle =: S_1 \\ & + \left\langle r^{(4)}_{\mathbf{k}}(0)\, t - \bar{\varepsilon}(t)_{\mathbf{k}}\, t\, , \Delta(t)\bar{\varepsilon}(t)_{\mathbf{l}}\, t \right\rangle =: S_2 \\ & + \left\langle r^{(4)}_{\mathbf{k}}(0)\, t - \bar{\varepsilon}(t)_{\mathbf{k}}\, t\, , (\Delta(t) - N_1(t))\, (r^{(4)}_{\mathbf{l}}(0)\, t - \bar{\varepsilon}(t)_{\mathbf{l}}\, t) \right\rangle =: S_3. \end{aligned}$$

Let $\rho := \|r^{(4)}(0)\|$. For the following computations, we recall that Θ is continuous and homogeneous of degree 2 on $\mathbb{R}^d$ and that for $t \in \mathbb{R}^d \backslash \{0\}$, $\Delta(t) = \Theta(t)^{-1}$. We introduce $\delta := \max_{v \in \mathbb{S}^{d-1}} \|\Delta(v)\|$. Thanks to Cauchy-Schwarz inequality, we may bound the first term S_1 and the second one S_2 as follows:

$$\begin{aligned} & |S_1| \leq \delta\rho\|\bar{\varepsilon}(t)\|, \\ & |S_2| \leq \delta\|r^{(4)}(0) - \bar{\varepsilon}(t)\|\|\bar{\varepsilon}(t)\| \leq \delta\rho\|\bar{\varepsilon}(t)\| + \delta\|\bar{\varepsilon}(t)\|^2. \end{aligned}$$

We now focus on the third term S_3. In order to bound it, we write a precise expansion of $N_1(t) - \Delta(t)$ around zero, based on formula (11). We have

$$\begin{aligned} N_1(t) &= \left(I_d - r''(t)^2\right)^{-1} = \left(\Theta(t) - \widehat{E}(t) + O(\|t\|^4)\right)^{-1} \\ &= \Delta(t)\, \left(I_d - \widehat{E}(t)\Delta(t) + O(\|t\|^4)\Delta(t)\right)^{-1} \end{aligned}$$

where $\widehat{E}(t)\Delta(t) + O(\|t\|^4)\Delta(t) = \left(\dfrac{\widehat{E}(t)}{\|t\|^2} + O(\|t\|^2)\right)\, \Delta(\dfrac{t}{\|t\|})$, which tends to 0.

If A is a $d \times d$ matrix, $(I_d - A)^{-1} = I_d + A + o(A)$ as $\|A\|$ tends to zero, so we get

$$N_1(t) = \Delta(t) + \Delta(t)\, \widehat{E}(t)\, \Delta(t) + \Delta(t)\, o\Big(\widehat{E}(t)\Delta(t)\Big) + O(1),$$

and hence

$$\|t\|^2\, (N_1(t) - \Delta(t)) = \Delta(\frac{t}{\|t\|})\, \frac{\widehat{E}(t)}{\|t\|^2}\, \Delta(\frac{t}{\|t\|}) + \Delta(\frac{t}{\|t\|})\, o\left(\frac{\widehat{E}(t)}{\|t\|^2}\right)\, \Delta(\frac{t}{\|t\|}) + O(\|t\|^2).$$

Since $\|\widehat{E}(t)\| \le 2\,\|\hat{\varepsilon}(t)\|\,\|t\|^2$ and $t \mapsto \Delta(\frac{t}{\|t\|})$ is bounded, there exists a neighbourhood $\mathcal{V}$ of zero in $\mathbb{R}^d$ and constants $c, d > 0$ such that, for any $t \in \mathcal{V}\backslash\{0\}$,

$$\|t\|^2\,\|N_1(t) - \Delta(t)\| \le c\,\delta^2\,\|\hat{\varepsilon}(t)\| + d\|t\|^2.$$

Consequently, since $\bar{\varepsilon}(t)$ and $\hat{\varepsilon}(t)$ tend to zero as t tends to zero, there exists a neighbourhood $\mathcal{V}'$ of zero in $\mathbb{R}^d$ and $c' > 0$ such that, for any $t \in \mathcal{V}'\backslash\{0\}$,

$$|S_3| \le (\delta + \|\bar{\varepsilon}(t)\|)^2(c\delta^2\|\hat{\varepsilon}(t)\| + d\|t\|^2) \le c'(\|\hat{\varepsilon}(t)\| + \|t\|^2).$$

So, for any $1 \le \mathbf{k}, \mathbf{l} \le K$, $\left|\Gamma^Z(t)_{\mathbf{k},\mathbf{l}} - \gamma(t)_{\mathbf{k},\mathbf{l}}\right|$ is bounded by a term proportional to $\|\bar{\varepsilon}(t)\| + \|\hat{\varepsilon}(t)\| + \|t\|^2$ in a neighbourhood of zero. This concludes the proof of Lemma 3.5. □

Let us now combine all our intermediate results to complete the proof of the theorem. Our aim is to prove that $G(v,\cdot) \in L^1(\mathcal{V}_0, \|t\|^{-d}dt)$ in order to conclude thanks to Lemma 2.1. We recall that Lemma 3.2 also allows us to write that $G(v,t) = Q_{(d)}(\Gamma^Z(t)) + o(\|t\|)$ as t tends to zero. Using Proposition 3.4, we get for $t \neq 0$

$$\begin{aligned} G(v,t) &= Q_{(d)}(\Gamma^Z(t)) - Q_{(d)}(\gamma(t)) + o(\|t\|) \\ &= \left\langle \left(Q_{(d)}\right)'(\gamma(t)), \Gamma^Z(t) - \gamma(t) \right\rangle + o\left(\|\Gamma^Z(t) - \gamma(t)\|\right) + o(\|t\|). \end{aligned}$$

Since $\left\|\left(Q_{(d)}\right)'(\gamma(t))\right\|$ is bounded for t in any compact set of $\mathbb{R}^d$, we deduce from Lemma 3.5 that there exists a neighbourhood of zero $\mathcal{W}$ in $\mathbb{R}^d$ and a positive constant c such that, for any $t \in \mathcal{W}$,

$$G(v,t) \le c\left(\|\bar{\varepsilon}(t)\| + \|\hat{\varepsilon}(t)\| + \|t\|\right).$$

A change of variable easily shows that condition **(G)** implies that $\bar{\varepsilon} \in L^1(\mathcal{V}_0, \|t\|^{-d}dt)$ and the same holds for $\hat{\varepsilon}$. Obviously, we also have $t \mapsto \|t\| \in L^1(\mathcal{V}_0, \|t\|^{-d}dt)$. Consequently, under condition **(G)**, $G(v,\cdot) \in L^1(\mathcal{V}_0, \|t\|^{-d}dt)$. The proof of Theorem 3.1 is complete. □

4. Conclusion and perspectives

In brief, our paper addresses the issue of the finiteness of the variance of $N^{X'}(T,v)$ in dimension $d > 1$, with no assumption of isotropy on X. We do not restrict ourselves to the number of stationary points $N^{X'}(T,0)$. In fact, a sufficient condition is established in Theorem 3.1. It is named **(G)** and appears as a generalization to higher dimensions of Geman condition. As in dimension one, it does not depend on the considered level $v \in \mathbb{R}^d$.

An open question is whether, in dimension $d > 1$, **(G)** remains a necessary condition for $N^{X'}(T,v)$ to admit a second moment. Another natural question concerns the finiteness of the moments of $N^{X'}(T,v)$ of order higher than two. In particular, sufficient conditions on the covariance function of X should be investigated. Note that in [12], the author deals with the higher moments of $N^Y(T,v)$, where $Y : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ is a multivariate random field, and an answer is given through a condition on the spectral density. The latter problem is not the same, but close to ours.

As it is done in dimension one in [11], our work could be extended to the study of the finiteness of the variance of $N^{X'}(T,\phi) := \{t \in T \ : \ X'(t) = \phi(t)\}$, where $\phi : \mathbb{R}^d \to \mathbb{R}^d$ is a function of class $\mathcal{C}^1$. Does our condition remain sufficient with some assumptions on ϕ?

We are convinced that the simple and explicit sufficient condition we have exhibited will be of interest for many applications, especially for statistic purposes.

Let us mention here briefly a specific consequence of our result. Actually, the random variable $N^{X'}(T,0)$ is involved in the computation of another random variable linked to the geometrical properties of X: the Euler characteristic of an excursion set. Let us explain why under our sufficient condition for $N^{X'}(T,0)$ to be in $L^2(\Omega)$, the Euler characteristic of any excursion set also admits a second moment. This question has been raised forty years ago in [1] and is still subject to investigation, see [13] for instance.

To simplify, we assume that T is a d-dimensional compact rectangle and we consider an excursion set of X above level u : $A_u(T) := \{t \in T \ : \ X(t) \geq u\}$. We refer to Section 9.4 in [2] and to [9] for precise definitions and explanations about the next formula. The modified Euler characteristic of the excursion set $A_u(T)$ is defined by

$$\varphi(T,u) = \sum_{i=0}^{d} (-1)^i \mu_i(T,u),$$

where $\mu_i(T,u) = \#\{t \in T \ : \ X(t) \geq u,\, X'(t) = 0,\, \mathrm{index}(X''(t)) = d-i\}$ and the "index" stands for the number of negative eigenvalues. Since each $\mu_i(T,u)$ is bounded by $N^{X'}(T,0)$, it is clear that $\varphi(T,u)$ is square integrable as soon as $N^{X'}(T,0)$ does. So condition **(G)** appears as a sufficient condition.

Appendix A. Computations in dimension $d = 2$

We introduce the following notations:

$$\mu_1 = r^{(4)}_{1111}(0)\ ;\ \nu_1 = r^{(4)}_{1112}(0)\ ;\ \nu = r^{(4)}_{1122}(0)\ ;\ \nu_2 = r^{(4)}_{1222}(0)\ ;\ \mu_2 = r^{(4)}_{2222}(0)\,.$$

Note that in the case $d = 2$, we have $K = 3$ and from now on, we use the lexicographic order to denote the "double" indices $\mathbf{k}$, i.e. $\mathbf{1} = (1,1)\,,\ \mathbf{2} = (1,2)\,,\ \mathbf{3} = (2,2)$.

Matrix $\gamma(t)$. The coefficients of the 3×3 symmetric matrix $\gamma(t)$ are defined by (8). For any $t = (t_1,t_2) \neq (0,0)$, we have

$$\begin{aligned}
&\gamma_{\mathbf{11}}(t) = \tfrac{\alpha}{D(t)}\, t_2^4\,; \quad &&\gamma_{\mathbf{12}}(t) = -\tfrac{\alpha}{D(t)}\, t_1 t_2^3\,; \quad &&\gamma_{\mathbf{13}}(t) = \tfrac{\alpha}{D(t)}\, t_1^2 t_2^2\,;\\
&\gamma_{\mathbf{22}}(t) = \tfrac{\alpha}{D(t)}\, t_1^2 t_2^2\,; \quad &&\gamma_{\mathbf{23}}(t) = -\tfrac{\alpha}{D(t)}\, t_1^3 t_2\,; \quad &&\gamma_{\mathbf{33}}(t) = \tfrac{\alpha}{D(t)}\, t_1^4\,.
\end{aligned}$$

where $\alpha = \mu_1\mu_2\nu - \mu_1\nu_2^2 - \mu_2\nu_1^2 - \nu^3 + 2\nu\nu_1\nu_2$
and $D(t) = \det \begin{pmatrix} \mu_1 t_1^2 + \nu t_2^2 + 2\nu_1 t_1 t_2 & \nu_1 t_1^2 + \nu_2 t_2^2 + 2\nu t_1 t_2 \\ \nu_1 t_1^2 + \nu_2 t_2^2 + 2\nu t_1 t_2 & \nu t_1^2 + \mu_2 t_2^2 + 2\nu_2 t_1 t_2 \end{pmatrix}$.

Polynomial function $Q_{(2)}$. Let $Y = (Y_1, Y_2, Y_3)$ be a centered Gaussian vector. Then

$$\mathbb{E}[\tilde{\det}(Y)^2] = \mathbb{E}[(Y_1Y_3 - (Y_2)^2)^2] = \mathbb{E}[(Y_1Y_3)^2] - 2\mathbb{E}[Y_1Y_3(Y_2)^2] + \mathbb{E}[(Y_2)^4]$$

$$= 2\mathbb{E}[Y_1Y_3]^2 + \mathbb{E}[(Y_1)^2]\mathbb{E}[(Y_3)^2] - 2\mathbb{E}[Y_1Y_3]\mathbb{E}[(Y_2)^2] - 4\mathbb{E}[Y_1Y_2]\mathbb{E}[Y_2Y_3] + 3\mathbb{E}[(Y_2)^2]^2$$

where we have used Wick's formula to get the last line. The polynomial function $Q_{(2)}$ is defined through the identity (7), so

$$Q_{(2)}(\gamma) \quad = \quad 2{\gamma_{\mathbf{13}}}^2 + \gamma_{\mathbf{11}}\gamma_{\mathbf{33}} - 2\gamma_{\mathbf{13}}\gamma_{\mathbf{22}} - 4\gamma_{\mathbf{12}}\gamma_{\mathbf{23}} + 3{\gamma_{\mathbf{22}}}^2.$$

Using the expression of $\gamma(t)$, we recover that $Q_{(2)}(\gamma(t)) = 0\,,\, \forall t \neq 0$.

Case of a separable covariance. Let us focus on the special case where $r(t_1, t_2) = R_1(t_1)R_2(t_2)$, R_1 and R_2 being two one-dimensional covariance functions, each of them of class $\mathcal{C}^4$. Then the fourth derivatives of r are such that $r^{(4)}_{\mathbf{kl}}(0) - r^{(4)}_{\mathbf{kl}}(t) = o(\|t\|)$ for any $(\mathbf{k},\mathbf{l}) \neq (\mathbf{1},\mathbf{1}),(\mathbf{3},\mathbf{3})$. Indeed, for instance $r^{(4)}_{\mathbf{12}}(t) = R_1^{(3)}(t_1)R_2'(t_2)$, and all the $R_i^{(j)}$'s for $i = 1,2$ and $j = 0,1,2,3$ are at least of class $\mathcal{C}^1$. Hence, r satisfies our Geman condition **(G)** if and only if R_1 and R_2 both satisfy the usual one-dimensional Geman condition.

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